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Cyclotomic double affine Hecke algebras

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CYCLOTOMIC DOUBLE AFFINE HECKE ALGEBRAS

BY ALEXANDER BRAVERMAN, PAVEL ETINGOF
AND MICHAEL FINKELBERG
WITH AN APPENDIX BY HIRAKU NAKAJIMA
AND DAISUKE YAMAKAWA

To Ivan Cherednik with admiration

ABSTRACT. – We show that the partially spherical cyclotomic rational Cherednik algebra (obtained from the full rational Cherednik algebra by averaging out the cyclotomic part of the underlying reflection group) has four other descriptions: (1) as a subalgebra of the degenerate DAHA of type A given by generators; (2) as an algebra given by generators and relations; (3) as an algebra of differential-reflection operators preserving some spaces of functions; (4) as equivariant Borel-Moore homology of a certain variety. Also, we define a new q -deformation of this algebra, which we call *cyclotomic DAHA*. Namely, we give a q -deformation of each of the above four descriptions of the partially spherical rational Cherednik algebra, replacing differential operators with difference operators, degenerate DAHA with DAHA, and homology with K-theory, and show that they give the same algebra. In addition, we show that spherical cyclotomic DAHA are quantizations of certain multiplicative quiver and bow varieties, which may be interpreted as K-theoretic Coulomb branches of a framed quiver gauge theory. Finally, we apply cyclotomic DAHA to prove new flatness results for various kinds of spaces of q -deformed quasiinvariants.

RÉSUMÉ. – Nous démontrons que l’algèbre rationnelle cyclotomique de Cherednik partiellement sphérique (obtenue à partir de l’algèbre rationnelle de Cherednik complète en effectuant la moyenne par la partie cyclotomique du groupe de réflexions sous-jacent) admet quatre autres descriptions: (1) comme une sous-algèbre de la DAHA dégénérée de type A donnée par générateurs; (2) comme une algèbre donnée par générateurs et relations; (3) comme une algèbre des opérateurs différentiels-réflexions préservant certains espaces des fonctions; (4) comme l’homologie de Borel-Moore équivariante d’une certaine variété. Aussi nous définissons une nouvelle q -déformation de cette algèbre que nous appelons *DAHA cyclotomique*. À savoir, nous donnons une q -déformation de chacune des descriptions ci-dessus de l’algèbre rationnelle de Cherednik partiellement sphérique, remplaçant les opérateurs différentiels par les opérateurs en différences, DAHA dégénérée par DAHA, et l’homologie par la K -théorie; et démontrons qu’ils donnent lieu à la même algèbre. En outre, nous montrons que les DAHA sphériques cyclotomiques sont les quantifications de certaines variétés de carquois et arc multiplicatives, qui peuvent être interprétées comme les branches de Coulomb K -théoriques d’une théorie de jauge de carquois encadrée. Enfin, nous appliquons la DAHA cyclotomique pour prouver de nouveaux résultats de platitude pour des types différents d’espaces de quasi-invariants q -déformés.

1. Introduction

Let $N \geq 0, l \geq 0$ be integers, $c_0, \dots, c_{l-1}, \hbar, k$ be parameters, and $c = (c_0, \dots, c_{l-1})$. Let $\mathbb{H}_N^{l, \text{cyc}}(c, \hbar, k)$ be the cyclotomic rational Cherednik algebra attached to the complex reflection group $W = S_N \ltimes (\mathbb{Z}/l\mathbb{Z})^N$. Let \mathbf{p} be the symmetrizer of the subgroup $(\mathbb{Z}/l\mathbb{Z})^N$, and $\mathbb{H}_N^{l, \text{psc}}(c, \hbar, k) := \mathbf{p}\mathbb{H}_N^{l, \text{cyc}}(c, \hbar, k)\mathbf{p}$ be the corresponding partially spherical subalgebra.

In this paper we give a geometric interpretation of $\mathbb{H}_N^{l, \text{psc}}$ as the equivariant Borel-Moore homology of a certain variety $\mathcal{R} = \mathcal{R}(N, l)$ equipped with a group action. This allows us to define a natural q -deformation HH_N^l of $\mathbb{H}_N^{l, \text{psc}}$ in terms of the equivariant K-theory of $\mathcal{R}(N, l)$, which we call the *cyclotomic double affine Hecke algebra* (DAHA).

The existence of this q -deformation may seem somewhat surprising from the viewpoint of classical algebraic theory of DAHA ([15]), since typically DAHA are attached to crystallographic reflection groups (Weyl groups), while the group W is not crystallographic for $l \geq 3$. Yet, we also give a purely algebraic definition of cyclotomic DAHA. Namely, we characterize the cyclotomic DAHA as the subalgebra of the usual Cherednik's DAHA for GL_N generated by certain elements, and also as the subalgebra preserving certain spaces of functions. Finally, we present cyclotomic DAHA by generators and relations. These three descriptions also make sense in the trigonometric limit $q \rightarrow 1$ (for partially spherical cyclotomic rational Cherednik algebras). We note that for $l = 1$, cyclotomic DAHA essentially appeared in [3].

We also connect cyclotomic DAHA with multiplicative quiver and bow varieties. Namely, we show that the spherical cyclotomic DAHA $\mathbf{e}HH_N^l(Z, 1, t)\mathbf{e}$ (where \mathbf{e} is the symmetrizer of the finite Hecke algebra) is commutative, and its spectrum for generic parameters is isomorphic to the algebra of regular functions on the multiplicative quiver variety for the cyclic quiver of length l with dimension vector (N, \dots, N) ; hence $\mathbf{e}HH_N^l(Z, q, t)\mathbf{e}$ is a quantization of this variety. In particular, we show that this multiplicative quiver variety is connected, and that $HH_N^l(Z, 1, t)$ is an Azumaya algebra of degree $N!$ over this variety. We also show that if t is not a root of unity then the algebra $\mathbf{e}HH_N^l(Z, 1, t)\mathbf{e}$ is an integrally closed Cohen-Macaulay domain isomorphic to the center $\mathcal{Z}(HH_N^l(Z, 1, t))$, while $HH_N^l(Z, 1, t)\mathbf{e}$ is a Cohen-Macaulay module over this algebra.

Finally, we provide some applications of cyclotomic DAHA to the theory of quasiinvariants. Namely, we show that natural q -deformations of various classes of spaces of quasiinvariants are flat, and therefore free modules over the algebra of symmetric polynomials. We also introduce a new type of quasiinvariants (namely, twisted quasiinvariants) and their q -deformation, and prove the freeness property for them.

We note that the degenerate cyclotomic DAHA were studied in a way similar to ours by R. Kodera and H. Nakajima in [30]. In fact, their paper was one of the starting points for our work.

The paper is organized as follows. In Section 2 we develop the theory of partially spherical cyclotomic rational Cherednik algebras as subalgebras in trigonometric (degenerate) DAHA, and give their presentation. In Section 3 we define cyclotomic DAHA as subalgebras of DAHA, and study their properties. We also give a presentation of cyclotomic DAHA, which allows us to find various bases in them and prove flatness results. In Section 4 we give a

geometric description of cyclotomic DAHA and their degenerate versions in terms of equivariant K-theory and Borel-Moore homology, and apply it to proving flatness of these algebras. In Section 5 we relate the spherical subalgebra of cyclotomic DAHA at $q = 1$ with certain multiplicative quiver and bow varieties; the latter are isomorphic to the K -theoretic Coulomb branch of framed quiver gauge theories of affine type A . We also study the properties of the spectrum of the spherical cyclotomic DAHA for $q = 1$. In Section 6 we give applications of cyclotomic DAHA to proving flatness of q -deformation of various spaces of quasi-invariants. Finally, Appendix , written by H. Nakajima and D. Yamakawa, explains the relations between multiplicative bow varieties and (various versions of) multiplicative quiver varieties for a cyclic quiver.

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2. Degenerate cyclotomic DAHA

2.1. Notation

In this paper, we will consider many different algebras depending on parameters. So let us clarify our conventions.

First of all, if an algebra depends on parameters, we will list the parameters explicitly when they are given numerical values, and omit them when they are indeterminates (i.e., we work over a commutative base algebra generated by them). Also, throughout the paper, we will use the following notation, to be defined below.

- $HH_{N,\text{deg}}(\hbar, k)$: the degenerate (trigonometric) DAHA, Definition 2.1;
- $HH_{N,\text{deg}}^l(z, \hbar, k)$, $z := (z_1, \dots, z_l)$: the degenerate cyclotomic DAHA, Definition 2.8;
- $\mathbb{H}_N^{l,\text{cyc}}(c, \hbar, k)$, $c := (c_0, \dots, c_{l-1})$: the cyclotomic rational Cherednik algebra for the group $S_n \times (\mathbb{Z}/l\mathbb{Z})^n$, Definition 2.16;
- $\mathbb{H}_N^{l,\text{psc}}(c, \hbar, k) := \mathfrak{p}\mathbb{H}_N^{l,\text{cyc}}(c, \hbar, k)\mathfrak{p}$: the partially spherical cyclotomic rational Cherednik algebra, Subsection 2.8;
- $HH_N(q, t)$: Cherednik’s DAHA, Definition 3.1;
- $HH_N^{\text{formal}}(\hbar, k)$: formal Cherednik’s DAHA over $\mathbb{C}[[\varepsilon]]$, with $q = e^{\varepsilon\hbar}$ and $t = e^{-\varepsilon k}$, Subsection 3.1;

- $HH_N^l(Z, q, t)$, $Z := (Z_1, \dots, Z_l)$: the cyclotomic DAHA, Definition 3.9;
- $HH_N^{l, \text{formal}}(z, \hbar, k)$, the formal cyclotomic DAHA, with $q = e^{\varepsilon \hbar}$, $Z_i = e^{\varepsilon z_i}$ and $t = e^{-\varepsilon k}$, Definition 3.9;
- $HH_N^{\text{rat}}(\hbar, k)$, the rational Cherednik algebra for S_N , Example 2.17;
- $\mathcal{HH}_{N, \text{deg}}^l$: the geometric version of the degenerate cyclotomic DAHA, $H_{\bullet}^{\mathbb{C}^\times \times T(W)_{\theta^\times} \mathcal{P} \times \mathbb{C}^\times}(\mathcal{R})$, Section 4;
- \mathcal{HH}_N^l : the geometric version of the cyclotomic DAHA, $K^{\mathbb{C}^\times \times T(W)_{\theta^\times} \mathcal{P} \times \mathbb{C}^\times}(\mathcal{R})$, Section 4.

2.2. Degenerate (trigonometric) DAHA

Let \hbar, k be variables.

DEFINITION 2.1. – The degenerate (or trigonometric) double affine Hecke algebra (DAHA) $HH_{N, \text{deg}}$ is generated over $\mathbb{C}[\hbar, k]$ by $\pi^{\pm 1}, s_0, \dots, s_{N-1}, y_1, \dots, y_N$ with defining relations

$$\begin{aligned} s_i^2 &= 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \text{ if } i - j \neq \pm 1, \quad \pi s_i = s_{i+1} \pi, \\ [y_i, y_j] &= 0, \quad \pi y_i = y_{i+1} \pi \text{ if } i \neq N, \quad \pi y_N = (y_1 - \hbar) \pi, \\ s_i y_i &= y_{i+1} s_i + k, \text{ if } i \neq 0, \quad s_0 y_N = (y_1 - \hbar) s_0 + k, \\ [s_i, y_j] &= 0 \text{ if } i - j \neq \pm 1, \end{aligned}$$

where addition is mod N .

PROPOSITION 2.2. – The algebra $HH_{N, \text{deg}}$ is generated by $S_N \ltimes \mathbb{Z}^N$ (generated by s_i and invertible commuting elements X_1, \dots, X_N) and elements y_1, \dots, y_N with commutation relations

$$\begin{aligned} s_i y_i &= y_{i+1} s_i + k, \quad s_i y_j = y_j s_i, \quad j \neq i, i + 1, \\ [y_i, y_j] &= 0, \\ [y_i, X_j] &= k X_j s_{ij}, \quad i > j, \\ [y_i, X_j] &= k X_i s_{ij}, \quad i < j, \\ [y_i, X_i] &= \hbar X_i - k \sum_{r < i} X_r s_{ir} - k \sum_{r > i} X_i s_{ir}, \end{aligned}$$

and the relations of $S_N \ltimes \mathbb{Z}^N$, where s_{ij} is the transposition of i and j . Namely, the transition between the two definitions is given by the formulas

$$\pi = X_1 s_1 \cdots s_{N-1}, \quad s_0 = X_N^{-1} X_1 s_{1N}.$$

Proof. – The proposition is standard, and the proof is by a direct computation; see e.g., [40], Section 2. \square

REMARK 2.3. – The commutation relations between y_i and X_j in Proposition 2.2 can be replaced by the relations

$$(2.1) \quad [y_i, \prod_j X_j] = \hbar \prod_j X_j \text{ or } [\sum_i y_i, X_j] = \hbar X_j,$$

and

$$(2.2) \quad [y_2, X_1] = kX_1s_1.$$

Indeed, the relation for $[y_i, X_j]$, $i \neq j$ can be obtained from (2.2) by the action of S_N , and then the relation for $[y_i, X_i]$ can be obtained by using one of the relations (2.1).

Note that $HH_{N,\text{deg}}$ is a bigraded algebra: $\deg(X_i) = (1, 0)$, $\deg(y_i) = \deg(\hbar) = \deg(k) = (0, 1)$, $\deg(s_i) = (0, 0)$. Moreover, the PBW theorem for the degenerate DAHA, which follows from the existence of its polynomial representation (see Subsection 2.3) implies that $HH_{N,\text{deg}}$ is a free bigraded module over $\mathbb{C}[\hbar, k]$.

Also, we have the specialization $HH_{N,\text{deg}}(\hbar, k)$ of the degenerate DAHA at $\hbar, k \in \mathbb{C}$, which is defined by the same generators and relations, where \hbar, k are numerical. The bigrading on $HH_{N,\text{deg}}$ induces a grading on this specialization defined by $\deg(X_i) = 1$, $\deg(y_i) = \deg(s_j) = 0$ and an increasing filtration F^\bullet compatible with this grading, defined by $\deg(X_i) = \deg(s_j) = 0$, $\deg(y_i) = 1$.

2.3. The polynomial representation

Let D_i be the *rational Dunkl operators*

$$D_i := \hbar \partial_i - \sum_{j \neq i} \frac{k}{X_i - X_j} (1 - s_{ij}),$$

where ∂_i is the derivative with respect to X_i , and s_{ij} is the permutation of i and j (so $s_i = s_{i,i+1}$). Define the *trigonometric Dunkl operators* by the formula

$$D_i^{\text{trig}} := X_i D_i - k \sum_{j < i} s_{ij}.$$

The following well known proposition is due to Cherednik ([14]; see also [40], Proposition 3.1).

PROPOSITION 2.4. – We have a representation ρ of $HH_{N,\text{deg}}(\hbar, k)$ on $\mathbf{P} := \mathbb{C}[X_1^{\pm 1}, \dots, X_N^{\pm 1}]$, defined by

$$\begin{aligned} \rho(s_i) &= s_i \text{ for } i \neq 0, \quad \rho(\pi) = X_1 s_1 \cdots s_{N-1}, \\ \rho(s_0) &= s_{1N} X_1^{-1} X_N, \\ \rho(y_i) &= D_i^{\text{trig}}. \end{aligned}$$

Proof. – The relations involving only π and s_i are easy. Let us prove that $[\rho(y_i), \rho(y_m)] = 0$ for $i < m$. Using that $[D_i, X_m] = k s_{im}$ and $[D_i, D_m] = 0$, we get

$$[\rho(y_i), \rho(y_m)] = k X_i s_{im} D_m - k X_m s_{im} D_i - k [X_i D_i, s_{im}] + k^2 \sum_{i < j} [s_{ij}, s_{im} + s_{jm}].$$

The first three summands cancel, and the last summand is zero, as desired.

Let us prove the commutation relations between $\rho(\pi)$ and $\rho(y_i)$. For $i < N$ we have

$$\begin{aligned} \rho(\pi)\rho(y_i) &= X_1 s_1 \cdots s_{N-1} (X_i D_i - k \sum_{j < i} s_{ij}) \\ &= X_1 (X_{i+1} D_{i+1} - k \sum_{j < i} s_{j+1,i+1}) s_1 \cdots s_{N-1} \\ &= (X_{i+1} D_{i+1} - k \sum_{j < i} s_{j+1,i+1}) X_1 s_1 \cdots s_{N-1} - k X_{i+1} s_{1,i+1} s_1 \cdots s_{N-1} \\ &= \rho(y_{i+1})\rho(\pi). \end{aligned}$$

Also $\rho(\pi^n) = X_1 \cdots X_n$, so $\rho(\pi^n)\rho(y_i) = (\rho(y_i) - \hbar)\rho(\pi^n)$, which implies that the relation $\pi y_N = (y_1 - \hbar)\pi$ is preserved.

Let us show that the relation $s_i y_i = y_{i+1} s_i + k$ for $i \neq 0$ is preserved. We have

$$\begin{aligned} \rho(s_i)\rho(y_i) &= s_{i,i+1} (X_i D_i - k \sum_{j < i} s_{ij}) \\ &= (X_{i+1} D_{i+1} - k \sum_{j < i} s_{i+1,j}) s_{i,i+1} = \rho(y_{i+1})\rho(s_i) + k. \end{aligned}$$

The relation $s_0 y_N - (y_1 - \hbar)s_0 + k$ is obtained from the previous relation by conjugation by π . Finally, the relation $[s_i, y_j] = 0$ for $i - j \neq \pm 1$ is easy for $i \neq 0$, and for $i = 0$ is obtained from the case $i = 1$ by conjugation by π . The proposition is proved. \square

DEFINITION 2.5. – The representation \mathbf{P} of $HH_{N,\text{deg}}(\hbar, k)$ is called the *polynomial representation*.

It is easy to see that the polynomial representation is faithful when $\hbar \neq 0$. Moreover, replacing $\hbar \partial_i$ with momentum variables p_i , we can make it faithful in the limit $\hbar = 0$ (see [23], 2.10).

Let \mathcal{D}_N be the algebra of differential operators in X_1, \dots, X_N with poles at $X_i = 0$ and $X_i = X_j$. Then the algebra $\mathbb{C}S_N \times \mathcal{D}_N$ acts naturally on $\mathbf{P}[\Delta^{-1}]$, where

$$\Delta := \prod_{i < j} (X_i - X_j),$$

and ρ defines an inclusion

$$HH_{N,\text{deg}}(1, k) \hookrightarrow \mathbb{C}S_N \times \mathcal{D}_N.$$

We will use this inclusion to view $HH_{N,\text{deg}}(1, k)$ as a subalgebra of $\mathbb{C}S_N \times \mathcal{D}_N$. Note that the filtration F^\bullet on $HH_{N,\text{deg}}(1, k)$ is induced under this inclusion by the order filtration on differential operators.

Let ψ_κ be the automorphism of the algebra $\mathbb{C}S_N \times \mathcal{D}_N$ fixing X_i and sending s_i to $-s_i$ and ∂_i to $\partial_i + \sum_{j \neq i} \frac{\kappa}{X_i - X_j}$, i.e., conjugation by $|\Delta|^\kappa \text{sign}(\Delta)$ on the real locus.

LEMMA 2.6. – *The algebra $\psi_{2k}(HH_{N,\text{deg}}(1, k)) \subset \mathbb{C}S_N \times \mathcal{D}_N$ preserves \mathbf{P} . In other words, $HH_{N,\text{deg}}(1, k)$ preserves $|\Delta|^{2k} \text{sign}(\Delta)\mathbf{P}$.*

Proof. – We need to check that

$$\begin{aligned} \psi_{2k}(D_1) &= D_1 - \sum_{j \neq 1} \frac{2k}{X_1 - X_j} \\ &= \partial_1 - \sum_{j \neq 1} \frac{k}{X_1 - X_j} (1 + s_{ij}) + \sum_{j \neq 1} \frac{2k}{X_1 - X_j} \\ &= \partial_1 + \sum_{j \neq 1} \frac{k}{X_1 - X_j} (1 - s_{ij}) \end{aligned}$$

preserves \mathbf{P} , which is straightforward. □

Let $\mathbf{e} = \frac{1}{N!} \sum_{s \in S_N} s$ be the symmetrizer of S_N . The algebra $\mathbf{e}HH_{N,\text{deg}}(\hbar, k)\mathbf{e}$ is called the *spherical subalgebra* of $HH_{N,\text{deg}}(\hbar, k)$; it has the polynomial representation $\mathbf{eP} = \mathbb{C}[X_1^{\pm 1}, \dots, X_N^{\pm 1}]^{S_N}$. Let $H := \mathbf{e}(\sum_i y_i^2)\mathbf{e} \in \mathbf{e}HH_{N,\text{deg}}(\hbar, k)\mathbf{e}$. This element acts on the polynomial representation by the trigonometric Calogero-Moser Hamiltonian.

LEMMA 2.7. – *The algebra $A := \mathbf{e}HH_{N,\text{deg}}(1, k)\mathbf{e}$ is generated by H and $\mathbb{C}[X_1^{\pm 1}, \dots, X_N^{\pm 1}]^{S_N}$.*

Proof. – The algebra A has a filtration F^\bullet given by $\text{deg}(X_i) = \text{deg}(s_j) = 0, \text{deg}(y_i) = 1$, and $\text{gr}(A) = \mathbb{C}[X_1^{\pm 1}, \dots, X_N^{\pm 1}, p_1, \dots, p_N]^{S_N}$ with Poisson bracket corresponding to the symplectic form $\sum_i dp_i \wedge dX_i / X_i$. Since the symbol of H is $H_0 := \sum_i p_i^2$, it suffices to check that $\text{gr}(A)$ is Poisson generated by $\mathbb{C}[X_1^{\pm 1}, \dots, X_N^{\pm 1}]^{S_N}$ and H_0 . For this it suffices to show that the Poisson algebra B generated by these elements contains $F_{r,s} := \sum_i X_i^r p_i^s$, where r is any integer and s a nonnegative integer, as such elements generate $\text{gr}(A)$ as a commutative algebra (by a theorem of H. Weyl). The Poisson bracket is given by $\{p_j, X_i\} = \delta_{ij} X_i$, and X_i are pairwise Poisson commutative, as are p_j . So for $r \neq 0$

$$F_{r,s} = \frac{1}{2r} \left\{ \sum_i p_i^2, F_{r,s-1} \right\},$$

and $F_{r,0} = \sum_i X_i^r \in \mathbb{C}[X_1^{\pm 1}, \dots, X_N^{\pm 1}]^{S_N} \subset B$. So it remains to show that $F_{0,s} \in B$. For this, it suffices to note that

$$F_{0,s} = \frac{1}{s+1} \{F_{-1,s+1}, F_{1,0}\}. \quad \square$$

2.4. Degenerate cyclotomic DAHA

DEFINITION 2.8. – Let $l \in \mathbb{Z}_{\geq 0}$, $z_1, \dots, z_l \in \mathbb{C}$, and $z = (z_1, \dots, z_l)$. The degenerate cyclotomic DAHA is the subalgebra $HH_{N,\text{deg}}^l(z, \hbar, k)$ of $HH_{N,\text{deg}}(\hbar, k)$ generated by $s_i, i = 1, \dots, N - 1, y_i, i = 1, \dots, N, \pi$, and the element

$$\pi_- := \pi^{-1} \prod_{i=1}^l (y_1 - z_i).$$

Similarly, if z_i, \hbar , and k are variables, we define $HH_{N,\text{deg}}^l$ to be the subalgebra of $HH_{N,\text{deg}}[z_1, \dots, z_l]$ generated by the same elements.

Note that by this definition

$$HH_{N,\text{deg}}^0(\hbar, k) = HH_{N,\text{deg}}(\hbar, k), \quad HH_{N,\text{deg}}^{l'}(z', \hbar, k) \subset HH_{N,\text{deg}}^l(z, \hbar, k)$$

if $l' \geq l$ and $z' \supset z$ as a multiset.

For $u \in \mathbb{C}$, let ϕ_u be the automorphism of $\mathbb{C}S_N \rtimes \mathcal{D}_N$ which preserves X_i, s_i , and sends ∂_i to $\partial_i + uX_i^{-1}$ (i.e., conjugation by $(\prod_i X_i)^u$).

PROPOSITION 2.9. – *The algebras $\phi_{z_i}(HH_{N,\text{deg}}^l(z, 1, k))$, $i \in [1, l]$, preserve the subspace $\mathbf{P}_+ := \mathbb{C}[X_1, \dots, X_N] \subset \mathbf{P}$. In other words, the algebra $HH_{N,\text{deg}}^l(z, 1, k)$ preserves $(\prod_j X_j)^{z_i} \mathbf{P}_+$ for all i .*

Proof. – We only need to show that $\pi^{-1}\phi_{z_i}((y_1 - z_1) \cdots (y_l - z_l))$ preserves \mathbf{P}_+ . Note that $y_1 = X_1 D_1$, so $\phi_{z_i}(y_1) = y_1 + z_i$. Thus, we need to check that $\pi^{-1}y_1 \prod_{j \neq i} (y_1 - z_j + z_i)$ preserves \mathbf{P}_+ . But this holds since $\pi^{-1}y_1 = s_{N-1} \cdots s_1 D_1$ preserves \mathbf{P}_+ (and, of course, so does y_1). □

A similar result holds when z_i, k are variables.

THEOREM 2.10. – (i) [27] *If $k \notin \mathbb{Z} + 1/2$ then $HH_{N,\text{deg}}(1, k)$ is the algebra of all elements of the algebra $S_N \rtimes \mathcal{D}_N$ which preserve \mathbf{P} and $|\Delta|^{2k} \text{sign}(\Delta) \mathbf{P}$.*

(ii) *Suppose $z_i - z_j$ are not integers and $k \in \mathbb{C}$ is Weil generic⁽¹⁾ (i.e., outside a countable set). Then the algebra $HH_{N,\text{deg}}^l(z, 1, k)$ is the subalgebra of all elements of $HH_{N,\text{deg}}(1, k)$ which preserve $(\prod_j X_j)^{z_i} \mathbf{P}_+$ for all i .*

(iii) *Under the assumption of (ii), the algebra $HH_{N,\text{deg}}^l(z, 1, k)$ is the subalgebra of all elements of $S_N \rtimes \mathcal{D}_N$ which preserve \mathbf{P} , $|\Delta|^k \text{sign}(\Delta) \mathbf{P}$, and $(\prod_j X_j)^{z_i} \mathbf{P}_+$ for all i .*

Theorem 2.10 is proved in the next two subsections.

2.5. The case $N = 1$

Let us first prove Theorem 2.10 for $N = 1$. Let $B_l(z) := HH_1^l(1, z)$ (the parameter k does not enter in this case). The algebra $B_l(z)$ is generated by $X = X_1$, the Euler element $E := X\partial = y_1$, and

$$L := \pi_- = X^{-1}(X\partial - z_1) \cdots (X\partial - z_l).$$

Then Theorem 2.10 for $N = 1$ reduces to the following statement.

PROPOSITION 2.11. – *If $z_i - z_j$ are not integers for $i \neq j$ then $B_l(z)$ is the algebra of all differential operators on \mathbb{C}^* which preserve $X^{z_i} \mathbb{C}[X]$ for all $i = 1, \dots, l$.*

Proof. – It is clear that every element of $B_l(z)$ preserves $X^{z_i} \mathbb{C}[X]$, since so do X, E and L . So we need to show that any operator M which preserves $X^{z_i} \mathbb{C}[X]$ belongs to $B_l(z)$.

Suppose that M is of degree r , i.e., $M = X^r g(E)$, where g is some polynomial. If $r \geq 0$, then $M \in B_l(z)$, as $E \in B_l(z)$. So it remains to consider the case $r = -q$, where $q > 0$. Then M has to annihilate $X^{z_i}, \dots, X^{z_i+q-1}$ for all i (which are all different thanks to the condition on the z_i), so the degree of g is at least ql . On the other hand, if we take $g(y) = P(y)P(y-1) \cdots P(y-q+1)h(y)$ where $P(y) := (y-z_1) \cdots (y-z_l)$, then $M := X^{-q}g(E) = L^q h(E)$, so $M \in B_l(z)$. This shows that M always belongs to $B_l(z)$

⁽¹⁾ We don't know if Theorem 2.10 (ii), (iii) actually fails for any value of $k \notin \mathbb{Z} + 1/2$.

(as we can subtract $L^q h(E)$ to make the degree of the polynomial $< ql$). The proposition is proved. □

REMARK 2.12. – We have $E = XL + z_1$ for $l = 1$ and $E = \frac{1}{2}([L, X] + z_1 + z_2 - 1)$ for $l = 2$, so one may ask if $B_l(z)$ is in fact generated by L, X (i.e., if E can be expressed via L, X). It is not hard to show that this is indeed the case for generic z_i . But if $l \geq 3$ and z_i are special, then the algebra $B_l(z)$ may not be generated by X, L . Indeed, let $z_1 = 0, z_2 = 1, z_3 = 2$, and the other z_i be arbitrary. Then $B_l(z)$ preserves $\mathbb{C}[X]$ and $X^2\mathbb{C}[X]$, so has a 2-dimensional representation $V = \mathbb{C}[X]/(X^2)$ where L acts by 0 and X acts nilpotently. Hence, if $B'_l(z) \subset B_l(z)$ is the subalgebra generated by L, X then every element of $B'_l(z)$ has only one eigenvalue on V . On the other hand, E has eigenvalues 0, 1 on V , so $E \notin B'_l(z)$.

We will also need the following “unsymmetrized” version of Proposition 2.11 for $l = 2$. Let $A(k)$ be the rational Cherednik algebra with parameter k attached to the group $\mathbb{Z}/2\mathbb{Z}$, i.e., generated by $x, s \in \mathbb{Z}/2\mathbb{Z}$ such that $sx = -xs$, and $D = \partial - \frac{k}{x}(1 - s)$.

LEMMA 2.13. – *If $k \notin \mathbb{Z} + \frac{1}{2}$ then $A(k)$ is the algebra of all elements of $\mathbb{C}\mathbb{Z}/2\mathbb{Z} \rtimes \mathcal{D}(\mathbb{C}^*)$ which preserve $\mathbb{C}[x]$ and $|x|^{2k} \text{sign}(x)\mathbb{C}[x]$.⁽²⁾*

Proof. – It is clear that $A(k)$ preserves these spaces, since so do its generators. So it remains to show that any element M preserving these spaces is in $A(k)$. We may assume that M is homogeneous. Note that $E := \frac{1}{2}(xD + Dx - 1 + 2k) = x\partial \in A(k)$. So if M is of nonnegative degree, then $M = x^r(g_1(E)(1 - s) + g_2(E)(1 + s))$, where g_1, g_2 are some polynomials, hence $M \in A(k)$ automatically. Now suppose $\text{deg}(M) = -q < 0$, i.e., $M = x^{-q}(g_1(E)(1 - s) + g_2(E)(1 + s))$. The operator M has to annihilate x^m and $\text{sign}(x)|x|^{2k}x^m$ for $0 \leq m \leq q - 1$, so $g_1(u)$ is divisible by $\prod_{j=0}^{q-1}(u - j - (1 + (-1)^j)k)$, while $g_2(u)$ is divisible by $\prod_{j=0}^{q-1}(u - j - (1 - (-1)^j)k)$ (here we use that these products have no repeated factors since $k \notin \mathbb{Z} + 1/2$). On the other hand, it is easy to see by acting on monomials x^p that

$$x^{-q} \prod_{j=0}^{q-1} (E - j - (1 + (-1)^j)k) \cdot (1 - s) = D^q(1 - s),$$

while

$$x^{-q} \prod_{j=0}^{q-1} (E - j - (1 - (-1)^j)k) \cdot (1 + s) = D^q(1 + s),$$

which are both in $A(k)$. This implies the lemma. □

⁽²⁾ Lemma 2.13 fails if $k \in \mathbb{Z} + 1/2$. For example, if $k = 1/2$ then the element $(x^{-1}\partial - x^{-2})(1 - s)$ preserves the required spaces but is not in $A(k)$.

2.6. Proof of Theorem 2.10

Let us prove (i). By Lemma 2.13 and Lemma 2.6, $HH_{N,\text{deg}}(1, k)$ is the space of elements of $S_N \times \mathcal{D}_N$ which upon formal completion at a generic point of each hyperplane $X_i = X_j$ (in the sense of [7]) lie in the formal completion of $A(k) \otimes \mathbb{W}_{N-1}$, where \mathbb{W}_{N-1} is the Weyl algebra of $N-1$ variables. So (i) follows from the results of [19] (see also the appendix to [7]).

It is clear that given (i), statements (ii) and (iii) are equivalent, so let us prove (ii) (assuming $l > 0$, as the case $l = 0$ is trivial). First consider the case $k = 0$. In this case, the result follows from the following lemma.

LEMMA 2.14. – *Let $L \in S_N \times \mathcal{D}(\mathbb{C}^*)^{\otimes N}$ be an element preserving the space $(\prod_j X_j)^{z_i} \mathbf{P}_+$ for all i . Then $L \in S_N \times B_l(z)^{\otimes N}$.*

Proof. – Let $L = \sum_{\sigma \in S_N} \sigma L_\sigma$, where L_σ are differential operators. Consider a generic point \mathbf{x} in the hyperplane $X_1 = 0$, and let E_i be the formal completion of the $\mathbb{C}[X_1, \dots, X_N]$ -module $(\prod_j X_j)^{z_i} \mathbf{P}_+$ near the S_N -orbit of \mathbf{x} . Then $E_i = \bigoplus_{\sigma \in S_N} E_{i,\sigma}$, where $E_{i,\sigma}$ is the completion of $(\prod_j X_j)^{z_i} \mathbf{P}_+$ at the point $\sigma \mathbf{x}$. It is clear that L preserves E_i for all i . This implies that for each σ , L_σ preserves $E_{i,1}$. Hence L_σ preserves $(\prod_j X_j)^{z_i} \mathbf{P}_+$ for all i and σ . Thus we may assume without loss of generality that $L \in \mathcal{D}(\mathbb{C}^*)^{\otimes N}$ is a differential operator.

It is clear from taking completions that L preserves the space $X_1^{z_i} \mathbb{C}[X_1, X_2^{\pm 1}, \dots, X_N^{\pm 1}]$ for all i . Therefore, by Proposition 2.11, for any $v \in \mathbb{C}[X_2^{\pm 1}, \dots, X_N^{\pm 1}]$, $\psi \in \mathbb{C}[X_2^{\pm 1}, \dots, X_N^{\pm 1}]^*$ the differential operator $(\text{Id} \otimes \psi)(L(\text{Id} \otimes v)) \in \mathcal{D}(\mathbb{C}^*)$ in fact belongs to $B_l(z)$. Let $\{a_i\}$ be a basis of $B_l(z)$, and $\{a'_j\}$ its extension to a basis of $\mathcal{D}(\mathbb{C}^*)$. We can uniquely write L as

$$L = \sum_i a_i \otimes L_i + \sum_j a'_j \otimes L'_j.$$

Thus we have

$$\sum_i \psi(L_i v) a_i + \sum_j \psi(L'_j v) a'_j \in B_l(z).$$

Hence $\psi(L'_j v) = 0$ for all j . Since this holds for all ψ , we have $L'_j v = 0$. Since this holds for all v , we have $L'_j = 0$. Thus $L = \sum_i a_i \otimes L_i \in B_l(z) \otimes \mathcal{D}(\mathbb{C}^*)^{\otimes N-1}$, i.e., the first component of L lies in $B_l(z)$. A similar argument applies to all the other components. Thus, $L \in B_l(z)^{\otimes N}$, as desired. \square

Now, consider the case $k \neq 0$. By Proposition 2.9, elements of $HH_{N,\text{deg}}^l(1, k)$ preserve the spaces $(\prod_j X_j)^{z_i} \mathbf{P}_+$. Thus we only have to show that $HH_{N,\text{deg}}^l(1, k)$ is “big enough”, i.e., coincides with the algebra $A_N(z, k)$ of all the elements preserving the spaces $(\prod_j X_j)^{z_i} \mathbf{P}_+$. But this follows for Weil generic k from the case $k = 0$ by a standard deformation argument (the algebra can only “get bigger” if we deform its generators). More precisely, recall that we have a grading on the algebra $S_N \times \mathcal{D}_N$ defined by $\deg(X_i) = 1$, $\deg(s_i) = 0$, $\deg(\partial_i) = -1$ inherited by $HH_{N,\text{deg}}(1, k)$, $A_N(z, k)$, and $HH_{N,\text{deg}}^l(z, 1, k)$, and the filtration F^\bullet by order of differential operators, and it is not hard to see that for each s , $F^s A_N(z, k)$ is a finitely generated graded $\mathbb{C}[X_1, \dots, X_N]$ -module. Thus, for each r, s , the space $F^s A_N(z, k)[r]$ is finite dimensional. Hence, for each r, s the set of $k \in \mathbb{C}$ for which $F^s HH_{N,\text{deg}}^l(z, 1, k)[r] \neq F^s A_N(z, k)[r]$ is finite.

2.7. Comparison to the cyclotomic rational Cherednik algebra for $N = 1$

Let us now consider the cyclotomic rational Cherednik algebra $\mathbb{H}_1^{l,\text{cyc}}$ of rank 1 with parameters \hbar and $c = (c_0, \dots, c_{l-1})$ (see [21, 23]. By definition, this algebra is generated over $\mathbb{C}[\hbar, c_0, \dots, c_{l-1}]$ by x and the cyclotomic Dunkl operator

$$D_{\text{cyc}} := \hbar \partial - x^{-1} \sum_{i=0}^{l-1} c_i \sigma^i,$$

where

$$\sigma(x) = \zeta x, \quad \zeta = e^{2\pi i/l}.$$

This algebra is bigraded by $\deg(x) = (1, 0)$, $\deg(D_{\text{cyc}}) = (-1, 1)$, $\deg(\sigma) = (0, 0)$, $\deg(\hbar) = \deg(c) = (0, 1)$, and by the PBW theorem (see [18, 22]) is a free bigraded module over $\mathbb{C}[c, \hbar]$. We also have the algebra $\mathbb{H}_1^{l,\text{cyc}}(c, \hbar)$ with numerical parameters, i.e., the specialization of $\mathbb{H}_1^{l,\text{cyc}}$, and it carries a grading and a compatible filtration F^\bullet .

We have $\mathbb{H}_1^{l,\text{cyc}}(c, 1) \subset \mathbb{C}\mathbb{Z}/l\mathbb{Z} \ltimes \mathcal{D}$, where $\mathcal{D} = \mathcal{D}_1$ is the algebra of differential operators on \mathbb{C}^* , and the filtration F^\bullet on $\mathbb{H}_1^{l,\text{cyc}}(c, 1)$ is induced by the order filtration on differential operators.

Let \mathbf{p} be the symmetrizer of $\mathbb{Z}/l\mathbb{Z}$. Then we have a spherical subalgebra $\mathbb{B}_l(c) := \mathbf{p}\mathbb{H}_1^{l,\text{cyc}}(c, 1)\mathbf{p}$. This algebra acts naturally on $\mathbb{C}[x^{\pm 1}]$, where $X = x^l$.

PROPOSITION 2.15. – *If $z_i - z_j$ are not integers then $B_l(z) = \mathbb{B}_l(c)$ inside $\text{End}\mathbb{C}[x^{\pm 1}]$, where c_i are related to z_j by the linear inhomogeneous change of variables*

$$(2.3) \quad z_i = \frac{1}{l}(l - i + \sum_j c_j \zeta^{ij}).$$

Proof. – Fix c and let us find values of u for which $X^u\mathbb{C}[X^{\pm 1}]$ is preserved by $\mathbb{B}_l(c)$. The condition is that there exists $1 \leq i \leq l$ such that $D_{\text{cyc}}(x^{i-l}(x^l)^u) = 0$, where we treat $(x^l)^u$ as a $\mathbb{Z}/l\mathbb{Z}$ -invariant. This gives the equation

$$i - l + lu - \sum_j c_j \zeta^{ij} = 0,$$

i.e.,

$$u = z_i := \frac{1}{l}(l - i + \sum_j c_j \zeta^{ij}).$$

This yields the desired change of variable.

Now, $\mathbb{B}_l(c)$ preserves the subspaces $X^{z_i}\mathbb{C}[X^{\pm 1}]$, so by Proposition 2.11 we have an inclusion $\mathbb{B}_l(c) \subset B_l(z)$. To show that this is actually an equality, it suffices to observe that the element L of $B_l(z)$ is proportional to $D_{\text{cyc}}^l \mathbf{p}$. □

2.8. Comparison to cyclotomic Cherednik algebra for general N

Let us now extend the result of the previous subsection to general N .

DEFINITION 2.16. – The cyclotomic rational Cherednik algebra for the group $S_N \ltimes (\mathbb{Z}/l\mathbb{Z})^N$, $\mathbb{H}_N^{l,\text{cyc}}$, is the algebra generated over $\mathbb{C}[c_0, \dots, c_{l-1}, \hbar, k]$ by the group $S_N \ltimes (\mathbb{Z}/l\mathbb{Z})^N$, elements x_i , and the cyclotomic Dunkl operators, also called Dunkl-Opdam operators, [18, Definition 3.2]:

$$D_{i,\text{cyc}} = \hbar \partial_i - \frac{1}{x_i} \sum_{j=0}^{l-1} c_j \sigma_i^j - k \sum_{r \neq i,m} \frac{1}{x_i - \zeta^m x_r} (1 - s_{ir} \sigma_i^m \sigma_r^{-m}),$$

for $i = 1, \dots, N$, where σ_i is σ acting in the i -th component.

As in the rank 1 case, this algebra is bigraded by $\deg(x_i) = (1, 0)$, $\deg(D_{i,\text{cyc}}) = (-1, 1)$, $\deg(\sigma_i) = (0, 0)$, $\deg(\hbar) = \deg(c) = \deg(k) = (0, 1)$, and by the PBW theorem (see [18]) is a free bigraded module over $\mathbb{C}[c, \hbar, k]$. We also have the algebra $\mathbb{H}_N^{l,\text{cyc}}(c, \hbar, k)$ with numerical parameters, i.e., the specialization of $\mathbb{H}_N^{l,\text{cyc}}$, and it carries a grading and a compatible filtration F^\bullet .

Let \mathbf{p} be the symmetrizer of the subgroup $(\mathbb{Z}/l\mathbb{Z})^N$, and $\mathbb{H}_N^{l,\text{psc}}(c, \hbar, k) = \mathbf{p} \mathbb{H}_N^{l,\text{cyc}}(c, \hbar, k) \mathbf{p}$ be the corresponding partially spherical subalgebra.

EXAMPLE 2.17. – Let $l = 1$. Then $\mathbb{H}_N^{l,\text{psc}}(c, \hbar, k)$ does not depend on c (up to a natural isomorphism), and is the rational Cherednik algebra $HH_N^{\text{rat}}(\hbar, k)$, generated by X_i, D_i , and $s \in S_N$ such that X_i, D_i are permuted by S_N and satisfy the relations

$$\begin{aligned} [X_i, X_j] &= [D_i, D_j] = 0, \\ [D_i, X_j] &= k s_{ij}, \quad [D_i, X_i] = \hbar - k \sum_{j \neq i} s_{ij}. \end{aligned}$$

Also in this case $\mathbf{p} = 1$.

THEOREM 2.18. – Suppose $z_i - z_j$ are not integers and k is Weil generic. Then we have a natural isomorphism $\mathbb{H}_N^{l,\text{psc}}(c, 1, k) \cong HH_{N,\text{deg}}^l(z, 1, k)$, where z_i are expressed via c_j by Formula (2.3). This isomorphism preserves the order filtration for differential operators.

Proof. – We have a natural faithful action of $\mathbb{H}_N^{l,\text{psc}}(c, 1, k)$ on \mathbf{P} . Moreover, it is easy to see that $\mathbb{H}_N^{l,\text{psc}}(c, 1, k)$ satisfies the conclusion of Theorem 2.10(iii) (this follows by taking formal completions at generic points of reflection hyperplanes, as in [19], and using Proposition 2.15). Therefore, by Theorem 2.10, $\mathbb{H}_N^{l,\text{psc}}(c, 1, k) = HH_{N,\text{deg}}^l(z, 1, k)$. \square

One of our main results is the following theorem.

THEOREM 2.19. – (i) We have a natural isomorphism of bigraded algebras $\theta : \mathbb{H}_N^{l,\text{psc}} \cong HH_{N,\text{deg}}^l$, where z_i are expressed via c_j and \hbar by the homogenization of Formula (2.3):

$$(2.4) \quad z_i = \frac{1}{l} (\hbar(l - i) + \sum_j c_j \zeta^{ij}).$$

(ii) $HH_{N,\text{deg}}^l$ is a free bigraded $\mathbb{C}[z_1, \dots, z_l, \hbar, k]$ -module, and $HH_{N,\text{deg}}^l(z, \hbar, k)$ are specializations of this algebra.

(iii) *The isomorphism θ induces an isomorphism*

$$\theta_{z,\hbar,k} : \mathbb{H}_N^{l,\text{psc}}(c, \hbar, k) \cong HH_{N,\text{deg}}^l(z, \hbar, k)$$

for all $z_i, \hbar, k \in \mathbb{C}$, which is compatible with the grading and the filtration.

Proof. – Let $\text{gr}(HH_{N,\text{deg}}^l(z, 1, k)) = HH_{N,\text{deg}}^l(0, 0, 0)$ be the associated graded algebra of $HH_{N,\text{deg}}^l(z, 1, k)$ with respect to the filtration by order of differential operators. Then $\text{gr}(HH_{N,\text{deg}}^l(z, 1, k))$ contains $S_N \times A^{\otimes N}$, where A is the algebra of functions on the A_{l-1} -singularity, generated by X, XP , and $X^{-1}(XP)^l$ (where P is the symbol of ∂). On the other hand, $\text{gr}(\mathbb{H}_N^{l,\text{psc}}(c, 1, k))$ clearly coincides with $S_N \times A^{\otimes N}$. Since by Theorem 2.18, $\mathbb{H}_N^{l,\text{psc}}(c, 1, k) = HH_{N,\text{deg}}^l(z, 1, k)$ for Weil generic parameters, we conclude that $\text{gr}(HH_{N,\text{deg}}^l(z, \hbar, k)) = S_N \times A^{\otimes N}$ for all z_i, \hbar, k . This implies Theorem 2.19. \square

REMARK 2.20. – 1. Since the degenerate DAHA has a \mathbb{G}_a -action given by $y_i \mapsto y_i + a$ and trivial on other generators, the algebra $HH_{N,\text{deg}}^l(z, 1, k)$ does not change under the transformation $z_i \rightarrow z_i + a$ (i.e., it depends only on the differences $z_i - z_{i+1}$). Under the isomorphism of Theorem 2.19, this symmetry transforms into the symmetry $c_0 \mapsto c_0 + la$.

2. Another proof of Theorem 2.19(ii) is given in the next subsection.

REMARK 2.21. – The isomorphism of spherical subalgebras of $\mathbb{H}_N^{l,\text{psc}}$ and $HH_{N,\text{deg}}^l$ is due to Kodera and Nakajima [30]. More precisely, the parameters of the cyclotomic rational Cherednik algebra in [30] are related to ours by the formula $\hbar^{KN} = -\hbar, c_m^{KN} = c_m(1 - \zeta^m)$ for $m \neq 0$ (and c_0^{KN} is not used in [30]). Also in [30], one has $\sum_{m=0}^{l-1} c_m = 0$ and thus $z_l = 0$, which is not really a restriction due to the symmetry $z_i \mapsto z_i + a, c_0 \mapsto c_0 + la$.

Also, this isomorphism is closely related to the new presentation of the full cyclotomic DAHA $\mathbb{H}_N^{l,\text{cyc}}(c, \hbar, k)$ given in [28, 3.3] and [44, Section 2]. Namely, the element $e'x_1^{\ell-1}\sigma e'$ in [44, (4.1)] (which in terms of the pictures is wrapping around the cylinder $\ell - 1$ times, and a little further so as to cross the seam ℓ times) matches up with our element π_- , and similarly $e'y_n^{\ell-1}\tau e'$ matches with our element π (with ℓ in [44] equal to our l).

2.9. A presentation of $HH_{N,\text{deg}}^l$ by generators and relations

Let us give a presentation of $HH_{N,\text{deg}}^l$ by generators and relations. As generators we will use the elements of $S_N, y_1, \dots, y_N, X_1, \dots, X_N$, and the elements

$$D_i^{(l)} := s_{1i} D_1^{(l)} s_{1i}, \text{ where } D_1^{(l)} := X_1^{-1}(y_1 - z_1) \cdots (y_1 - z_l).$$

Obviously, these elements generate $HH_{N,\text{deg}}^l$, so we only need to write down the relations.

First of all, the elements $s \in S_N, y_i$ and X_i satisfy the relations of Proposition 2.2, except that X_i are no longer invertible.

We also claim that

$$(2.5) \quad [D_i^{(l)}, D_j^{(l)}] = 0.$$

Indeed, it suffices to check it for $i = 1, j = 2$. Then we have

$$\begin{aligned} D_1^{(l)} D_2^{(l)} &= X_1^{-1} (y_1 - z_1) \cdots (y_1 - z_l) s_{12} X_1^{-1} (y_1 - z_1) \cdots (y_1 - z_l) s_{12} \\ &= X_1^{-1} s_{12} (y_2 + k s_{12} - z_1) \cdots (y_2 + k s_{12} - z_l) X_1^{-1} (y_1 - z_1) \cdots (y_1 - z_l) s_{12} \\ &= X_1^{-1} X_2^{-1} s_{12} (y_2 - z_1) \cdots (y_2 - z_l) (y_1 - z_1) \cdots (y_1 - z_l) s_{12}. \end{aligned}$$

This expression commutes with s_{12} , which implies the statement.

We also see by direct computation that

$$[y_1, D_1^{(l)}] = -\hbar D_1^{(l)} + k \sum_{i>1} s_{1i} D_1^{(l)}$$

and

$$[y_j, D_1^{(l)}] = -k s_{1j} D_1^{(l)}, \quad j > 1.$$

Finally, we write down the commutation relations between $D_i^{(l)}$ and X_j .

LEMMA 2.22. – *We have*

$$[D_1^{(l)}, X_1] = \sum_{r=1}^l \prod_{i=1}^{r-1} (y_1 - z_i + \hbar - k \sum_{j>1} s_{1j}) (\hbar - k \sum_{j>1} s_{1j}) \prod_{i=r+1}^l (y_1 - z_i),$$

and for $m > 1$

$$[D_1^{(l)}, X_m] = k \sum_{r=1}^l \prod_{i=1}^{r-1} (y_1 - z_i + \hbar - k \sum_{j>1} s_{1j}) s_{1m} \prod_{i=r+1}^l (y_1 - z_i).$$

Proof. – The first relation holds because

$$[D_1^{(l)}, X_1] = P(y_1 + \hbar - k \sum_{j>1} s_{1j}) - P(y_1),$$

where $P(y) = (y - z_1) \cdots (y - z_l)$. To prove the second relation, note that

$$\begin{aligned} [D_1^{(l)}, X_m] &= X_1^{-1} (P(y_1) - P(y_1 - k s_{1m})) X_m \\ &= k \sum_{r=1}^l X_1^{-1} \prod_{i=1}^{r-1} (y_1 - z_i) s_{1m} \prod_{i=r+1}^l (y_1 - z_i - k s_{1m}) X_m \\ &= k \sum_{r=1}^l X_1^{-1} \prod_{i=1}^{r-1} (y_1 - z_i) X_1 s_{1m} \prod_{i=r+1}^l (y_1 - z_i) \\ &= k \sum_{r=1}^l \prod_{i=1}^{r-1} (y_1 - z_i + \hbar - k \sum_{j>1} s_{1j}) s_{1m} \prod_{i=r+1}^l (y_1 - z_i), \end{aligned}$$

as desired. \square

Note that the commutation relations between $D_i^{(l)}$ and X_m for $i > 1$ can now be obtained by conjugating the relations of Lemma 2.22 by S_N .

PROPOSITION 2.23. – Let $l \geq 1$. Let M_X be a monomial in X_i , M_D a monomial in $D_i^{(l)}$, M_y a monomial in y_i with degrees of all the y_i at most $l - 1$, and $s \in S_N$. Then the elements of the form $M_X M_y s M_D$ form a basis in $HH_{N,\text{deg}}^l$; in particular, $HH_{N,\text{deg}}^l$ is a free $\mathbb{C}[\hbar, k, z_1, \dots, z_l]$ -module.

Proof. – It is easy to see by looking at the polynomial representation that these elements are linearly independent, so we only need to establish the spanning property. Since the generators are monomials of this form, it suffices to show that any (unordered) monomial in $s, X_i, y_i, D_i^{(l)}$ can be reduced to a linear combination of such standard monomials.

Let us introduce a filtration by setting $\text{deg}(S_N) = 0, \text{deg}(y_i) = 2, \text{deg}(X_i) = \text{deg}(D_i^{(l)}) = l$. Using the above commutation relations, we can reduce any monomial to ordered form by adding corrections of lower degree. Further, by using the relation

$$X_1 D_1^{(l)} = P(y_1)$$

and its conjugates, we can reduce powers of y_i to $0, \dots, l - 1$. This implies the statement. \square

Thus, we obtain the following proposition.

PROPOSITION 2.24. – The degenerate cyclotomic DAHA $HH_{N,\text{deg}}^l$ is generated by S_N and elements $y_i, X_i, D_i, i = 1, \dots, N$, with the following defining relations:

$$\begin{aligned} s_i y_i &= y_{i+1} s_i + k, s_i y_j = y_j s_i, j \neq i, i + 1, \\ [y_i, y_j] &= 0, \\ s X_i &= X_{s(i)} s, s \in S_N, [X_i, X_j] = 0, \\ [y_i, X_1] &= k X_1 s_{1i}, i > 1, \\ [y_1, X_1] &= \hbar X_1 - k \sum_{i>1} X_1 s_{1i}, \\ s D_i &= D_{s(i)} s, [D_i, D_j] = 0, \\ [y_j, D_1] &= -k s_{1j} D_1, j > 1, \\ [y_1, D_1] &= -\hbar D_1 + k \sum_{i>1} s_{1i} D_1, \\ [D_1, X_1] &= \sum_{r=1}^l \prod_{i=1}^{r-1} (y_1 - z_i + \hbar - k \sum_{j>1} s_{1j}) (\hbar - k \sum_{j>1} s_{1j}) \prod_{i=r+1}^l (y_1 - z_i), \\ [D_1, X_m] &= k \sum_{r=1}^l \prod_{i=1}^{r-1} (y_1 - z_i + \hbar - k \sum_{j>1} s_{1j}) s_{1m} \prod_{i=r+1}^l (y_1 - z_i), m > 1, \\ X_1 D_1 &= (y_1 - z_1) \cdots (y_1 - z_l). \end{aligned}$$

Proof. – This follows from Proposition 2.23 (with $D_i = D_i^{(l)}$). \square

REMARK 2.25. – It is easy to check using this presentation that we have an involution ϕ on $HH_{N,\text{deg}}^l$ given by

$$\phi(\hbar) = -\hbar, \phi(k) = -k, \phi(X_i) = D_i, \phi(D_i) = X_i, \phi(s_{ij}) = s_{ij},$$

$$\phi(y_i) = y_i + \hbar - k \sum_{j \neq i} s_{ij}.$$

The existence of this involution is also clear from the isomorphism of $HH_{N,\text{deg}}^l$ with $\mathbb{H}\mathbb{H}_N^{l,\text{psc}}$, since the latter algebra is well known to have such an involution (coming from the corresponding involution of the cyclotomic rational Cherednik algebra exchanging the coordinates with the Dunkl operators).

The proof of Proposition 2.23 in fact shows that for any $l \geq 0$ ordered products of M_X, M_y, s, M_D in any of the 24 possible orders are a spanning set for $HH_{N,\text{deg}}^l$, and those of them with degrees of y_i at most $l - 1$ are a basis for $l \geq 1$. This implies that we also have another basis of this algebra, formed by monomials $M_X M_D s M_y$ without restriction on the degree of y_i , but with the restriction that for each i either X_i is missing in M_X or D_i is missing in M_D . Indeed, if this restriction is not satisfied, we may use the relation $X_1 D_1 = P(y_1)$ and its permutations to lower the number of X_i and D_i , and it is easy to see by looking at the polynomial representation that monomials with this restriction are linearly independent. Thus we obtain the following proposition.

PROPOSITION 2.26. – *The elements $M_X M_D$ which miss either X_i or D_i for each i form a basis of $HH_{N,\text{deg}}^l$ as a left or right module over the degenerate affine Hecke algebra $H_{N,\text{deg}}$ generated by $s \in S_N$ and $y_i, 1 \leq i \leq n$; in particular, $HH_{N,\text{deg}}^l$ is a free module over this subalgebra.*

Note that the basis of Proposition 2.26 is labeled by N -tuples of integers, (m_1, \dots, m_N) . Namely, if $M_X M_D$ contains X_i^p then we set $m_i = p$, and if it contains D_i^p then we set $m_i = -p$.

We note that Propositions 2.23 and 2.26 also follow from Theorem 2.19. Yet another, geometric proof of Proposition 2.26 will be given in Section 4.

3. Cyclotomic DAHA

3.1. DAHA and formal DAHA

Recall the definition of Cherednik's double affine Hecke algebra (DAHA), [15]. Let $q, \mathbf{t} \in \mathbb{C}^*$, and $t = \mathbf{t}^2$.

DEFINITION 3.1. – The DAHA $HH_N(q, t)$ is generated by invertible elements $X_i, Y_i, i = 1, \dots, N$, and $T_i, i = 1, \dots, N - 1$, with relations⁽³⁾

$$(T_i - \mathbf{t})(T_i + \mathbf{t}^{-1}) = 0, \tag{R1}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \tag{R2}$$

$$T_i T_j = T_j T_i \ (|i - j| \geq 2), \tag{R3}$$

$$T_i X_i T_i = X_{i+1}, \tag{R4}$$

$$T_i X_j = X_j T_i \ (j \neq i, i + 1), \tag{R5}$$

⁽³⁾ This algebra really depends on \mathbf{t} rather than $t = \mathbf{t}^2$, but it is traditional to use the parameter t , implying that a square root of this parameter has been chosen, see [15]. While somewhat clumsy, this convention turns out to be more natural from the viewpoint of Macdonald theory.

$$T_i Y_i T_i = Y_{i+1}, \tag{R6}$$

$$T_i Y_j = Y_j T_i \quad (j \neq i, i + 1), \tag{R7}$$

$$X_1^{-1} Y_2^{-1} X_1 Y_2 = T_1^2, \tag{R8}$$

$$Y_i \tilde{X} = q \tilde{X} Y_i, \tag{R9}$$

$$X_i \tilde{Y} = q^{-1} \tilde{Y} X_i, \tag{R10}$$

$$[X_i, X_j] = 0, \tag{R11}$$

$$[Y_i, Y_j] = 0. \tag{R12}$$

where $\tilde{X} := \prod_i X_i$ and $\tilde{Y} = \prod_i Y_i$.

We can define the element

$$T_0 := T_1^{-1} \cdots T_{N-1}^{-1} \cdots T_1^{-1} X_1^{-1} X_N$$

which together with $T_i, i = 1, \dots, N - 1$ generates the affine Hecke algebra of type A_{N-1} in the Coxeter presentation (i.e., relations (R1),(R2) are satisfied for all $i, j \in \mathbb{Z}/N\mathbb{Z}$).

Similarly one defines the algebra HH_N over $\mathbb{C}[q^{\pm 1}, \mathbf{t}^{\pm 1}]$.

We will also consider a formal version of DAHA over $\mathbb{C}[[\varepsilon]]$, in which $q = e^{\varepsilon \hbar}$ and $t = e^{-\varepsilon k}$ for $k \in \mathbb{C}$. Namely, set $T_i = s_i e^{-\varepsilon k s_i / 2}$, $Y_i = e^{\varepsilon y_i}$, and let $HH_N^{\text{formal}}(\hbar, k)$ be the ε -adically complete algebra generated over $\mathbb{C}[[\varepsilon]]$ by s_i, X_i and y_i with the relations of Definition 3.1. We can also treat \hbar, k as indeterminates, working over $\mathbb{C}[\hbar, k]$.

Note that using (R6), relation (R8) can be written as

$$X_1^{-1} T_1^{-1} Y_1^{-1} T_1^{-1} X_1 T_1 Y_1 T_1 = T_1^2,$$

or

$$X_1 T_1 Y_1 = T_1 Y_1 T_1 X_1 T_1. \tag{R8a}$$

This shows that HH_N has the Cherednik involution φ defined by $\varphi(q) = q^{-1}, \varphi(\mathbf{t}) = \mathbf{t}^{-1}, \varphi(X_i) = Y_i^{-1}, \varphi(Y_i) = X_i^{-1}, \varphi(T_i) = T_i^{-1}$.

3.2. The quasiclassical limit of the formal DAHA

The following proposition is well known; for instance, in the rank 1 case it appears in [15].

PROPOSITION 3.2. – *The algebra $HH_N^{\text{formal}}(\hbar, k)/(\varepsilon)$ is isomorphic to the trigonometric DAHA $HH_{N, \text{deg}}(\hbar, k)$, and $HH_N^{\text{formal}}(\hbar, k)$ is a flat deformation of $HH_{N, \text{deg}}(\hbar, k)$.*

Proof. – To prove the first statement, we need to show that the DAHA relations of Definition 3.1 degenerate to the relations of the trigonometric DAHA.

Clearly, relation (R1) yields $s_i^2 = 1$, and relations (R2,R3) yield $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ for $1 \leq i, j \leq N - 1$ and $|i - j| \geq 2$. Relations (R4,R5) give $s_i X_j = X_j s_i$ if $j \neq i, i + 1$, and $s_i X_i = X_{i+1} s_i$. Relation (R6) gives a trivial relation in zeroth order, but in the first order it gives

$$-k s_i + s_i y_i s_i = y_{i+1},$$

which yields

$$s_i y_i = y_{i+1} s_i + k.$$

Relation (R7) gives

$$[s_i, y_j] = 0, \quad j \neq i, i + 1.$$

Relation (R8) yields

$$y_2 - X_1^{-1} y_2 X_1 = -k s_1,$$

which is equivalent to

$$[y_2, X_1] = k X_1 s_1.$$

Relations (R9,R10) yield

$$[y_i, \prod_j X_j] = \hbar \prod_j X_j, \quad [\sum_i y_i, X_j] = \hbar X_j.$$

Finally, relations (R11, R12) yield

$$[X_i, X_j] = 0, \quad [y_i, y_j] = 0.$$

It is easy to see that these relations are exactly the relations of $HH_{N,\text{deg}}(\hbar, k)$ given in Proposition 2.2 (see Remark 2.3).

The second statement of the proposition follows from the first one and the PBW theorem for DAHA ([15]). \square

3.3. The polynomial representation of DAHA

PROPOSITION 3.3. – [15] *We have an action of $HH_N(q, t)$ on \mathbf{P} given by*

$$\begin{aligned} \rho(X_i) &= X_i, \\ \rho(T_i) &= \mathbf{t} s_i + \frac{\mathbf{t} - \mathbf{t}^{-1}}{X_i/X_{i+1} - 1} (s_i - 1), \\ \rho(Y_i) &= \mathbf{t}^{N-1} \rho(T_i^{-1} \cdots T_{N-1}^{-1}) \omega \rho(T_1 \cdots T_{i-1}), \end{aligned}$$

where $(\omega f)(X_1, \dots, X_N) := f(qX_N, \dots, X_{N-1})$.

The same formulas define a representation of $HH_N^{\text{formal}}(\hbar, k)$. The representation ρ of DAHA on \mathbf{P} is called the *polynomial representation* of DAHA.

PROPOSITION 3.4. – *The quasiclassical limit (i.e., reduction modulo ε) of the polynomial representation of $HH_N^{\text{formal}}(\hbar, k)$ coincides with the polynomial representation of the trigonometric DAHA given by Proposition 2.4.*

Proof. – The proof is by a direct calculation. \square

3.4. The cyclotomic DAHA

Let $\pi \in HH_N(q, t)$ be the element given by the formula

$$\pi = X_1 T_1 \cdots T_{N-1}.$$

Let $Z_1, \dots, Z_l \in \mathbb{C}^*$, and $Z = (Z_1, \dots, Z_l)$.

DEFINITION 3.5. – The subalgebra $HH_N^l(Z, q, t)$ of $HH_N(q, t)$ is generated by $T_i, i = 1, \dots, N - 1, Y_i, i = 1, \dots, N, \pi$, and the element

$$\pi_- := \pi^{-1} \prod_{i=1}^l (Y_1 - Z_i).$$

Let $z_1, \dots, z_l \in \mathbb{C}$, and $Z_i = q^{z_i}$ (for some choice of branches).

PROPOSITION 3.6. – The algebra $HH_N^l(Z, q, t)$ preserves the space $(\prod_j X_j)^{z_i} \mathbf{P}_+$ for all i .

Proof. – We only need to check that π_- preserves this space. For this, it is enough to prove that for any $u \in \mathbb{C}$, the element $X_1^{-1}(Y_1 - q^u)$ preserves the space $(\prod_j X_j)^u \mathbf{P}_+$. To this end, note that

$$\rho(T_i^{-1}) = \mathbf{t}^{-1} s_i + \frac{(\mathbf{t}^{-1} - \mathbf{t}) X_i}{X_{i+1} - X_i} (s_i - 1).$$

Now consider

$$\begin{aligned} (3.1) \quad \rho(Y_1) &= \mathbf{t}^{N-1} \rho(T_1^{-1}) \cdots \rho(T_{N-1}^{-1}) \omega \\ &= \left(1 + \frac{(1-t)X_1}{X_2 - X_1} (1 - s_{12})\right) \cdots \left(1 + \frac{(1-t)X_1}{X_N - X_1} (1 - s_{1N})\right) \tau_1, \end{aligned}$$

where τ_j replaces X_j with qX_j and keeps X_i fixed for $i \neq j$. By opening the brackets, this product can be written as a sum of 2^N terms (as in each factor, we can take the first or the second summand). If we take the first summand from all factors, we get τ_1 , and $X_1^{-1}(\tau_1 - q^u)$ clearly preserves $(\prod_j X_j)^u \mathbf{P}_+$. So it suffices to show that for each of the remaining $2^N - 1$ terms T , the operator $X_1^{-1}T$ preserves $(\prod_j X_j)^u \mathbf{P}_+$. But all of these terms have a factor X_1 on the left (as so does the second summand in each factor in (3.1)), which implies the desired statement. \square

Let $HH_N^{l,\text{formal}}(z, \hbar, k)$ be the formal version of $HH_N^l(Z, q, t)$, namely the subalgebra of $HH_N^{\text{formal}}(\hbar, k)$ generated by $T_i, i = 1, \dots, N - 1, y_i, i = 1, \dots, N, \pi$, and the element

$$\varepsilon^{-l} \pi^{-1} \prod_{i=1}^l (Y_1 - e^{\varepsilon z_i}) = \pi^{-1} \prod_{i=1}^l \frac{e^{\varepsilon y_1} - e^{\varepsilon z_i}}{\varepsilon}.$$

COROLLARY 3.7. – Let $z_i - z_j \notin \mathbb{Z}$, and k be Weil generic. Then the algebra $HH_N^{l,\text{formal}}(z, 1, k)$ is the algebra of all elements of $HH_N^{\text{formal}}(1, k)$ which preserve $(\prod_j X_j)^{z_i} \mathbf{P}_+$ for all i .

Proof. – Recall that by Proposition 3.2, $HH_N^{\text{formal}}(\hbar, k)$ is a flat deformation of $HH_{N, \text{deg}}(\hbar, k)$, and by Proposition 3.4, the same applies to the polynomial representations of these algebras. Thus the result follows by a deformation argument from Proposition 3.6 and the fact that a similar statement holds in the trigonometric case (Theorem 2.10(ii)). Namely, the algebra $HH_N^{l, \text{formal}}(z, 1, k)$ is a priori “at least as big” as $HH_{N, \text{deg}}^l(z, 1, k)$, as its generators are deformations of generators of $HH_{N, \text{deg}}^l(z, 1, k)$. At the same time, the subalgebra of elements of $HH_N^{l, \text{formal}}(z, 1, k)$ preserving $(\prod_j X_j)^{z_i} \mathbf{P}_+$ for all i is “at most as big” as $HH_{N, \text{deg}}^l(z, 1, k)$, as by Theorem 2.10(ii), this condition cuts out $HH_{N, \text{deg}}^l(z, 1, k)$ inside $HH_{N, \text{deg}}(1, k)$. But by Theorem 3.6, the former subalgebra is contained in the latter one. This implies the corollary. \square

THEOREM 3.8. – *For any z_1, \dots, z_l , the algebra $HH_N^{l, \text{formal}}(z, \hbar, k)$ is a flat deformation of $HH_{N, \text{deg}}^l(z, \hbar, k)$.*

Proof. – Since $HH_N^{l, \text{formal}}(z, \hbar, k)$ is generated by deformations of generators of $HH_{N, \text{deg}}^l(z, \hbar, k)$, it suffices to prove this statement for Weil generic z_i, \hbar, k , but in this case it follows from Corollary 3.7. \square

Another proof of Theorem 3.8 is obtained from the presentation of HH_N^l given below.

DEFINITION 3.9. – The algebra $HH_N^l(Z, q, t)$ is called the *cyclotomic DAHA* and $HH_N^{l, \text{formal}}(z, \hbar, k)$ is called the *formal cyclotomic DAHA*.

As usual, one can similarly define versions of these algebras where the parameters are indeterminates. Note also that by this definition $HH_N^0(q, t) = HH_N(q, t)$, and $HH_N^{l'}(Z', q, t) \subset HH_N^l(Z, q, t)$ if $l' \geq l$ and $Z' \supset Z$ as a multiset.

Thus, the cyclotomic DAHA is a q -deformation of the partly spherical cyclotomic rational Cherednik algebra. More precisely, it follows from Theorem 2.19 and Theorem 3.8 that for any $z_1, \dots, z_l, \hbar, k$, the formal cyclotomic DAHA $HH_N^{l, \text{formal}}(z, \hbar, k)$ is a flat deformation of $\mathbb{H}_N^{l, \text{psc}}(c, \hbar, k)$, where c is related to z by Equation (2.4). In particular, the cyclotomic DAHA is interesting already for $l = 1$, as it provides a q -deformation of the rational Cherednik algebra $\mathbb{H}_N^{\text{rat}}(\hbar, k)$ attached to S_N and its permutation representation.

REMARK 3.10. – Since the DAHA has a \mathbb{G}_m -action given by $Y_i \mapsto aY_i$ and trivial on other generators, the algebra $HH_N^l(Z, q, t)$ does not change under the transformation $Z_i \rightarrow aZ_i$ (i.e., it depends only on the ratios Z_i/Z_{i+1}).

EXAMPLE 3.11. – Let $N = 1$. Then there is no dependence on t , and $HH_1(q, t) = HH_1(q)$ is the quantum torus algebra with invertible generators X, Y and relation $YX = qXY$. Let q be not a root of unity. The polynomial representation is $\mathbf{P} = \mathbb{C}[X^{\pm 1}]$ with Y acting by shift, $(Yf)(X) = f(qX)$, so that $HH_1(q)$ is the algebra of polynomial q -difference operators. The subalgebra $HH_1^1(Z, q, t) = HH_1^1(Z, q)$ inside $HH_1(q)$ is generated by $X, Y^{\pm 1}$, and

$$L := X^{-1}(Y - q^{z_1}) \cdots (Y - q^{z_l}),$$

where $Z_i = q^{z_i}$. We claim that if $Z_i/Z_j \notin q^{\mathbb{Z}}$ then $HH_1^1(Z, q)$ is exactly the subalgebra of all difference operators preserving $X^{z_i} \mathbb{C}[X]$ for all i . Indeed, if we set $\deg(Y) = 0$, $\deg(X) = 1$, then any difference operator of nonnegative degree is in both subalgebras,

while an operator M of degree $-d < 0$ has the form $X^{-d}g(Y)$, where g is a Laurent polynomial, and applying this operator to X^{z_i+j} , $j < d$, we must get zero, so we get $g(q^{z_i+j}) = 0$, $i = 1, \dots, l, j = 0, \dots, d - 1$. Thus $M = L^d h(Y)$ for some polynomial h .

REMARK 3.12. – We expect that Corollary 3.7 holds in the non-formal setting, i.e., $HH_N^l(Z, q, t)$ can be characterized as the algebra of elements of $HH_N(q, t)$ preserving the spaces $(\prod X_j)^{z_i} \mathbf{P}_+$, as in Proposition 3.6. For $N = 1$ this is demonstrated in Example 3.11. Moreover, recall that Ginzburg, Kapranov, and Vasserot ([27]) characterized DAHA (for any Weyl group W) as the algebra of difference-reflection operators $L = \sum_{w \in W} L_w w$ (where L_w are difference operators) satisfying some residue conditions. These conditions are equivalent to the conditions that L preserves \mathbf{P} and $\Delta_{q,t} \mathbf{P}$, where $\Delta_{q,t}$ is an appropriate meromorphic function. Therefore, we expect that $HH_N^l(Z, q, t)$ can be characterized as the algebra of difference operators preserving the spaces \mathbf{P} , $\Delta_{q,t} \mathbf{P}$, and $(\prod X_j)^{z_i} \mathbf{P}_+$ for $i = 1, \dots, l$.

We note that this approach to DAHA-type algebras in the more general elliptic setting is developed in the ongoing work [38]. In the one-variable case $N = 1$ such (spherical) algebras generated by a given set of difference-reflection operators have been studied in [39].

3.5. The case $l = 1$

Let us study the algebra HH_N^l for $l = 1$ and give its presentation. These results can be derived from the case of general l considered below, but the special case $l = 1$ is especially nice, and it is instructive to do it separately first.

By rescaling Y_i by the same scalar, we may assume without loss of generality that $Z_1 = 1$. Then the algebra $HH_N^1(q, t) := HH_N^1(1, q, t)$ is generated inside $HH_N(q, t)$ by $T_i, X_i, Y_i^{\pm 1}$, and $X_1^{-1}(Y_1 - 1)$.

This algebra actually appeared a long time ago in the paper [3]. Let us describe it in more detail, following [3]. (We note that our conventions are slightly different from those of [3]).

Define the Dunkl elements $D_i := T_{i-1}^{-1} \cdots T_1^{-1} D_1 T_1^{-1} \cdots T_{i-1}^{-1} \in HH_N^1(q, t)$, where $D_1 = X_1^{-1}(Y_1 - 1)$. It is easy to check that

$$D_i = T_i D_{i+1} T_i, [T_i, D_j] = 0 \text{ for } j \neq i, i + 1.$$

LEMMA 3.13. – We have $[D_1, D_2] = 0$.

Proof. – By definition, we have $D_2 = T_1^{-1} D_1 T_1^{-1}$, so our job is to show that

$$T_1^{-1} D_1 T_1^{-1} D_1 = D_1 T_1^{-1} D_1 T_1^{-1}.$$

In other words, we must show that

$$T_1^{-1} X_1^{-1} (Y_1 - 1) T_1^{-1} X_1^{-1} (Y_1 - 1) = X_1^{-1} (Y_1 - 1) T_1^{-1} X_1^{-1} (Y_1 - 1) T_1^{-1}.$$

Since $T_1 X_1 T_1 X_1 = X_2 X_1 = X_1 X_2 = X_1 T_1 X_1 T_1$, it suffices to prove two identities in $HH_N(q, t)$:

$$(3.2) \quad T_1^{-1} X_1^{-1} Y_1 T_1^{-1} X_1^{-1} Y_1 = X_1^{-1} Y_1 T_1^{-1} X_1^{-1} Y_1 T_1^{-1}$$

and

(3.3)

$$T_1^{-1}X_1^{-1}T_1^{-1}X_1^{-1}Y_1 + T_1^{-1}X_1^{-1}Y_1T_1^{-1}X_1^{-1} = X_1^{-1}T_1^{-1}X_1^{-1}Y_1T_1^{-1} + X_1^{-1}Y_1T_1^{-1}X_1^{-1}T_1^{-1}.$$

Identity (3.2) actually holds already in the braid group. Indeed, we have

$$\begin{aligned} T_1^{-1}X_1^{-1}Y_1T_1^{-1}X_1^{-1}Y_1 &= T_1^{-1}X_1^{-1}T_1^{-1}Y_2T_1^{-2}X_1^{-1}Y_1 = T_1^{-1}X_1^{-1}T_1^{-1}X_1^{-1}Y_2Y_1 \\ &= X_1^{-1}T_1^{-1}X_1^{-1}T_1^{-1}Y_2Y_1 = X_1^{-1}T_1^{-1}X_1^{-1}Y_2Y_1T_1^{-1} \\ &= X_1^{-1}T_1^{-1}Y_2T_1^{-2}X_1^{-1}Y_1T_1^{-1} = X_1^{-1}Y_1T_1^{-1}X_1^{-1}Y_1T_1^{-1}. \end{aligned}$$

It remains to establish (3.3). Note that because $T_1^{-1} = T_1 - \mathbf{t} + \mathbf{t}^{-1}$, (3.3) is equivalent to

(3.4)

$$T_1X_1^{-1}T_1^{-1}X_1^{-1}Y_1 + T_1^{-1}X_1^{-1}Y_1T_1^{-1}X_1^{-1} = X_1^{-1}T_1^{-1}X_1^{-1}Y_1T_1 + X_1^{-1}Y_1T_1^{-1}X_1^{-1}T_1^{-1}.$$

On the other hand, (3.4) holds already in the group algebra of the braid group (i.e., termwise). Indeed, we have

$$\begin{aligned} T_1^{-1}X_1^{-1}Y_1T_1^{-1}X_1^{-1} &= T_1^{-1}X_1^{-1}T_1^{-1}Y_2T_1^{-2}X_1^{-1} = T_1^{-1}X_1^{-1}T_1^{-1}X_1^{-1}Y_2 \\ &= X_1^{-1}T_1^{-1}X_1^{-1}T_1^{-1}Y_2 = X_1^{-1}T_1^{-1}X_1^{-1}Y_1T_1, \end{aligned}$$

and

$$\begin{aligned} T_1X_1^{-1}T_1^{-1}X_1^{-1}Y_1 &= X_1^{-1}T_1^{-1}X_1^{-1}T_1Y_1 = X_1^{-1}T_1^{-1}X_1^{-1}Y_2T_1^{-1} \\ &= X_1^{-1}T_1^{-1}Y_2T_1^{-2}X_1^{-1}T_1^{-1} = X_1^{-1}Y_1T_1^{-1}X_1^{-1}T_1^{-1}. \quad \square \end{aligned}$$

COROLLARY 3.14. – *One has $[D_i, D_j] = 0$ for all i, j .*

Proof. – For any $j > 1$, we have

$$[D_1, D_j] = [D_1, T_{j-1}^{-1} \cdots T_2^{-1}D_2T_2^{-1} \cdots T_{j-1}^{-1}] = 0,$$

since D_1 commutes with every factor by Lemma 3.13. Hence for $i < j$,

$$[D_i, D_j] = [T_{i-1}^{-1} \cdots T_1^{-1}D_1T_1^{-1} \cdots T_{i-1}^{-1}, D_j] = 0,$$

again because D_j commutes with every factor. □

Thus, T_i and D_i generate the “positive part” of an affine Hecke algebra.

When $q \rightarrow 1$ and $t = q^{-k}$, where k is fixed, the algebra $HH_N^1(q, t)$ degenerates to the rational Cherednik algebra $HH_N^{\text{rat}}(1, k)$ for S_N (associated to the permutation representation), and $X_i, \frac{D_i}{q-1}$ degenerate to the standard generators of $HH_N^{\text{rat}}(1, k)$. Thus, let us compute the commutation relations between D_i and X_j which deform the corresponding relations of $HH_N^{\text{rat}}(1, k)$.

LEMMA 3.15. – *One has*

$$X_1D_2 = D_2T_1^2X_1 + (\mathbf{t} - \mathbf{t}^{-1})T_1^{-1}$$

and

$$D_1X_2 = X_2T_1^{-2}D_1 - (\mathbf{t} - \mathbf{t}^{-1})T_1.$$

Proof. – We have

$$D_2 X_1^{-1} = T_1^{-1} X_1^{-1} Y_1 T_1^{-1} X_1^{-1} - T_1^{-1} X_1^{-1} T_1^{-1} X_1^{-1}.$$

Thus, by the proof of Lemma 3.13, we obtain

$$\begin{aligned} D_2 X_1^{-1} &= X_1^{-1} T_1^{-1} X_1^{-1} Y_1 T_1 - X_1^{-1} T_1^{-1} X_1^{-1} T_1^{-1} \\ &= X_1^{-1} D_2 T_1^2 + X_1^{-1} T_1^{-1} X_1^{-1} T_1 - X_1^{-1} T_1^{-1} X_1^{-1} T_1^{-1} \\ &= X_1^{-1} D_2 T_1^2 + (\mathbf{t} - \mathbf{t}^{-1}) X_1^{-1} T_1^{-1} X_1^{-1}. \end{aligned}$$

Now the first relation of the lemma is obtained by multiplying both sides by X_1 on the left and on the right. To obtain the second relation of the lemma from the first one, it suffices to multiply the first relation by T_1 on both sides, and apply the commutation relations between D_i , X_i and T_j . \square

COROLLARY 3.16. – *If $i < j$, one has*

$$X_i D_j = D_j T_{j-1} \cdots T_{i+1} T_i^2 T_{i+1}^{-1} \cdots T_{j-1}^{-1} X_i + (\mathbf{t} - \mathbf{t}^{-1}) T_{j-1}^{-1} \cdots T_i^{-1} \cdots T_{j-1}^{-1}.$$

If $i > j$, one has

$$D_j X_i = X_i T_{i-1}^{-1} \cdots T_{j+1}^{-1} T_j^{-2} T_{j+1} \cdots T_{i-1} D_j - (\mathbf{t} - \mathbf{t}^{-1}) T_{i-1} \cdots T_j \cdots T_{i-1}.$$

Proof. – First consider the case $i = 1$. Then

$$X_1 D_j = X_1 T_{j-1}^{-1} \cdots T_2^{-1} D_2 T_2^{-1} \cdots T_{j-1}^{-1} = T_{j-1}^{-1} \cdots T_2^{-1} X_1 D_2 T_2^{-1} \cdots T_{j-1}^{-1}.$$

By the first relation of Lemma 3.15, this implies

$$\begin{aligned} X_1 D_j &= T_{j-1}^{-1} \cdots T_2^{-1} D_2 T_1^2 X_1 T_2^{-1} \cdots T_{j-1}^{-1} + (\mathbf{t} - \mathbf{t}^{-1}) T_{j-1}^{-1} \cdots T_2^{-1} T_1^{-1} T_2^{-1} \cdots T_{j-1}^{-1} \\ &= D_j T_{j-1} \cdots T_2 T_1^2 T_2^{-1} \cdots T_{j-1}^{-1} X_1 + (\mathbf{t} - \mathbf{t}^{-1}) T_{j-1}^{-1} \cdots T_2^{-1} T_1^{-1} T_2^{-1} \cdots T_{j-1}^{-1}, \end{aligned}$$

as claimed. Now consider the general case. We have

$$\begin{aligned} X_i D_j &= T_{i-1} \cdots T_1 X_1 T_1 \cdots T_{i-1} D_j = T_{i-1} \cdots T_1 X_1 D_j T_1 \cdots T_{i-1} \\ &= T_{i-1} \cdots T_1 D_j T_{j-1} \cdots T_2 T_1^2 T_2^{-1} \cdots T_{j-1}^{-1} X_1 T_1 \cdots T_{i-1} \\ &\quad + (\mathbf{t} - \mathbf{t}^{-1}) T_{i-1} \cdots T_1 T_{j-1}^{-1} \cdots T_2^{-1} T_1^{-1} T_2^{-1} \cdots T_{j-1}^{-1} T_1 \cdots T_{i-1} \\ &= D_j T_{j-1} \cdots T_{i+1} T_i^2 T_{i+1}^{-1} \cdots T_{j-1}^{-1} X_1 + (\mathbf{t} - \mathbf{t}^{-1}) T_{j-1}^{-1} \cdots T_{i+1}^{-1} T_i^{-1} T_{i+1}^{-1} \cdots T_{j-1}^{-1} \end{aligned}$$

(here we repeatedly used the braid relations between the T_i). This proves the first relation of the corollary.

The second relation is proved similarly, using the second relation of Lemma 3.15. \square

Finally, let us generalize the commutation relations between D_i and X_i (which are the only relations containing q). It turns out that it is convenient to write instead the commutation relations between D_i and $\sum_{j=1}^N X_j$.

LEMMA 3.17. – *We have*

$$[D_i, \sum_{j=1}^N X_j] = (1 - q^{-1})(D_i X_i + 1) T_{i-1}^{-1} \cdots T_1^{-2} \cdots T_{i-1}^{-1}.$$

Proof. – First assume $i = 1$. Looking at the polynomial representation and using that $\sum_{j=1}^N X_j$ is symmetric, we find that

$$[D_1, \sum_{j=1}^N X_j] = (1 - q^{-1})(D_1 X_1 + 1).$$

The general case is now obtained by multiplying both sides by $T_{i-1}^{-1} \cdots T_1^{-1}$ on the left and by $T_1^{-1} \cdots T_{i-1}^{-1}$ on the right. □

We obtain the following theorem.

THEOREM 3.18. – (i) *Let $HH_N^{1,+}(q, t)$ be the subalgebra of $HH_N^1(q, t)$ generated by T_i , $1 \leq i \leq N - 1$, X_i, D_i , $1 \leq i \leq N$. Then the defining relations for $HH_N^{1,+}(q, t)$ are*

$$\begin{aligned} (T_i - \mathbf{t})(T_i + \mathbf{t}^{-1}) &= 0, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \\ T_i T_j &= T_j T_i \quad (|i - j| \geq 2), \\ T_i X_i T_i &= X_{i+1}, \quad [T_i, X_j] = 0 && \text{for } j \neq i, i + 1, \\ D_i &= T_i D_{i+1} T_i, \quad [T_i, D_j] = 0 && \text{for } j \neq i, i + 1, \\ [X_i, X_j] &= 0, \\ [D_i, D_j] &= 0, \\ X_i D_j &= D_j T_{j-1} \cdots T_{i+1} T_i^2 T_{i+1}^{-1} \cdots T_{j-1}^{-1} X_i + (\mathbf{t} - \mathbf{t}^{-1}) T_{j-1}^{-1} \cdots T_i^{-1} \cdots T_{j-1}^{-1}, \quad i < j, \\ D_j X_i &= X_i T_{i-1}^{-1} \cdots T_{j+1}^{-1} T_j^{-2} T_{j+1} \cdots T_{i-1} D_j - (\mathbf{t} - \mathbf{t}^{-1}) T_{i-1} \cdots T_j \cdots T_{i-1}, \quad i > j, \\ [D_i, \sum_{j=1}^N X_j] &= (1 - q^{-1})(D_i X_i + 1) T_{i-1}^{-1} \cdots T_1^{-2} \cdots T_{i-1}^{-1}. \end{aligned}$$

(ii) *(the PBW theorem) For any values of parameters, the elements $\prod_i X_i^{m_i} \cdot T_w \cdot \prod_i D_i^{n_i}$ form a basis of $HH_N^{1,+}(q, t)$.*

(iii) *The algebra $HH_N^1(q, t)$ is obtained from $HH_N^{1,+}(q, t)$ by inverting the element $Y_1 := 1 + X_1 D_1$.*

Proof. – We have shown in Lemma 3.14, Corollary 3.16, Lemma 3.17 that the claimed relations between T_i, X_i, D_i are satisfied. These relations allow us to order any monomial as claimed in (ii). Since $HH_N^{1,\text{formal}}$ is a flat deformation of the rational Cherednik algebra $HH_N^{\text{rat}}(\hbar, k)$, the relations in (i) are defining, and the monomials in (ii) are linearly independent. This implies both (i) and (ii).

Now, by definition, $HH_N^1(q, t)$ is generated inside $HH_N(q, t)$ by $HH_N^{1,+}(q, t)$ and Y_1^{-1} , where $Y_1 = 1 + X_1 D_1$. This implies (iii). □

The following proposition shows that there is a symmetry between X_i and D_i . Let us regard q and \mathbf{t} as variables.

PROPOSITION 3.19. – *There is an involutive automorphism ϕ of HH_N^1 such that $\phi(X_i) = D_i$, $\phi(D_i) = X_i$, $\phi(T_i) = T_i^{-1}$, $\phi(\mathbf{t}) = \mathbf{t}^{-1}$, $\phi(q) = q^{-1}$. This automorphism preserves the subalgebra $HH_N^{1,+}$.*

Proof. – It is easy to see that all the relations of Theorem 3.18 except the last one are invariant under ϕ . It is sufficient to impose the last relation for $i = 1$, so let us consider this case. The relation has the form

$$\begin{aligned} q^{-1}(D_1 X_1 + 1) - (X_1 D_1 + 1) &= (\mathbf{t} - \mathbf{t}^{-1}) \sum_{i>1} (X_i T_{i-1}^{-1} \cdots T_2^{-1} T_1^{-1} T_2 \cdots T_{i-1} D_1 + T_{i-1} \cdots T_1 \cdots T_{i-1}) \\ &= (\mathbf{t} - \mathbf{t}^{-1}) \left(\sum_{i>1} T_{i-1} \cdots T_1 \cdots T_{i-1} \right) (X_1 D_1 + 1), \end{aligned}$$

i.e.,

$$D_1 X_1 + 1 = q(1 + (\mathbf{t} - \mathbf{t}^{-1}) \sum_{i>1} T_{i-1} \cdots T_1 \cdots T_{i-1}) (X_1 D_1 + 1).$$

Using braid relations, we see that $T_i \cdots T_1 \cdots T_i = T_1 \cdots T_i \cdots T_1$, so this can be also written as

$$D_1 X_1 + 1 = q(1 + (\mathbf{t} - \mathbf{t}^{-1}) \sum_{i>1} T_1 \cdots T_{i-1} \cdots T_1) (X_1 D_1 + 1).$$

Now let $J_N := T_1 \cdots T_{N-1}^2 \cdots T_1$ be the braid-like Jucys-Murphy element. Then, using the quadratic relation for T_{N-1} , we get

$$J_N = (\mathbf{t} - \mathbf{t}^{-1}) T_1 \cdots T_{N-1} \cdots T_1 + J_{N-1}.$$

This yields

$$J_N = 1 + (\mathbf{t} - \mathbf{t}^{-1}) \sum_{i>1} T_1 \cdots T_{i-1} \cdots T_1,$$

so our relation can be simplified to

$$(3.5) \quad D_1 X_1 + 1 = q J_N (X_1 D_1 + 1).$$

The invariance of this relation under ϕ reduces to the identity $J_N \phi(J_N) = 1$, which is obvious from the definitions of ϕ and J_N . This shows that ϕ is a well defined automorphism of $HH_N^{1,+}$, which is obviously involutive.

We also see from (3.5) that $\phi(Y_1) = q J_N Y_1$, which implies that ϕ extends to HH_N^1 . \square

COROLLARY 3.20. – (i) *The last defining relation of $HH_N^{1,+}(q, t)$ may be replaced with*

$$\left[X_i, \sum_{j=1}^N D_j \right] = (1 - q)(X_i D_i + 1) T_{i-1} \cdots T_1^2 \cdots T_{i-1}.$$

(ii) *The last defining relation of $HH_N^{1,+}(q, t)$ may be replaced with (3.5).*

Proof. – (i) follows immediately from Theorem 3.18, and (ii) from its proof. \square

3.6. A commutative subalgebra in HH_N^l

In this subsection, we will construct a commutative subalgebra inside HH_N^l . In the case $l = 1$, this subalgebra will reduce to the subalgebra $\mathbb{C}[D_1, \dots, D_N]$ constructed in the previous subsection (so we will obtain an alternative construction of this subalgebra).

Let $f \in \mathbb{C}[X]$ be any polynomial. Define the elements

$$Y_i(f) := Y_i T_{i-1}^{-1} \cdots T_1^{-1} f(X_1^{-1}) T_1 \cdots T_{i-1} \in HH_N.$$

Also let \mathbf{e} be the symmetrizer of the finite Hecke algebra generated by the T_i . (To define \mathbf{e} , we need to invert $[N]_t!$).

LEMMA 3.21. – (i) *The elements $Y_i(f)$ are pairwise commuting:*

$$[Y_i(f), Y_j(f)] = 0.$$

(ii) *For $r \geq 1$ the element $\mathcal{M}_r(f) = (\sum_{i=1}^N e_r(Y_1(f), \dots, Y_N(f)))\mathbf{e}$ (where e_r is the r -th elementary symmetric function) commutes with T_i , i.e., $\mathcal{M}_r(f) = \mathbf{e}\mathcal{M}_r(f)$.*

Proof. – (i) We may specialize the variables to numerical values. Let us compute the action of $Y_i(f)$ in the polynomial representation. Using Proposition 3.3, we get

$$\rho(Y_i(f)) = \mathbf{t}^{N-1} \rho(T_i^{-1} \cdots T_{N-1}^{-1}) \omega f(X_1) \rho(T_1 \cdots T_{i-1}).$$

Let $g(X)$ be a meromorphic function such that $g(q^{-1}X) = g(X)f(X^{-1})$. Let us extend the polynomial representations to meromorphic functions, and conjugate it by the function $G(X_1, \dots, X_N) := g(X_1) \cdots g(X_N)$. This gives a representation ρ_G such that

$$\begin{aligned} \rho_G(Y_i) &= G \rho(Y_i) G^{-1} = \mathbf{t}^{N-1} \rho(T_i^{-1} \cdots T_{N-1}^{-1}) G \omega G^{-1} \rho(T_1 \cdots T_{i-1}) \\ &= \mathbf{t}^{N-1} \rho(T_i^{-1} \cdots T_{N-1}^{-1}) \omega \frac{G(q^{-1}X_1)}{G(X_1)} \rho(T_1 \cdots T_{i-1}) \\ &= \mathbf{t}^{N-1} \rho(T_i^{-1} \cdots T_{N-1}^{-1}) \omega f(X_1^{-1}) \rho(T_1 \cdots T_{i-1}) = \rho(Y_i(f)). \end{aligned}$$

Thus $\rho(Y_i(f))$ are pairwise commuting, hence so are $Y_i(f)$.

(ii) It suffices to show that $\mathcal{M}_r(f)$ maps symmetric polynomials to symmetric ones. But this follows from the fact that $\rho(\mathcal{M}_r(f)) = \rho_G(\mathcal{M}_r)$, where $\mathcal{M}_r = \sum_{i=1}^N e_r(Y_1, \dots, Y_N)$, and \mathcal{M}_r act in the polynomial representation by Macdonald difference operators, [15]. \square

Now let $f(X) = (X - Z_1) \cdots (X - Z_l)$. Let us apply the Cherednik involution φ to the elements $Y_i(f)$. Namely, let

$$D_i^{(l)} := \varphi(Y_i(f)) = T_{i-1}^{-1} \cdots T_1^{-1} D_1^{(l)} T_1^{-1} \cdots T_{i-1}^{-1} \in HH_N,$$

where, $D_1^{(l)} := X_1^{-1}(Y_1 - Z_1) \cdots (Y_1 - Z_l)$.

COROLLARY 3.22. – (i) *The elements $D_i^{(l)}$, $i = 1, \dots, N$, are pairwise commuting.*

(ii) *The elements $\mathbf{M}_r(f) := \varphi(\mathcal{M}_r(f))$ commute with T_i .*

Proof. – This follows from Lemma 3.21. \square

EXAMPLE 3.23. – Let $l = 1$ and $f(X) = X - 1$. Then we get $D_i^{(1)} = D_i$. Thus, we recover the q -deformed Dunkl operators of [3] described in the previous subsection and therefore obtain another proof of Corollary 3.14.

Corollary 3.22(ii) implies that the elements $\mathbf{M}_r(f)$ act on symmetric functions by some commuting symmetric difference operators $M_r(f) =: M_r^{(l)}$. Thus for each l we obtain a family of quantum integrable systems $\{M_1^{(l)}, \dots, M_N^{(l)}\}$ depending on l parameters Z_1, \dots, Z_l (and also q, t). This system is a q -deformation of the cyclotomic Calogero-Moser system.

EXAMPLE 3.24 ([3], Lemma 5.3). – We have

$$M_1^{(1)} = \sum_{j=1}^N \left(\prod_{i \neq j} \frac{X_i - tX_j}{X_i - X_j} \right) \frac{1}{X_j} (\tau_j - 1).$$

Thus, this operator defines a quantum integrable system.

REMARK 3.25. – J. F. Van Diejen and S. Ruijsenaars have explained to us that the system defined by the operator $M_1 := M_1^{(1)}$ from Example 3.24 may be obtained as a limit of the system from [17] defined by the Hamiltonian (3.13a). Namely, conjugating by the Gaussian $\exp(\sum_i \frac{(\log X_i)^2}{2 \log q})$ and rescaling X_i and M_1 , we can reduce the operator M_1 to the form

$$M'_1 = \sum_{j=1}^N \left(\prod_{i \neq j} \frac{X_i - tX_j}{X_i - X_j} \right) \tau_j - \sum_{j=1}^N X_j^{-1} = M - \sum_{j=1}^N X_j^{-1},$$

where M is the first Macdonald operator. (Here we use the identity at the beginning of [17, p. 1621]). On the other hand, let us multiply the Hamiltonian [17, (3.13a)] by \hat{t}_0 and then put $\hat{t}_1 = \hat{t}_0^{-1}$ and send \hat{t}_0 to 0. Conjugating the resulting operator by an appropriate function, and again using the identity at the beginning of [17, p. 1621], one obtains the operator M'_1 .

REMARK 3.26. – It is interesting to compute the joint eigenfunctions of $D_1^{(l)}, \dots, D_N^{(l)}$ and symmetric joint eigenfunctions of $M_1^{(l)}, \dots, M_N^{(l)}$. For $l = 1$, this is done in [3]. Let us sketch how this can be done for any l in the nonsymmetric case (without working out any details).

For simplicity, assume that $Z_1 \cdots Z_l = (-1)^l$ (this can be assumed without loss of generality, as we can simultaneously rescale the Z_i). Then $f(0) = 1$, so for $q > 1$ we can set $g(X) = \prod_{m=1}^{\infty} f(q^{-m} X^{-1})$. Given a collection $\Lambda = (\lambda_1, \dots, \lambda_N)$ of eigenvalues, let $F(X_1, \dots, X_N, \Lambda)$ be the joint eigenfunction of Y_i (i.e., the nonsymmetric Macdonald function). Then

$$\tilde{F}(X_1, \dots, X_N, \Lambda) := g(X_1) \cdots g(X_N) F(X_1, \dots, X_N, \Lambda)$$

is a joint eigenfunction of $Y_i(f)$. So the joint eigenfunctions of $D_i^{(l)}$ can be obtained by applying Cherednik’s difference Fourier transform ([15]) to $\tilde{F}(X_1, \dots, X_N, \Lambda)$:

$$(3.6) \quad \hat{F}(X_1, \dots, X_N, \Lambda) = \mathcal{F}_{\text{Cherednik}}(\tilde{F}(X_1, \dots, X_N, \Lambda)).$$

For $l = 1$ this should recover the formulas of [3].

Also, the following observation was made by O. Chalykh when we sent him a preliminary version of this paper. Consider the case where $Z_1 = \dots = Z_l = 0$ (this violates our restriction that $Z_i \in \mathbb{C}^*$, but this restriction is not essential here). In this case, $f(X) = X^l$, and one can take $g(X)$ to be the l -th power of the Gaussian $g(X) = \exp(\frac{(\log X)^2}{2 \log q})$ (this function is not single-valued but this is not important if we restrict to the locus $q, X \in \mathbb{R}_+$).

Therefore, the symmetrized version of Formula (3.6) turns into formula (7.5) in O. Chalykh’s appendix to [10] (up to changing q to q^{-1}). This shows that the Hamiltonians of the twisted Macdonald-Ruijsenaars model of Theorem 7.1 in the appendix to [10] (for type A_{N-1}) are a special case of the commuting Hamiltonians $e_r(D_1^{(l)}, \dots, D_N^{(l)})$ when $Z_1 = \dots = Z_l = 0$.

3.7. Presentation of cyclotomic DAHA by generators and relations

Let us give a presentation of $HH_N^l(Z, q, t)$ by generators and relations. We will use as generators the elements $T_i, i = 1, \dots, N - 1$, and $X_i, Y_i^{\pm 1}, D_i^{(l)}, i = 1, \dots, N$.

First of all, the elements $T_i, X_i, Y_i^{\pm 1}$ satisfy the relations of DAHA, (R1-R12) from Definition 3.1. More precisely, we need to rewrite relations (R8-R10) to account for non-invertibility of X_i . Namely, from (R8-R10) we get

$$(3.7) \quad X_i Y_j = Y_j X_i T_{j-1}^{-1} \cdots T_{i+1}^{-1} T_i^2 T_{i+1} \cdots T_{j-1}, \quad i < j.$$

Similarly, we have

$$(3.8) \quad Y_i X_j = T_{j-1}^{-1} \cdots T_{i+1}^{-1} T_i^2 T_{i+1} \cdots T_{j-1} X_j Y_i, \quad i < j.$$

Finally, we have

$$(3.9) \quad Y_i T_{i-1}^{-1} \cdots T_1^{-2} \cdots T_{i-1}^{-1} X_i = q X_i T_i \cdots T_{N-1}^2 \cdots T_i Y_i.$$

Secondly, we have similar relations between $D_j^{(l)}$ (instead of X_j) and T_i, Y_i . Namely, we know from Corollary 3.22 that

$$(3.10) \quad [D_i^{(l)}, D_j^{(l)}] = 0,$$

and it follows from the definition that

$$(3.11) \quad T_i^{-1} D_i^{(l)} T_i^{-1} = D_{i+1}^{(l)}, \quad [T_j, D_i^{(l)}] = 0 \text{ for } |i - j| \geq 2.$$

Also since $X_1^{-1} Y_2 = Y_2 T_1^{-2} X_1^{-1}$, we have

$$D_1^{(l)} Y_2 = Y_2 T_1^{-2} D_1^{(l)}.$$

This implies that

$$(3.12) \quad D_i^{(l)} Y_j = Y_j T_{j-1}^{-1} \cdots T_{i+1}^{-1} T_i^{-2} T_{i+1} \cdots T_{j-1} D_i^{(l)}, \quad i < j.$$

Similarly, we have

$$D_2^{(l)} T_1^2 Y_1 = Y_1 D_2^{(l)},$$

which implies that

$$(3.13) \quad D_j^{(l)} T_{j-1} \cdots T_{i+1} T_i^2 T_{i+1}^{-1} \cdots T_{j-1}^{-1} Y_i = Y_i D_j^{(l)}, \quad i < j.$$

Finally, we have

$$D_1^{(l)} Y_1 = q T_1 \cdots T_{N-1}^2 \cdots T_1 Y_1 D_1^{(l)},$$

which gives

$$(3.14) \quad D_i^{(l)} Y_i T_{i-1}^{-1} \cdots T_1^{-2} \cdots T_{i-1}^{-1} = T_i \cdots T_{N-1}^2 \cdots T_i Y_i D_i^{(l)}.$$

Finally, we write commutation relations between X_i and $D_j^{(l)}$. First of all, we have

$$(3.15) \quad X_1 D_1^{(l)} = (Y_1 - Z_1) \cdots (Y_1 - Z_l), \quad D_1^{(l)} X_1 = (q J_N Y_1 - Z_1) \cdots (q J_N Y_1 - Z_l),$$

where $J_N = T_1 \cdots T_{N-1}^2 \cdots T_1$. Also we have

$$\begin{aligned}
 [D_1^{(l)}, X_2] &= \sum_{r=1}^l X_1^{-1}(Y_1 - Z_1) \cdots (Y_1 - Z_{r-1}) [Y_1, X_2] (Y_1 - Z_{r+1}) \cdots (Y_1 - Z_l) \\
 &= \sum_{r=1}^l X_1^{-1}(Y_1 - Z_1) \cdots (Y_1 - Z_{r-1}) (T_1^{-2} - 1) X_2 Y_1 (Y_1 - Z_{r+1}) \cdots (Y_1 - Z_l) \\
 &= (\mathbf{t}^{-1} - \mathbf{t}) \sum_{r=1}^l X_1^{-1}(Y_1 - Z_1) \cdots (Y_1 - Z_{r-1}) T_1^{-1} X_2 Y_1 (Y_1 - Z_{r+1}) \cdots (Y_1 - Z_l) \\
 &= (\mathbf{t}^{-1} - \mathbf{t}) \sum_{r=1}^l X_1^{-1}(Y_1 - Z_1) \cdots (Y_1 - Z_{r-1}) X_1 T_1 Y_1 (Y_1 - Z_{r+1}) \cdots (Y_1 - Z_l) \\
 &= (\mathbf{t}^{-1} - \mathbf{t}) \sum_{r=1}^l (q J_N Y_1 - Z_1) \cdots (q J_N Y_1 - Z_{r-1}) T_1 Y_1 (Y_1 - Z_{r+1}) \cdots (Y_1 - Z_l).
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 (3.16) \quad [D_1^{(l)}, X_2] &= (\mathbf{t}^{-1} - \mathbf{t}) \sum_{r=1}^l (q J_N Y_1 - Z_1) \cdots (q J_N Y_1 - Z_{r-1}) Y_2 T_1^{-2} (Y_2 T_1^{-2} - Z_{r+1}) \cdots (Y_2 T_1^{-2} - Z_l) T_1.
 \end{aligned}$$

Now the commutation relations between $D_i^{(l)}$ and X_j can be obtained from this by applying the elements T_m .

Thus, we have the following proposition. Let $HH_N^{l,+}$ be the “unlocalized” version of the cyclotomic DAHA, generated by $T_i, X_i, D_i^{(l)}$, and Y_i (without Y_i^{-1}).

PROPOSITION 3.27. – *Let M_X be a monomial in X_i , M_D a monomial in $D_i^{(l)}$, M_Y a monomial in Y_i with powers of Y_i being $\leq l - 1$. Then the monomials of the form $M_X M_Y T_s M_D$, $s \in S_N$ form a basis in $HH_N^{l,+}$.*

Proof. – It is easy to see from considering the polynomial representation that these monomials are linearly independent. Therefore, it remains to establish the spanning property. We consider a filtration with $\deg(T_i) = 0, \deg(X_i) = \deg(D_i^{(l)}) = l, \deg(Y_i) = 2$. We can use the above commutation relations to order any monomial without increasing its degree. Then we can reduce the degrees of Y_i below l by using the relation $X_1 D_1^{(l)} = (Y_1 - Z_1) \cdots (Y_1 - Z_l)$ and its conjugates by the T_i . Namely, we have the relation

$$X_i D_i^{(l)} = (Y_i T_{i-1}^{-1} \cdots T_1^{-2} \cdots T_{i-1}^{-1} - Z_1) \cdots (Y_i T_{i-1}^{-1} \cdots T_1^{-2} \cdots T_{i-1}^{-1} - Z_l),$$

which can be used to express the monomial Y_i^l as

$$Y_i^l = X_i D_i^{(l)} T_{i-1} \cdots T_1^2 \cdots T_{i-1} + \cdots$$

where \cdots is a linear combination of ordered monomials of degrees $< l$ in Y_i, T_i and monomials of degree l involving Y_i in degree $< l$ and some Y_j with $j < i$. For example, for $i = 2, l = 2$ we get

$$X_2 D_2^{(l)} = (Y_2 T_1^{-2} - Z_1)(Y_2 T_1^{-2} - Z_2),$$

so we get

$$Y_2 T_1^{-2} Y_2 = X_2 D_2^{(l)} T_1^2 + \text{lower degree terms} .$$

But

$$Y_2 T_1^{-2} Y_2 = Y_2^2 + (\mathbf{t}^{-1} - \mathbf{t}) Y_2 Y_1 T_1 ,$$

so

$$Y_2^2 = X_2 D_2^{(l)} T_1^2 + (\mathbf{t} - \mathbf{t}^{-1}) Y_2 Y_1 T_1 + \text{lower degree terms} .$$

This implies the required spanning property. □

THEOREM 3.28. – (i) *The algebra $HH_N^{l,+}$ is generated by $T_i, X_i, D_i := D_i^{(l)}$, and Y_i with the following defining relations:*

(1) *relations (R1-R7), (R11), (R12) of DAHA;*

(2) *the relations (3.7)–(3.16).*

The algebra HH_N^l is defined by the same generators and relations, adding the condition that Y_i are invertible.

Proof. – We have shown that these relations hold. Moreover, it was shown in the proof of Proposition 3.27 that using these relations, we can reduce every monomial to a linear combination of basis monomials. This implies that the relations are defining. □

PROPOSITION 3.29. – *There is an involutive automorphism ϕ of HH_N^l such that $\phi(X_i) = D_i, \phi(D_i) = X_i, \phi(T_i) = T_i^{-1}, \phi(\mathbf{t}) = \mathbf{t}^{-1}, \phi(q) = q^{-1}$ and*

$$\phi(Y_i) = q T_i \cdots T_{N-1}^2 \cdots T_i Y_i T_{i-1}^{-1} \cdots T_1^{-2} \cdots T_{i-1}^{-1} .$$

This automorphism preserves the subalgebra $HH_N^{l,+}$.

Proof. – We check that ϕ preserves the defining relations. Relations (R1-R3) go to themselves. Relations (R4,R5) get exchanged with (3.11). Relations (R6,R7) go to themselves once (R1-R3) are imposed. Relations (R11) get exchanged with (3.10) and (R12) with themselves once (R1-R3,R6,R7) are imposed. Relation (3.7) gets exchanged with (3.12), (3.8) with (3.13), (3.9) with (3.14). Relations (3.15) go to themselves. So it remains to see that relation (3.16) is preserved.

Let $K_N = T_2 \cdots T_{N-1}^2 \cdots T_2$. We have

$$\begin{aligned} [D_2, X_1] &= \sum_{r=1}^l X_2^{-1} (Y_2 T_1^{-2} - Z_l) \\ &\quad \cdots (Y_2 T_1^{-2} - Z_{r+1}) [Y_2 T_1^{-2}, X_1] (Y_2 T_1^{-2} - Z_{r-1}) \cdots (Y_2 T_1^{-2} - Z_1) \\ &= \sum_{r=1}^l X_2^{-1} (Y_2 T_1^{-2} - Z_l) \\ &\quad \cdots (Y_2 T_1^{-2} - Z_{r+1}) Y_2 (T_1^{-2} - 1) X_1 (Y_2 T_1^{-2} - Z_{r-1}) \cdots (Y_2 T_1^{-2} - Z_1) \\ &= (\mathbf{t}^{-1} - \mathbf{t}) \sum_{r=1}^l X_2^{-1} (Y_2 T_1^{-2} - Z_l) \\ &\quad \cdots (Y_2 T_1^{-2} - Z_{r+1}) Y_2 T_1^{-1} X_1 (Y_2 T_1^{-2} - Z_{r-1}) \cdots (Y_2 T_1^{-2} - Z_1) \end{aligned}$$

$$\begin{aligned}
 &= (\mathbf{t}^{-1} - \mathbf{t}) \sum_{r=1}^l X_2^{-1} (Y_2 T_1^{-2} - Z_l) \\
 &\quad \cdots (Y_2 T_1^{-2} - Z_{r+1}) Y_2 T_1^{-2} X_2 T_1^{-1} (Y_2 T_1^{-2} - Z_{r-1}) \cdots (Y_2 T_1^{-2} - Z_1).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 (3.17) \quad &[D_2, X_1] \\
 &= (\mathbf{t}^{-1} - \mathbf{t}) \sum_{r=1}^l (qK_N Y_2 - Z_l) \cdots (qK_N Y_2 - Z_{r+1}) qK_N Y_2 (Y_1 - Z_{r-1}) \cdots (Y_1 - Z_1) T_1^{-1}.
 \end{aligned}$$

Now it is easy to see that ϕ maps (3.16) to (3.17) (as Y_1 commutes with K_N and Y_2 , hence with $K_N Y_2$). □

REMARK 3.30. – It is easy to see that as $q \rightarrow 1$, the involution ϕ degenerates to the involution ϕ on $H_{N,\text{deg}}^l$ constructed in Remark 2.25.

COROLLARY 3.31. – *The algebra $HH_N^{l,+}(Z_1, \dots, Z_l, q, \mathbf{t})$ is a free module over $\mathbb{C}[X_1, \dots, X_N] \otimes \mathbb{C}[D_1, \dots, D_N]$ of rank $N! \cdot l^N$, where the first factor acts by left multiplication and the second one by right multiplication.*

Proof. – This follows immediately from Proposition 3.27. □

The proof of Proposition 3.27 in fact shows that for any $l \geq 0$ ordered products of M_X, M_Y, T_s, M_D in any of the 24 possible orders are a spanning set for $HH_N^{l,+}$, and those of them with degrees of Y_i at most $l - 1$ are a basis for $l \geq 1$. This implies that we also have another basis of this algebra, formed by monomials $M_X M_D T_s M_Y$ without restriction on the degree of Y_i , but with the restriction that for each i either X_i is missing in M_X or D_i is missing in M_D . Indeed, if this restriction is not satisfied, we may use the relation $X_i D_i = f(Y_i)$ and its permutations to lower the number of X_i and D_i , and it is easy to see by looking at the polynomial representation that monomials with this restriction are linearly independent. Thus we obtain the following proposition.

PROPOSITION 3.32. – *The elements $M_X M_D$ which miss either X_i or D_i for each i form a basis of $HH_N^{l,+}$ as a left or right module over the positive part of the affine Hecke algebra H_N^+ generated by $T_s, s \in S_N$ and Y_i , and a basis of HH_N^l as a left or right module over the affine Hecke algebra H_N generated by $T_s, s \in S_N$ and $Y_i^{\pm 1}$; in particular, $HH_N^{l,+}$ is a free module over H_N^+ and HH_N^l is a free module over H_N .*

Note that the basis of Proposition 3.32 is labeled by N -tuples of integers, (m_1, \dots, m_N) . Namely, if $M_X M_D$ contains X_i^p then we set $m_i = p$, and if it contains D_i^p then we set $m_i = -p$.

Another geometric proof of Proposition 3.32 (for a geometric version of HH_N^l) will be given in Section 4.

4. Geometric realization

4.1. A variety of triples

We consider a quiver with a set of vertices I and a set of arrows Ω . Let $V = \bigoplus_{i \in I} V_i$, $W = \bigoplus_{i \in I} W_i$ be I -graded finite dimensional \mathbb{C} -vector spaces; $d_i := \dim V_i$. Given a length ℓ sequence $\mathbf{i} = (i_1, \dots, i_\ell) \in I^\ell$ and a length ℓ sequence $\mathbf{a} = (a_1, \dots, a_\ell)$ of positive integers such that $\sum_{n:i_n=i} a_n = d_i$ for any $i \in I$, we choose an I -graded flag in V : $V = V^0 \supset V^1 \supset \dots \supset V^\ell = 0$ such that V^{n-1}/V^n is an a_n -dimensional vector space supported at the vertex i_n for any $n = 1, \dots, \ell$.

We set $\mathcal{X} = \mathbb{C}((z)) \supset \mathbb{C}[[z]] = \mathcal{O}$. We consider the following flag of I -graded lattices in $V_{\mathcal{X}} = V \otimes \mathcal{X}: \dots \supset L_{-1} \supset L_0 \supset L_1 \supset \dots$, where $L_{r+\ell} = zL_r$ for any $r \in \mathbb{Z}$; $L_0 = V_{\mathcal{O}}$; $L_n/L_\ell = V^n \subset V = L_0/L_\ell$ for any $n = 1, \dots, \ell$. Let $GL(V) := \prod_{i \in I} GL(V_i)$, and let $\mathcal{P} \subset GL(V)_{\mathcal{O}} \subset GL(V)_{\mathcal{X}}$ be the stabilizer of the flag L_\bullet . Then $GL(V)_{\mathcal{X}}/\mathcal{P}$ is the set of points of the ind-projective moduli space $\mathcal{F}\ell$ of flags of I -graded lattices $\dots \supset M_{-1} \supset M_0 \supset M_1 \supset \dots$ in $V_{\mathcal{X}}$ such that $M_{r+\ell} = zM_r$ for any $r \in \mathbb{Z}$, and M_{n-1}/M_n is an a_n -dimensional vector space supported at the vertex i_n for any $n = 1, \dots, \ell$. This is a partial affine flag variety of the reductive group $GL(V)$. Note that the set of connected components $\pi_0(\mathcal{F}\ell)$ is naturally identified with \mathbb{Z}^I (the virtual graded dimension of M_0).

Let \mathcal{R} be the moduli space of the following data (cf. [31, Section 1]):

- (a) $M_\bullet \in \mathcal{F}\ell$;
- (b) a \mathcal{X} -linear homomorphism $p_i: W_{i,\mathcal{X}} \rightarrow V_{i,\mathcal{X}}$ for any $i \in I$;
- (c) a \mathcal{X} -linear homomorphism $b_{i \rightarrow j}: V_{i,\mathcal{X}} \rightarrow V_{j,\mathcal{X}}$ for any $i \rightarrow j \in \Omega$; such that
 - (1) $b := \sum_{i \rightarrow j \in \Omega} b_{i \rightarrow j}$ takes L_r to L_{r+1} and M_r to M_{r+1} for any $r \in \mathbb{Z}$;
 - (2) $p := \sum_{i \in I} p_i$ takes $W_{\mathcal{O}}$ to $L_0 \cap M_0$.

Note that when $\ell = 1$, we have $\mathcal{P} = GL(V)_{\mathcal{O}}$, and \mathcal{R} is nothing but the variety of triples $\mathcal{R}_{GL(V),\mathbf{N}}$ associated in [8, 2(i)] to a $GL(V)$ -module

$$\mathbf{N} = \bigoplus_{i \rightarrow j \in \Omega} \text{Hom}(V_i, V_j) \oplus \bigoplus_{i \in I} \text{Hom}(W_i, V_i).$$

The definition of equivariant Borel-Moore homology $H_\bullet^{GL(V)_{\mathcal{O}} \times \mathbb{C}^\times}(\mathcal{R}_{GL(V),\mathbf{N}})$ (respectively, equivariant K -theory $K^{GL(V)_{\mathcal{O}} \times \mathbb{C}^\times}(\mathcal{R}_{GL(V),\mathbf{N}})$) and the construction of convolution product on it in [8, Sections 2,3] work without any changes in our situation, and produce the convolution algebras $H_\bullet^{\mathcal{P} \times \mathbb{C}^\times}(\mathcal{R})$ and $K^{\mathcal{P} \times \mathbb{C}^\times}(\mathcal{R})$. Moreover, if we choose a Cartan torus $T(W_i) \subset GL(W_i)$ and set $T(W) := \prod_{i \in I} T(W_i) \subset \prod_{i \in I} GL(W_i) =: GL(W)$ (a flavor symmetry group), we obtain the convolution algebras $H_\bullet^{\mathbb{C}^\times \times T(W)_{\mathcal{O}} \times \mathcal{P} \times \mathbb{C}^\times}(\mathcal{R})$ and $K^{\mathbb{C}^\times \times T(W)_{\mathcal{O}} \times \mathcal{P} \times \mathbb{C}^\times}(\mathcal{R})$. Here the first factor \mathbb{C}^\times acts by dilations in the fibers of the projection

$$(4.1) \quad \varpi: \mathcal{R} \rightarrow \mathcal{F}\ell.$$

The following easy result will be important in the future.

LEMMA 4.1. – *The algebra $H_\bullet^{\mathbb{C}^\times \times T(W)_{\mathcal{O}} \times \mathcal{P} \times \mathbb{C}^\times}(\mathcal{R})$ is free as a module over the equivariant point cohomology $H_{\mathbb{C}^\times \times T(W)_{\mathcal{O}} \times \mathcal{P} \times \mathbb{C}^\times}^\bullet(pt)$. Similarly, the algebra $K^{\mathbb{C}^\times \times T(W)_{\mathcal{O}} \times \mathcal{P} \times \mathbb{C}^\times}(\mathcal{R})$ is free as a module over $K_{\mathbb{C}^\times \times T(W)_{\mathcal{O}} \times \mathcal{P} \times \mathbb{C}^\times}(pt)$.*

Proof. – It is enough to show that \mathcal{R} has an algebraic cell decomposition, which is invariant under the maximal torus of the group $\mathbb{C}^\times \times T(W)_\theta \times \mathcal{P} \rtimes \mathbb{C}^\times$. For this we need to choose an Iwahori subgroup \mathcal{J} of $GL(V)_\mathcal{R}$ which is contained in \mathcal{P} . Then the \mathcal{J} -orbits on $\mathcal{F}\ell$ are affine spaces; hence their preimages in \mathcal{R} are (infinite-dimensional) affine spaces as well, which are clearly invariant under $\mathbb{C}^\times \times T(W)_\theta \times \mathcal{J} \rtimes \mathbb{C}^\times$, hence under some maximal torus of $\mathbb{C}^\times \times T(W)_\theta \times \mathcal{P} \rtimes \mathbb{C}^\times$. \square

In case $W = W' \oplus W''$, we denote the variety of triples corresponding to W' (resp. W'') by \mathcal{R}' (resp. \mathcal{R}''), and we have an evident closed embedding $\mathbf{z}: \mathcal{R}' \hookrightarrow \mathcal{R}$. The argument of [8, Lemma 5.11] goes through word for word in our situation and proves that

$$(4.2) \quad \mathbf{z}^*: H_\bullet^{\mathbb{C}^\times \times T(W)_\theta \times \mathcal{P} \rtimes \mathbb{C}^\times}(\mathcal{R}) \hookrightarrow H_\bullet^{\mathbb{C}^\times \times T(W')_\theta \times \mathcal{P} \rtimes \mathbb{C}^\times}(\mathcal{R}') \otimes H_{T(W'')}^\bullet(pt)$$

and

$$(4.3) \quad \mathbf{z}^*: K^{\mathbb{C}^\times \times T(W)_\theta \times \mathcal{P} \rtimes \mathbb{C}^\times}(\mathcal{R}) \hookrightarrow K^{\mathbb{C}^\times \times T(W')_\theta \times \mathcal{P} \rtimes \mathbb{C}^\times}(\mathcal{R}') \otimes K_{T(W'')}(\mathcal{P}t)$$

are the convolution algebra homomorphisms.

4.2. Jordan quiver

In what follows we consider a special case of the construction of Section 4.1 where I consists of a single vertex and Ω consists of a single loop, $W = \mathbb{C}^l$ with a basis e_1, \dots, e_l (hence the diagonal torus $T(W) \subset GL(W)$), and $V = \mathbb{C}^N$ with a basis v_1, \dots, v_N . Moreover, $\ell = N$, $a_n = 1$ for any $n = 1, \dots, N$, and V^\bullet is a complete flag $V^n := \mathbb{C}v_1 \oplus \dots \oplus \mathbb{C}v_{N-n}$. We fix a flag of lattices $L_i \subset V((z))$, $i \in \mathbb{Z}$, such that $L_0 = V[[z]]$, $L_{j+N} = zL_j$, and $L_j = zV[[z]] \oplus \mathbb{C}v_1 \oplus \dots \oplus \mathbb{C}v_{N-j}$, $j = 0, \dots, N-1$.

The space of triples of Section 4.1 is the moduli space of the following data:

- (a) a sequence of $\mathbb{C}[[z]]$ -lattices $M_i \subset V((z))$, $i \in \mathbb{Z}$, such that $M_i \supsetneq M_{i+1}$ and $M_{j+N} = zM_j$;
- (b) a $\mathbb{C}((z))$ -linear map $b: V((z)) \rightarrow V((z))$; and
- (c) a $\mathbb{C}((z))$ -linear map $p: W((z)) \rightarrow V((z))$; such that
 - (1) b strongly preserves L and M , i.e., $bL_i \subset L_{i+1}$ and $bM_i \subset M_{i+1}$; and
 - (2) $pW[[z]] \subset L_0 \cap M_0$.

The basis of V gives rise to the diagonal torus $T(V) \subset GL(V)$, and we denote by y_1, \dots, y_N the generators of $H_{T(V)}^\bullet(pt)$. Also, we denote by z_1, \dots, z_l (resp. $\hbar, -k$) the generators of $H_{T(W)}^\bullet(pt)$ (resp. $H_{\mathbb{C}^\times}^\bullet(pt)$) for the loop rotation \mathbb{C}^\times , $H_{\mathbb{C}^\times}^\bullet(pt)$ for the dilation \mathbb{C}^\times). We shall denote the corresponding generators of $K_{T(V)}(pt)$, $K_{T(W)}(pt)$, $K_{\mathbb{C}^\times}(pt)$, $K_{\mathbb{C}^\times}(pt)$ by Y_1, \dots, Y_N ; Z_1, \dots, Z_l ; $q; t$. The algebra $H_\bullet^{\mathbb{C}^\times \times T(W)_\theta \times \mathcal{P} \rtimes \mathbb{C}^\times}(\mathcal{R})$ (resp. $K^{\mathbb{C}^\times \times T(W)_\theta \times \mathcal{P} \rtimes \mathbb{C}^\times}(\mathcal{R})$) will be denoted $\mathcal{H}\mathcal{H}_{N,\text{deg}}^l$ (resp. $\mathcal{H}\mathcal{H}_N^l$). We shall also denote $\mathcal{H}\mathcal{H}_{N,\text{deg}}^0$ by $\mathcal{H}\mathcal{H}_{N,\text{deg}}$, and $\mathcal{H}\mathcal{H}_N^0$ by $\mathcal{H}\mathcal{H}_N$. According to (4.2) and (4.3) we have algebra embeddings $\mathbf{z}^*: \mathcal{H}\mathcal{H}_{N,\text{deg}}^l \hookrightarrow \mathcal{H}\mathcal{H}_{N,\text{deg}}[z_1, \dots, z_l]$ and $\mathbf{z}^*: \mathcal{H}\mathcal{H}_N^l \hookrightarrow \mathcal{H}\mathcal{H}_N[Z_1^{\pm 1}, \dots, Z_l^{\pm 1}]$.

Note that for $W = 0$ the variety of triples \mathcal{R} is nothing but the affine Steinberg variety of $GL(V)$, and $\mathcal{F}\ell$ is nothing but the affine flag variety of $GL(V)$. It is well known that $HH_{N,\text{deg}} \simeq \mathcal{H}\mathcal{H}_{N,\text{deg}}$ (see e.g., [37]), and one can check that $HH_N \simeq \mathcal{H}\mathcal{H}_N$, cf. [43]

(see [8, Remark 3.9(2)]). More precisely, for $n = 0, \dots, N - 1$, we denote by $\mathbb{P}_n^1 \subset \mathcal{F}l$ the projective line formed by all the flags M_\bullet of lattices (see Section 4.1) such that $M_m = L_m$ for $m \neq n \pmod N$. Each \mathbb{P}_n^1 contains the base point $L_\bullet \in \mathcal{F}l$, and we denote by $\mathbb{A}_n^1 \subset \mathbb{P}_n^1$ the complement. The restriction of the projection $\varpi: \mathcal{R} \rightarrow \mathcal{F}l$ to \mathbb{A}_n^1 is a (profinite dimensional) vector bundle, and the closure of $\varpi^{-1}(\mathbb{A}_n^1)$ in \mathcal{R} is still a vector bundle over \mathbb{P}_n^1 , to be denoted by $\widetilde{\mathbb{P}}_n^1$. We define $s_n \in \mathcal{H}_N$ as $1 + [\widetilde{\mathbb{P}}_n^1]$ for $n = 0, \dots, N - 1$ (the fundamental cycle of $\widetilde{\mathbb{P}}_n^1$). Also, let $\bar{\pi}$ (resp. $\bar{\pi}^{-1}$) be a point-orbit of the Iwahori group \mathcal{P} in $\mathcal{F}l$ consisting of the flag M_\bullet such that $M_n = L_{n+1}$ (resp. $M_n = L_{n-1}$) for any $n \in \mathbb{Z}$. Finally, we define $\pi^{\pm 1} \in \mathcal{H}_{N,\text{deg}}$ as $[\varpi^{-1}(\bar{\pi}^{\pm 1})]$. Now the desired isomorphism $HH_{N,\text{deg}} \xrightarrow{\sim} \mathcal{H}_{N,\text{deg}}$ takes the generators of $HH_{N,\text{deg}}$ to the same named elements of $\mathcal{H}_{N,\text{deg}}$, except $\hbar \mapsto -\hbar$.

Furthermore, we define $T_n \in \mathcal{H}_N$ as $-1 - [\mathcal{O}_{\mathbb{P}_n^1}(-2)]$ for $n = 0, \dots, N - 1$, and we define $\pi^{\pm 1} \in \mathcal{H}_N$ as $[\mathcal{O}_{\varpi^{-1}(\bar{\pi}^{\pm 1})}]$. The desired isomorphism $HH_N \xrightarrow{\sim} \mathcal{H}_N$ takes the generators of HH_N to the same named elements of \mathcal{H}_N , except $q \mapsto q^{-1}$.

For arbitrary $W = \mathbb{C}^l$, in order to distinguish from the $l = 0$ case, we will denote by $\varpi_l: \mathcal{R}^l \rightarrow \mathcal{F}l$ the corresponding projection. Let us set

$$\mathcal{H}_{N,\text{deg}}^l \ni \pi_- = [\varpi_l^{-1}(\bar{\pi}^{-1})]; \quad \mathcal{H}_N^l \ni \pi_- = (-q^{-1}t^{-1})^l Z_1 \cdots Z_l [\mathcal{O}_{\varpi_l^{-1}(\bar{\pi}^{-1})}].$$

Then we have

$$(4.4) \quad \mathbf{z}^* \pi_- = \prod_{m=1}^l (y_N - z_m - k) \cdot \pi^{-1} = \pi^{-1} \prod_{m=1}^l (y_1 - z_m - k + \hbar) \in \mathcal{H}_{N,\text{deg}},$$

and

$$(4.5) \quad \begin{aligned} \mathbf{z}^* \pi_- &= (-q^{-1}t^{-1})^l Z_1 \cdots Z_l \prod_{m=1}^l (1 - Y_N Z_m^{-1}t) \cdot \pi^{-1} \\ &= \pi^{-1} \prod_{m=1}^l (Y_1 - Z_m q^{-1}t^{-1}) \in \mathcal{H}_N. \end{aligned}$$

Indeed, $\mathbf{z}^* \pi_-$ is obtained from π^{-1} by multiplication with the Euler class of the finite dimensional quotient space $\mathcal{F}_{\bar{\pi}^{-1}}^l / \mathcal{R}_{\bar{\pi}^{-1}}^l$, where $\mathcal{R}_{\bar{\pi}^{-1}}^l$ is the fiber of \mathcal{R}^l at the point $\bar{\pi}^{-1}$, and $\mathcal{F}_{\bar{\pi}^{-1}}^l$ is defined similarly, but the condition (2) above: $pW[[z]] \subset L_0 \cap M_0 = L_0 \cap L_{-1} = L_0$ is relaxed to the condition $pW[[z]] \subset M_0 = L_{-1}$, cf. [8, 4(vi)].

We preserve the name π for $[\varpi_l^{-1}(\bar{\pi})] \in \mathcal{H}_{N,\text{deg}}^l$ (resp. $[\mathcal{O}_{\varpi_l^{-1}(\bar{\pi})}] \in \mathcal{H}_N^l$) since \mathbf{z}^* takes this π to the one in $\mathcal{H}_{N,\text{deg}}$ (resp. in \mathcal{H}_N). For the same reason we preserve the names s_n (resp. T_n), $n = 1, \dots, N - 1$, for the corresponding elements of $\mathcal{H}_{N,\text{deg}}^l$ (resp. \mathcal{H}_N^l). Finally, we set $z_m = z_m + k - \hbar$, and $Z_m = Z_m q^{-1}t^{-1}$. So the following diagrams

commute:

$$(4.6) \quad \begin{array}{ccc} HH_{N,\text{deg}}^l & \xrightarrow{\quad} & \mathcal{HH}_{N,\text{deg}}^l \\ \downarrow & & \downarrow \mathbf{z}^* \\ HH_{N,\text{deg}}[z_1, \dots, z_l] & \xrightarrow{\sim} & \mathcal{HH}_{N,\text{deg}}[z_1, \dots, z_l], \end{array}$$

$$(4.7) \quad \begin{array}{ccc} HH_N^l \mathcal{HH}_N^l & & \\ \downarrow \mathbf{z}^* & & \\ HH_N[Z_1^{\pm 1}, \dots, Z_l^{\pm 1}] & \xrightarrow{\sim} & \mathcal{HH}_N[Z_1^{\pm 1}, \dots, Z_l^{\pm 1}]. \end{array}$$

Note that in diagram (4.6) we have bigradings defined by $\text{deg}(X_i) = (1, 0)$, $\text{deg}(y_i) = \text{deg}(z, \hbar, k) = (0, 1)$, which are preserved by all the maps.

The following theorem is independently obtained in [44, Lemma 4.2].

THEOREM 4.2. – *The map $\xi : HH_{N,\text{deg}}^l \rightarrow \mathcal{HH}_{N,\text{deg}}^l$ in diagram (4.6) is an isomorphism of bigraded algebras.*

Proof. – We may assume that $l > 0$. Recall from Lemma 4.1 that $\mathcal{HH}_{N,\text{deg}}^l$ is a free $\mathbb{C}[z_1, \dots, z_l, \hbar, k]$ -module. Thus it suffices to show that ξ_0 is an isomorphism, where ξ_0 is the specialization of ξ at $\hbar = k = 0, z = 0$. It is clear that ξ_0 is injective, hence so are ξ and all its specializations $\xi_{z,\hbar,k}$. Since the bigraded components $HH_{N,\text{deg}}^l[r, s]$ are finite dimensional (as $l > 0$), it suffices to show that the specialization $\xi_{z,\hbar,k}$ is an isomorphism for *Weil generic* z, \hbar, k .

We will now use the following easy lemma about unital rings.

LEMMA 4.3. – *Let B be a unital ring and $A \subset B$ a unital subring. If $\mathbf{e} \in A$ is an idempotent such that $\mathbf{AeA} = A$ and $\mathbf{eAe} = \mathbf{eBe}$, then $A = B$.*

Proof. – Since $\mathbf{AeA} = A$, we have $\mathbf{AeB} = \mathbf{AeAB} = \mathbf{AB} = B$. Similarly $\mathbf{BeA} = B$. Thus $\mathbf{AeBeA} = \mathbf{BeA} = B$. But since $\mathbf{eAe} = \mathbf{eBe}$ and $\mathbf{AeA} = A$, we have $\mathbf{AeBeA} = \mathbf{AeAeA} = \mathbf{AeA} = A$. Thus $A = B$. □

Now let \mathbf{e} be the symmetrizer of S_N , and let us apply Lemma 4.3 to $A = HH_{N,\text{deg}}^l(z, 1, k)$ and $B = \mathcal{HH}_{N,\text{deg}}^l(z, -1, k)$. Using Theorem 2.19, we see that the condition $\mathbf{AeA} = A$ is satisfied for generic z, k , namely when the corresponding parameters of the cyclotomic rational Cherednik algebras are not aspherical (see e.g., [7], Subsection 4.1). In fact, it suffices to consider the case $z = 0, k = 0$, when this is easy, since $HH_{N,\text{deg}}^l(0, 1, 0) = S_n \times (\mathcal{D}(\mathbb{C})^{\mathbb{Z}/l\mathbb{Z}})^{\otimes N}$ is a simple algebra (by [32, Theorem 2.3]). Also, by [9, Proposition 3.24], the theorem holds for spherical subalgebras, i.e., $\mathbf{e}\xi_0\mathbf{e}$ is an isomorphism. Hence so are $\mathbf{e}\xi\mathbf{e}$ and all its specializations $\mathbf{e}\xi_{z,\hbar,k}\mathbf{e}$, which yields the condition $\mathbf{eAe} = \mathbf{eBe}$ for all z, \hbar, k . Thus Lemma 4.3 applies and Theorem 4.2 follows. □

We expect that an analog of Theorem 4.2 also holds in the K-theoretic setting, i.e., for diagram (4.7). Let us prove a formal version of this statement. Let $q = e^{\varepsilon h}$, $t = e^{-\varepsilon k}$, $Z_i = e^{\varepsilon z_i}$. Let $\mathcal{H}\mathcal{H}_N^{l,\text{formal}}$ be the corresponding formal completion of $\mathcal{H}\mathcal{H}_N^l$ (obtained by viewing equivariant K-theory as a formal deformation of equivariant Borel-Moore homology⁽⁴⁾). Diagram (4.7) furnishes a map $\widehat{\xi} : \mathcal{H}\mathcal{H}_N^{l,\text{formal}} \rightarrow \mathcal{H}\mathcal{H}_N^{l,\text{formal}}$.

COROLLARY 4.4. – *The map $\widehat{\xi}$ is an isomorphism.*

Proof. – By Lemma 4.1, $\mathcal{H}\mathcal{H}_N^{l,\text{formal}}$ is a flat formal deformation of $\mathcal{H}\mathcal{H}_{N,\text{deg}}^l$ over $\mathbb{C}[[\varepsilon]]$. Finally, $\widehat{\xi}|_{\varepsilon=0} = \xi$. This implies the corollary. \square

In particular, this gives another, geometric proof of the facts that $\mathcal{H}\mathcal{H}_{N,\text{deg}}^l$ is a free bigraded $\mathbb{C}[z_1, \dots, z_l, h, k]$ -module, and the algebra $\mathcal{H}\mathcal{H}_N^{l,\text{formal}}(z, h, k)$ is its flat formal deformation.

We also have

THEOREM 4.5. – *Let \mathbf{e} be the symmetrizer of the finite Hecke algebra generated by T_i , $1 \leq i \leq N - 1$. Then the natural map*

$$\mathbf{e}\xi_{Z,q,t}\mathbf{e} : \mathbf{e}\mathcal{H}\mathcal{H}_N^l(Z, q, t)\mathbf{e} \rightarrow \mathbf{e}\mathcal{H}\mathcal{H}_N^l(Z, q, t)\mathbf{e}$$

is an isomorphism when $q = t = 1$ and $Z_i = 1$ for all i .

Proof. – Same as the proof of [9, Proposition 3.24]. \square

THEOREM 4.6. – *Let $\widehat{\mathcal{H}\mathcal{H}}_N^l$, $\widehat{\mathcal{H}\mathcal{H}}_N^l$ be the completions of $\mathcal{H}\mathcal{H}_N^l$, $\mathcal{H}\mathcal{H}_N^l$ at $q = t = Z_i = 1$ (as modules over $\mathbb{C}[Z_1^{\pm 1}, \dots, Z_l^{\pm 1}, q^{\pm 1}, t^{\pm 1}]$). Then the map*

$$\widehat{\xi} : \widehat{\mathcal{H}\mathcal{H}}_N^l \rightarrow \widehat{\mathcal{H}\mathcal{H}}_N^l$$

is an isomorphism.

Proof. – The proof is analogous to the proof of Theorem 4.2, using Theorem 4.5. Namely, the identity $A\mathbf{e}A = A$ for $A = \mathcal{H}\mathcal{H}_N^l(Z, q, t)$ is established for generic q and $Z = 1$, $t = 1$ using the fact that in this case A is a simple algebra by [32, Theorem 2.3]. \square

⁽⁴⁾ Indeed, the Borel-Moore homology is the associated graded of the γ -filtration on the K-theory. On the equivariant K-theory of the point, this is the filtration by the order of vanishing at the neutral element. See [2] for the topological situation, and [42] for the comparison with the algebraic situation.

5. Cyclotomic DAHA and multiplicative quiver and bow varieties

5.1. Multiplicative quiver varieties

Let $t \in \mathbb{C}^*$ be not a root of unity, and $Z_1, \dots, Z_l \in \mathbb{C}^*$ be such that Z_i/Z_j is not an integer power of t for $i \neq j$. Let Q_l be the cyclic quiver \hat{A}_{l-1} with vertices $1, \dots, l$ and an additional ‘‘Calogero-Moser vertex’’ 0 attached to the vertex 1 . Let $\mathcal{M}_N^l(Z, t)$ be the multiplicative quiver variety for Q_l with dimension vector $d_1 = \dots = d_l = N$ and $d_0 = 1$, see [16]. Namely, given complex vector spaces $V_i, i = 1, \dots, l$, with $\dim V_i = N$, $\mathcal{M}_N^l(Z, t)$ is the variety of collections of linear maps $\mathbf{X}_i : V_{i+1} \rightarrow V_i$ and $\mathbf{D}_i : V_i \rightarrow V_{i+1}$ (where addition is mod l) satisfying the equations

$$(5.1) \quad Z_i(1 + \mathbf{X}_i\mathbf{D}_i) = Z_{i-1}(1 + \mathbf{D}_{i-1}\mathbf{X}_{i-1}), 2 \leq i \leq l$$

and

$$(5.2) \quad Z_1(1 + \mathbf{X}_1\mathbf{D}_1)T = Z_l(1 + \mathbf{D}_l\mathbf{X}_l),$$

where $T : V_1 \rightarrow V_1$ is an operator conjugate to $\text{diag}(t^{-1}, \dots, t^{-1}, t^{n-1})$, modulo simultaneous conjugation (i.e., the corresponding categorical quotient).

EXAMPLE 5.1. – Let $l = 1$. Then there is no dependence on Z_1 , and $\mathcal{M}_N^1(t)$ is the variety of pairs (\mathbf{X}, \mathbf{D}) of N by N matrices such that

$$(1 + \mathbf{X}\mathbf{D})T = (1 + \mathbf{D}\mathbf{X}),$$

where $1 + \mathbf{X}\mathbf{D}$ is invertible, and T is as above, modulo simultaneous conjugation.

Let $\mathbf{X} := \mathbf{X}_1 \cdots \mathbf{X}_l, \mathbf{D} := \mathbf{D}_l \cdots \mathbf{D}_1, \mathbf{Y} := Z_1(1 + \mathbf{X}_1\mathbf{D}_1)$. Consider the operators $L_+ := Z_1 \cdots Z_l \mathbf{X}\mathbf{D}, L_- := Z_1 \cdots Z_l \mathbf{D}\mathbf{X}$.

LEMMA 5.2. – We have

$$\begin{aligned} L_+ &= (\mathbf{Y} - Z_1) \cdots (\mathbf{Y} - Z_l), \\ L_- &= (\mathbf{Y}T - Z_1) \cdots (\mathbf{Y}T - Z_l). \end{aligned}$$

Proof. – We prove the formula for L_+ ; the formula for L_- is proved in a similar way.

It suffices to prove by induction in r that

$$Z_1 \cdots Z_r \mathbf{X}_1 \cdots \mathbf{X}_r \mathbf{D}_r \cdots \mathbf{D}_1 = (\mathbf{Y} - Z_1) \cdots (\mathbf{Y} - Z_r).$$

The base $r = 0$ is obvious. For $r > 0$, we have, using the induction assumption:

$$Z_1 \cdots Z_r \mathbf{X}_1 \cdots \mathbf{X}_r \mathbf{D}_r \cdots \mathbf{D}_1 = \mathbf{X}_1 \cdots \mathbf{X}_{r-1} (Z_r \mathbf{X}_r \mathbf{D}_r) \mathbf{X}_{r-1}^{-1} \cdots \mathbf{X}_1^{-1} (\mathbf{Y} - Z_1) \cdots (\mathbf{Y} - Z_{r-1}).$$

But

$$\begin{aligned} \mathbf{X}_1 \cdots \mathbf{X}_{r-1} (Z_r \mathbf{X}_r \mathbf{D}_r) \mathbf{X}_{r-1}^{-1} \cdots \mathbf{X}_1^{-1} &= \mathbf{X}_1 \cdots \mathbf{X}_{r-1} (Z_{r-1} \mathbf{D}_{r-1} \mathbf{X}_{r-1} + Z_{r-1} - Z_r) \mathbf{X}_{r-1}^{-1} \cdots \mathbf{X}_1^{-1} \\ &= Z_{r-1} - Z_r + \mathbf{X}_1 \cdots \mathbf{X}_{r-2} (Z_{r-1} \mathbf{X}_{r-1} \mathbf{D}_{r-1}) \mathbf{X}_{r-2}^{-1} \cdots \mathbf{X}_1^{-1}. \end{aligned}$$

Thus,

$$\mathbf{X}_1 \cdots \mathbf{X}_{r-1} (Z_r \mathbf{X}_r \mathbf{D}_r) \mathbf{X}_{r-1}^{-1} \cdots \mathbf{X}_1^{-1} = Z_1 - Z_r + Z_1 \mathbf{X}_1 \mathbf{D}_1 = \mathbf{Y} - Z_r.$$

This implies the induction step. □

Thus we have

$$(5.3) \quad \mathbf{XD} = (Z_1^{-1}\mathbf{Y} - 1) \cdots (Z_l^{-1}\mathbf{Y} - 1), \quad \mathbf{DX} = (Z_1^{-1}\mathbf{YT} - 1) \cdots (Z_l^{-1}\mathbf{YT} - 1),$$

$$(5.4) \quad \mathbf{YX} = \mathbf{XYT}, \mathbf{YTD} = \mathbf{DY}.$$

LEMMA 5.3. – *We have an isomorphism*

$$\Phi : \mathcal{M}_N^l(Z_1, \dots, Z_l, t) \rightarrow \mathcal{M}_N^l(Z_l, \dots, Z_1, t^{-1})$$

given by $\Phi(\mathbf{X}_i) = \mathbf{D}_{l+1-i}$, $\Phi(\mathbf{D}_i) = \mathbf{X}_{l+1-i}$, $\Phi(Z_i) = Z_{l+1-i}$.

Proof. – It is easy to check that the relations defining these varieties are matched by Φ . \square

5.2. Quadruple varieties and their connection to multiplicative quiver varieties

Let $\mathbf{M}_N^l(Z, t)$ be the variety of quadruples of matrices $(\mathbf{X}, \mathbf{D}, \mathbf{Y}, T)$ satisfying (5.3), (5.4) such that \mathbf{Y} is invertible, modulo simultaneous conjugation (i.e., the categorical quotient). We have a natural map

$$\psi : \mathcal{M}_N^l(Z, t) \rightarrow \mathbf{M}_N^l(Z, t)$$

sending $\mathbf{X}_i, \mathbf{D}_i$, $1 \leq i \leq l$ to $(\mathbf{X}, \mathbf{D}, \mathbf{Y}, T)$.

PROPOSITION 5.4. – *Any collection $(\mathbf{X}, \mathbf{D}, \mathbf{Y}, T) \in \mathbf{M}_N^l(Z, t)$ acts irreducibly on \mathbb{C}^N .*

Proof. – Assume the contrary. Then there is an invariant subspace or quotient V for $(\mathbf{X}, \mathbf{D}, \mathbf{Y}, T)$ of dimension $1 \leq n \leq N - 1$ on which T acts by t^{-1} . So by Lemma 5.2, on V we have

$$\mathbf{XD} = (Z_1^{-1}\mathbf{Y} - 1) \cdots (Z_l^{-1}\mathbf{Y} - 1), \quad \mathbf{DX} = ((tZ_1)^{-1}\mathbf{Y} - 1) \cdots ((tZ_l)^{-1}\mathbf{Y} - 1),$$

$$t\mathbf{YX} = \mathbf{XY},$$

and

$$(5.5) \quad \mathbf{YD} = t\mathbf{DY}.$$

Equation (5.5) implies that \mathbf{D} cannot be invertible (otherwise taking determinants of both sides gives a contradiction). Thus there is a nonzero vector $v \in V$ such that $\mathbf{D}v = 0$. Since (again by (5.5)) $\text{Ker } \mathbf{D}$ is \mathbf{Y} -stable, we can choose v so that $\mathbf{Y}v = \lambda v$ for some $\lambda \neq 0$. Since $\mathbf{XD}v = 0$, we have $\lambda = Z_j$ for some j .

We may assume that V is irreducible for the action of $(\mathbf{X}, \mathbf{D}, \mathbf{Y}, T)$. Then V has a basis $v, \mathbf{X}v, \dots, \mathbf{X}^{n-1}v$ with $\mathbf{YX}^i v = t^{-i}\lambda\mathbf{X}^i v$, and

$$\mathbf{DX}^i v = (Z_1^{-1}\lambda t^{-i} - 1) \cdots (Z_l^{-1}\lambda t^{-i} - 1)\mathbf{X}^i v.$$

In particular, since $\mathbf{X}^n v = 0$, we must have $\lambda = t^n Z_m$ for some m . Thus, we have $t^n Z_m = Z_j$, which contradicts our assumption on the Z_i . \square

By Schur's lemma, Proposition 5.4 implies that any operator A commuting with $\mathbf{X}, \mathbf{D}, \mathbf{Y}, T$ has to be a scalar, so $\mathbf{M}_N^l(Z, t)$ is, in fact, the ordinary quotient of the set of quadruples $(\mathbf{X}, \mathbf{D}, \mathbf{Y}, T)$ satisfying (5.3), (5.4) by the free action of $PGL_N(\mathbb{C})$.

COROLLARY 5.5. – *Every collection of endomorphisms $A_i : V_i \rightarrow V_i$ which commute with $(\mathbf{X}_1, \dots, \mathbf{X}_l, \mathbf{D}_1, \dots, \mathbf{D}_l)$ satisfying (5.1),(5.2) is a scalar (the same at all vertices).*

Proof. – Suppose we have such a collection. Then A_1 commutes with $(\mathbf{X}, \mathbf{D}, \mathbf{Y}, T)$, so by Proposition 5.4, it has to be a scalar. So by shifting A_i by the same scalar we may assume that $A_1 = 0$. Our job is to show that $A_i = 0$ for all i . Assume the contrary, i.e., that $A_s \neq 0$ for some s .

Let $V'_i = \text{Im} A_i$. Then $V'_1 = 0$, so there exist $1 \leq i < j \leq l$ such that $V'_{i+1} = V'_{i+2} = \dots = V'_j \neq 0$, while $V'_i = V'_{j+1} = 0$. The collection of nonzero spaces $V'_s, i < s \leq j$ is invariant under the operators $\mathbf{X}_s, \mathbf{D}_s$, and these operators satisfy on V'_s the equations

$$Z_{i+1}(1 + \mathbf{X}_{i+1}\mathbf{D}_{i+1}) = Z_i, \quad Z_{i+2}(1 + \mathbf{X}_{i+2}\mathbf{D}_{i+2}) = Z_{i+1}(1 + \mathbf{D}_{i+1}\mathbf{X}_{i+1}), \dots,$$

$$Z_j = Z_{j-1}(1 + \mathbf{D}_{j-1}\mathbf{X}_{j-1}).$$

(if $j = i + 1$, then we get just one equation $Z_i = Z_{i+1}$). If $j \geq i + 2$, this implies that any nonzero vector $v \in V'_{i+1}$ is an eigenvector of the operator $Z_{i+1}(1 + \mathbf{X}_{i+1}\mathbf{D}_{i+1})$ with eigenvalue Z_i . Since $Z_{i+1} \neq Z_i$, this implies that $\mathbf{D}_{i+1}v$ is an eigenvector of the operator

$$Z_{i+1}(1 + \mathbf{D}_{i+1}\mathbf{X}_{i+1}) = Z_{i+2}(1 + \mathbf{X}_{i+2}\mathbf{D}_{i+2})$$

with eigenvalue Z_i . Continuing like this, we find that $Z_{j-1}(1 + \mathbf{D}_{j-1}\mathbf{X}_{j-1})$ has eigenvector $\mathbf{D}_{j-1} \cdots \mathbf{D}_{i+1}v$ with eigenvalue Z_i , hence $Z_i = Z_j$. This is a contradiction, which proves the corollary. □

Corollary 5.5 implies that $\mathcal{M}_N^l(Z, t)$ is also an ordinary quotient. A similar argument shows that Equations (5.1),(5.2) define a smooth complete intersection, i.e., the multiplicative quiver variety $\mathcal{M}_N^l(Z, t)$ is smooth (in fact, both of these statements follow from the results of [16]).

Let $\mathbf{M}_N^l(Z, t)^\circ$ be the open subset of $\mathbf{M}_N^l(Z, t)$ where \mathbf{X} is invertible. On this set, \mathbf{D} is redundant (i.e., expresses in terms of \mathbf{X} and \mathbf{Y}), and the only equation we are left with is

$$\mathbf{YX} = \mathbf{XYT}.$$

Thus, $\mathbf{M}_N^l(t)^\circ = \mathbf{M}_N^l(Z, t)^\circ$ is independent of the Z_i and is the multiplicative Calogero-Moser space considered in [35] (the phase space of the Ruijsenaars integrable system). In particular, as explained in [35], $\mathbf{M}_N^l(t)^\circ$ is smooth and connected.

Now let us study the properties of the map ψ . Note that if $l = 1$, ψ is tautologically an isomorphism. Moreover, we claim that ψ is an isomorphism $\psi^{-1}(\mathbf{M}_N^l(t)^\circ) \rightarrow \mathbf{M}_N^l(t)^\circ$. Indeed, if \mathbf{X} is invertible then we can set $V_i = \mathbb{C}^N$ and $\mathbf{X}_i = 1$ for $i = 1, \dots, l - 1$, while $\mathbf{X}_l = \mathbf{X}$. Then we get that $Z_i(1 + \mathbf{D}_i) = \mathbf{Y} = Z_l(1 + \mathbf{D}_l\mathbf{X}_l)T^{-1}$ for $i = 2, \dots, l$. Thus $\mathbf{D}_i = Z_i^{-1}\mathbf{Y} - 1$ for $i = 1, \dots, l - 1$, and $\mathbf{D}_l = \mathbf{X}^{-1}(Z_1^{-1}\mathbf{Y} - 1)$.

PROPOSITION 5.6. – *ψ is a closed embedding.*

Proof. – Our job is to show that the map

$$\psi^* : \mathcal{O}(\mathbf{M}_N^l(Z, t)) \rightarrow \mathcal{O}(\mathcal{M}_N^l(Z, t))$$

is surjective. By the Fundamental Theorem of invariant theory, $\mathcal{O}(\mathbf{M}_N^l(Z, t))$ is generated by the elements $\text{Tr}(w)$, where w are words in $\mathbf{X}, \mathbf{D}, \mathbf{Y}^{\pm 1}$, and $T^{\pm 1}$. So it suffices to show that $\mathcal{O}(\mathcal{M}_N^l(Z, t))$ is generated by the elements $\psi^*\text{Tr}(w)$. We know that this algebra is

generated by expressions $\text{Tr}(u)$, where u is any cyclic word (i.e., closed path) consisting of \mathbf{D}_i , \mathbf{X}_i , $(1 + \mathbf{X}_i \mathbf{D}_i)^{\pm 1}$ and $T^{\pm 1}$, so it suffices to show that any such cyclic word can be expressed as a cyclic word w in \mathbf{X} , \mathbf{D} , $\mathbf{Y}^{\pm 1}$, and $T^{\pm 1}$.

Let Λ be the deformed multiplicative preprojective algebra of the Calogero-Moser quiver Q_l defined in [16]; thus, $\mathcal{O}(\mathcal{M}_N^l(Z, t))$ is the representation variety of Λ for the dimension vector (d_i) . Let $\mathbf{e}_1 \in \Lambda$ be the idempotent of the vertex 1.

LEMMA 5.7. – *One has $\Lambda = \Lambda \mathbf{e}_1 \Lambda$.*

Proof. – Assume the contrary. Then we have a nonzero Λ -module V such that $V_1 = 0$ (namely, any nonzero module over $\Lambda/\Lambda \mathbf{e}_1 \Lambda$), and the argument in the proof of Corollary 5.5 gives a contradiction with the condition $Z_i \neq Z_j$. \square

Lemma 5.7 implies that the closed path u in question may be assumed to pass through the vertex 1. So it remains to show that using the relations of Λ , one may reduce u to a product w of \mathbf{X} , \mathbf{D} , $\mathbf{Y}^{\pm 1}$, $T^{\pm 1}$.

To this end, we may assume that the path u begins and ends at the vertex 1. The proof is by induction in the length ℓ of u . The base of induction ($\ell = 0$) is clear. Let us make the induction step from $\ell - 1$ to ℓ . Suppose that u ends with \mathbf{D}_1 (the case of \mathbf{X}_i is similar). If all the factors in u are \mathbf{D}_i then u is a power of \mathbf{D} , and we are done. So let m be the number of factors \mathbf{D}_i in u until the first \mathbf{X}_i , counting from the end. We may assume that $m < l$, since otherwise we can split away the factor \mathbf{D} at the end of u and pass to smaller length. If $m = 1$, we can split away the factor $\mathbf{X}_1 \mathbf{D}_1 = Z_1^{-1} \mathbf{Y} - 1$ at the end and reduce to smaller length. On the other hand, if $m > 1$, then

$$u = \bar{u} \mathbf{X}_m \mathbf{D}_m \mathbf{D}_{m-1} \cdots \mathbf{D}_1$$

for some word \bar{u} . By adding smaller length words, we may replace u with

$$u' := \bar{u}(1 + \mathbf{X}_m \mathbf{D}_m) \mathbf{D}_{m-1} \cdots \mathbf{D}_1 = Z_{m-1} Z_m^{-1} \bar{u}(1 + \mathbf{D}_{m-1} \mathbf{X}_{m-1}) \mathbf{D}_{m-1} \cdots \mathbf{D}_1.$$

By adding words of smaller length and rescaling we can replace u' by

$$u'' := \bar{u} \mathbf{D}_{m-1} \mathbf{X}_{m-1} \mathbf{D}_{m-1} \cdots \mathbf{D}_1,$$

which is a word of the same type as u but with m replaced by $m - 1$. Proceeding in this way, we will eventually reach the case $m = 1$, which has already been considered. This implies the claim. \square

Let $\mathcal{M}_N^l(Z, t)_*$ be the closure of $\psi^{-1}(\mathbf{M}_N^l(t)^\circ)$ and $\mathbf{M}_N^l(Z, t)_*$ be the closure of $\mathbf{M}_N^l(t)^\circ$.

PROPOSITION 5.8. – *The map $\psi : \mathcal{M}_N^l(Z, t)_* \rightarrow \mathbf{M}_N^l(Z, t)_*$ is an isomorphism of smooth connected affine varieties.*

Proof. – This follows from Proposition 5.6. \square

We will also see that the multiplicative quiver variety $\mathcal{M}_N^l(Z, t)$ is connected, i.e., $\mathcal{M}_N^l(Z, t)_* = \mathcal{M}_N^l(Z, t)$ (Theorem 5.17).

5.3. Connection to cyclotomic DAHA

Now let us connect multiplicative quiver varieties with cyclotomic DAHA. Let e_N be the symmetrizer of the finite Hecke algebra of S_N generated by T_i , and consider the spherical subalgebra $e_N HH_N^l(Z, 1, t)e_N$. This is a subalgebra of the commutative domain $e_N HH_N(1, t)e_N$ (see [35, Theorem 5.1(1),(2)]), so it is also a commutative domain. Consider the module $HH_N^l(Z, 1, t)e_N$ over this algebra. Let $\mathbb{M}_N^l(Z, t) = \text{Specm}(e_N HH_N^l(Z, 1, t)e_N)$.

PROPOSITION 5.9. – For any Z_i the algebra $e_N HH_N^l(Z, 1, t)e_N$ is finitely generated and Cohen-Macaulay (i.e., $\mathbb{M}_N^l(Z, t)$ is an irreducible Cohen-Macaulay variety) and the module $HH_N^l(Z, 1, t)e_N$ is Cohen-Macaulay. In particular, $HH_N^l(Z, 1, t)e_N$ is projective of rank $N!$ on the smooth locus $\mathbb{M}_N^l(Z, t)_{\text{smooth}}$ of $\mathbb{M}_N^l(Z, t)$.

Proof. – The proof is analogous to the proof of [35, Theorem 5.1(2),(3)]. Namely, the statements follow from the fact that by Proposition 3.31, $HH_N^{l,+}(Z, 1, t)e_N$ and $e_N HH_N^{l,+}(Z, 1, t)e_N$ are free modules of finite rank over the subalgebra $\mathbb{C}[X_1, \dots, X_N]^{S_N} \otimes \mathbb{C}[D_1, \dots, D_N]^{S_N}$, and $HH_N^l(Z, 1, t)e_N, e_N HH_N^l(Z, 1, t)e_N$ are obtained from these by inverting the element $Y_1 \cdots Y_N e_N$. □

PROPOSITION 5.10. – For any Z_i , the variety $\mathbb{M}_N^l(Z, t)$ is smooth outside a set of codimension two.

Proof. – Consider the open set $\mathbb{M}_N^l(Z, t)_X = \mathbb{M}_N^l(t)_X \subset \mathbb{M}_N^l(Z, t)$ where $\prod_i X_i$ is invertible. On this set, the localization of the spherical cyclotomic DAHA to $\mathbb{M}_N^l(t)_X$ is the usual spherical DAHA with $q = 1$, so $\mathbb{M}_N^l(t)_X$ is smooth by the result of [35]. Similarly, consider the open set $\mathbb{M}_N^l(Z, t)_D = \mathbb{M}_N^l(t)_D \subset \mathbb{M}_N^l(Z, t)$ where D_i are invertible. By Proposition 3.29, the localization of the spherical cyclotomic DAHA to $\mathbb{M}_N^l(t)_D$ is also isomorphic to the usual spherical DAHA, as there is an involution ϕ of the cyclotomic DAHA exchanging X_i and D_i . Thus $\mathbb{M}_N^l(t)_D$ is also smooth, by [35]. But it is easy to see that the complement of $\mathbb{M}_N^l(t)_X \cup \mathbb{M}_N^l(t)_D$ has codimension at least two. □

COROLLARY 5.11. – The variety $\mathbb{M}_N^l(Z, t)$ is normal.

Proof. – The proof is similar to the proof of [35, Theorem 5.1(2)]. Namely, the statement follows from Proposition 5.9 and Proposition 5.10, since by the Serre criterion, a Cohen-Macaulay variety smooth outside a set of codimension 2 is normal. □

Let $\mathcal{Z}(HH_N^l(Z, 1, t))$ be the center of $HH_N^l(Z, 1, t)$.

PROPOSITION 5.12. – For any Z_i : (i) The natural map

$$HH_N^l(Z, 1, t) \rightarrow \text{End}_{e_N HH_N^l(Z, 1, t)e_N}(HH_N^l(Z, 1, t)e_N)$$

is an isomorphism.

(ii) The natural map $\mathcal{Z}(HH_N^l(Z, 1, t)) \rightarrow e_N HH_N^l(Z, 1, t)e_N$ given by $\mathbf{z} \mapsto \mathbf{z}e_N$ is an isomorphism.

Proof. – The proof is analogous to the proof of [35, Theorem 5.1(4),(5)], replacing Y_i with D_i and the Cherednik involution with the involution ϕ , and using [35, Theorem 5.1]. □

We can now define a regular map $\xi : \mathbb{M}_N^l(Z, t) \rightarrow \mathbf{M}_N^l(Z, t)_*$ as follows. Given $\chi \in \mathbb{M}_N^l(Z, t)_{\text{smooth}}$, consider the representation $I(\chi) := HH_N^l(Z, 1, t)\mathbf{e}_N \otimes_{\mathbf{e}_N} HH_N^l(Z, 1, t)\mathbf{e}_N \chi$. By Proposition 5.9, this representation has dimension $N!$ and is the regular representation of the finite Hecke algebra generated by T_i . Let \mathbf{e}_{N-1} be the symmetrizer of the subalgebra generated by T_2, \dots, T_{N-1} , and let $V(\chi) := \mathbf{e}_{N-1}I(\chi)$. This is an N -dimensional space, and it carries an action of the operators $\mathbf{X} := X_1, \mathbf{D} = D_1 = X_1^{-1}(Y_1 - Z_1) \cdots (Y_1 - Z_l), \mathbf{Y} := Y_1$ and $T := T_1 \cdots T_{N-1}^2 \cdots T_1$.

PROPOSITION 5.13. – *The operators $\mathbf{X}, \mathbf{D}, \mathbf{Y}, T$ satisfy Equations (5.3),(5.4), i.e., define a point of $\mathbf{M}_N^l(Z, t)$.*

Proof. – Relations (5.4) follow from relation (3.5). The first relation of (5.3) is easy, and the second one follows from the first one and (3.5). □

REMARK 5.14. – On the open set where X_1 is invertible, Proposition 5.13 reduces to the result of [35].

Proposition 5.13 allows us to set $\xi(\chi) := (\mathbf{X}, \mathbf{D}, \mathbf{Y}, T)$, which defines the map ξ on the smooth locus $\mathbb{M}_N^l(Z, t)_{\text{smooth}}$. By Corollary 5.11, ξ then uniquely extends from the smooth locus to the whole variety $\mathbb{M}_N^l(Z, t)$. It is also clear that this map lands in $\mathbf{M}_N^l(Z, t)_*$ (as $\mathbb{M}_N^l(Z, t)$ is irreducible).

Thus, altogether we obtain a map $\kappa := \psi^{-1} \circ \xi : \mathbb{M}_N^l(Z, t) \rightarrow \mathcal{M}_N^l(Z, t)_*$.

PROPOSITION 5.15. – *κ is an isomorphism.*

Proof. – Consider the restriction κ_X of κ to the open set $\mathbb{M}_N^l(t)_X \subset \mathbb{M}_N^l(Z, t)$ where X_i are invertible. As shown above, κ_X is an isomorphism $\mathbb{M}_N^l(t)_X \rightarrow \mathcal{M}_N^l(Z, t)_X$ onto the open set $\mathcal{M}_N^l(Z, t)_X \subset \mathcal{M}_N^l(Z, t)_*$ where \mathbf{X} is invertible. Similarly, by using the involution ϕ of Proposition 3.29 and involution Φ of Lemma 5.3 and the fact that $\kappa \circ \phi = \Phi \circ \kappa$, we see that the restriction κ_D of κ to $\mathbb{M}_N^l(t)_D$ is an isomorphism $\mathbb{M}_N^l(t)_D \rightarrow \mathcal{M}_N^l(Z, t)_D$ onto the open set $\mathcal{M}_N^l(Z, t)_D \subset \mathcal{M}_N^l(Z, t)_*$ where \mathbf{D} is invertible.

Consider the morphism $\kappa^* : \mathcal{O}(\mathcal{M}_N^l(Z, t)_*) \rightarrow \mathcal{O}(\mathbb{M}_N^l(Z, t))$. Obviously, it is an inclusion which becomes an isomorphism after passing to the fields of fractions. Let $F \in \mathcal{O}(\mathbb{M}_N^l(Z, t))$. Then F is a rational function on $\mathcal{M}_N^l(Z, t)_*$. As shown above, this function is regular on $\mathcal{M}_N^l(Z, t)_X \cup \mathcal{M}_N^l(Z, t)_D \subset \mathcal{M}_N^l(Z, t)$, an open subset whose complement has codimension ≥ 2 . But we know that $\mathcal{M}_N^l(Z, t)_*$ is smooth, hence normal. Thus, F extends to a regular function on the whole $\mathcal{M}_N^l(Z, t)_*$. This implies that κ^* and hence κ is an isomorphism. □

Thus, we obtain the following theorem.

THEOREM 5.16. – *Under the above assumptions on the Z_i , the following statements hold.*

- (i) *The variety $\mathbb{M}_N^l(Z, t)$ is smooth.*
- (ii) *The module $HH_N^l(Z, 1, t)\mathbf{e}_N$ over $\mathcal{Z}(HH_N^l(Z, 1, t)) \cong \mathbf{e}_N HH_N^l(Z, 1, t)\mathbf{e}_N$ is projective of rank $N!$.*

(iii) $HH_N^l(Z, 1, t)$ is a split Azumaya algebra over $\mathcal{Z}(HH_N^l(Z, 1, t))$ of rank $N!$, namely the endomorphism algebra of the vector bundle $HH_N^l(Z, 1, t)\mathbf{e}_N$. Thus, all irreducible representations of $HH_N^l(Z, 1, t)$ have dimension $N!$ and are parametrized by points of $\mathbb{M}_N^l(Z, t)$.

Thus, we see that irreducible representations of $HH_N^l(Z, 1, t)$ are parametrized by points of a connected component of the multiplicative quiver variety. In fact, it turns out that this is the only connected component. Namely, we have

THEOREM 5.17. – *The variety $\mathcal{M}_N^l(Z, t)$ is connected, i.e., $\mathcal{M}_N^l(Z, t) = \mathcal{M}_N^l(Z, t)_*$. Thus, $\kappa : \mathbb{M}_N^l(Z, t) \rightarrow \mathcal{M}_N^l(Z, t)$ is an isomorphism.*

Proof. – See Subsection 5.6 below. □

REMARK 5.18. – 1. Recall that the multiplicative quiver variety $\mathcal{M}_N^l(Z, t)$ carries a Poisson structure (symplectic for generic parameters and generically symplectic for any parameters), coming from the quasi-Hamiltonian reduction procedure ([6]). Our results imply that the algebra $\mathbf{e}_N HH_N^l(Z, q, t)\mathbf{e}_N$ (where $q = e^\varepsilon$ and ε is a formal parameter) is a deformation quantization of this Poisson variety (namely, the matching of Poisson brackets may be checked on the open set where \mathbf{X} is invertible, using the results of [35]).

2. We expect that the results of this subsection can be lifted to the quantum level. Namely, we expect that the algebra $\mathbf{e}_N HH_N^l(Z, q, t)\mathbf{e}_N$ is isomorphic to the quantization of the multiplicative quiver variety $\mathcal{M}_N^l(Z, t)$ defined by D. Jordan in [29], via a quantization of the map κ .⁽⁵⁾ We note that this is known in the degenerate setting, see [36, 20].

REMARK 5.19. – The functions $\text{Tr}(\mathbf{D}^r)$, $r = 1, \dots, N$, form a classical integrable system on the symplectic variety $\mathcal{M}_N^l(Z, t)$. This system is the classical limit of the quantum integrable system $\{D_1^r + \dots + D_N^r, r = 1, \dots, N\}$ in the (spherical) cyclotomic DAHA discussed in Subsection 3.6. These classical integrable systems have been studied independently by O. Chalykh and M. Fairon ([11]).

5.4. Multiplicative bow varieties

In this section we follow the notations of [34]. Given a bow diagram with a balanced dimension vector as in [34, (6.1)] we consider the auxiliary diagram of vector spaces and linear maps as in [34, 6.1]. Let $Z_{r,i}$, $i = 0, \dots, n - 1$; $1 \leq r \leq \mathbf{w}_i$, be a collection of nonzero complex numbers, and let t be another nonzero complex number.

Let $\mathbb{M}_{Z,t}^\times(\mathbf{y}, \mathbf{w})$ be the variety of collections $(A_i, B_i, B'_i, a_i, b_i, D_{r,i}, C_{r,i})$ such that (cf. [34, (6.3), 2.2])

- (i) B_i, B'_i are invertible for all $i = 0, \dots, n - 1$;
- (a) $B'_i A_{i-1} - A_{i-1} B_{i-1} + a_i b_{i-1} = 0$;
- (b) $(B'_i)^{-1} = t^{-1} Z_{1,i}^{-1} (1 + D_{1,i} C_{1,i})$;
- (c) $(1 + C_{k,i} D_{k,i}) = Z_{k,i} Z_{k+1,i}^{-1} (1 + D_{k+1,i} C_{k+1,i})$;

⁽⁵⁾ We note that for $l = 1$ we would need a slightly less localized version than that of [29], not requiring \mathbf{X} to be invertible, which should produce not the full DAHA but its subalgebra.

(d) $B_i^{-1} = Z_{\mathbf{w}_i, i}^{-1}(1 + C_{\mathbf{w}_i, i} D_{\mathbf{w}_i, i});$

(S1) There is no nonzero subspace $0 \neq S \subset V_i^{\mathbf{w}_i}$ with $B_i(S) \subset S$, $A_i(S) = 0 = b_i(S);$

(S2) There is no proper subspace $T \subsetneq V_{i+1}^0$ with $B'_{i+1}(T) \subset T$, $\text{Im } A_i + \text{Im } a_{i+1} \subset T.$

Then $\mathbb{M}_{\underline{Z}, t}^{\times}(\mathbf{v}, \mathbf{w})$ is an affine algebraic variety acted upon by $G = \prod_{i=0}^{n-1} \prod_{k=0}^{\mathbf{w}_i} GL(V_i^k),$ cf. [34, Proposition 3.2].

We define a multiplicative bow variety $\mathcal{M}_{\underline{Z}, t}^{\times}(\mathbf{v}, \mathbf{w})$ as the categorical quotient $\mathbb{M}_{\underline{Z}, t}^{\times}(\mathbf{v}, \mathbf{w}) // G,$ cf. [34, 2.2].

5.5. K -theoretic Coulomb branch

Recall [8, Section 2] that given a representation \mathbf{N} of a reductive group G one can consider the variety of triples $\mathcal{R}.$ ⁽⁶⁾ According to [8, Remark 3.9(3)], the equivariant K -theory $K^{G\theta}(\mathcal{R})$ is a commutative ring with respect to convolution. Moreover, if the G -action on \mathbf{N} extends to an action of a larger group \tilde{G} containing G as a normal subgroup, then the equivariant K -theory $K^{\tilde{G}\theta}(\mathcal{R})$ is a commutative ring with respect to convolution, a deformation of $K^{G\theta}(\mathcal{R})$ over $\text{Spec}(K^{G_F}(\text{pt}))$ where $G_F = \tilde{G}/G$ is the flavor symmetry group. The affine variety $\text{Spec}(K^{\tilde{G}\theta}(\mathcal{R}))$ is denoted $\mathcal{M}_{\mathbb{C}}^{\times}(\tilde{G}, \mathbf{N})$ and called the K -theoretic Coulomb branch.

A framed oriented quiver representation gives rise to a representation

$$\mathbf{N} = \bigoplus_{i \rightarrow j} \text{Hom}(V_i, V_j) \oplus \bigoplus_i \text{Hom}(W_i, V_i)$$

of $G = GL(V) := \prod_i GL(V_i).$ Choosing a maximal torus $T(W_i) \subset GL(W_i),$ we consider the natural action of $\hat{G} := \mathbb{C}^{\times} \times G \times \prod_i T(W_i)$ on \mathbf{N} where \mathbb{C}^{\times} acts by dilation on the component $\mathbf{N}_{\text{hor}} := \bigoplus_{i \rightarrow j} \text{Hom}(V_i, V_j)$ of $\mathbf{N}.$ If $W := \bigoplus_i W_i \neq 0,$ the action of scalars $\mathbb{C}^{\times} \subset T(W) := \prod_i T(W_i)$ coincides with the action of scalars $\mathbb{C}^{\times} \subset G.$ Hence if $W \neq 0,$ the action of \hat{G} on \mathbf{N} factors through the action of $\tilde{G} := \mathbb{C}^{\times} \times (G \times T(W))/\mathbb{C}^{\times}.$ If $W = 0,$ we denote $\tilde{G} := \hat{G}.$ The corresponding K -theoretic Coulomb branch $\mathcal{M}_{\mathbb{C}}^{\times}(\tilde{G}, \mathbf{N})$ of a framed quiver gauge theory will be denoted by $\mathcal{M}_{\mathbb{C}}^{\times}$ for short.

Similarly to [34, Theorem 6.18] one can construct an isomorphism $\mathcal{M}_{\mathbb{C}}^{\times} \xrightarrow{\sim} \mathcal{M}_{\underline{Z}, t}^{\times}(\mathbf{v}, \mathbf{w})$ from the K -theoretic Coulomb branch of a framed quiver gauge theory of affine type A_{n-1} with dimension vectors $\mathbf{v}, \mathbf{w}.$ Here \underline{Z}, t in the RHS are parameters corresponding to the equivariant flavor symmetry parameters in the LHS, cf. [34, 6.8.2]. The proof of the above isomorphism in particular shows that $\mathcal{M}_{\underline{Z}, t}^{\times}(\mathbf{v}, \mathbf{w})$ is connected (and $\mathcal{M}_{\mathbb{C}}^{\times}$ is connected similarly to [8, Corollary 5.22]).

5.6. Example

We consider a special case when a bow diagram has 1 cross and l circles (we allow $l = 0$), that is $n = 1,$ $\mathbf{w}_0 = l,$ and $\dim V_0^k = N$ for any $k = 0, \dots, l.$ Then according to [34, Lemma 3.1], $A_0: V_0^l \rightarrow V_0^0$ is an isomorphism. We identify $V_0^l \equiv V_0^0$ with the help of $A_0,$ and then the definition of $\mathcal{M}_{\underline{Z}, t}^{\times}(N, l)$ of Section 5.4 becomes nothing but the definition

⁽⁶⁾ It is different from the one considered in Section 4.1.

of $\mathcal{M}_N^l(Z, t)$ of Section 5.1, that is $\mathcal{M}_{\mathbb{Z}, t}^{\times}(N, l) \simeq \mathcal{M}_N^l(Z, t)$. In particular, we conclude that $\mathcal{M}_N^l(Z, t)$ is connected, which proves Theorem 5.17.

REMARK 5.20. – The relations between general multiplicative bow varieties and (various versions of) multiplicative quiver varieties for a cyclic quiver are explained in Appendix .

6. Application to q -deformed m -quasiinvariants

6.1. q -deformed quasiinvariants

Let m be a nonnegative integer, and $q \in \mathbb{C}^*$.

DEFINITION 6.1. – [12] We call $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_N^{\pm 1}]$ a q -deformed m -quasiinvariant if $(1 - s_{ij})F$ is divisible by $\prod_{p=-m}^m (X_i - q^p X_j)$ for any $i < j$.

The algebra of q -deformed m -quasiinvariant Laurent (or trigonometric) polynomials will be denoted by $Q_{m,q}^{\text{trig}}$.

Let $Q_{m,q} \subset Q_{m,q}^{\text{trig}}$ be the graded algebra of q -deformed quasiinvariants inside $\mathbb{C}[X_1, \dots, X_N]$. By the Hilbert basis theorem, $Q_{m,q}$ is a finitely generated module over the ring of symmetric polynomials $\mathbb{C}[X_1, \dots, X_N]^{S_N}$. Note that $Q_{m,1} = Q_m$, the usual space of m -quasiinvariants defined by Chalykh and Veselov [13], i.e., polynomials F such that $(1 - s_{ij})F$ is divisible by $(X_i - X_j)^{2m+1}$.

THEOREM 6.2. – *For all except countably many values of q , the algebra $Q_{m,q}$ has the same Hilbert series as Q_m and is Cohen-Macaulay, i.e., a free module over $\mathbb{C}[X_1, \dots, X_N]^{S_N}$.*

In other words, Theorem 6.2 says that any quasiinvariant polynomial can be q -deformed. Note that the Hilbert series of Q_m is known (see [26], [5]).

REMARK 6.3. – Theorem 6.2 was conjectured by P.E. and E. Rains on the basis of a computer calculation.

Theorem 6.2 is proved in the next subsection.

REMARK 6.4. – The algebra of q -deformed trigonometric quasiinvariants $Q_{m,q}^{\text{trig}}(R)$ may be defined for any reduced root system R with Weyl group W and a W -invariant multiplicity function $m : R \rightarrow \mathbb{Z}_+$, see [12]. Namely, it is the algebra of regular functions F on the corresponding torus T such that for each $\alpha \in R_+$ the function $F(X) - F(s_\alpha X)$ is divisible by $\prod_{i=-m_\alpha}^{m_\alpha} (e^\alpha - q^i)$. Moreover, this algebra is Cohen-Macaulay for generic q . To see this, note that by using the exponential map $\exp : \text{Lie}(T) \rightarrow T$ and rescaling in $\text{Lie}(T)$, we may identify formal neighborhoods of closed points of $\text{Spec}(Q_{m,q}^{\text{trig}}(R))$ with those of $\text{Spec}(Q_m^{\text{trig}}(R))$, the usual trigonometric (a.k.a. non-homogeneous) quasiinvariants for R , see e.g., [24], Remark 6.4. But the algebra $Q_m^{\text{trig}}(R)$ is Cohen-Macaulay, see [24], Proposition 6.5. Hence so is $Q_{m,q}^{\text{trig}}(R)$, as desired. Note that this result also holds for $q = 1$ since in this case formal neighborhoods are the same as for usual (rational) quasiinvariants $Q_m(R)$.

Theorem 6.2 is a refinement of this result for $R = A_{N-1}$, as $Q_{m,q}^{\text{trig}}$ is a localization of $Q_{m,q}$.

6.2. Proof of Theorem 6.2

The Cohen-Macaulayness statement of Theorem 6.2 holds for $q = 1$ (i.e., for Q_m) by the results of [21], [5] (conjectured earlier in [25]). Thus it suffices to show that $Q_{m,q}$ is a flat deformation of Q_m when $q = e^\varepsilon$ and ε is a formal parameter.

Let \mathbf{e} be the symmetrizer of the finite Hecke algebra generated by T_i , and consider the action of the spherical subalgebra $\mathbf{e}HH_N(q, t)\mathbf{e}$ on $\mathbb{C}[X_1^{\pm 1}, \dots, X_N^{\pm 1}]^{S_N}$. This action is by q -difference operators. Consider the element $Y_1 + \dots + Y_N$. It is a central element of the affine Hecke algebra generated by T_i and Y_i , hence commutes with \mathbf{e} . Thus

$$(Y_1 + \dots + Y_N)\mathbf{e} \in \mathbf{e}HH_N(q, t)\mathbf{e}.$$

We will need the following lemma, due to Cherednik ([15]).

LEMMA 6.5. – *The element $\mathcal{M} := (Y_1 + \dots + Y_N)\mathbf{e}$ acts on $\mathbb{C}[X_1^{\pm 1}, \dots, X_N^{\pm 1}]^{S_N}$ by the first Macdonald operator*

$$M := \sum_{j=1}^N \left(\prod_{i \neq j} \frac{X_i - tX_j}{X_i - X_j} \right) \tau_j,$$

where $\tau_i X_j = q^{\delta_{ij}} X_j$.

Proof. – Denote the q -difference operator by which $(Y_1 + \dots + Y_N)\mathbf{e}$ acts by L . Clearly, we have $L = \sum_j f_j \tau_j$, where f_j are rational functions such that $s_r f_j = f_j$ for $r \neq j, j+1$, and $s_j f_j = f_{j+1}$. Thus it suffices to show that

$$f_1 = \prod_{i \neq 1} \frac{X_i - tX_1}{X_i - X_1}.$$

To this end, treat n as a variable, and note that

$$L(X_1^n + \dots + X_N^n) = \sum_j f_j(X_1, \dots, X_N)(q^n X_j^n + \sum_{i \neq j} X_i^n),$$

which we can treat as a polynomial in q^n, X_1^n, \dots, X_N^n with coefficients in $\mathbb{C}(X_1, \dots, X_N)$. Thus, f_1 may be obtained by extracting the coefficient of $q^n X_1^n$.

On the other hand, we have

$$\rho(Y_i) = t^{i-1} \prod_{j=i+1}^N \left(1 + \frac{(1-t)X_i}{X_j - X_i} (1 - s_{ij}) \right) \tau_i \prod_{j=1}^{i-1} \left(1 + \frac{(1-t^{-1})X_i}{X_j - X_i} (1 - s_{ij}) \right).$$

An easy direct computation using this formula shows that

$$\begin{aligned} & (Y_1 + \dots + Y_n)(X_1^n + \dots + X_N^n) \\ &= \left(\sum_{i=1}^N t^{i-1} \right) (X_1^n + \dots + X_N^n) + (q^n - 1) \sum_{i=1}^N t^{i-1} \prod_{j=i+1}^N \left(1 + \frac{(1-t)X_i}{X_j - X_i} (1 - s_{ij}) \right) X_i^n. \end{aligned}$$

Thus, only the second summand contributes to the coefficient of q^n , and in the second summand only $i = 1$ contributes to the coefficient of X_1^n . Moreover, in this contribution, the transposition terms (involving s_{ij}) don't contribute. Thus

$$f_1(X_1, \dots, X_N) = \prod_{j=2}^N \left(1 + \frac{(1-t)X_1}{X_j - X_1}\right) = \prod_{j=2}^N \frac{X_j - tX_1}{X_j - X_1},$$

as desired. □

We will also need the following lemma, which is a special case of Proposition 2.1 of [12] (but we give a proof for reader's convenience). Assume that q is not a root of unity.

LEMMA 6.6. – *If $t = q^{-m}$ then the operator M preserves $Q_{m,q}^{\text{trig}}$.*

Proof. – Let $r > l$, and $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_N^{\pm 1}]$. A direct computation shows that

$$\begin{aligned} (1 - s_{rl})(MF)(X_1, \dots, X_N) &= \prod_{j \neq r} \frac{X_j - q^{-m}X_r}{X_j - X_r} (F(\dots X_l \dots qX_r \dots) - F(\dots qX_r \dots X_l \dots)) \\ &= - \prod_{j \neq l} \frac{X_j - q^{-m}X_l}{X_j - X_l} (F(\dots X_r \dots qX_l \dots) - F(\dots qX_l \dots X_r \dots)). \end{aligned}$$

If $F \in Q_{m,q}^{\text{trig}}$ is a quasiinvariant then $F(\dots X_l \dots qX_r \dots) - F(\dots qX_r \dots X_l \dots)$ vanishes for $X_l = q^p X_r$ with $-m - 1 \leq p \leq m - 1$, while $F(\dots X_r \dots qX_l \dots) - F(\dots qX_l \dots X_r \dots)$ vanishes for $X_l = q^p X_r$ with $-m + 1 \leq p \leq m + 1$. Hence, both terms in the formula for $(1 - s_{rl})MF$ vanish when $X_l = q^p X_r$ with $-m \leq p \leq m$, $p \neq 0$ and are defined for $p = 0$ (vanishing in the extremal cases $p = m, -m$ follows from vanishing of the prefactors). Thus, $(1 - s_{rl})MF(X_1, \dots, X_N)$ is a polynomial which is antisymmetric in X_l, X_r , so divisible by $X_r - X_l$. This takes care of the case $p = 0$. □

Now note that since the algebra $\mathbf{e}HH_N(q, t)\mathbf{e}$ acts on $\mathbb{C}[X_1^{\pm 1}, \dots, X_N^{\pm 1}]$ by q -difference operators, this action can be extended tautologically to non-symmetric rational functions $\mathbb{C}(X_1^{\pm 1}, \dots, X_N^{\pm 1})$ (by the same difference operators).

COROLLARY 6.7. – *If $t = q^{-m}$ for $m \in \mathbb{Z}_{\geq 0}$ then the spherical cyclotomic DAHA $\mathbf{e}HH_N^1(q, t)\mathbf{e}$ (with $Z_1 = 1$) preserves the subspace of q -deformed m -quasiinvariants $Q_{m,q} \subset \mathbb{C}(X_1^{\pm 1}, \dots, X_N^{\pm 1})$.*

Proof. – By Proposition 3.6, the difference-reflection operators $\rho(L)$, $L \in HH_N^1(q, t)$ do not create poles at $X_i = 0$. Therefore, the action of $\mathbf{e}HH_N^1(q, t)\mathbf{e} \subset \mathbf{e}HH_N(q, t)\mathbf{e}$ on $\mathbb{C}(X_1, \dots, X_N)$ preserves the subspace $\mathbb{C}(X_1, \dots, X_N)_{\text{reg}}$ of functions regular at the hyperplanes $X_i = 0$, $i = 1, \dots, N$.

We now claim that the algebra $\mathbf{e}HH_N(q, t)\mathbf{e}$ preserves $Q_{m,q}^{\text{trig}}$. It suffices to prove this in the formal setting $q = e^\varepsilon$, $t = e^{-m\varepsilon}$. By Lemma 6.5 and Lemma 6.6, the element \mathcal{M} preserves $Q_{m,q}^{\text{trig}}$. Also, $\mathbb{C}[X_1^{\pm 1}, \dots, X_N^{\pm 1}]^{S_N}$ and $\tilde{Y} := \prod_i Y_i$ preserve $Q_{m,q}^{\text{trig}}$. But we claim that the element $\widehat{H} := \varepsilon^{-2}(\mathcal{M} - N - \varepsilon(\tilde{Y} - 1))$ and $\mathbb{C}[X_1^{\pm 1}, \dots, X_N^{\pm 1}]^{S_N}$ (topologically) generate $\mathbf{e}HH_N^{\text{formal}}(1, k)\mathbf{e}$. Indeed, the quasiclassical limit of \widehat{H} is the trigonometric Calogero-Moser

operator $H = \mathbf{e} \sum_i y_i^2 \mathbf{e} = \sum_i y_i^2 \mathbf{e}$, and by Lemma 2.7 H and $\mathbb{C}[X_1^{\pm 1}, \dots, X_N^{\pm 1}]^{S_N}$ generate $\mathbf{e}HH_{N,\text{deg}}(1, k)\mathbf{e}$. This implies the claim.

Thus, $\mathbf{e}HH_N^1(q, t)\mathbf{e}$ preserves $\mathbb{C}(X_1, \dots, X_N)_{\text{reg}} \cap Q_{m,q}^{\text{trig}} = Q_{m,q}$, as desired. □

Now we can complete the proof of the theorem. Since every polynomial divisible by $\prod_{i < j} \prod_{p=-m}^m (X_i - q^p X_j)$ is automatically a q -deformed m -quasiinvariant, the leading coefficient of the Hilbert polynomial of $Q_{m,q}$ is the same as that of Q_m . On the other hand, working in the formal setting ($q = e^\varepsilon$) and reducing Corollary 6.7 modulo ε , we find that $Q_{m,q}/(\varepsilon)$ is a submodule of the module Q_m over the spherical rational Cherednik algebra $\mathbf{e}HH_N^{\text{rat}}(1, m)\mathbf{e}$. But as shown in [5], $Q_m = \bigoplus_\lambda \mathbf{e}M_m(\lambda) \otimes \pi_\lambda$, where λ runs over partitions of N , π_λ is the Specht module for S_N corresponding to λ , and $M_m(\lambda)$ is the Verma module over the rational Cherednik algebra $HH_N^{\text{rat}}(1, m)$ attached to λ . Since the $\mathbf{e}HH_N^{\text{rat}}(1, m)\mathbf{e}$ -modules $\mathbf{e}M_m(\lambda)$ are irreducible (see [5]), any proper $\mathbf{e}HH_N^{\text{rat}}(1, m)\mathbf{e}$ -submodule of Q_m would have a strictly smaller leading coefficient of the Hilbert series than that of Q_m . This implies that $Q_{m,q}/(\varepsilon) = Q_m$, i.e., $Q_{m,q}$ is a flat deformation of Q_m (i.e., has the same Hilbert series). By standard abstract nonsense, this applies also to numerical values of q , excluding a countable set. The theorem is proved.

6.3. Generalization: q -deformed cyclotomic quasiinvariants

Let us now use the cyclotomic DAHA to generalize Theorem 6.2 to the cyclotomic case.

Let $m \geq 0, l \geq 1, \mathbf{q} \in \mathbb{C}^*$, and $q = \mathbf{q}^l$. Let us first define the algebra of \mathbf{q} -deformed cyclotomic trigonometric quasiinvariants $Q_{m,\mathbf{q}}^{l,\text{trig}}$. We introduce variables x_i such that $X_i = x_i^l$. We define $Q_{m,\mathbf{q}}^{l,\text{trig}} \subset \mathbb{C}[x_1^{\pm 1}, \dots, x_N^{\pm 1}]$ to be the subalgebra of Laurent polynomials F such that for every i, j, r , the Laurent polynomial

$$F(\dots, x_i, \dots, x_j, \dots) - F(\dots, \zeta^r x_j, \dots, \zeta^{-r} x_i, \dots)$$

is divisible by $\prod_{p=-m}^m (x_i - \zeta^r \mathbf{q}^p x_j)$.

Now let m, m_1, \dots, m_{l-1} be nonnegative integers. For $0 \leq r \leq l - 1$ let \mathbf{p}_r be the homogeneous projector $\mathbb{C}[x] \rightarrow x^r \mathbb{C}[x^l]$, and $\mathbf{p}_r^{(i)}$ denotes \mathbf{p}_r acting with respect to x_i . By analogy with [4], let $Q_{m,m_1,\dots,m_{l-1},\mathbf{q}}^{l,\text{trig}} \subset Q_{m,\mathbf{q}}^{l,\text{trig}}$ be the graded space of all \mathbf{q} -deformed cyclotomic trigonometric quasiinvariants F inside $\mathbb{C}[x_1, \dots, x_N]$ such that

$$(6.1) \quad \mathbf{p}_r^{(i)} F \text{ is divisible by } x_i^{r+m_r l} \text{ for } i = 1, \dots, N \text{ and } r = 1, \dots, l - 1.$$

DEFINITION 6.8. – The space $Q_{m,m_1,\dots,m_{l-1},\mathbf{q}}^l$ is called the space of \mathbf{q} -deformed cyclotomic (m, m_1, \dots, m_{l-1}) -quasiinvariants.

By the Hilbert basis theorem, $Q_{m,m_1,\dots,m_{l-1},\mathbf{q}}^l$ is a finitely generated module over the ring of symmetric polynomials $\mathbb{C}[X_1, \dots, X_N]^{S_N}$. Note that $Q_{m,m_1,\dots,m_{l-1}}^l := Q_{m,m_1,\dots,m_{l-1},1}^l$ is the usual space of (m, m_1, \dots, m_{l-1}) -quasiinvariants for the complex reflection group $S_N \ltimes (\mathbb{Z}/l\mathbb{Z})^N$ defined in [4]. Note also that similarly to [4], $Q_{m,m_1,\dots,m_{l-1},\mathbf{q}}^l$ is not necessarily an algebra.

Our main result about cyclotomic q -deformed quasiinvariants is the following theorem, which is a generalization ⁽⁷⁾ of Theorem 6.2.

⁽⁷⁾ Theorem 6.2 is a special case of Theorem 6.9 for $l = 1$, but it is convenient for us to treat this special case first, and then pass to the general case.

THEOREM 6.9. – *For all except countably many values of \mathbf{q} , the space $Q_{m,m_1,\dots,m_{l-1},\mathbf{q}}^l$ has the same Hilbert series as $Q_{m,m_1,\dots,m_{l-1}}^l$, and is a free module over $\mathbb{C}[X_1, \dots, X_N]^{S_N}$.*

In other words, every quasiinvariant for $S_N \times (\mathbb{Z}/l\mathbb{Z})^N$ can be \mathbf{q} -deformed.

We note that the Hilbert series of $S_N \times (\mathbb{Z}/l\mathbb{Z})^N$ is computed in [4].

Theorem 6.9 is proved in the next subsection.

6.4. Proof of Theorem 6.9

The proof of Theorem 6.9 is parallel to the proof of Theorem 6.2, using the results of [4]. In fact, most of the technical statements we'll need have already been obtained in the proof of Theorem 6.2.

The freeness statement of Theorem 6.9 holds for $Q_{m,m_1,\dots,m_{l-1}}^l$ (i.e., for $\mathbf{q} = 1$) by the results of [4]. Thus it suffices to show that $Q_{m,m_1,\dots,m_{l-1},\mathbf{q}}^l$ is a flat deformation of $Q_{m,m_1,\dots,m_{l-1}}^l$ when $\mathbf{q} = e^{\varepsilon/l}$ and ε is a formal parameter.

To this end, recall from [4] that the space $Q_{m,m_1,\dots,m_{l-1}}^l$ carries an action of the spherical cyclotomic Cherednik algebra $\mathbf{e}\mathbb{H}_N^{l,\text{psc}}(c, 1, m)\mathbf{e}$, where \mathbf{e} is the symmetrizer for S_N and c_i are certain linear functions of m_j (in fact, this action is the main tool in [4] for proving that $Q_{m,m_1,\dots,m_{l-1}}^l$ is a free $\mathbb{C}[X_1, \dots, X_N]$ -module). Therefore, by Theorem 2.19, this space carries an action of $\mathbf{e}HH_N^l(z, 1, m)\mathbf{e}$, and it is easy to compute that $z_i = m_i/l, i = 1, \dots, l-1$ and $z_l = 0$.

The main idea of the proof is to show that this representation can be \mathbf{q} -deformed to a representation of the spherical cyclotomic DAHA $\mathbf{e}HH_N^l(Z, q, t)\mathbf{e}$ on $Q_{m,m_1,\dots,m_{l-1},\mathbf{q}}^l$, where $t = q^{-m}$ and $Z_i = q^{z_i} = \mathbf{q}^{m_i}$. Then, similarly to the proof of Theorem 6.2, the result will follow by looking at the leading coefficient of the Hilbert series and using [4, Theorem 8.2], which gives a decomposition of the $\mathbf{e}HH_N^l(z, 1, m)\mathbf{e}$ -module $Q_{m,m_1,\dots,m_{l-1}}^l$ into a direct sum of irreducible modules.

Finally, let us show that the representation of $\mathbf{e}HH_N^l(z, 1, m)\mathbf{e}$ on the space $Q_{m,m_1,\dots,m_{l-1}}^l$ admits a \mathbf{q} -deformation. Recall from the proof of Theorem 6.2 that the algebra $\mathbf{e}HH_N(q, t)\mathbf{e}$ acts on $\mathbb{C}(x_1^l, \dots, x_N^l)$ by difference operators. This action can be straightforwardly extended to the field extension $\mathbb{C}(x_1, \dots, x_N)$ by using the same formulas, where now $\tau_i x_j = \mathbf{q}^{\delta_{ij}} x_j$. Hence, the subalgebra $\mathbf{e}HH_N^l(Z, q, t)\mathbf{e} \subset \mathbf{e}HH_N(q, t)\mathbf{e}$ acts on $\mathbb{C}(x_1, \dots, x_N)$.

Since $X_i - q^p X_j = \prod_{r=0}^{l-1} (x_i - \zeta^r \mathbf{q}^p x_j)$, the argument in the proof of Theorem 6.2 implies that $\mathbf{e}HH_N(q, t)\mathbf{e}$ preserves the subspace $Q_{m,\mathbf{q}}^{l,\text{trig}} \subset \mathbb{C}(x_1, \dots, x_N)$. Hence, so does the subalgebra $\mathbf{e}HH_N^l(Z, q, t)\mathbf{e}$. Also, by Proposition 3.6, the algebra $\mathbf{e}HH_N^l(Z, q, t)\mathbf{e}$ preserves the space $(x_1 \cdots x_N)^{m_i} \mathbb{C}(X_1, \dots, X_N)_{\text{reg}}$, where, as before, the subscript “reg” means functions regular at the generic points of the hyperplanes $X_i = 0$. Therefore, the algebra $\mathbf{e}HH_N^l(Z, q, t)\mathbf{e}$ preserves quasiinvariance conditions (6.1).

Thus, we see that the algebra $\mathbf{e}HH_N^l(Z, q, t)\mathbf{e}$ preserves the space $Q_{m,m_1,\dots,m_{l-1},\mathbf{q}}^l$. Moreover, it is easy to see that the classical limit of this representation as $\mathbf{q} \rightarrow 1$ is exactly the representation of $\mathbf{e}HH_N^l(z, 1, m)\mathbf{e}$ on the space $Q_{m,m_1,\dots,m_{l-1}}^l$ constructed in [4]. This completes the proof of Theorem 6.9.

6.5. Twisted quasiinvariants

Let $a_1, \dots, a_N \in \mathbb{C}$, $m \in \mathbb{Z}_+$. Let $Q_m(a_1, \dots, a_N)$ be the space of polynomials $F \in \mathbb{C}[X_1, \dots, X_N]$ such that the function

$$\tilde{F}(X_1, \dots, X_N) := \left(\prod_i X_i^{a_i} \right) F(X_1, \dots, X_N)$$

(regarded as a function of $X_i > 0$) is m -quasiinvariant, in the sense that $(1 - s_{ij})\tilde{F}$ vanishes to order $2m + 1$ at $X_i = X_j$ for all $i < j$. Note that $Q_m(a_1, \dots, a_N) = Q_m(a_1 - a, \dots, a_N - a)$, and $Q_m(a, \dots, a) = Q_m$ for all a . Obviously, $Q_m(a_1, \dots, a_N)$ is a graded $\mathbb{C}[X_1, \dots, X_N]^{S_N}$ -module. By the Hilbert basis theorem, this module is finitely generated.

DEFINITION 6.10. – We will call $Q_m(a_1, \dots, a_N)$ the module of *twisted quasiinvariants*.

THEOREM 6.11. – *If $a_i - a_j \notin \mathbb{Z} \setminus \{0\}$ for all $i < j$, then $Q_m(a_1, \dots, a_N)$ is a free $\mathbb{C}[X_1, \dots, X_N]^{S_N}$ -module (of rank $N!$).*

EXAMPLE 6.12. – Let $N = 2$. Then $Q_m(a, 0)$ is a free module over $R := \mathbb{C}[X_1, X_2]^{S_2}$ for any $a \in \mathbb{C}$. We show this by induction in m . The base case $m = 0$ is obvious. Let $m > 0$. Then it is easy to show that the lowest degree d of a nonzero element $P_{a,m}$ in $Q_m(a, 0)$ is a if $a = 0, 1, \dots, m - 1$, and m otherwise. Moreover, we can uniquely choose $P_{a,m}$ so that $P_{a,m}(X, X) = X^d$; for example, if $0 \leq a \leq m$ is an integer, then $P_{a,m} = X_2^a$, and

$$P_{a,1}(X_1, X_2) = \frac{(a - 1)X_1 + (a + 1)X_2}{2a}$$

for $a \neq 0$. Therefore,

$$(6.2) \quad Q_m(a, 0) = RP_{a,m} + (X_1 - X_2)^2 Q_{m-1}(a, 0).$$

Indeed, given a nonzero homogeneous $F \in Q_m(a, 0)$ such that $F(X, X) = \beta X^r$, consider $F' := F - 2^{d-r} \beta (X_1 + X_2)^{r-d} P_{a,m}$. Then $F' \in Q_m(a, 0)$ and is divisible by $X_1 - X_2$, so it is in $(X_1 - X_2)^2 Q_{m-1}(a, 0)$.

Consider first the case when $0 \leq a \leq m - 1$ is an integer. Then by the induction assumption, the module $Q_{m-1}(a, 0)$ is free (of rank 2), so it is generated by X_2^a and some homogeneous polynomial $f_{a,m}$ of degree $2m - 1 - a$. So by (6.2), $Q_m(a, 0)$ is generated by X_2^a and $f_{a,m+1} := (X_1 - X_2)^2 f_{a,m}$, which validates the induction step.

Now consider the case $a \neq 0, \dots, m - 1$. Then $Q_{m-1}(a, 0)$ is free by the induction assumption, so it is generated by some homogeneous polynomials of $P_{a,m-1}$, $T_{a,m-1}$ of degrees $m - 1$ and m , respectively, such that $T_{a,m}(X, X) = 0$ (as one can easily check that one always has such generators). It is easy to see that there exists $\alpha_{a,m} \in \mathbb{C}$ such that

$$P_{a,m} := T_{a,m-1} + \frac{1}{2} \alpha_{a,m} (X_1 + X_2) P_{a,m-1} \in Q_m(a, 0),$$

and $T_{a,m} := (X_1 - X_2)^2 P_{a,m-1} \in Q_m(a, 0)$. Moreover, by (6.2), these elements generate $Q_m(a, 0)$. This completes the induction step.

This argument also implies that the Hilbert series of $Q_m(a, 0)$ is $\frac{t^a + t^{2m+1-a}}{(1-t)(1-t^2)}$ if $0 \leq a \leq m - 1$ is an integer, and $\frac{t^m}{(1-t)^2}$ otherwise.

EXAMPLE 6.13. – In spite of Example 6.12, for $N \geq 3$ the condition on the a_i cannot be dropped. Indeed, for $N = 3$, the computer calculation shows that the Hilbert series of $Q_2(1, 0, 0)$ has the form

$$\begin{aligned} h(t) &= t^2(1 + t + 2t^2 + 3t^3 + 5t^4 + 7t^5 + 10t^6 + 15t^7 + 20t^8 + 26t^9 + 33t^{10} + \dots) \\ &= \frac{t^2 + t^6 + t^7 + 2t^9 + t^{10} - t^{12} + \dots}{(1-t)(1-t^2)(1-t^3)} \end{aligned}$$

and the minus sign in the numerator shows that $Q_2(1, 0, 0)$ cannot be a free module over symmetric polynomials. Indeed, if this module were free, the numerator would have been the Hilbert polynomial of the generators.

Proof of Theorem 6.11. – By permuting a_i we may assume that

$$(a_1, \dots, a_N) = (z_1, \dots, z_1, \dots, z_l, \dots, z_l),$$

where $z_i \neq z_j$ and z_i occurs N_i times, where $N = N_1 + \dots + N_l$. By simultaneously shifting a_i we may assume that $z_l = 0$.

Assume first that $z_r = \frac{r}{l} + m_r$, $1 \leq r \leq l-1$. Let χ be the character of $(\mathbb{Z}/l\mathbb{Z})^N$ given by $\chi(\sigma_i) = \zeta^{la_i}$. Denote by $Q_{m,m_1,\dots,m_{l-1}}^{l,\chi}$ the χ -eigenspace of $(\mathbb{Z}/l\mathbb{Z})^N$ in $Q_{m,m_1,\dots,m_{l-1}}^l$. Then it is easy to see that $F \in Q_{m,m_1,\dots,m_{l-1}}^{l,\chi}$ if and only if it has the form

$$F(x_1, \dots, x_N) = X_1^{a_1} \dots X_N^{a_N} f(X_1, \dots, X_N),$$

where $X_i = x_i^l$, and satisfies the quasiinvariance condition saying that $(1 - s_{ij})F$ vanishes to order $2m + 1$ at $X_i = X_j$ (for $X_i > 0$), i.e., if and only if $f \in Q_m(a_1, \dots, a_N)$.

Let $G := S_N \ltimes (\mathbb{Z}/l\mathbb{Z})^N$. By [4, Theorem 8.2],

$$(6.3) \quad Q_{m,m_1,\dots,m_{l-1}}^{l,\chi} = \bigoplus_{\tau \in \text{Irrep}(G): \tau^\chi \neq 0} \mathbf{e}_G M_{c,k}(\tau^*) \otimes \tau^\chi,$$

where \mathbf{e}_G is the symmetrizer for G , $M_{c,k}(\tau)$ denotes the Verma module over the cyclotomic rational Cherednik algebra $\mathbb{H}_N^{l,\text{cyc}}(c, 1, k)$ for appropriate c, k , and the superscript χ denotes the χ -eigenspace. Here the grading on $Q_{m,m_1,\dots,m_{l-1}}^{l,\chi}$ is obtained from the grading on the right hand side of (6.3) (given by the scaling element \mathbf{h} of the rational Cherednik algebra) by dividing by l and shifting by $m \frac{N(N-1)}{2} + \frac{N}{2l}$.

Thus, $Q_m(a_1, \dots, a_N)$ is a free module over $\mathbb{C}[X_1, \dots, X_N]^{S_N}$, with Hilbert series independent of the numbers m_r . Since this holds for a Zariski dense set of vectors (z_1, \dots, z_{l-1}) (namely, $z_r = \frac{r}{l} + m_r$, $m_r \in \mathbb{Z}_+$), this holds for Weil generic (z_1, \dots, z_{l-1}) .

It remains to show that the statement holds if $z_i - z_j$ is not a nonzero integer for $1 \leq i < j \leq l$. Using Theorem 2.19 and Formula (6.3), we see that $Q_{m,m_1,\dots,m_{l-1}}^{l,\chi}$ is a module over the spherical subalgebra $\mathbf{e} \mathbb{H}_{N,\text{deg}}^l(z, 1, m) \mathbf{e}$, where $z_r = \frac{r}{l} + m_r$ for $1 \leq i \leq l-1$ and $z_l = 0$. Interpolating to arbitrary complex values of z_i , we get that for any $z_1, \dots, z_{l-1} \in \mathbb{C}$, the algebra $\mathbf{e} \mathbb{H}_{N,\text{deg}}^l(z, 1, m) \mathbf{e}$ with $z := (z_1, \dots, z_{l-1}, 0)$ acts on $Q_m(a_1, \dots, a_N)$. If $z_i - z_j$ is not a nonzero integer, then the category \mathcal{O} for the algebra $\mathbf{e} \mathbb{H}_{N,\text{deg}}^l(z, 1, m) \mathbf{e}$ is semisimple (see [4, Theorem 6.6]). Moreover, by a deformation argument, $Q_m(a_1, \dots, a_N)$ must contain representation (6.3). Since every irreducible representation in \mathcal{O} has full support, this implies that $Q_m(a_1, \dots, a_N)$ coincides with (6.3). This proves the theorem. \square

Formula (6.3) allows us to easily compute the character of $Q_m(a_1, \dots, a_N)$ as a graded $S_{N_1} \times \dots \times S_{N_l}$ -module. Namely, given irreducible representations π_r of S_{N_r} , we have from (6.3):

$$\text{Hom}_{\prod_r S_{N_r}}(\pi_1 \otimes \dots \otimes \pi_l, Q_m(a_1, \dots, a_N)) = \mathbf{e}_G M_{c,k}(\tau(\pi_1, \dots, \pi_l, \chi)^*),$$

where $\tau(\pi_1, \dots, \pi_l, \chi) = \text{Ind}_{(\mathbb{Z}/l\mathbb{Z})^N \rtimes \prod_r S_{N_r}}^{(\mathbb{Z}/l\mathbb{Z})^N \rtimes S_N} \chi$, and the grading is modified as explained above. Therefore, we have

PROPOSITION 6.14. – *If $(a_1, \dots, a_N) = (z_1, \dots, z_1, \dots, z_l, \dots, z_l)$ and $z_i - z_j$ are not nonzero integers, then the Hilbert series of the graded vector space $\text{Hom}_{\prod_r S_{N_r}}(\pi_1 \otimes \dots \otimes \pi_l, Q_m(a_1, \dots, a_N))$ equals*

$$h_{\pi_1, \dots, \pi_l}(t) = t^{m(\frac{N(N-1)}{2} - \sum_{r=1}^l \text{cont}(\pi_r))} h_{\pi_1}(t) \dots h_{\pi_l}(t),$$

where for an irreducible representation π of S_n , $\text{cont}(\pi)$ is the content of the Young diagram of π , and $h_\pi(t)$ is the Hilbert series of the graded space $(\pi \otimes \mathbb{C}[X_1, \dots, X_n])^{S_n}$, i.e.,

$$h_\pi(t) = \frac{K_\pi(t)}{(1-t) \dots (1-t^n)},$$

where $K_\pi(t)$ is the Kostka polynomial associated to π .

EXAMPLE 6.15. – 1. If $l = 1$ (i.e., $a_i = 0$), we recover the standard formula for the character of Q_m , see e.g., [26].

2. Let $N = 2, l = 2, N_1 = N_2 = 1$. We get from Proposition 6.14 that the Hilbert series of $Q_m(a_1, a_2)$ is $h(t) = \frac{t^m}{(1-t)^2}$, i.e., we recover the formula of Example 6.12 for the case of generic a .

3. Let $N = 3, l = 3, N_1 = N_2 = N_3 = 1$. Then we get from Proposition 6.14 that the Hilbert series of $Q_m(a_1, a_2, a_3)$ is $h(t) = \frac{t^{3m}}{(1-t)^3}$.

4. Let $N = 3, l = 2, N_1 = 1, N_2 = 2$. The space $Q_m(a, 0, 0)$ splits into the direct sum

$$Q_m(a, 0, 0) = Q_m(a, 0, 0)_+ \oplus Q_m(a, 0, 0)_-,$$

the symmetric and antisymmetric part under s_{23} . Denoting the Hilbert series of these spaces by h_+ and h_- , we get from Proposition 6.14:

$$h_+(t) = \frac{t^{2m}}{(1-t)^2(1-t^2)},$$

$$h_-(t) = \frac{t^{4m+1}}{(1-t)^2(1-t^2)}.$$

6.6. q -deformed twisted quasiinvariants

Keep the notation of the previous subsection, and let $q > 0$. Define the module of q -deformed twisted quasiinvariants $Q_{m,q}(a_1, \dots, a_N)$ to be the space of polynomials $F \in \mathbb{C}[X_1, \dots, X_N]$ such that the function

$$\widetilde{F}(X_1, \dots, X_N) := \left(\prod_i X_i^{a_i} \right) F(X_1, \dots, X_N)$$

(regarded as a function of $X_i > 0$) is a q -deformed m -quasiinvariant, in the sense that $(1 - s_{ij})\widetilde{F}$ is divisible by $\prod_{p=-m}^m (X_i - q^p X_j)$ in the ring of analytic functions. If

$q \neq 1$, this is equivalent to saying that $(1 - s_{ij})\tilde{F}$ vanishes if $X_i = q^p X_j, -m \leq p \leq m$. Note that

$$\begin{aligned} Q_{m,1}(a_1, \dots, a_N) &= Q_m(a_1, \dots, a_N), \\ Q_{m,q}(a_1, \dots, a_N) &= Q_{m,q}(a_1 - a, \dots, a_N - a), \\ \text{and } Q_{m,q}(a, \dots, a) &= Q_{m,q} \text{ for all } a. \end{aligned}$$

Obviously, $Q_{m,q}(a_1, \dots, a_N)$ is a graded $\mathbb{C}[X_1, \dots, X_N]^{S_N}$ -module. By the Hilbert basis theorem, this module is finitely generated.

THEOREM 6.16. – *If $a_i - a_j$ are not nonzero integers then for all but countably many q , the Hilbert series of $Q_{m,q}(a_1, \dots, a_N)$ coincides with the one for $Q_m(a_1, \dots, a_N)$ (i.e., $Q_{m,q}(a_1, \dots, a_N)$ is a flat q -deformation of $Q_m(a_1, \dots, a_N)$). Moreover, $Q_{m,q}(a_1, \dots, a_N)$ is a free $\mathbb{C}[X_1, \dots, X_N]^{S_N}$ -module (of rank $N!$).*

Proof. – The proof is parallel to the proof of Theorem 6.9, using Theorem 6.11. Namely, the second statement follows from the first one, and it suffices to prove the first statement in the formal setting $q = e^\varepsilon$. In this case, one shows as in the proof of Theorem 6.9 that the action of $\mathbf{e}HH_{N,\text{deg}}^l(z, 1, m)\mathbf{e}$ on $Q_m(a_1, \dots, a_N)$ constructed in the proof of Theorem 6.11 can be q -deformed to an action of $\mathbf{e}HH_N^{l,\text{formal}}(z, 1, m)\mathbf{e}$ on $Q_{m,q}(a_1, \dots, a_N)$. The rest of the argument is the same as in the proof of Theorem 6.9. \square

REMARK 6.17. – We expect that $Q_{m,q}(a_1, \dots, a_N)$ is a flat deformation of $Q_m(a_1, \dots, a_N)$ for all values of a_i , in particular when the module $Q_m(a_1, \dots, a_N)$ is not necessarily free. This has been confirmed by a computer calculation in low degrees in Example 6.13.

Also, we expect that Theorems 6.2, 6.9, 6.16 hold for all nonzero values of q .

Appendix

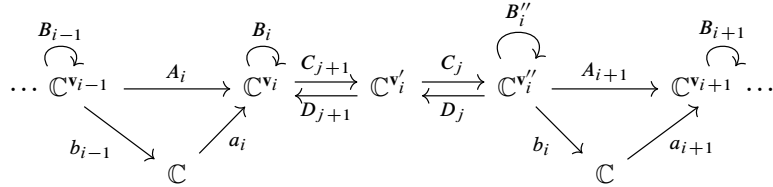
More general multiplicative quiver and bow varieties by Hiraku Nakajima and Daisuke Yamakawa

In this appendix, we study more general multiplicative quiver and bow varieties, as multiplicative analog of results in [34].

Multiplicative bow varieties are defined as in Section 5.4 corresponding to more general bow diagrams with dimension vectors not necessarily satisfying the balanced condition:

$$(A.1) \quad \begin{array}{ccccccc} & \mathbf{v}_{i-1} & \mathbf{v}_i & \mathbf{v}'_i & \mathbf{v}''_i & \mathbf{v}_{i+1} & \\ & \times & \circ & \circ & \times & & \\ \cdots & & x_i & h_{j+1} & h_j & x_{i+1} & \cdots \end{array}$$

Recall that the balanced condition is $v_i = v'_i = v''_i$ for all i . We put A, B, a, b at \mathbf{x} , and C, D at \mathbf{O} as



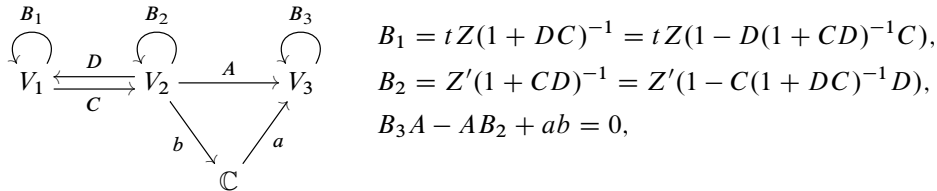
We only consider the stability parameter $v^{\mathbb{R}} = 0$ for brevity, as generalization to arbitrary $v^{\mathbb{R}}$ is straightforward.

The definition of multiplicative quiver varieties is more delicate when $\dim W \neq 1$, hence will be given in Section A.3.

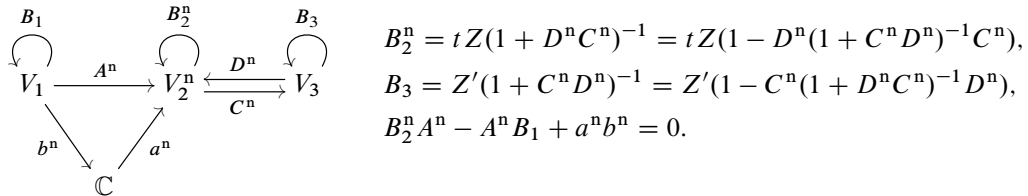
A.1. Hanany-Witten transition

We first introduce Hanany-Witten transition for multiplicative bow varieties.

Consider the following part of bow data:



where t, Z, Z' are fixed nonzero scalars. We consider another bow data with the same V_1, V_3, B_1, B_3 :



We take quotients by $GL(V_2), GL(V_2^n)$ respectively. (But not by $GL(V_1), GL(V_3)$.)

PROPOSITION A.1. – *There exists a $GL(V_1) \times GL(V_3)$ -equivariant isomorphism between two varieties above.*

The isomorphism is explicitly given during the proof.

Proof. – We consider a three term complex

$$V_2 \xrightarrow{\alpha = \begin{bmatrix} D \\ A \\ b \end{bmatrix}} V_1 \oplus V_3 \oplus \mathbb{C} \xrightarrow{\beta = [AB_2C \ (B_3 - Z') \ a]} V_3,$$

where $\beta\alpha = 0$ follows from one of the defining equation. We claim that α is injective. We consider $S = \text{Ker}\alpha$. Then $A(S) = 0 = b(S)$ and $B_2(S) = (1 - C(1 + DC)^{-1}D)(S) = S$. Therefore the condition (S1) implies $S = 0$.

Let

$$V_2^n := \text{Coker } \alpha.$$

We define new bow data as

- A^n, D^n, a^n are composition of inclusions of V_1, V_3, \mathbb{C} , to $V_1 \oplus V_3 \oplus \mathbb{C}$ and the projection $V_1 \oplus V_3 \oplus \mathbb{C} \rightarrow V_2^n$ respectively.
- $b^n = bCB_1$, and C^n is a homomorphism induced from $-B_3^{-1}\beta$.
- $B_2^n = tZ(1 + D^n C^n)^{-1} = tZ(1 - D^n(Z'^{-1}B_3)C^n)$.

The last definition requires checking of the invertibility of $1 + D^n C^n$. Let us consider $1 + C^n D^n$. From the definition it is $1 - B_3^{-1}(B_3 - Z') = Z' B_3^{-1}$. This is the second of the defining equation. Thus $1 + C^n D^n$ is invertible, and hence $1 + D^n C^n$ is also invertible. (In fact, $(1 + D^n C^n)^{-1} = 1 - D^n(1 + C^n D^n)^{-1} C^n$.) Hence the above B_2^n is well defined.

In order to check the remaining defining equation, we consider

$$V_1 \xrightarrow{\alpha^n = \begin{bmatrix} tZ - B_1 \\ -C^n B_2^n A^n \\ b^n \end{bmatrix}} V_1 \oplus V_3 \oplus \mathbb{C} \xrightarrow{\beta^n = [A^n \ D^n \ a^n]} V_2^n.$$

This is a complex if and only if the last defining equation holds. Observe that β^n is nothing but the natural projection. We also have

$$(A.2) \quad \alpha^n = \begin{bmatrix} tZ - B_1 \\ -tZZ'^{-1}B_3C^nA^n \\ bCB_1 \end{bmatrix} = \begin{bmatrix} DCB_1 \\ tZZ'^{-1}AB_2C \\ bCB_1 \end{bmatrix} = \alpha CB_1,$$

where we have used $CB_1 = tZZ'^{-1}B_2C$, $C^n B_2^n = tZZ'^{-1}B_3C^n$. Therefore $\beta^n \alpha^n = 0$.

Let us check the condition (S1) for new data. Take a subspace $S \subset V_1$ such that $B_1(S) \subset S$, $A^n(S) = 0 = b^n(S)$. Observe that $A^n(S) = 0$ means $S \oplus 0 \oplus 0 \subset \text{Im } \alpha$. Let us consider $\tilde{S} = \alpha^{-1}(S \oplus 0 \oplus 0)$. Then $D(\tilde{S}) = S$ and $A(\tilde{S}) = 0 = b(\tilde{S})$. Therefore

$$\alpha(Z' - B_2)(\tilde{S}) = \alpha B_2 C D(\tilde{S}) = \alpha B_2 C(S) = \alpha CB_1(S) = \alpha^n(S) = \begin{bmatrix} (Z - B_1)(S) \\ 0 \\ 0 \end{bmatrix}.$$

The condition $B_1(S) \subset S$ implies $B_2(\tilde{S}) \subset \tilde{S}$. Hence $\tilde{S} = 0$ thanks to (S1) for the original data. We have $S = 0$ as well.

Let us check the condition (S2). Suppose we have a subspace $T \subset V_2^n$ such that $B_2^n(T) \subset T$, $\text{Im } A^n + \text{Im } a^n \subset T$. We take its inverse image $\tilde{T} = (\beta^n)^{-1}(T)$ in $V_1 \oplus V_3 \oplus \mathbb{C}$. By the second assumption, it contains $V_1 \oplus \{0\} \oplus \mathbb{C}$. Hence \tilde{T} is a form of $V_1 \oplus \tilde{T} \oplus \mathbb{C}$ for $\tilde{T} \subset V_3$. We also have $\text{Im } \alpha \subset \tilde{T}$. Hence $A(V_2) \subset \tilde{T}$. As $B_2^n = tZ(1 - D^n(Z'^{-1}B_3)C^n)$, the condition $B_2^n(T) \subset T$ implies $0 \oplus \beta(\tilde{T}) \oplus 0 \subset \tilde{T}$, i.e., $AB_2C(V_1) + (B_3 - Z')(\tilde{T}) + a(\mathbb{C}) \subset \tilde{T}$. Hence $\tilde{T} = V_3$ thanks to (S2) for the original data. We have $T = V_2^n$ as well.

The inverse construction is clear. The original vector space V_2 is recovered from the new data as $\text{Ker } \beta^n$. Note also β^n is surjective thanks to (S2). Then A, b, D are given as restrictions of projections $V_1 \oplus V_3 \oplus \mathbb{C}$ to V_3, \mathbb{C}, V_1 to $\text{Ker } \beta^n$ respectively, a is $-B_3 C^n a^n$, and C is $\alpha^n B_1^{-1}$ by (A.2). Finally we set $B_2 = Z'(1 + CD)^{-1}$.

The conditions (S1),(S2) for (A, B_2, B_3, a, b) follow from the conditions (S1),(S2) for $(A^n, B_1, B_2^n, a^n, b^n)$. We leave the details to the reader as an exercise. \square

Once this isomorphism is established, the remaining arguments of [34, §7] only use dimension vectors, hence work for the multiplicative case. To state the result, we recall invariants (see [34, §7.3]): Let N_{x_i} be the difference (left minus right) of entries of the dimension vector at x_i . Then set $N(x_i, x_{i+1}) = N_{x_i} - N_{x_{i+1}}$ plus the number of \circ between $x_i \rightarrow x_{i+1}$. It is invariant under Hanany-Witten transition. Now by [34, Prop. 7.20] we have

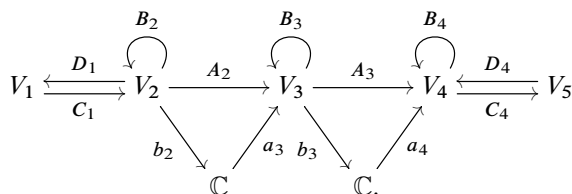
PROPOSITION A.2. – *If $N(x_i, x_{i+1}) \geq 0$ for any i , the multiplicative bow variety is isomorphic to another multiplicative bow variety with a cobalanced dimension vector by successive applications of Hanany-Witten transitions of Proposition A.1.*

Recall the cobalanced condition is $N_{x_i} = 0$ for all i .

Recall (Section 5.5) a K -theoretic Coulomb branch for a framed quiver gauge theory of an affine type with dimension vectors $\underline{v}, \underline{w}$ is isomorphic to a multiplicative bow variety with the balanced condition. In this case, the condition $N(x_i, x_{i+1}) \geq 0$ for all i is equivalent to the dominance condition $\underline{w} - C\underline{v} \in \mathbb{Z}_{\geq 0}^n$, as the number of \circ between $x_i \rightarrow x_{i+1}$ is w_i .

A.2. Multiplicative bow varieties with cobalanced dimension vector

Now we study a multiplicative bow variety with cobalanced condition. Consider the following part of a bow diagram:



We assume $\dim V_2 = \dim V_3 = \dim V_4$ by the cobalanced condition. By [41, Lemma 2.18], A_2, A_3 are isomorphisms. So we normalize $A_2 = A_3 = \text{id}$ by the action of $GL(V_3) \times GL(V_4)$. Then the defining equation becomes

$$\begin{aligned} 1 + C_1 D_1 &= Z_2 B_2^{-1} = Z_2 (B_3 + a_3 b_2)^{-1} = Z_2 (1 + B_3^{-1} a_3 b_2)^{-1} B_3^{-1} \\ &= t^{-1} Z_2 (1 + B_3^{-1} a_3 b_2)^{-1} (1 + B_4^{-1} a_4 b_3)^{-1} B_4^{-1} \\ &= t^{-2} Z_4^{-1} Z_2 (1 + B_3^{-1} a_3 b_2)^{-1} (1 + B_4^{-1} a_4 b_3)^{-1} (1 + D_4 C_4). \end{aligned}$$

This can be regarded as the defining equation of a multiplicative quiver variety with

(A.3)

where we do not take the quotient by $\mathbb{C}^\times \times \mathbb{C}^\times$ at bottom vertices. We can further make

$$(A.4) \quad \begin{array}{ccc} V_1 & \xleftrightarrow{D_1} & V_2 & \xleftrightarrow{D_4} & V_5 \\ & \searrow C_1 & \uparrow C_4 & & \\ & \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} & \begin{bmatrix} \tilde{a}_3 \\ \tilde{a}_4 \end{bmatrix} & & \\ & \mathbb{C}^2 & & & \end{array}$$

where $\tilde{a}_3 = (1 + B_4^{-1}a_4b_3)B_3^{-1}a_3$, $\tilde{a}_4 = B_4^{-1}a_4$. For additive quiver varieties, (A.3) and (A.4) give isomorphic varieties. But in the multiplicative case, (A.3) gives an open subvariety in (A.4) as the invertibility of both $(1 + B_3^{-1}a_3b_2)$, $(1 + B_4^{-1}a_4b_3)$ is stronger than the invertibility of $(1 + \tilde{a}_3b_2 + \tilde{a}_4b_3)$. In fact, we have an additional requirement that $(1 + \tilde{a}_4b_3)$ is invertible. Note that the difference between (A.3) and (A.4) disappears when we only have one \mathbb{C} .

A.3. Multiplicative quiver varieties

Before generalizing the definition of multiplicative quiver varieties given in Section 5.1, let us recall the notion of quasi-Hamiltonian spaces and fusion/reduction procedures introduced in [1].

Let G be a complex reductive group with Lie algebra \mathfrak{g} and fix a non-degenerate Ad_G -invariant symmetric bilinear form (\dots) on \mathfrak{g} . Let θ (resp. $\bar{\theta}$) be the left (resp. right) invariant Maurer-Cartan form on G .

A *quasi-Hamiltonian G -space* is a smooth G -variety M equipped with a G -invariant two-form ω on M and a G -equivariant morphism $\mu: M \rightarrow G$ (where G acts on G by conjugation) satisfying the following three axioms:

(QH1): $12d\omega = -\mu^*(\theta \wedge [\theta \wedge \theta])$.

(QH2): $2\iota(\xi^*)\omega = \mu^*(\theta + \bar{\theta}, \xi)$ for all $\xi \in \mathfrak{g}$. Here ξ^* is the fundamental vector field:

$$\xi_x^* = \left. \frac{d}{dt} \exp(t\xi) \cdot x \right|_{t=0} \quad (x \in M).$$

(QH3): $\text{Ker}\omega_x = \{ \xi_x^* \mid \xi \in \text{Ker}(\text{Ad}_{\mu(x)} + 1) \}$ for all $x \in M$.

The map μ is called the *group-valued moment map*.

EXAMPLE A.3. – Any conjugacy class \mathcal{C} of G has a structure of quasi-Hamiltonian G -space whose group-valued moment map is just the inclusion $\mathcal{C} \hookrightarrow G$. This is a multiplicative analogue of a coadjoint orbit of \mathfrak{g}^* .

Also, the *double $G \times G$* is a quasi-Hamiltonian $G \times G$ -space, which is a multiplicative analogue of T^*G .

EXAMPLE A.4. – In [6] Van den Bergh introduced a multiplicative analogue of the Hamiltonian $GL(V) \times GL(W)$ -space $T^*\text{Hom}(V, W)$, where V, W are finite-dimensional \mathbb{C} -vector spaces. It is defined to be the quasi-Hamiltonian $GL(V) \times GL(W)$ -space

$$\mathcal{B}(V, W) = \{ (X, Y) \in \text{Hom}(V, W) \times \text{Hom}(W, V) \mid \det(1 + XY) \neq 0 \},$$

where the two-form and the group-valued moment map are given by

$$\omega = \frac{1}{2} \operatorname{Tr} \left((1 + XY)^{-1} dX \wedge dY \right) - \frac{1}{2} \operatorname{Tr} \left((1 + YX)^{-1} dY \wedge dX \right),$$

$$\mu(X, Y) = ((1 + YX)^{-1}, 1 + XY).$$

This is called the *Van den Bergh space*.

Next we introduce the internal fusion of quasi-Hamiltonian spaces. Let H be another complex reductive group equipped with a non-degenerate Ad_H -invariant symmetric bilinear form on its Lie algebra \mathfrak{h} .

THEOREM A.5 ([1]). – *Let $(M, \omega, (\mu_1, \mu_2, \nu))$ be a quasi-Hamiltonian $G \times G \times H$ -space. Let $G \times H$ act on M through the diagonal embedding $(g, h) \mapsto (g, g, h)$. Then M equipped with the two-form*

$$\omega_{\text{fus}} := \omega + \frac{1}{2} (\mu_1^* \theta \wedge \mu_2^* \bar{\theta})$$

and the map

$$(\mu_{\text{fus}}, \nu) := (\mu_1 \cdot \mu_2, \nu): M \rightarrow G \times H$$

is a quasi-Hamiltonian $G \times H$ -space.

The quasi-Hamiltonian $G \times H$ -space $(M, \omega_{\text{fus}}, (\mu_{\text{fus}}, \nu))$ is called the *internal fusion* and denoted by M_{fus} .

The internal fusion procedure is associative in the following sense: if M is a quasi-Hamiltonian $G \times G \times G \times H$ -space, then two quasi-Hamiltonian $G \times H$ -spaces $M_{(12)3}$ obtained by first fusing the first two G -factors and $M_{1(23)}$ obtained by first fusing the last two G -factors are identical. Therefore, if I is a non-empty totally ordered finite set and M is a quasi-Hamiltonian $G^I \times H$ -space with group-valued moment map $((\mu_i)_{i \in I}, \nu)$, then we can define its internal fusion M_{fus} as a quasi-Hamiltonian $G \times H$ -space in a canonical way so that its group-valued moment map is (μ, ν) , where $\mu := \prod_{i \in I}^< \mu_i$.

A multiplicative analogue of the Marsden-Weinstein theorem is the following theorem:

THEOREM A.6 ([1]). – *Let $(M, \omega, (\mu, \nu))$ be a quasi-Hamiltonian $G \times H$ -space and \mathcal{C} be a conjugacy class of G . If the G -action on $\mu^{-1}(\mathcal{C})$ is free, then $\mu^{-1}(\mathcal{C})$ is a smooth subvariety of M , and if furthermore the action has a geometric quotient $\mu^{-1}(\mathcal{C})/G$, then ω and ν induce a quasi-Hamiltonian H -structure on $\mu^{-1}(\mathcal{C})/G$.*

The above quasi-Hamiltonian H -space $\mu^{-1}(\mathcal{C})/G$ is called the *quasi-Hamiltonian reduction* of M by G along \mathcal{C} and denoted by $M//_{\mathcal{C}} G$.

Note that in the above situation if H is abelian then $M//_{\mathcal{C}} G$ is symplectic. The following fact is also useful.

THEOREM A.7. – *Let $(M, \omega, (\mu, \nu))$ be a quasi-Hamiltonian $G \times H$ -space. Assume that H is abelian and M has a good quotient $\pi: M \rightarrow M/G$. Then for any open $U \subset M/G$ and $f \in \Gamma(U, \mathcal{O}_{M/G})$, there exists a unique vector field $v_f \in \Gamma(\pi^{-1}(U), \Theta_M)$ such that*

$$\iota(v_f)\omega = d\pi^* f, \quad \iota(v_f)\mu^* \theta = 0.$$

Each v_f is G -invariant and preserves ω and μ . Also, $\pi^* \{f, g\} = v_f(\pi^* g)$ ($f, g \in \mathcal{O}_{M/G}$) defines an H -invariant Poisson structure on M/G .

See [1, Proposition 4.6], where H is assumed to be trivial but the arguments in the proof work in the general case.

Since each v_f preserves μ , for any conjugacy class $\mathcal{C} \subset G$ the closed subvariety $M//_{\overline{\mathcal{C}}}G := \mu^{-1}(\overline{\mathcal{C}})/G \subset M/G$ is Poisson. We call it the quasi-Hamiltonian reduction of M by G along $\overline{\mathcal{C}}$. On the other hand, if there exists an open $U \subset \mu^{-1}(\overline{\mathcal{C}})/G$ such that $\mu(\pi^{-1}(U)) \subset \mathcal{C}$ and G acts freely on $\pi^{-1}(U)$, then Theorem A.6 shows that U is a quasi-Hamiltonian H -space, in particular symplectic. By the definition, the Poisson structure on U given by Theorem A.7 coincides with one induced from the symplectic structure.

Now we introduce the multiplicative quiver varieties. Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver. Let $V = \bigoplus_{i \in Q_0} V_i$ and $W = \bigoplus_{i \in Q_0} W_i$ be two finite-dimensional Q_0 -graded \mathbb{C} -vector spaces with dimension vectors $\underline{v} = (v_i)_{i \in Q_0}$ and $\underline{w} = (w_i)_{i \in Q_0}$, respectively. For each $i \in Q_0$ fix a decomposition of W_i into one-dimensional pieces

$$W_i = \bigoplus_{j=1}^{w_i} W_{i,j}, \quad \dim W_{i,j} = 1,$$

and let $T(W) \subset GL(W) := \prod_{i \in Q_0} GL(W_i)$ be the associated maximal torus. Define

$$\widetilde{\mathcal{B}}_Q(V, W) = \prod_{h \in Q_1} \mathcal{B}(V_{s(h)}, V_{t(h)}) \times \prod_{i \in Q_0} \prod_{j=1}^{w_i} \mathcal{B}(W_{i,j}, V_i),$$

which is a quasi-Hamiltonian $\mathbb{G} \times T(W)$ -space, where

$$\mathbb{G} := \prod_{h \in Q_1} (GL(V_{s(h)}) \times GL(V_{t(h)})) \times \prod_{i \in Q_0} GL(V_i)^{w_i}.$$

Using the double \overline{Q} of Q with involution $*$: $\overline{Q}_1 \rightarrow \overline{Q}_1$ (so $\overline{Q}_1 = Q_1 \sqcup Q_1^*$ and $s \circ * = t$, $t \circ * = s$), we denote an element of $\widetilde{\mathcal{B}}_Q(V, W)$ by (C, a, b) , where $C = (C_h)_{h \in \overline{Q}_1}$, $a = (a_{i,j})$, $b = (b_{i,j})$ and

$$\begin{aligned} (C_h, C_{h^*}) &\in \mathcal{B}(V_{s(h)}, V_{t(h)}) \quad (h \in Q_1), \\ (a_{i,j}, b_{i,j}) &\in \mathcal{B}(W_{i,j}, V_i) \quad (i \in Q_0, j = 1, \dots, w_i). \end{aligned}$$

(In other places, C_h is denoted by C_i or D_i according to whether h is in Q_1 or not.) Fix a total ordering $<$ on \overline{Q}_1 and define a quasi-Hamiltonian $GL(V) \times T(W)$ -space by

$$\mathcal{B}_Q(V, W) = \widetilde{\mathcal{B}}_Q(V, W)_{\text{fus}, <},$$

where the internal fusion is taken with respect to the diagonal embedding $GL(V) \hookrightarrow \mathbb{G}$ so that the resulting group-valued moment map $(\mu, \nu) = ((\mu_i), (v_{i,j}))$ is

$$\begin{aligned} \mu_i(B, a, b) &= \prod_{h \in \overline{Q}_1; t(h)=i}^{<} (1 + C_h C_{h^*})^{\epsilon(h)} \prod_{j=1}^{w_i} (1 + a_{i,j} b_{i,j}), \\ v_{i,j}(C, a, b) &= (1 + b_{i,j} a_{i,j})^{-1}, \end{aligned}$$

where $\epsilon(h) = 1$ if $h \in Q_1$ and $\epsilon(h) = -1$ if $h \in Q_1^*$.

DEFINITION A.8. – For $\gamma = (\gamma_i)_{i \in Q_0} \in (\mathbb{C}^*)^{Q_0}$, put $\gamma_V = (\gamma_i 1_{V_i})_{i \in Q_0} \in GL(V)$ and define

$$\mathcal{M}_\gamma^\times(\underline{v}, \underline{w}) = \mu^{-1}(\gamma_V) / GL(V),$$

where the quotient is taken as the affine GIT quotient. We call $\mathcal{M}_\gamma^\times(\underline{\mathbf{v}}, \underline{\mathbf{w}})$ the *multiplicative quiver variety*.

By Theorem A.7, $\mathcal{M}_\gamma^\times(\underline{\mathbf{v}}, \underline{\mathbf{w}}) = \mathcal{B}_Q(V, W) //_{\gamma_V} GL(V)$ is a Poisson variety. Also, as in the case of additive quiver varieties, the $GL(V)$ -action on the open subset $\mu^{-1}(\gamma_V)^s \subset \mu^{-1}(\gamma_V)$ consisting of stable points has stabilizer $\mathbb{C}^* 1_V$ everywhere (where the definition of stability is exactly the same as in the additive case), and Theorem A.6 implies that the open subset $\mathcal{M}_\gamma^{\times \text{reg}}(\underline{\mathbf{v}}, \underline{\mathbf{w}}) := \mu^{-1}(\gamma_V)^s / GL(V)$ of $\mathcal{M}_\gamma^\times(\underline{\mathbf{v}}, \underline{\mathbf{w}})$ is a quasi-Hamiltonian $T(W)$ -space. We also call $\mathcal{M}_\gamma^{\times \text{reg}}(\underline{\mathbf{v}}, \underline{\mathbf{w}})$ the multiplicative quiver variety.

EXAMPLE A.9. – The multiplicative quiver variety $\mathcal{M}_N^l(Z, t)$ introduced in Section 5.1 coincides with $\mathcal{M}_\gamma^\times(\underline{\mathbf{v}}, \underline{\mathbf{w}})$ with $Q = \hat{A}_{l-1}$ (the orientation is $1 \rightarrow l \rightarrow \dots \rightarrow 2 \rightarrow 1$) and

$$\underline{\mathbf{v}} = (N, N, \dots, N), \quad \underline{\mathbf{w}} = (1, 0, \dots, 0), \quad \gamma = \left(\frac{Z_l}{Z_1} t, \frac{Z_1}{Z_2}, \dots, \frac{Z_{l-1}}{Z_1} \right).$$

REMARK A.10. – Definition A.8 is close to the definition of *framed multiplicative quiver variety* introduced in [45]. Let us identify all $W_{i,j}$ with \mathbb{C} . Then the framed multiplicative quiver variety is obtained by replacing $\mathcal{B}_Q(V, W)$ with its internal fusion for the diagonal subgroup $\mathbb{C}^* \subset T(W)$ in the definition of $\mathcal{M}_\gamma^\times(\underline{\mathbf{v}}, \underline{\mathbf{w}})$. It is the same as $\mathcal{M}_\gamma^\times(\underline{\mathbf{v}}, \underline{\mathbf{w}})$ as a variety, but the Poisson brackets are different in general.

REMARK A.11. – There is another multiplicative analogue of quiver variety. It is obtained by replacing $\widetilde{\mathcal{B}}_Q(V, W)$ with the quasi-Hamiltonian $\mathbb{G} \times GL(W)$ -space

$$\widetilde{\mathcal{B}}'_Q(V, W) := \prod_{h \in Q_1} \mathcal{B}(V_{s(h)}, V_{t(h)}) \times \prod_{i \in Q_0} \mathcal{B}(W_i, V_i)$$

in the definition of $\mathcal{M}_\gamma^\times(\underline{\mathbf{v}}, \underline{\mathbf{w}})$. But note that in general the two-form on $\widetilde{\mathcal{B}}'_Q(V, W)_{\text{fus}, <}$ does not induce a Poisson structure on the resulting variety. It is why we decompose each W_i into one-dimensional pieces. Of course if $\dim W_i \leq 1$ for all $i \in Q_0$ then the two definitions coincide.

EXAMPLE A.12. – Consider $Q = A_{\ell-1}$ with $\underline{\mathbf{w}} = (\ell, 0, \dots, 0)$. The usual quiver variety $\mathfrak{M}_\zeta(\underline{\mathbf{v}}, \underline{\mathbf{w}})$, if it is not \emptyset , is a semisimple coadjoint orbit of $\mathfrak{gl}(\ell)$ for generic ζ . The multiplicative quiver variety in the definition in Remark A.11 is a conjugacy class of $GL(\ell)$. (See e.g., [16, §8].) On the other hand, Definition A.8 gives its open subset, the intersection with a big Bruhat cell of $GL(\ell)$. To see this, take a decomposition $W = \bigoplus_{j=1}^\ell W_j$ into one-dimensional spaces, and write $a = (a_j)$, $b = (b_j)$. Then $\tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_\ell)$, $\tilde{b} = {}^t(\tilde{b}_1, \dots, \tilde{b}_\ell)$ with $\tilde{a}_j = (1 + a_1 b_1) \cdots (1 + a_{j-1} b_{j-1}) a_j$, $\tilde{b}_j = b_j$ satisfy the defining equation in Remark A.11 as

$$1 + \sum_{j=1}^\ell \tilde{a}_j \tilde{b}_j = (1 + a_1 b_1) \cdots (1 + a_\ell b_\ell).$$

The isomorphism to a conjugacy class is given by the group-valued moment map $(1 + \tilde{b}\tilde{a})^{-1}$. If an element $1 + \tilde{b}\tilde{a}$ is coming from (a_j) , (b_j) , we have additional constraint $\det(1 + a_j b_j) \neq 0$ for $j = 1, \dots, \ell$, which is equivalent to require that every leading principal minor of $1 + \tilde{b}\tilde{a}$ is nonzero.

Now we understand the correct definition of multiplicative quiver varieties, and the argument in Section A.2 gives the following.

THEOREM A.13. – Consider a multiplicative bow variety with a cobalanced dimension vector. It is isomorphic to a multiplicative quiver variety $\mathcal{M}_\gamma^\times(\mathbf{v}, \mathbf{w})$ such that \mathbf{v}_j (resp. \mathbf{w}_j) is the dimension of vector spaces (resp. the number of \times) between h_j and h_{j+1} , and $\gamma_j = t^{\mathbf{w}_j} Z_j / Z_{j+1}$.

Combining this with Proposition A.2, we obtain an isomorphism between a K -theoretic Coulomb branch and a multiplicative quiver variety when the dominance condition is satisfied. We conjecture that this is an isomorphism of Poisson varieties. More generally we conjecture

CONJECTURE A.14. – (1) The space consisting of $A_{i-1}, B_{i-1}, B'_i, a_i, b_{i-1}$ with (i), (a), (S1), (S2) in Section 5.4 is a quasi-Hamiltonian $GL(V_{i-1}) \times GL(V_i) \times \mathbb{C}^*$ -space with the group-valued moment map $B_{i-1}^{-1}, B'_i, \det B_{i-1} \det B_i'^{-1}$. Therefore a multiplicative bow variety is a quasi-Hamiltonian reduction.

(2) The Hanany-Witten transition is a quasi-Hamiltonian $GL(V_1) \times GL(V_3) \times \mathbb{C}^*$ -spaces isomorphism.

(3) The isomorphism of Theorem A.13 is an isomorphism of quasi-Hamiltonian $T(W)$ -spaces.

REMARK A.15. – Both homological and K -theoretic Coulomb branches have a torus action induced from $\text{Hom}(\pi_1(G), \mathbb{Z})$. See [33, 4(iii)(c)] and the original physics literature therein. It is expected that the torus action extends to a nonabelian group action for homological Coulomb branches [33, 4(iii)(d)], and proved for ADE quiver gauge theories in [9, Remark 3.12]. On the other hand, it is not *true* for K -theoretic Coulomb branches as Example A.12 shows.

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