Population monotonicity in fair division of multiple indivisible goods[1]

Emre Doğan[§]

August 24, 2020

Abstract

We consider the fair division of a set of indivisible goods where each agent can receive more than one good, and monetary transfers are allowed. We show that if there are at least three goods to allocate, no efficient solution is population monotonic (PM) on the superadditive Cartesian product preference domain, and the Shapley solution is not PM even on the submodular domain. Moreover, the incompatibility between efficiency and PM prevails in the case of at least four goods on the subadditive Cartesian product domain. We also show that in case there are only two goods to allocate, the Shapley solution and the constrained egalitarian solution are PM on the subadditive preference domain but not on the full preference domain. For the two-good case, we provide a new tool (the hybrid solutions) to construct efficient solutions that are PM on the entire monotone preference domain. The hybrid Shapley solution and the hybrid constrained egalitarian solution are two important examples of such solutions.

Key Words: *Population monotonicity, fair division, indivisible goods*

1 Introduction

We consider the fair division problem where individuals have equal claims on a set of indivisible goods, or "objects", each agent can receive more than one object, and balanced monetary transfers between agents are feasible. Distributing inheritance to heirs and donations to the needy are some examples of such problems.

The population monotonicity (PM) principle was first introduced by Thomson (1983) as a strong solidarity idea:[2] if we are allocating a fixed bundle of resources and additional agents arrive at the scene, none of the existing agents should be made better off. It has been studied in a variety of models with common ownership (Alkan 1994, Beviá 1996a and 1996b, Kim 2004, Moulin 1990 and 1992, and Thomson 1995). Several impossibility results have proved that PM is a strong axiom. For example, no solution simultaneously satisfies PM and the central no-envy fairness criterion even in the simplest case where a single object is allocated

[1] This work is partially completed during my doctoral studies at Rice University under the supervision of Professor Hervé Moulin, and the final version is completed at Higher School of Economics. I thank both institutions for supporting my work. I am also grateful to Hervé Moulin, William Thomson, Fuad Aleskerov, an anonymous associate editor and two anonymous referees for their valuable comments.

[§] National Research University Higher School of Economics, Moscow, Russia. Email: edogan@hse.ru

[2] We use PM as an abbreviation for both population monotonicity and population monotonic.

(Moulin 1990). Critically so, when monetary transfers between agents are allowed, Moulin (1992) and Beviá (1996a) showed respectively in models with divisible and indivisible goods that achieving PM together with efficiency is impossible on the full preference domain. Beviá's (1996a) negative finding, unlike Moulin's (1992), is based on a 4-agent, 3-object example where some agents have non-monotone preferences (no free disposal). Consider the following simplest 2-agent, 2-object example:

$$u_1(\alpha) = u_2(\beta) = 3,\ u_1(\beta) = u_2(\alpha) = 0,\ u_1(\alpha\beta) = u_2(\alpha\beta) = 2$$

As transfers are allowed, efficiency dictates assigning the objects in a way that maximizes the total utility, giving α to agent 1 and β to agent 2. Then they will receive six units of total utility at any efficient allocation. Hence, one of the agents has at least three units of final utility. However, each agent gets two units of utility at the efficient allocation when they are the only claimant, ruling out PM. The source of incompatibility between PM and efficiency in this example concerns non-monotone preferences.

When there are incompatibilities between desired properties on the full preference domain, a natural approach is to seek narrower meaningful domains where possibility results hold or negative results prevail. In this regard, Beviá (1996a) provides a positive result: if the domain satisfies substitutability (see Section 3), the Shapley solution (Shapley 1953) is PM. We note that substitutability (a condition, known for allowing the existence of a competitive equilibrium for economies with indivisibilities (see Kelso and Crawford 1982)) is not a restriction on individual preferences (though it requires those to be submodular (decreasing marginal utility)) but a restriction on the preference profiles. A more realistic assumption is that the admissibility of an agent's preference does not depend on the preference of others in a given profile, i.e., the preference domain is a Cartesian product domain.

In this work, we revisit the question tackled in Beviá (1996a) and provide a more extensive analysis on the compatibility of efficiency and PM by examining this compatibility under different combinations of two important parameters of the model: the fixed number of objects available (l) and the set of admissible (and necessarily monotone) preferences for each agent. It is well-known for the case $l = 1$ that the transferable utility (TU) game induced by any given problem is concave, implying that the Shapley solution and the constrained egalitarian solution (CES) (also known as the dual Dutta-Ray solution (see Dutta and Ray 1989, and Klijn et al 2001)) are efficient and PM (respectively, by Moulin 1992, Dutta 1990, and Klijn et al 2001). Therefore, we concentrate on the problems with $l \geq 2$.

For $l \geq 3$, we provide only negative results. First, the incompatibility of efficiency and PM prevails not only on the full monotone preference domain (Corollary 3) but also on the superadditive Cartesian product domain for $l \geq 3$ and the subadditive Cartesian product domain for $l \geq 4$ (Theorem 1). Moreover, the Shapley solution is not PM on submodular Cartesian product domain (the domain of concave individual preferences) even in the case of $l = 3$ (Proposition 2).

For the case $l = 2$ we have positive results. We first show that the subadditive Cartesian product domain induces concave TU games (Proposition 3). Therefore, the Shapley solution and CES are PM on this domain. For our existence result on the full domain, we introduce a procedure that decomposes any problem with $l = 2$ into two problems: a two-object problem with subadditive preferences and an "imaginary" single-object problem. We can write a (hybrid) solution to any problem with $l = 2$ as the sum of two solutions to the decomposed portions of the problem (by Lemma 6). We show in Theorem 2 that a hybrid solution is PM if both solutions are PM and the solution to the single-object problem satisfies constant shift monotonicity (if the utility of all agents increase by the same constant, nobody is worse-off). Existence of several efficient and PM solutions on the full domain of monotone preferences follows as a corollary.

The previous positive results in the fair division literature on PM introduce domains which induce concave games. Therefore, they implicitly provide a menu of efficient solutions satisfying PM, including the Shapley solution and CES, for those who want to adopt a PM solution for such problems. The notion of hybrid solutions has two more important aspects to it apart from solely helping us provide an existence result. First, although the full monotone domain does not induce concave games in the two-object case, by utilizing the hybrid solution concept and Theorem 2, it is possible to construct a variety of PM solutions by choosing from a similar menu of efficient solutions for the two decomposed portions of the original problem. Moreover, it allows for obeying certain fairness principles (at least partially) without violating PM. The Shapley solution and CES represent respectively two important fairness principles: 1) higher marginal contributions to the total surplus should be awarded with higher shares from the surplus (marginalism), and 2) shares of agents should be as equal as possible under relevant participation constraints (a fairness constraint called the stand-alone test in this setting (see Section 3)) (egalitarianism). We show in Proposition 4 that the Shapley solution and CES fail PM on the full domain for $l = 2$. However, by Theorem 2, the hybrid Shapley solution (which applies this solution to both portions of the decomposed problem) and the hybrid CES are efficient and PM on the full domain when $l = 2$ (Corollary 5). Therefore, although fully applying the marginalism or the egalitarianism principle is not possible under PM, we can still partially adhere to those principles while keeping PM on the full domain.

In Section 2, we present the general setting. In Section 3, we define preference domains, basic properties, and solutions. Section 4 is devoted to the negative results for the case of $l \geq 3$. In Section 5, we study the two-object case. Section 6 provides some concluding remarks and a table summarizing the results.

2 The setting

A finite set of indivisible goods or "objects" denoted by Ω is allocated to a set of agents $N \in \mathcal{N}$, who have equal claims on it, where $\mathcal{N}$ is the set of all finite potential societies. The number of objects is denoted by $l = |\Omega|$ and $|N| = n$. Agents can receive multiple objects, monetary transfers between agents are allowed, and preferences are quasilinear in money. For a fixed (N, Ω), $u_i(A) \geq 0$ denotes the utility $i \in N$ gets from the bundle $A \subseteq \Omega$, while $u_i(A, m) = u_i(A) + m$ denotes the utility i gets when i receives an additional $m \in \mathbb{R}$ units of monetary transfer. We also allow for free disposability, which amounts to assuming that preferences are monotone, i.e., $u_i(A) \leq u_i(B)$ for all $A \subseteq B$. By convention $u_i(\emptyset) = 0$. A list of preferences $\{u_i\}_{i \in N}$ is denoted by u, and u^S denotes the list restricted to $S \subseteq N$. The full domain of preference (utility) profiles is $U(N, \Omega) = \{u : u_i \text{ is monotone } \forall i \in N\}$, and we simply use U by abuse of notation. A *fair division problem* is a triple (N, Ω, u), $\mathcal{E} = \{(N, \Omega, u) : N \in \mathcal{N}, \Omega \text{ is finite}, u \in U\}$ is the set (or domain) of all feasible problems and $\mathcal{E}_l = \{(N, \Omega, u) \in \mathcal{E} : |\Omega| = l\}$ is the set of problems with $|\Omega| = l$.

An allocation for (N, Ω, u) consists of two components: An assignment σ of the objects to the agents and a list (vector) m of balanced monetary transfers between the agents. $\sigma: N \to 2^{\Omega}$ is a mapping such that $\sigma_i \cap \sigma_j = \emptyset$ for all $i, j \in N$, $\bigcup_{i \in N} \sigma_i = \Omega$, and $\sigma_i = \emptyset$ indicates that $i \in N$ does not receive any object. $m \in \mathbb{R}^n$ is such that $\sum_{i \in N} m_i = 0$.

By quasilinearity, an allocation (σ, m) is *efficient* if and only if σ maximizes the total utility, i.e., $\sum_{i \in N} u_i(\sigma_i) \geq \sum_{i \in N} u_i(\sigma'_i)$ for all assignments $\sigma': N \to 2^{\Omega}$. It is individually rational (IR) if $u_i(\sigma_i) + m_i \geq 0$ for all $i \in N$. A *solution* φ defined on $\mathcal{E}' \subseteq \mathcal{E}$ is a mapping that yields a set of allocations for each problem in $\mathcal{E}'$. φ is *efficient* (EFF) on $\mathcal{E}'$ if it yields only efficient allocations for each problem on $\mathcal{E}'$, and it is *essentially single-valued* on $\mathcal{E}'$ if for all $(N, \Omega, u) \in \mathcal{E}'$, for all $(\sigma, m), (\sigma', m') \in \varphi(N, \Omega, u)$ we have $u_i(\sigma_i) + m_i = u_i(\sigma'_i) + m'_i$ for all $i \in N$. If φ is essentially single-valued, $\varphi_i(N, \Omega, u) \in \mathbb{R}$ denotes the final utility of agent i.[3]

Each (N, Ω, u) induces a *stand-alone transferable utility cooperative game (TU game)* (N, v) or simply $v: 2^N \to \mathbb{R}_+$ with $v(S) = \sum_{i \in S} u_i(\sigma_i)$ where σ maximizes the total utility at the problem (S, Ω, u^S) $(v(\emptyset) = 0)$. We write $v(S, A) = \sum_{i \in S} u_i(\sigma_i)$ if σ maximizes the total utility at (S, A, u^S) with $A \subseteq \Omega$. Therefore, we call $v(S)$ and/or $v(S, A)$ the *efficient surplus*. The nonnegative *dual core* (stand-alone core) of v is $\mathcal{C}(v) = \{x \in \mathbb{R}^N_+: \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \leq v(S) \text{ for all } S \subseteq N\}$. v is *concave* if for all $S \subseteq N$ and for all distinct $i, j \in N \setminus S$, we have $v(S \cup \{i, j\}) - v(S \cup \{j\}) \leq v(S \cup \{i\}) - v(S)$. The Shapley value of a TU game (N, v) is defined as $Sh_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!} \big(v(S \cup \{i\}) - v(S)\big)$.

For $N \in \mathcal{N}$ and $x \in \mathbb{R}^n$, let $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$ be the vector obtained by re-arranging the coordinates of x in a non-decreasing order, i.e., $\tilde{x}_1 \leq \cdots \leq \tilde{x}_n$. For any $x, y \in \mathbb{R}^n$, x *Lorenz dominates* y $(x \succ_L y)$ if $\sum_{h=1}^{k} \tilde{x}_h \geq \sum_{h=1}^{k} \tilde{y}_h$ for all $k = 1, \dots, n$ with at least one strict inequality. The constrained egalitarian solution for the game v is $CES(v) = \{x \in \mathcal{C}(v): \nexists\, y \in \mathcal{C}(v) \text{ such that } y \succ_L x\}$. As $\succ_L$ is a partial order and $\mathcal{C}(v)$ is compact, $CES(v)$ is non-empty. If v is concave, $CES(v)$ is singleton and equivalent to the dual Dutta-Ray solution.

3 Preferences, basic properties and solutions

Here we consider the following well-known types of preferences and the corresponding Cartesian product preference domains.

Definition 1: A utility function $u_i: 2^{\Omega} \to \mathbb{R}_+$

(i) is *submodular* if for all $A, B \subseteq \Omega$, $u_i(A \cup B) + u_i(A \cap B) \leq u_i(A) + u_i(B)$, or equivalently for all $A \subseteq B \subseteq \Omega$, $\alpha \in A$, $u_i(B) - u_i(B \setminus \alpha) \leq u_i(A) - u_i(A \setminus \alpha)$.

(ii) is *subadditive* if for all $A, B \subseteq \Omega$, $u_i(A \cup B) \leq u_i(A) + u_i(B)$.

(iii) is *additively separable* if for all $A \subseteq \Omega$, $u_i(A) = \sum_{\alpha \in A} u_i(\alpha)$.

(iv) is *superadditive* if for all $A, B \subseteq \Omega$ such that $A \cap B = \emptyset$, $u_i(A) + u_i(B) \leq u_i(A \cup B)$.

[3] $\varphi_i(N, \Omega, u)$ is a natural candidate to denote the set of all allocations that i can get under $\varphi(N, \Omega, u)$, which is $\{(\sigma_i, m_i)\}_{(\sigma, m) \in \varphi(N, \Omega, u)}$. However, since we never refer to this set in this work, we find it convenient to denote i's utility by $\varphi_i(N, \Omega, u)$ at an essentially single-valued solution φ.

A submodular utility function is subadditive. An additively separable utility function is both subadditive and superadditive. If $l = 1$, all properties trivially hold. If $l = 2$, subadditivity and submodularity coincide.

Cartesian product domain of submodular preferences is defined as $U^{subm} = \{u \in U\colon u_i \text{ is submodular } \forall i \in N\}$, $\mathcal{E}^{subm} = \{(N, \Omega, u)\colon N \in \mathcal{N}, \Omega \text{ is finite}, u \in U^{subm}\}$, and $\mathcal{E}_l^{subm} = \{(N, \Omega, u) \in \mathcal{E}^{subm}\colon |\Omega| = l\}$. U^{sub}, U^{add}, U^{sup}; $\mathcal{E}^{sub}$, $\mathcal{E}^{add}, \mathcal{E}^{sup}$ are defined similarly for the preferences defined by (ii), (iii), and (iv) above.

We call a domain $\mathcal{E}' \subseteq \mathcal{E}$ *regular* if for all $(N, \Omega, u) \in \mathcal{E}'$ and all $S \subseteq N$, we have $(S, \Omega, u^S) \in \mathcal{E}'$. We define some properties and state some of the results on regular domains to avoid the arbitrary exclusion of the formation of certain societies ex ante. This restriction does not affect the essence of our results as all the domains we analyze are regular.

Definition 2: φ is *population monotonic* on $\mathcal{E}' \subseteq \mathcal{E}$ if for all $(N, \Omega, u), (S, \Omega, u^S) \in \mathcal{E}'$ with $S \subset N$, for all $(\sigma, m) \in \varphi(N, \Omega, u)$, and for all $(\sigma', m') \in \varphi(S, \Omega, u^S)$, we have $u_i(\sigma_i, m_i) \leq u_i(\sigma_i', m_i')$ for all $i \in S$.

Definition 3: An essentially single-valued solution φ is *population monotonic* on $\mathcal{E}' \subseteq \mathcal{E}$ if for all $(N, \Omega, u), (S, \Omega, u^S) \in \mathcal{E}'$ with $S \subset N$, we have $\varphi_i(N, \Omega, u) \leq \varphi_i(S, \Omega, u^S)$ for all $i \in S$.

If φ is PM on $\mathcal{E}' \subseteq \mathcal{E}$, any essentially single-valued solution $\overline{\varphi}$ derived by choosing an allocation from φ at each problem is PM. This argument proves the following Lemma which implies that we can use Definitions 2 and 3 interchangeably for our impossibility result (Theorem 1). We concentrate on essentially single-valued solutions for the existence results. Thus, φ is assumed to be essentially single-valued for the Definitions 6, 7, 8, and Lemma 3, which are auxiliary to Theorem 2 and Corollary 5. Moreover, we assume φ to be essentially single-valued for the Definitions 4, 5, and Lemma 2 solely for the simplicity of the argument.

Lemma 1: A regular domain $\mathcal{E}' \subseteq \mathcal{E}$ admits a PM solution, i.e., there exists a PM solution on $\mathcal{E}'$, if and only if it admits an essentially single-valued solution that is PM.

By the help of the following example we first discuss a reasonable critique of the above version of the population monotonicity axiom we use in this paper and then provide a reason for our choice.

Example 1: $N = \{1,2,3\}$, $\Omega = \{\alpha\}$, $u_i(\alpha) = 2$ for $i \in \{1,2\}$ and $u_3(\alpha) = 999$.

Suppose initially the set of claimants is $S = \{1,2\}$. Then, $\varphi_1(S) + \varphi_2(S) = 2$ for any efficient φ. When agent 3 arrives, we have $\varphi_1(N) + \varphi_2(N) + \varphi_3(N) = 999$ for any efficient φ. The possibility of monetary transfers allows the efficient surplus to increase tremendously by the arrival of an additional agent even though the set of resources is fixed. Then it may be reasonable to consider the following weaker solidarity notion for the underlying model.

Definition 4: φ satisfies *population solidarity* (PS) on $\mathcal{E}' \subseteq \mathcal{E}$ if for all $(N, \Omega, u), (S, \Omega, u^S) \in \mathcal{E}'$ with $S \subset N$, we have either for all $i \in S$, $\varphi_i(N, \Omega, u) \leq \varphi_i(S, \Omega, u^S)$; or for all $i \in S$, $\varphi_i(S, \Omega, u^S) \leq \varphi_i(N, \Omega, u)$.[4]

---

[4] This criterion was first introduced by Chun (1986) as the solidarity axiom in a different setting. It appears as population solidarity in Moulin (1992) and as population monotonicity in numerous other papers (e.g., Klaus 2001).

The equal surplus sharing rule $\hat{\varphi}_i(N,\Omega,u) = v(N)/n$ (for all $(N,\Omega,u) \subseteq \mathcal{E}$ and for all $i \in N$) satisfies PS and it is efficient on $\mathcal{E}$. If we think of the population monotonicity principle solely as a solidarity notion, $\hat{\varphi}$ performs well. For the above example, we have $\hat{\varphi}_i(N,\Omega,u) = 333$ for all $i \in N$. However, $S = \{1,2\}$ could only enjoy a total utility of 2 units in the absence of agent 3. By quasilinearity, we can interpret the preferences as individual technologies transforming the resources into money. When agent 3 with a very efficient technology arrives at the scene, she heavily subsidizes the coalition S of agents with inferior technologies under equal surplus sharing rule. Therefore, it is reasonable to think of population monotonicity as a broader fairness principle which, in addition to the solidarity idea, captures the no subsidization principle: no society is entitled to benefit from the arrival of additional agents with superior technologies, which amounts to imposing the following criterion introduced in Moulin (1992) to justify the use of PM instead of PS.

Definition 5: φ passes the stand-alone test (SAT) on $\mathcal{E}' \subseteq \mathcal{E}$ if $\left(\varphi_i(N,\Omega,u)\right)_{i\in N} \in \mathcal{C}(N,v)$ for all $(N,\Omega,u) \in \mathcal{E}'$, where (N,v) is the game induced by (N,Ω,u).

We skip the straightforward argument for the following lemma which establishes the fact that PM is the conjunction of the population solidarity and no subsidization principles under the efficiency assumption.

Lemma 2: An efficient solution φ satisfies PS and passes SAT on a regular domain $\mathcal{E}' \subseteq \mathcal{E}$ if and only if φ is PM on $\mathcal{E}' \subseteq \mathcal{E}$.

Definition 6: φ satisfies the *dummy axiom (DUM)* on $\mathcal{E}'$ if for all $(N,\Omega,u) \in \mathcal{E}'$, $\varphi_i(N,\Omega,u) = 0$ whenever $u_i(A) = 0$ for all $A \subseteq \Omega$.

Definition 7: φ is *dummy invariant (DI)* on a regular domain $\mathcal{E}'$ if for all $(N,\Omega,u) \in \mathcal{E}'$, $u_i(A) = 0$ for all $A \subseteq \Omega$ implies that $\varphi_j(N,\Omega,u) = \varphi_j(N \setminus \{i\},\Omega,u^{N\setminus\{i\}})$ for all $j \neq i$.

Lemma 3: If φ is EFF and PM on a regular domain $\mathcal{E}'$, then φ satisfies DUM on $\mathcal{E}'$ and φ is DI on $\mathcal{E}'$.

Proof: Let φ be EFF and PM, and $(N,\Omega,u) \in \mathcal{E}'$, $i \in N$ be such that $u_i(A) = 0$ for all $A \subseteq \Omega$. Suppose $\varphi_i(N,\Omega,u) \neq 0$. Then, $2 \leq |N|$ by EFF. $\varphi_i(N,\Omega,u) > 0$ contradicts that φ is PM as $\varphi_i(\{i\},\Omega,u^{\{i\}}) = 0$. $\varphi_i(N,\Omega,u) < 0$ contradicts that φ is PM as $v(N) = v(N \setminus \{i\})$ and EFF would then require $\varphi_j(N \setminus \{i\},\Omega,u^{N\setminus\{i\}}) < \varphi_j(N,\Omega,u)$ to hold for some $j \in N$. Hence, we have $\varphi_i(N,\Omega,u) = 0$. Suppose $\varphi_j(N,\Omega,u) \neq \varphi_j(N \setminus \{i\},\Omega,u^{N\setminus\{i\}})$ for $j \in N \setminus \{i\}$. As φ is PM, $\varphi_j(N,\Omega,u) < \varphi_j(N \setminus \{i\},\Omega,u^{N\setminus\{i\}})$ and the weak inequality holds for all other j', contradicting that $v(N) = v(N \setminus \{i\})$ and φ is EFF. □

The Shapley solution (Sh). Given (N,Ω,u), $(\sigma,m) \in Sh(N,\Omega,u)$ if and only if for all $i \in N$, $u_i(\sigma_i) + m_i = Sh_i(v)$, where v is the induced TU game.

Constrained egalitarian solution (CES). Given (N,Ω,u), $(\sigma,m) \in CES(N,\Omega,u)$ if and only if there is $x \in CES(v)$ such that $u_i(\sigma_i) + m_i = x_i$ for all $i \in N$, where v is the induced TU game.

By definition, both solutions are efficient and CES is IR. The Shapley solution is essentially single-valued and it is IR as all the induced games (v) are monotone. Now, let $\Gamma = \{(N,v): N \in \mathcal{N}, v: 2^N \to \mathbb{R}\}$ and $\Gamma_{con} = \{(N,v) \in \Gamma: v \text{ is concave}\}$. If $\mathcal{E}'$ is such that all the problems in $\mathcal{E}'$ induce concave games, CES is essentially single-valued on $\mathcal{E}'$ as CES is single-valued on Γ_{con}. A single-valued TU game solution ψ satisfies *dual* *population monotonicity* (*dPM*) on Γ' if for all $(N, v), (S, v_S) \in \Gamma'$ (where $v_S(T) = v(T)$ for all $T \subseteq S$) and for all $i \in S$ we have $\psi_i(N, v) \leq \psi_i(S, v_S)$.[5] Both the Shapley value (Moulin 1992) and CES satisfy dPM on the class of concave games.[6]

Lemma 4: Let $\mathcal{E}' \subseteq \mathcal{E}$ be such that each problem in $\mathcal{E}'$ induces a game in Γ' and ψ be a single-valued TU game solution satisfying dPM on Γ'. Then the solution φ defined on $\mathcal{E}'$ is PM where φ is as follows: for any $(N, \Omega, u) \in \mathcal{E}'$, $(\sigma, m) \in \varphi(N, \Omega, u)$ if and only if $u_i(\sigma_i) + m_i = \psi_i(v)$ for all $i \in N$, where v is the TU game induced by (N, Ω, u).

Proof: Let $\mathcal{E}' \subseteq \mathcal{E}$, ψ, Γ', and φ be as in the statement of the Lemma. Fix any $(N, \Omega, u), (S, \Omega, u^S) \in \mathcal{E}'$ with $S \subseteq N$. Let (N, Ω, u) induce the game (N, v) and (S, Ω, u^S) induce the game (S, w). By definition $w = v_S$, which is the restriction of game (N, v) on the agent set S. Since $(N, v), (S, w) \in \Gamma'$ and ψ satisfies dPM on Γ', we have $\varphi_i(N, \Omega, u) = \psi_i(N, v) \leq \psi_i(S, v_S) = \varphi_i(S, \Omega, u^S)$ for all $i \in S$.

Corollary 1: Let $\mathcal{E}' \subseteq \mathcal{E}$ be such that each problem in $\mathcal{E}'$ induces a game in Γ_{con}. Then the Shapley solution and CES are PM on $\mathcal{E}'$.

To make our contribution clear, we present a sufficient condition for the existence of EFF and PM solutions and the related positive result in Beviá (1996a).

Substitutability. Given (N, Ω, u), and $S \subseteq N$, $v(S,\cdot)$ satisfies *substitutability* if for all $\alpha \in A \subseteq B \subseteq \Omega$, $v(S, B) - v(S, B \setminus \{\alpha\}) \leq v(S, A) - v(S, A \setminus \{\alpha\})$. $\mathcal{E}' \subseteq \mathcal{E}$ is a *substitutable domain* if for all $(N, \Omega, u) \in \mathcal{E}'$, $v(S,\cdot)$ satisfies substitutability for all $S \subseteq N$.

Note that if $\mathcal{E}'$ is a substitutable domain and $(N, \Omega, u) \in \mathcal{E}'$, u_i is submodular for all $i \in N$ (set $S = \{i\}$).

Proposition 1: (*Beviá 1996a*). If $\mathcal{E}' \subseteq \mathcal{E}$ is a substitutable domain, the induced game v is concave for all $(N, \Omega, u) \in \mathcal{E}'$.

Corollary 2: The Shapley solution and CES are PM on a substitutable domain $\mathcal{E}'$.

4 Problems with $l \geq 3$

Cartesian product domain of two special types of submodular preferences constitutes substitutable domains. The first one is $\mathcal{E}^{add}$. For all $(N, \Omega, u) \in \mathcal{E}^{add}$, and all $S \subseteq N$, the marginal contribution of α to the

[5] The established standard interpretation of the PM principle in the TU game literature is "positive", requiring that no agent is hurt by the arrival of new agents, contrary to the "negative" interpretation in the fair division literature, as the main focus in the former literature is the environments (e.g., those inducing convex games) with increasing returns to cooperation. This is not the case for the environments inducing concave TU games. Although dPM for TU games and PM for fair division problems are conceptually the same, we choose to call it dPM to avoid the possible confusion resulting from calling two different properties PM in the TU games setting. See Thomson (1995) and Sprumont (2008) for different formulations of the PM principle for a variety of environments.

[6] The argument for the Shapley value is provided in Moulin (1992). Dutta (1990) proved that the Dutta-Ray solution satisfies PM (no one is hurt by the arrival of new agents) on convex games (Lemma 5.5, Dutta 1990). Klijn et al (2001) provided a dual algorithm to compute the dual of this solution which coincides $CES(v)$ on Γ_{con}. One can easily adapt the argument in Lemma 5.5 to prove that $CES(v)$ satisfies dPM on Γ_{con} using this dual algorithm.

efficient surplus $v(S,\cdot)$, i.e. $v(S,A) - v(S, A \setminus \{\alpha\})$ for $\alpha \in A$, is constant for all $A \ni \alpha$ and equal to $\max_{i \in S} u_i(\alpha)$. However, in this case, instead of the general approach of distributing many objects at once, we can just think of distributing the objects separately. A solution to (N, Ω, u) can be defined as the sum of solutions to l distinct problems. If at each single object allocation problem the solution is PM, then their summation is also PM. Hence, a variety of solutions are both EFF and PM on $\mathcal{E}^{add}$ (e.g., the Shapley solution and CES). The second one is the domain of unit demand preferences, i.e., $u_i(A) = \max_{\alpha \in A} u_i(\alpha)$ for all i (Moulin 1992).[7] In this paper, we focus on the superadditive, subadditive, and submodular Cartesian product domains.

Theorem 1: Let $\hat{\mathcal{E}} \supseteq \mathcal{E}_l^{sup}$ and $\tilde{\mathcal{E}} \supseteq \mathcal{E}_{l'}^{sub}$. No solution is both EFF and PM on $\hat{\mathcal{E}}$ for $l \geq 3$, and on $\tilde{\mathcal{E}}$ for $l' \geq 4$.

Proof: It suffices to show that no such essentially single-valued solution exists by Lemma 1. Let $(N, \Omega, u) \in \mathcal{E}' \subseteq \mathcal{E}$ be such that v induced by (N, Ω, u) satisfies the inequality $(*)$ $v(12) + v(23) + v(34) < v(123) + v(234)$, and $\mathcal{E}'$ be regular. Then no solution is both EFF and PM on $\mathcal{E}'$. To see that, consider the following inequalities required by population monotonicity: $\varphi_i(12) \geq \varphi_i(123)$ for $i = 1,2$, $\varphi_i(34) \geq \varphi_i(234)$ for $i = 3,4$, $\varphi_3(23) \geq \varphi_3(123)$, and $\varphi_2(23) \geq \varphi_2(234)$. If φ is efficient, when we sum up all six inequalities, we get the inequality sign reversed and replaced with the weak sign in $(*)$. Note that both $\mathcal{E}_l^{sup}$ and $\mathcal{E}_{l'}^{sub}$ are regular.

To prove the statement on $\mathcal{E}_l^{sup}$, let $\Omega = \{\alpha, \beta, \gamma\}$ and consider the following preferences: $u_1(A) = 2$ if $\gamma \in A$ and $u_1(A) = 0$ otherwise; $u_2(A) = 2$ if $\alpha\gamma \in A$ and $u_2(A) = 0$ otherwise; $u_3(A) = 2$ if $\alpha\beta \in A$ and $u_3(A) = 0$ otherwise; $u_4(A) = 2$ if $\beta \in A$ and $u_4(A) = 0$ otherwise. Note that $v(12) = v(23) = v(34) = 2$ and $v(123) = v(234) = 4$. For $l > 3$, use the same profile and add dummy objects that bring 0 extra utility to all bundles for all agents. The argument trivially extends to $\hat{\mathcal{E}} \supseteq \mathcal{E}_l^{sup}$.

To prove the statement on $\mathcal{E}_{l'}^{sub}$, let $\Omega = \{\alpha, \beta, \gamma, \delta\}$ and consider the utility profile u': $u_1'(A) = 4$ if $\gamma \in A$ and $u_1'(A) = 2$ otherwise; $u_2'(A) = 4$ if $\alpha\gamma \in A$ and $u_2'(A) = 2$ otherwise; $u_3'(A) = 4$ if $\alpha\beta \in A$ and $u_3'(A) = 2$ otherwise; $u_4'(A) = 4$ if $\beta \in A$ and $u_4'(A) = 2$ otherwise. Note that $v(12) = v(23) = v(34) = 6$ and $v(123) = v(234) = 10$. For $l' > 4$ and $\tilde{\mathcal{E}} \supseteq \mathcal{E}_{l'}^{sub}$ the argument is the same as in the above paragraph. □

Corollary 3: No efficient solution satisfies PM on $\mathcal{E}_l$ for $l \geq 3$.

It is well known that concavity of a TU game is a sufficient but not a necessary condition for the Shapley value of the TU game to be PM. Our next result shows that concavity of the preferences (submodularity) is not sufficient for the Shapley solution to be PM.

Proposition 2: The Shapley solution is not PM on $\mathcal{E}_l^{subm}$ for $l \geq 3$.

Proof: Let $N = \{1,2,3,4\}$ and $\Omega = \{\alpha, \beta, \gamma\}$ and consider the problem below where $u = \{u_i\}_{i \in N}$. We have $Sh_3(\{1,2,3\}, \Omega, u^{\{1,2,3\}}) \cong 4.67$ while $Sh_3(N, \Omega, u) \cong 4.92$.

[7] Moulin (1992) proved that preferences satisfy substitutability if the problem is such that agents can receive at most 1 object, hence preferences over sets of objects are irrelevant. Instead, let all agents have unit demand preferences and allow them to get more than 1 object. Then for any (N, Ω, u), $v(S,\cdot)$ will be the same in two different frameworks discussed above for all S.

A	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\gamma$
$u_1(A)$	8	4	8	12	14	8	14
$u_2(A)$	0	1	4	1	4	4	4
$u_3(A)$	8	0	0	8	8	0	8
$u_4(A)$	0	8	0	8	0	8	8

Open Question: Does there exist an efficient solution that is PM on $\mathcal{E}_l^{subm}$ for some $l \geq 3$?

5 Problems with $l \leq 2$

Here our primary focus is the compatibility of efficiency and PM on $\mathcal{E}_2$. First, we briefly discuss some existing results and well-known facts about the domain $\mathcal{E}_1$ as those will help us provide a positive result for the domain $\mathcal{E}_2$ together with a new property that we define below for solutions on $\mathcal{E}_1$.

5.1 The case $l = 1$

Preference of agent i is represented by a number u_i and the tuple (N, u) represents a problem for this case. Without loss of generality, fix an ordered profile $u_1 \leq \cdots \leq u_n$. Efficiency dictates assigning the object to an agent (for example, n here) with the highest u_i. Then, for any efficient, individually rational, and essentially single-valued solution, we have $\sum_{i \in N} \varphi_i(N, u) = u_n$, and $\varphi_i \geq 0$ for all $i \in N$. It is well-known and easy to argue that all problems in $\mathcal{E}_1$ induce a concave TU game. Therefore, we have the following:

Remark 1: The Shapley solution and the constrained egalitarian solution are PM on $\mathcal{E}_1$. Moreover, the simple nature of $\mathcal{E}_1$ allows for many other efficient and population monotonic solutions.[8]

The following property requires the solution to be monotonic with respect to a shift in the entire utility profile by a positive constant.

Definition 8: φ is *constant shift monotonic* (*CSM*) if for any $(N, u), (N, u') \in \mathcal{E}_1$ such that for some $t > 0$, $u'_i = u_i + t$ for all $i \in N$, we have $\varphi_i(N, u) \leq \varphi_i(N, u')$ for all $i \in N$.

Lemma 5: The Shapley solution and the constrained egalitarian solution are CSM on $\mathcal{E}_1$.

Proof: Fix (N, u), (N, u') as in the premises of CSM. Let v, v' be the induced games. Note that $v(S) = \max_{i \in S} u_i$ and $v(S \cup i) - v(S) \leq v'(S \cup i) - v'(S)$ for all $i \in N$, and all $S \subseteq N$. As both the Shapley value and *CES* meet weak contribution monotonicity *(WCM)*, agents will not be worse-off in v' compared to v.[9] □

Constant shift monotonicity is a fairly weak property met by many reasonable solutions. Another trivial example of a PM solution that is also CSM is the equal distribution of the efficient surplus u_n among the agents with the highest valuation.

---

[8] See Thomson (2007) for a survey on the well-known, technically equivalent, airport (cost sharing) problem.

[9] A TU game solution ψ meets *WCM* if for all $(N, v), (N, v')$ with $v(S \cup i) - v(S) \leq v'(S \cup i) - v'(S)$ for all $i \in N$, $S \subseteq N$; $\psi_i(N, v) \leq \psi_i(N, v')$ for all $i \in N$. Hokari and Gellekom (2002) showed that both solutions satisfy WCM on convex games. It is easy to check that the conclusion is true for the concave games induced by problems in $\mathcal{E}_1$.

5.2 The case $l = 2$

Since there are only two objects, say α and β, preference of an agent is either superadditive ($u_i(\alpha) + u_i(\beta) \leq u_i(\alpha\beta)$), or subadditive ($\max\{u_i(\alpha), u_i(\beta)\} \leq u_i(\alpha\beta) \leq u_i(\alpha) + u_i(\beta)$), or both (additively separable, i.e., $u_i(\alpha) + u_i(\beta) = u_i(\alpha\beta)$). Subadditivity and submodularity coincide and as we show below, they are not only necessary but also sufficient for substitutability.

Proposition 3: $\mathcal{E}_2^{sub} = \mathcal{E}_2^{subm}$ is a substitutable domain.

Proof: Let $(N, \Omega, u) \in \mathcal{E}_2^{sub}$ and $S \subseteq N$. To show that $v(S,\cdot)$ satisfies substitutability, the only relevant case is $A = \{\alpha\}$, and $B = \{\alpha\beta\}$. Hence, it suffices to show that $v(S, \{\alpha\beta\}) - v(S, \{\beta\}) \leq v(S, \{\alpha\})$. Note that $v(S, \{\alpha\}) = \max_{i \in S} u_i(\alpha)$, and either $v(S, \{\alpha\beta\}) = u_j(\alpha) + u_k(\beta)$ for some distinct $j, k \in S$ or $v(S, \{\alpha\beta\}) = u_h(\alpha\beta)$ for some $h \in S$. Then, subadditivity implies $v(S, \{\alpha\beta\}) \leq \max_{i \in S} u_i(\alpha) + \max_{i \in S} u_i(\beta)$. □

Corollary 4: For all $(N, \Omega, u) \in \mathcal{E}_2^{sub}$ the induced TU game v is concave. The Shapley solution and the constrained egalitarian solution are PM on $\mathcal{E}_2^{sub}$.

Proposition 4: Neither the Shapley solution nor the constrained egalitarian solution is PM on $\mathcal{E}_2$.

Proof: Let $N = \{1,2,3\}$ and $\Omega = \{\alpha, \beta\}$ and consider the problem below where, $r > t + \epsilon$, $t > \epsilon > 0$, and $u = \{u_i\}_{i \in N}$. We have $Sh_1(\{1,2\}, \Omega, u^{\{1,2\}}) \cong t - \epsilon/2$ while $Sh_1(N, \Omega, u) \cong t - \epsilon/6$

A	α	β	$\alpha\beta$
$u_1(A)$	0	t	t
$u_2(A)$	t	0	$t+\epsilon$
$u_3(A)$	r	0	r

Now, let $N = \{1,2,3,4\}$, $\Omega = \{\alpha, \beta\}$, and consider the problem below. $CES(N, \Omega, u) = (3,3,3,3)$ while $CES(\{1,2,3\}, \Omega, u^{\{1,2,3\}}) = (0,6,6,.)$

A	α	β	$\alpha\beta$
$u_1(A)$	0	0	6
$u_2(A)$	6	0	6
$u_3(A)$	0	6	6
$u_4(A)$	0	0	12

□

The first profile in the above proof shows that a very slight departure from the subadditive domain might result in the failure of PM for the Shapley solution. Now, we introduce a construction through which we can write any problem $(N, \Omega, u) \in \mathcal{E}_2$ as the sum of two problems.

We first introduce some extra notation. Given a monotone function u_i, define $\bar{u}_i$ as follows: $\bar{u}_i(\alpha\beta) \stackrel{\text{def}}{=} \min\{u_i(\alpha\beta), u_i(\alpha) + u_i(\beta)\}$ and $\bar{u}_i(c) = u_i(c)$ for $c = \alpha, \beta$. Hence, the construction renders $\bar{u}_i$ subadditive. Therefore, any problem $(N, \Omega, u) \in \mathcal{E}_2$ with the associated game v induces a unique problem $(N, \Omega, \bar{u}) \in \mathcal{E}_2^{sub}$ with the associated concave game $\bar{v}$. u_i and $\bar{u}_i$ differs only if i has a strictly superadditive utility at the original problem. For such an agent, the only difference is $\bar{u}_i(\alpha\beta) = u_i(\alpha) + u_i(\beta) < u_i(\alpha\beta)$, and hence, $\bar{v}(N) \leq v(N)$.

Given (N, Ω, u) and the induced $(N, \Omega, \bar{u})$, we can define (and think of) $(N, \tilde{u})$ as an "imaginary" single object problem as follows: $\tilde{u}_i = \max\{u_i(\alpha\beta) - \bar{v}(N), 0\}$. Denote the efficient surplus at problem $(N, \tilde{u})$ by $M(N, \tilde{u})$, i.e., $M(N, \tilde{u}) = \max_{i \in N} \tilde{u}_i$.

Lemma 6: For any $(N, \Omega, u) \in \mathcal{E}_2$, we have $v(N) = \bar{v}(N) + M(N, \tilde{u})$.

Proof: Fix $(N, \Omega, u) \in \mathcal{E}_2$, and consider first the case $v(N) = u_j(\alpha) + u_k(\beta)$ for some distinct $j, k \in N$. Then, $u_i(\alpha\beta) \leq u_j(\alpha) + u_k(\beta)$ for all $i \in N$, and $\bar{v}(N) = u_j(\alpha) + u_k(\beta)$. Hence, $\tilde{u}_i = 0$ for all $i \in N$ and $M(N, \tilde{u}) = 0$. Now, consider the case $v(N) = u_j(\alpha\beta)$ for some $j \in N$. If $u_j(\alpha\beta) = \bar{u}_j(\alpha\beta)$, again we have $\bar{v}(N) = v(N)$ and $M(N, \tilde{u}) = 0$. If $u_j(\alpha\beta) > \bar{u}_j(\alpha\beta)$, $M(N, \tilde{u}) = u_j(\alpha\beta) - \bar{v}(N)$. □

By Lemma 6, we can define a solution to (N, Ω, u) as the sum of solutions to the decomposed portions of the original problem: $(N, \Omega, \bar{u})$ and $(N, \tilde{u})$ in the following way.

Definition 9: Given any two essentially single-valued solutions $\overline{\varphi}$ on $\mathcal{E}_2^{sub}$, and $\tilde{\varphi}$ on $\mathcal{E}_1$, φ is a hybrid (of $\overline{\varphi}$ and $\tilde{\varphi}$) solution on $\mathcal{E}_2$ if for all $(N, \Omega, u) \in \mathcal{E}_2$:

$$(\sigma, m) \in \varphi(N, \Omega, u) \Leftrightarrow u_i(\sigma_i, m_i) = \overline{\varphi}_i(N, \Omega, \bar{u}) + \tilde{\varphi}_i(N, \tilde{u}) \ \forall i \in N.$$

Remark 2: φ is EFF if and only if both $\overline{\varphi}$ and $\tilde{\varphi}$ are EFF and φ is IR if both $\overline{\varphi}$ and $\tilde{\varphi}$ are IR. Moreover, φ is essentially single-valued by construction.

To clarify our construction, consider the 4-person problem in Table 1. $u_i(\alpha\beta)$ differs from $\bar{u}_i(\alpha\beta)$ for agents 1, 2, and 4 as only their preferences are strictly superadditive. Note that $v(N) = u_1(\alpha\beta) = 10$, $\bar{v}(N) = u_3(\alpha) + u_2(\beta) = 8$ and $M(N, \tilde{u}) = \tilde{u}_1 = 2$. In order to understand the proof of Theorem 2 below, it is critical to understand the effect of absence of some agents on the problems $(N, \Omega, \bar{u})$ and $(N, \tilde{u})$. Given (N, Ω, u), in the absence of agents $N \setminus S$, the subproblem for agents in S is (S, Ω, u^S). (S, Ω, u^S) induces the subadditive problem $(S, \Omega, \bar{u}^S)$ which is equivalent to restriction of $(N, \Omega, \bar{u})$ on agents in S. A similar argument is not true for $(N, \tilde{u})$, $(S, \tilde{u}^S)$. The latter is the restriction of the former on S with $\tilde{u}^S{}_i = \tilde{u}_i = \max\{u_i(\alpha\beta) - \bar{v}(N), 0\}$ for all $i \in S$. However, in the absence of agents $N \setminus S$, the relevant imaginary single object problem $\big(S, \tilde{u}(S)\big)$ is the one induced by (S, Ω, u^S) where $\tilde{u}(S)_i \stackrel{\text{def}}{=} \max\{u_i(\alpha\beta) - \bar{v}(S), 0\}$. Note that $v(S) = \bar{v}(S) + M\big(S, \tilde{u}(S)\big)$ for all (N, Ω, u) and $S \subseteq N$. For example, when agent 2 and 3 leaves the problem above, for $S = \{1,4\}$, $v(S) = u_1(\alpha\beta) = 10$, $\bar{v}(S) = u_1(\alpha) + u_4(\beta) = 4$, $\tilde{u}(S)_1 = 6 = M\big(S, \tilde{u}(S)\big) \neq \tilde{u}_1^S = \tilde{u}_1 = 2$ and $\tilde{u}(S)_4 = 2$.

i	$u_i(\alpha) = \bar{u}_i(\alpha)$	$u_i(\beta) = \bar{u}_i(\beta)$	$u_i(\alpha\beta)$	$\bar{u}_i(\alpha\beta)$	$\tilde{u}_i$
1	2	2	10	4	2
2	0	4	9	4	1
3	4	4	7	7	0
4	2	2	6	4	0

Table 1

Theorem 2: Let φ be a hybrid solution such that for all $(N, \Omega, u) \in \mathcal{E}_2$, $i \in N$, $\varphi_i = \overline{\varphi}_i + \tilde{\varphi}_i$ where $\overline{\varphi}$ is a solution to $(N, \Omega, \bar{u})$, and $\tilde{\varphi}$ is a solution to $(N, \tilde{u})$. φ is PM if both $\overline{\varphi}$ and $\tilde{\varphi}$ are PM and $\tilde{\varphi}$ is CSM.

Proof: Let $\overline{\varphi}$ and $\tilde{\varphi}$ be population monotonic, and $\tilde{\varphi}$ be CSM. Fix $(N,\Omega,u) \in \mathcal{E}_2$, $S \subseteq N$. Note that $\varphi(S,\Omega,u^S) = \overline{\varphi}(S,\Omega,\bar{u}^S) + \tilde{\varphi}(S,\tilde{u}(S))$. As $\overline{\varphi}$ and $\tilde{\varphi}$ are PM, $\overline{\varphi}_i(N,\Omega,\bar{u}) \leq \overline{\varphi}_i(S,\Omega,\bar{u}^S)$ and $\tilde{\varphi}_i(N,\tilde{u}) \leq \tilde{\varphi}_i(S,\tilde{u}^S)$ for all $i \in S$. Therefore, it suffices to show that for all $i \in S$, $\tilde{\varphi}_i(S,\tilde{u}^S) \leq \tilde{\varphi}_i(S,\tilde{u}(S))$.

Define $K \overset{\text{def}}{=} \{i \in S : \bar{v}(N) < u_i(\alpha\beta)\}$, $L \overset{\text{def}}{=} \{i \in S \setminus K : \bar{v}(S) < u_i(\alpha\beta)\}$. First of all note that $\tilde{u}^S{}_i = \tilde{u}(S)_i = 0$ for all $i \in S \setminus (K \cup L)$; $\tilde{u}(S)_i = \tilde{u}^S{}_i + \bar{v}(N) - \bar{v}(S)$ and $\tilde{u}^S{}_i > 0$ for all $i \in K$; $\tilde{u}(S)_i > 0$ and $\tilde{u}^S{}_i = 0$ for all $i \in L$. As $\tilde{\varphi}$ is PM, these imply the following by Lemma 3: $\tilde{\varphi}_i(S,\tilde{u}^S) = 0$ for all $i \in S \setminus K$, and $\tilde{\varphi}_i(S,\tilde{u}(S)) = 0$ for all $i \in S \setminus (K \cup L)$ as $\tilde{\varphi}$ satisfies DUM; $\tilde{\varphi}_i(S,\tilde{u}^S) = \tilde{\varphi}_i(K,\tilde{u}^K)$ for all $i \in K$ and $\tilde{\varphi}_i(S,\tilde{u}(S)) = \tilde{\varphi}_i(K \cup L,\tilde{u}(S)^{K \cup L})$ for all $i \in K \cup L$ as $\tilde{\varphi}$ is DI. Then, to prove the proposition, it suffices to show that the inequality $\tilde{\varphi}_i(K,\tilde{u}^K) \leq \tilde{\varphi}_i(K \cup L,\tilde{u}(S)^{K \cup L})$ $(*)$ holds for all $i \in K$, and $0 \leq \tilde{\varphi}_i(K \cup L,\tilde{u}(S)^{K \cup L})$ for all $i \in L$.

Let $\{L_1,\dots,L_h\}$ be the partition of L such that $u_i(\alpha\beta) = u_j(\alpha\beta)$ for all $r \in \{1,\dots,h\}$, $i,j \in L_r$, and $u_i(\alpha\beta) > u_j(\alpha\beta)$ for all $r < r'$, $i \in L_r$, $j \in L_{r'}$. Finally, $t_0 \overset{\text{def}}{=} \bar{v}(N) - u_j(\alpha\beta)$ for $j \in L_1$; $t_r \overset{\text{def}}{=} u_j(\alpha\beta) - u_k(\alpha\beta)$ for $j \in L_r$, $k \in L_{r+1}$ and $r < h$; and $t_h \overset{\text{def}}{=} u_j(\alpha\beta) - \bar{v}(S)$ for $j \in L_h$.

Consider first the case $L = \emptyset$. $\tilde{u}(S)_i^K = \tilde{u}(S)_i = \tilde{u}_i^K + \bar{v}(N) - \bar{v}(S)$ for all $i \in K$. Then, as $\tilde{\varphi}$ is CSM, $(*)$ holds for all $i \in K$. Now, let $L \neq \emptyset$.

Now, set $Q_0 = K$, $\hat{u}^0 = \tilde{u}^K$, and define $P_r = (Q_r,\hat{u}^r)$ and $P_r' = (Q_r',\hat{u}'^r)$ recursively as follows for

$r = 0,\dots,h$: $\quad Q_r' = Q_r$, $\hat{u}_i'^r = \hat{u}^r + t_r$ for all $i \in Q_r$

$r = 0,\dots,h-1$: $\quad Q_{r+1} = Q_r \cup L_{r+1}$, $\hat{u}_i^{r+1} = \hat{u}_i'^r$ for all $i \in Q_r$, and $\hat{u}_i^{r+1} = 0$ for all $i \in L_{r+1}$

Note that $P_0 = (K,\tilde{u}^K)$, by construction $P_h' = (K \cup L,\tilde{u}(S)^{K \cup L})$, and $K \subseteq Q_r$ for all r.

As $\tilde{\varphi}$ is CSM, $\tilde{\varphi}_i(P_r) \leq \tilde{\varphi}_i(P_r')$ for all r, all $i \in Q_r$. Since $\tilde{\varphi}$ is DI (Lemma 3), we have $\tilde{\varphi}_i(P_r) \leq \tilde{\varphi}_i(P_r') = \tilde{\varphi}_i(P_{r+1})$ for all r, all $i \in Q_r$, and since $\tilde{\varphi}$ satisfies DUM, (Lemma 3) $\tilde{\varphi}_i(P_r) = 0$ for all r, all $i \in L_r$. □

Corollary 5*:* There exist several solutions both efficient and population monotonic on $\mathcal{E}_2$. Two examples are the hybrid Shapley solution ($\widehat{Sh}(N,\Omega,u) = Sh(N,\Omega,\bar{u}) + Sh(N,\tilde{u})$) and the hybrid constrained egalitarian solution ($\widehat{CES}(N,\Omega,u) = CES(N,\Omega,\bar{u}) + CES(N,\tilde{u})$).

6 Concluding remarks

The following table summarizes previous results, our results and the open questions we pose:

The *monotone* preference	Result/Argument	l	EFF and PM solutions
Full (non − monotone)	Beviá (1996a)	$l \geq 3$	No
Substitutable	Beviá (1996a)	$l \geq 2$	Sh, CES
Additive	Section 4	$l \geq 2$	Many
Unit demand	Section 4	$l \geq 2$	Sh, CES
Full	Cor 3	$l \geq 3$	No
Superadditive	Thm 1	$l \geq 3$	No
Subadditive	Thm 1	$l \geq 4$	No
Submodular	Prop 2	$l \geq 3$	?, not Sh
Subadditive (= submodular)	Cor 4	$l = 2$	Sh, CES
Full	Thm 2, Cor 5	$l = 2$	Many: The hybrid Sh, the hybrid CES, ...
Full	Prop 4	$l = 2$	Not Sh, not CES

We have strengthened Beviá (1996a)'s negative result and shown that PM is incompatible with efficiency on the superadditive and subadditive Cartesian product domains, while on the intersection of those two domains - the additively separable domain (there is no utility gain or loss in the conjunction of two bundles of objects) - several efficient and population monotonic solutions exist. Although the proper use of the terms requires the context of demand theory, one can roughly interpret superadditivity representing the complementarity and subadditivity representing the substitutability of two distinct bundles. Hence, our results can be interpreted as the prevalence of incompatibility between efficiency and PM even in case all distinct bundles of objects are unilaterally complements or substitutes for all agents.

We have also shown that PM and efficiency are compatible on the full preference domain in the case of two objects. Unlike many other works in the literature (e.g., Moulin 1992 and Bevia 1996a), this positive result does not rely on finding a sufficiently narrow domain so that the induced TU game is concave and hence the Shapley solution is PM. We provide an original constructive method to define various population monotonic solutions using the notion of hybrid solutions and Theorem 2.

Finally, constructive arguments similar to those in the two-object case fail to hold in the case of three or more objects. Therefore, it is hard to comment on the open question we posed ("Does there exist an efficient solution that is PM on ε_l^{subm} for some $l \geq 3$?") without referring to the class of TU games that the submodular preference domain induces. The source of the difficulty lies in the computational complexity of finding the efficient allocations and hence the efficient surplus. The very same reason renders describing the induced class of TU games or finding some relevant characteristics of these games a computationally complex procedure.

References

Alkan, A. (1994): "Monotonicity and envy-free assignments", *Economic Theory*, 4 (4), 605-616.

Beviá, C. (1996a): "Population monotonicity in a general model with indivisible goods", *Economic Letters*, 50 (1), 91-97.

Beviá, C. (1996b): "Population monotonicity in economies with one indivisible good", *Mathematical Social Sciences*, 32 (2), 125-137.

Chun, Y. (1986): "The solidarity axiom for quasi-linear social choice problems", *Social Choice and Welfare*, 3 (4), 297-310.

Dutta, B., and D. Ray (1989): "A concept of egalitarianism under participation constraints", *Econometrica*, 57 (3), 615-635.

Dutta, B. (1990): "The egalitarian solution and the reduced game properties in convex games", *International Journal of Game Theory*, 19, 153-169.

Feige, U. (2009): "On maximizing welfare when utility functions are subadditive", *SIAM Journal on Computing*, 39 (1), 122-142.

Hokari, T., and A. van Gellekom (2002): "Population monotonicity and consistency in convex games: Some logical relations", *International Journal of Game Theory*, 31, 593-607.

Kelso A.S., V.P. Crawford (1982): "Job matching, coalition formation, and gross substitutes", *Econometrica*, 50 (6), 1483-1504.

Kim, H. (2004): “Population monotonic rules for fair allocation problems”, *Social Choice and Welfare*, 23, 59-70.

Klaus, B. (2001): “Population-monotonicity and separability for economies with single-dipped preferences and the assignment of an indivisible object”, *Economic Theory*, 17 (3), 675-692.

Klijn, F., M. Slikker, and S. Tijs (2001): “A dual egalitarian solution”, *Economics Bulletin*, 10 (3), 1-8.

Moulin, H. (1990): “Fair division under joint ownership: Recent results and open problems”, *Social Choice and Welfare*, 7 (2), 149-170.

Moulin, H. (1992): “An application of the Shapley value to fair division with money” *Econometrica*, 60 (6), 1331-1349.

Shapley, L.S. (1953): A Value for n-person games. In: Tucker, A., and H. Kuhn (Eds.), *Contributions to the Theory of Games*, vol. II., Princeton University Press, Princeton, pp. 307-317.

Sprumont, Y. (2008): Monotonicity and solidarity axioms in economics and game theory. In: Pattanaik PK, Tadenuma K, Xu Y, Naoki Y (Eds.) Rational Choice and Social Welfare. Springer, pp. 71-94.

Thomson, W. (1983): The fair division of a fixed supply among a growing population. Mathematics of Operations Research 8 (3), 319-326.

Thomson, W. (1995): Population monotonic allocation rules. In: Barnett, WA, Moulin H., Salles M, Schofield N (Eds.) Social Choice, Welfare, and Ethics. Cambridge University Press, pp. 79–124.

Thomson, W. (2007): Cost allocation and airport problems. Mimeo, University of Rochester.