CLASS OF STABLE CONNECTIVITY OF SOURCE-SINK DIFFEOMORPHISM ON TWO-DIMENSIONAL SPHERE

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We consider the class of gradient-like diffeomorphisms possessing an attractor and a repeller separated by a circle on a 2-sphere. For any diffeomorphism in this class we construct a stable arc connecting it with the source-sink system. Bibliography: 14 *titles. Illustrations*: 13 *figures.*

1 Introduction

The notion of a stable arc connecting two structurally stable systems on a manifold was introduced in [1]. Such an arc does not change qualitative properties under small perturbations. As proved in [2], there exists a simple arc (containing only elementary bifurcations) between any two Morse–Smale flows. By the results of [3], such a simple arc can be always replaced with a stable arc. For Morse–Smale diffeomorphisms defined on manifolds of any dimension there are examples of systems that cannot be connected by a stable arc. Respectively, the following question naturally arises: find an invariant that uniquely determines the equivalence class of a Morse–Smale diffeomorphism with respect to the connection relation by a stable arc (a component of stable connection).

A circle is a unique closed manifold for which this problem is completely solved. As shown in [4], for orientation-preserving rough transformations of a circle the component of stable connection is determined by the Poincaré rotation number k/m, $(k,m)=1$, while all orientation-changing diffeomorphisms lie in the same component of stable connection.

For Morse–Smale diffeomorphisms on a two-dimensional sphere necessary conditions for the existence of a connecting stable arc were found in [5], where sufficient conditions were not

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discussed. The conditions found in [5] imply that even on a two-dimensional sphere there are infinitely many components of stable connection. To clarify this fact, we regard $\mathbb{S}^1$ as the equator of the sphere $\mathbb{S}^2$. Then the diffeomorphism of the circle with exactly two periodic orbits of period m and rotation number k/m can be extended to a diffeomorphism $F_{k/m} : \mathbb{S}^2 \to \mathbb{S}^2$ possessing two fixed sources at the north and south poles. Moreover, the diffeomorphisms $F_{k/m}$ and $F_{k'/m'}$ for $m = 2^r \cdot q$ and $m' = 2^{r'} \cdot q'$, where $r, r' \geqslant 0$ are integers and $q \neq q'$ are natural numbers, are not connected by a stable arc (cf. Figure 1 for the phase portraits of diffeomorphisms of the 2-spheres $F_1/2$ and $F_1/3$).

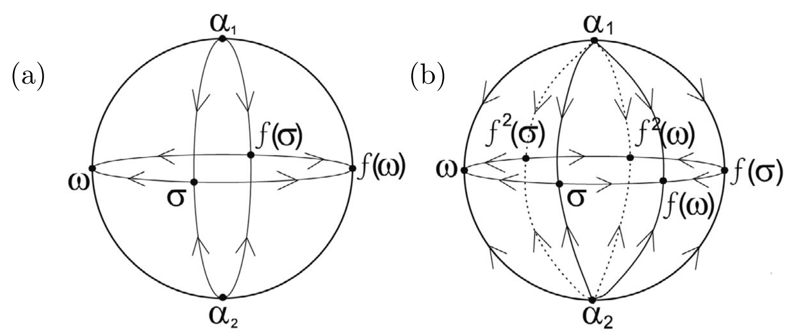

Figure 1. Phase portraits of diffeomorphisms of 2-sphere (a) $F_{1/2}$, (b) $F_{1/3}$.

The simplest structurally stable two-dimensional diffeomorphism is presented by a source-sink system on a two-dimensional sphere. All such systems are pairwise topologically conjugate, but it is not trivial to prove that there exists a stable path between two source-sink systems [6] (moreover, this assertion is false in the general case of source-sink systems on an n-dimensional sphere [7]). By the results of [5], the stable connectivity class for a source-sink diffeomorphism on a 2-sphere does not contain diffeomorphisms $F_{k/m}$ with odd $m > 1$.

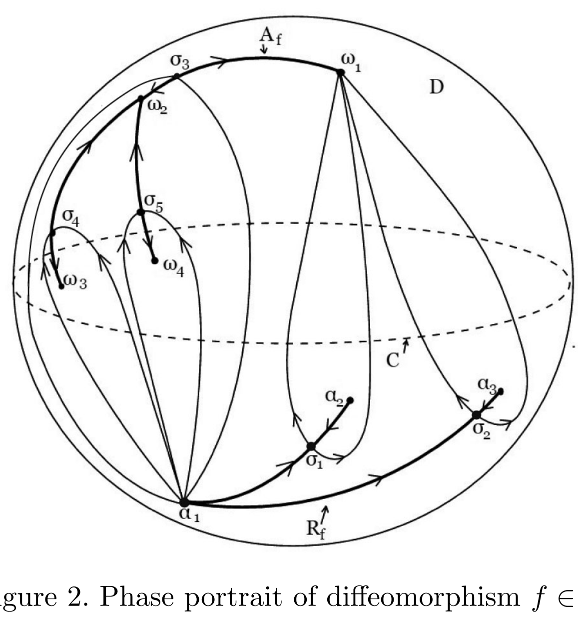

Figure 2. Phase portrait of diffeomorphism $f \in G$.

In this paper, we find sufficient conditions for a gradient-like diffeomorphism of a 2-sphere to belong to a component of stable connection of a source-sink diffeomorphism. Namely, we deal with orientation-preserving gradient-like diffeomorphisms $f : \mathbb{S}^2 \to \mathbb{S}^2$. We denote by Ω^0, Ω^1, and Ω^2 the sets of sinks, saddles, and sources of the diffeomorphism f respectively. Let $\Sigma \subset \Omega^1$ be a subset (possibly, empty) of saddle orbits. With each Σ we can associate the dual pair attractor–repeller A_Σ, R_Σ defined by

$$A_\Sigma = \Omega^0 \cup W^u_\Sigma, \quad R_\Sigma = \Omega^2 \cup W^s_{\Omega^1 \setminus \Sigma}.$$

We say that a diffeomorphism f belongs to the class G if there exists a set Σ and a circle $C \subset \mathbb{S}^2$ such that A_Σ and R_Σ belong to different connected components of the set $\mathbb{S}^2 \setminus C$. We denote by A_f and R_f the attractor and repeller possessing the above properties for a diffeomorphism $f \in G$ (cf. Figure 2). We formulate the main result of the paper.

Theorem 1.1. *Any diffeomorphism $f \in G$ is connected with a source-sink diffeomorphism by a stable arc.*

2 Preliminaries

2.1. Morse–Smale diffeomorphisms. Let a diffeomorphism $f : M^n \to M^n$ be defined on a smooth closed (compact, without boundary) n-manifold ($n \geqslant 1$) M^n with metric d.

A point $x \in M^n$ is said to be *wandering* for f if there exists an open neighborhood U_x of the point x such that $f^n(U_x) \cap U_x = \varnothing$ for all $n \in \mathbb{N}$. Otherwise, the point x is referred to as *nonwandering*. The set of nonwandering points for f is called the *nonwandering set* and is denoted by Ω_f.

For example, all limit points of a diffeomorphism are nonwandering. We recall that $y \in M^n$ is an *ω-limit* point for $x \in M^n$ if there exists a sequence $t_k \to +\infty$, $t_k \in \mathbb{Z}$, such that $\lim\limits_{t_k \to +\infty} d(f^{t_k}(x), y) = 0$. The set $\omega(x)$ of all ω-limit points for a point x is called the *ω-limit set*. Replacing $+\infty$ with $-\infty$, we define the *α-limit set* $\alpha(x)$ of the point x. The set $L_f = cl(\bigcup\limits_{x \in M^n} \omega(x) \cup \alpha(x))$ is called the *limit set* of the diffeomorphism f.

If the set Ω_f is finite, then each point $p \in \Omega_f$ is periodic. We denote by $m_p \in \mathbb{N}$ the period of a periodic point p. With any periodic point p we associate the *stable* and *unstable* manifolds by

$$W^s_p = \{x \in M^n : \lim_{k \to +\infty} d(f^{km_p}(x), p) = 0\},$$

$$W^u_p = \{x \in M^n : \lim_{k \to +\infty} d(f^{-km_p}(x), p) = 0\}.$$

Stable and unstable manifolds are called *invariant manifolds*. We say that periodic orbits $\mathscr{O}_1, \ldots, \mathscr{O}_k$ form a *cycle* if $W^s_{\mathscr{O}_i} \cap W^u_{\mathscr{O}_{i+1}} \neq \varnothing$ for $i \in \{1, \ldots, k\}$ and $\mathscr{O}_{k+1} = \mathscr{O}_1$.

A periodic point $p \in \Omega_f$ is said to be *hyperbolic* if the Jacobi matrix $\left(\dfrac{\partial f^{m_p}}{\partial x}\right)\Big|_p$ has no eigenevalues equal to 1 in absolute value. If all of the eigenvalues are less (greater) than 1 in absolute value, then the point p is called a *sink* (*source*). Sink and source points are called *nodes*. A hyperbolic periodic point that is not a node is called a *saddle*.

By the hyperbolic structure of periodic points p, the stable W^u_p and unstable W^s_p manifolds are injective immersions of the space $\mathbb{R}^{q_p}$ and $\mathbb{R}^{n-q_p}$, where q_p is the number of eigenvalues of the Jacobi matrix that are greater than 1 in absolute value. The number ν_p equal to $+1$ (-1) if the mapping $f^{m_p}|_{W^u_p}$ preserves (change) the orientation of W^u_p is called the *orientation type* of the point p. The path connected component of the set $W^u_p \setminus p$ $(W^s_p \setminus p)$ is called the *unstable* (*stable*) *separatrix* of the point p.

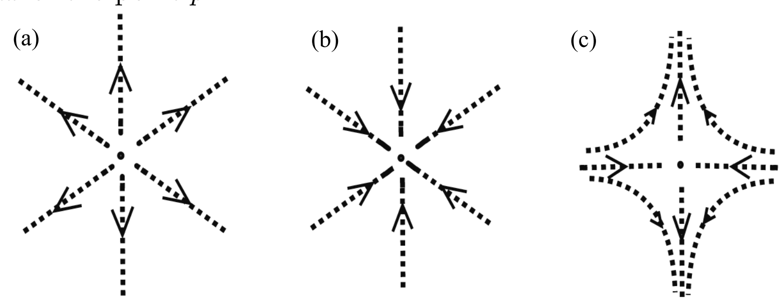

Figure 3. (a) source, (b) sink, (c) saddle.

A closed f-invariant set $A \subset M^n$ is called an *attractor* of a discrete dynamical system f if there is a compact neighborhood U_A of A such that $f(U_A) \subset \text{int}\, U_A$ and $A = \bigcap\limits_{k\geqslant 0} f^k(U_A)$. The neighborhood U_A is called *isolating.* The *repeller* is the attractor for f^{-1}. The complement to the isolating neighborhood of an attractor is the isolating neighborhood of the dual repeller.

A diffeomorphism $f : M^n \to M^n$ is called a *Morse–Smale diffeomorphism* if

1) the nonwandering set Ω_f consists of finitely many hyperbolic orbits,

2) the manifolds W^s_p and W^u_q transversally intersect for any nonwandering points p and q.

A Morse–Smale diffeomorphism is called a *gradient-like diffeomorphism* if the condition $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \varnothing$ for different points $\sigma_1, \sigma_2 \in \Omega_f$ implies $\dim W^u_{\sigma_1} < \dim W^u_{\sigma_2}$.

In a similar way, one can introduce a *Morse–Smale flow* on a manifold M^n which is called *gradient-like* if there are no periodic trajectories.

In the case $n = 2$, the dynamics of gradient-like diffeomorphisms is closely connected with the dynamics of periodic homeomorphisms. We recall that a homeomorphism $\varphi : M^2 \to M^2$ is said to be *periodic of order* $m \in \mathbb{N}$ if $\varphi^m = \text{id}$ and $\varphi^\mu \neq \text{id}$ for any natural number $\mu < m$.

Proposition 2.1 (cf. [8] and [9, Theorem 3.3])**.** *Any orientation-preserving gradient-like diffeomorphism $f : M^2 \to M^2$ is topologically conjugate to the composition of a periodic homeomorphism with a gradient-like flow shifted by the time unit.*

According to the classification [10], an orientation-preserving periodic homeomorphism of order m of a two-dimensional sphere has periodic points of only two periods 1 and m; moreover, the set of its fixed points is nonempty. Then Proposition 2.1 implies the following assertion.

Proposition 2.2. *Any orientation-preserving gradient-like diffeomorphism of a* 2*-sphere has periodic points only of two periods* 1 *and m* (*possibly,* $m = 1$); *moreover, the set of its fixed points is nonempty.*

Furthermore, for any orientation-preserving gradient-like diffeomorphism $f : M^2 \to M^2$ the following assertion holds.

Proposition 2.3 (cf. [9, Lemmas 3.1 and 3.3])**.** *Assume that $f : M^2 \to M^2$ is an orientation-preserving gradient-like diffeomorphism and m_f is the least natural number such that $\Omega_{f^{m_f}}$ consists of fixed points with positive type orientation. Then the period of any saddle separatrix of the diffeomorphism f is equal to m_f.*

Summarizing the results of Propositions 2.2 and 2.3, we obtain the following fact about the structure of periodic data of gradient-like diffeomorphisms of a 2-sphere.

Proposition 2.4. *For any orientation-preserving gradient-like diffeomorphism of a* 2*-sphere the following assertions hold*:

1) $m_f = m$,

2) *any saddle point with negative orientation type is a fixed point,*

3) *any saddle point with positive orientation type has period* m.

2.2. Stable arcs in the space of diffeomorphisms. We consider a one-parameter family of diffeomorphisms (an *arc*) $\varphi_t : M^n \to M^n$, $t \in [0,1]$. Denote by $\mathscr{Q}$ the set of arcs $\{\varphi_t\}$ that start and terminate at Morse–Smale diffeomorphisms and possess the following properties:

1) φ_t has the finite limit set for all $t \in [0,1]$,

2) $\{\varphi_t\}$ contains a finite set of bifurcational diffeomorphisms $b_1, \dots, b_m \in (0,1)$.

By [11], an arc $\{\varphi_t\}$ is said to be *stable* if it is an interior point of the equivalence class with respect to the following relation: arcs $\{\varphi_t\}$, $\{\varphi'_t\} \in \mathscr{Q}$ are *conjugate* if there exist homeomorphisms $h : [0,1] \to [0,1]$ and $H_t : M^n \to M^n$ such that $h(b_i) = b'_i$, $i \in \{1, \dots, m\}$, $H_t\varphi_t = \varphi'_{h(t)}H_t$, $t \in [0,1]$, and H_t continuously depends on t.

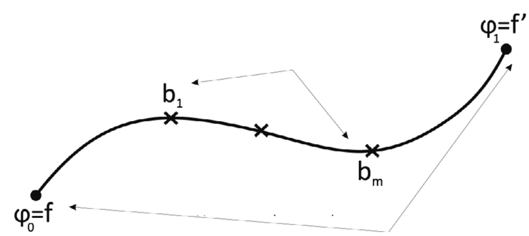

Figure 4. Arc in the set $\mathscr{Q}$.

It is also established in [11] that an arc $\{\varphi_t\} \in \mathscr{Q}$ is stable if and only if it possesses the following properties:

1) the diffeomorphism φ_{b_i}, $i \in \{1, \dots, m\}$, has no cycles and possesses exactly one nonhyperbolic periodic orbit (namely, a flip or a noncritical saddle–node); moreover, the arc unfolds generically through the bifurcational value,

2) the stable and unstable manifolds of an periodic point of the diffeomorphism φ_t, $t \in [0,1]$, transversally intersect (cf. Figure 4).

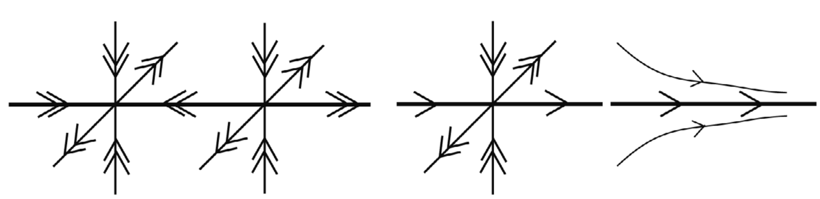
Figure 5. Saddle–node bifurcation.

We say that an arc $\{\varphi_t\} \in \mathscr{Q}$ *unfolds generically through a saddle-node bifurcation* φ_{b_i} (cf. Figures 5 and 6) if, in some neighborhood of the nonhyperbolic point (p, b_i), the arc φ_t is conjugate to the arc

$$\widetilde{\varphi}_{\widetilde{t}}(x_1, x_2, \dots, x_{1+n_u}, x_{2+n_u}, \dots, x_n) = \Big(x_1 + \frac{1}{2}x_1^2 + \widetilde{t}, \pm 2x_2, \dots, \pm 2x_{1+n_u}, \frac{\pm x_{2+n_u}}{2}, \dots, \frac{\pm x_n}{2}\Big),$$

where $(x_1, \dots, x_n) \in \mathbb{R}^n$, $|x_i| < 1/2$, $|\widetilde{t}| < 1/10$.

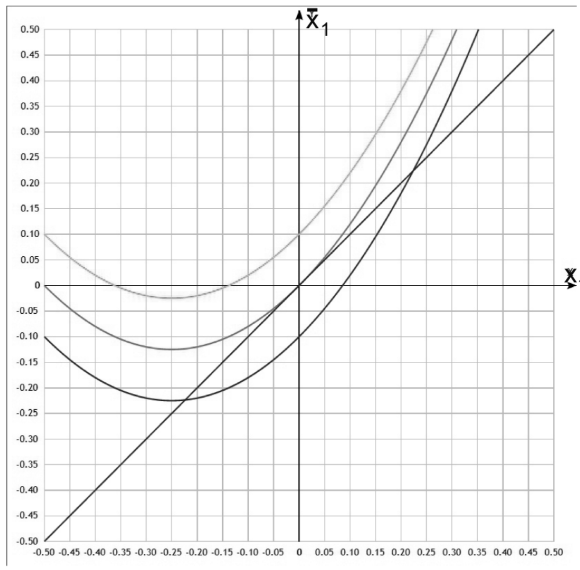

Figure 6. Graph of the mapping of the first coordinate of saddle-node bifurcation.

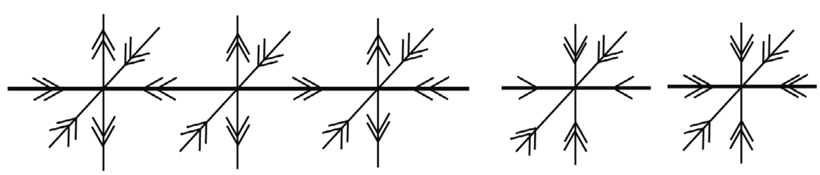

Figure 7. Period doubling bifurcation (flip bifurcation).

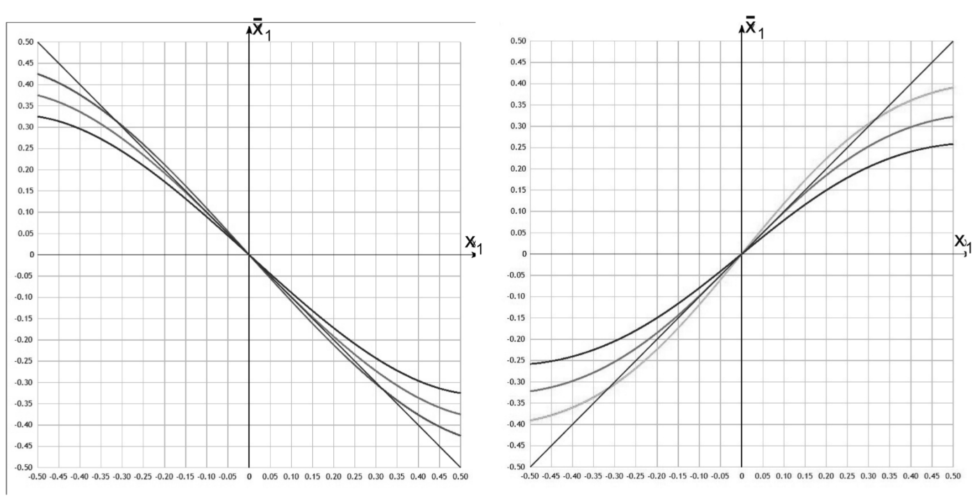

Figure 8. Graph of the mapping of the first coordinate and the squared mapping for period doubling bifurcation (flip bifurcation).

We say that an arc $\{\varphi_t\} \in \mathscr{Q}$ *unfolds generically through a period doubling bifurcation* (a *flip bifurcation*) φ_{b_i} (cf. Figures 7 and 8) if, in some neighborhood of the nonhyperbolic point (p, b_i), the arc φ_t is conjugate to the arc

$$\widetilde{\varphi}_{\widetilde{t}}(x_1, x_2, \dots, x_{1+n_u}, x_{2+n_u}, \dots, x_n) = \Big(-x_1(1 \pm \widetilde{t}) + x_1^3, \pm 2x_2, \dots, \pm 2x_{1+n_u}, \frac{\pm x_{2+n_u}}{2}, \dots, \frac{\pm x_n}{2}\Big),$$

where $(x_1, \dots, x_n) \in \mathbb{R}^n$, $|x_i| < 1/2$, $|\widetilde{t}| < 1/10$.

3 Dynamics of Diffeomorphisms of Class G

Lemma 3.1. *For any diffeomorphism $f \in G$ the attractor A_f either consists of a single sink point or is a connected one-dimensional complex without cycles.*

Proof. We denote by D the connected component of the set $\mathbb{S}^2 \setminus C$ containing A_f. Since A_f is an attractor and the disk D lies in its basin of attraction, there exists a natural number l such that $f^l(\text{cl } D) \subset \text{int } D$. Then $A_f = \bigcap\limits_{k \geqslant 0} f^{kl}(\text{cl } D)$, which implies that the attractor A_f is connected (cf., for example, [9, Proposition 10.1]). We show that A_f contains no cycles.

Assume the contrary. Let A_f contain a cycle formed by the closures of unstable manifolds of saddle points $\sigma_1, \dots, \sigma_r$. Then the closed curve $R = \bigcup\limits_{i=1}^{r} \text{cl } W_i^u$ bounds the disk $d \subset D$, which means that for each saddle σ_i one of its stable separatrices lies in d. Consequently, the disk d also contains the closure of this separatrix. Thus, $R_f \cap d \neq \varnothing$, which contradicts the assumptions on G. Thus, A_f does not contain cycles. □

Lemma 3.2. *If the attractor A_f of a diffeomorphism $f \in G$ is different from a sink, then exactly one of the following assertions holds*:

1) $A_f = \text{cl } W_\sigma^u$, *where* $q_\sigma = m_\sigma = 1, \nu_\sigma = -1$,

2) *there exist points* $\sigma, \omega \in A_f$ *such that* $m_\sigma = m_\omega$, $q_\omega = 0, q_\sigma = \nu_\sigma = 1$, $\omega \in \text{cl } W_\sigma^u$ *and* $W_\omega^s \cap A_f$ *consists of exactly one unstable separatrix of the saddle* σ *and the sink* ω.

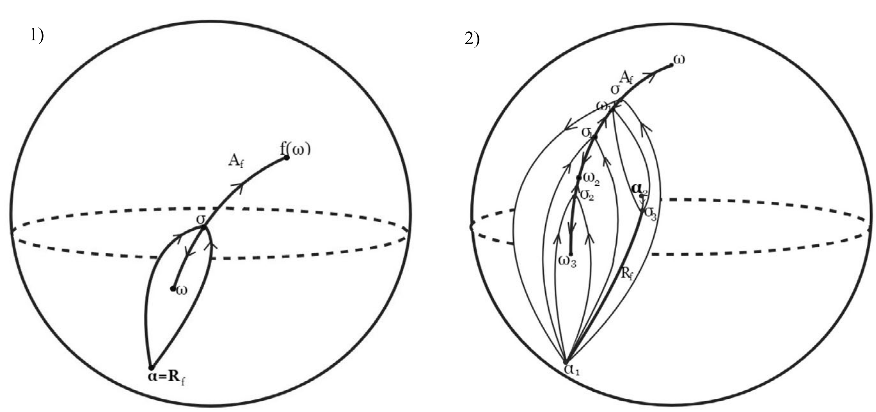

Figure 9. Illustration for Lemma 3.2.

Proof. Assume that the attractor A_f of a diffeomorphism $f \in G$ is not a sink. By Lemma 3.1, in this case, the one-dimensional complex A_f is a tree and, consequently, has the so-called pendant vertex, i.e., there exist points $\sigma,\omega\in A_f$ such that $q_\omega=0$, $q_\sigma=1$, $\omega\in \mathrm{cl}\, W^u_\sigma$, and $W^s_\omega\cap A_f$ consists of exactly one unstable separatrix of the saddle σ and the sink ω. Two cases are possible: 1) $\nu_\sigma=-1$ and 2) $\nu_\sigma=1$ (cf. Figure 9).

Case 1: $\nu_\sigma=-1$. By Proposition 2.4, $m_\sigma=1$ and the period of the separatrix of the saddle σ is equal to 2. Since A_f has no cycles, we have $\omega\neq f(\omega)$ and, consequently, $m_\omega=2$. Then $W^s_{f(\omega)}\cap A_f$ consists of exactly one unstable separatrix of the saddle σ and the sink $f(\omega)$. This implies $A_f=\omega\cup W^u_\sigma\cup f(\omega)$.

Case 2: $\nu_\sigma=1$. By Proposition 2.4, $m_\sigma=m$, the period of the separatrix of the saddle σ is equal to m, and for the sink ω there are two possibilities: 2a) $m_\omega=m$ and 2b) $m_\omega=1$. In case 2a), we have the required assertion of the lemma. In case 2b), all m unstable separatrices of the saddle σ lie in the basin of the sink ω. Since $W^s_\omega\cap A_f$ consists of one unstable separatrix of the saddle σ and the sink ω, we have $m=1$, and the required assertion follows. □

4 Construction of Stable Arc

We divide the class G into paiwise disjoint subsets $G_{\lambda,\mu}$, where $\lambda,\in\mathbb{N}$ ($\mu\in\mathbb{N}$) is the number of sinks (sources) in the attractor A_f (the repeller R_f). We note that the class $G_{1,1}$ consists of source-sink diffeomorphisms. To prove Theorem 1.1, it suffices to construct a stable arc $\Gamma_{f_{\lambda,\mu},f_{\lambda-1,\mu}}$, $\lambda>1$, connected the diffeomorphisms $f_{\lambda,\mu}\in G_{\lambda,\mu}$ and $f_{\lambda-1,\mu}\in G_{\lambda-1,\mu}$ (which is done in Lemma 4.2 below). Indeed, in this case, the stable arc $\Gamma_{f_{\lambda,\mu},f_{1,\mu}}=\Gamma_{f_{2,\mu},f_{1,\mu}}*\dots*\Gamma_{f_{\lambda,\mu},f_{\lambda-1,\mu}}$ connects the diffeomorphisms $f_{\lambda,\mu}$ and $f_{1,\mu}$. If c_1 and c_2 are paths in the topological space X such that $c_1(1)=c_2(0)$, then we introduce the *product of paths* c_1 and c_2 as the path c_1*c_2 defined by

$$(c_1*c_2)(t)=\begin{cases}c_1(2t), & 0\leqslant t\leqslant 1/2,\\ c_2(2t-1), & 1/2\leqslant t\leqslant 1.\end{cases} \tag{4.1}$$

Since the diffeomorphism $f^{-1}_{1,\mu}$ belongs to $G_{\mu,1}$, the stable arc $\Gamma_{f^{-1}_{1,\mu},f^{-1}_{1,1}}=\Gamma_{f^{-1}_{2,\mu},f^{-1}_{1,\mu}}*\dots*\Gamma_{f^{-1}_{1,\mu},f^{-1}_{1,\mu-1}}=\{\gamma_t\}$ connects the diffeomorphisms $f^{-1}_{1,\mu}$ and $f^{-1}_{1,1}$. Then the stable arc $\widetilde{\Gamma}_{f_{1,\mu},f_{1,1}}=\{\gamma_t^{-1}\}$ connects the diffeomorphisms $f_{1,\mu}$ and $f_{1,1}$. Thus, the required arc connecting the diffeomorphism $f_{\lambda,\mu}\in G_{\lambda,\mu}$ with some source-sink diffeomorphism has the form $\widetilde{\Gamma}_{f_{1,\mu},f_{1,1}}*\Gamma_{f_{\lambda,\mu},f_{1,\mu}}$ (cf. Figure 10).

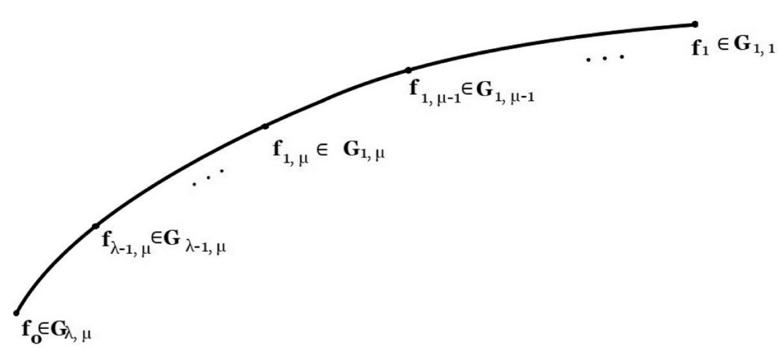

Figure 10. Scheme for constructing the arc.

It is possible to reduce the number of saddle points on the attractor A_f by constructing an arc unfolds generically through a saddle–node bifurcation or a flip. To realize such a scenario, it is necessary to reduce the merging objects to the canonical form. Namely, in Lemma 4.2, we present the dynamics in a neighborhood of a sink to the canonical contraction and, in Lemma 4.1, we prove that the unstable saddle separatrix in the basin of the canonical sink can be put on a smooth arc. The following classical result is an important tool of all our constructions.

Proposition 4.1 (cf. [12, Theorem 5.8])**.** *Assume that Y is a smooth manifold without boundary, X is a smooth compact submanifold of Y, and $\{f_t : X \to Y, t \in [0,1]\}$ is a smooth isotopy such that f_0 is the mapping of inclusion of X to Y. Then for any compact set $A \subset Y$ containing* $\operatorname{supp}\{f_t\}$ *there exists a smooth isotopy $\{g_t \in \operatorname{Diff}(Y), t \in [0,1]\}$, such that $g_0 = \mathrm{id}$, $g_t|_X = f_t|_X$ for any $t \in [0,1]$ and* $\operatorname{supp}\{g_t\}$ *lies in A.*

By $\operatorname{supp}\{f_t\}$ of an isotopy $\{f_t\}$ we mean the closure of the set $\{x \in X : f_t(x) \neq f_0(x)$ for some $t \in [0,1]\}$.

Proposition 4.2 (cf. [13, Lemma 1.1])**.** *Assume that θ is a finite set of points of a manifold M^n, $\varphi : M^n \to M^n$ is a diffeomorphism, and*

$$T = \bigcup_{x\in\theta} TM^n_x, \quad T' = \bigcup_{x\in\theta} TM^n_{\varphi(x)}.$$

Then there exists a neighborhood $U(\theta) \supset \theta$ and a number $\varepsilon > 0$ such that for any isomorphism $G : T \to T'$ such that $\|G - D\varphi\| < \varepsilon/10$ there exists a diffeomorphism $\psi : M^n \to M^n$ that is ε-close to φ in the C^1 topology and such that $D\psi = G$ on T and $\psi = f$ outside $U(\theta)$.

Since, between any hyperbolic automorphisms of the same index (the number of eigenvalues greater than 1 in modulus), there exists a path of hyperbolic automorphisms, Lemma 4.2 admits the following generalization.

Proposition 4.3. *Let a diffeomorphism $\varphi_0 : M^n \to M^n$ have a hyperbolic point r_0 of period m_0, and let (U_0, h) be a local chart of the manifold M^n such that $r_0 \in U_0$, $h(r_0) = O$. Then for any hyperbolic automorphism G possessing the same index as the automorphism $(D\varphi_0^{m_0})_{r_0}$ there exist neighborhoods U_1 and U_2 of the point r_0, $U_2 \subset U_1 \subset U_0$, and an arc $\varphi_t : M^n \to M^n$, $t \in [0,1]$, without bifurcation such that*

1) *the diffeomorphism φ_t, $t \in [0,1]$, coincides with the diffeomorphism φ_0 outside $\bigcup\limits_{k=0}^{m-1} \varphi_0^k(U_1)$, and $\bigcup\limits_{k=0}^{m-1} \varphi_0^k(r_0)$ is the hyperbolic orbit of period m_0 for every φ_t,*

2) *the diffeomorphism $h\varphi_1^m h^{-1}$ coincides with the diffeomorphism G on the set $h(U_2)$.*

Now, we describe the construction in detail. We denote by $O(0,0)$ the origin in the plane $\mathbb{R}^2$. For any $r > 0$ we set $B_r = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}$. We denote by $g : \mathbb{R}^2 \to \mathbb{R}^2$ the diffeomorphism given by $g(x,y) = (x/2, y/2)$.

Lemma 4.1. *Assume that a diffeomorphism $\varphi_0 : M^2 \to M^2$ has a hyperbolic sink ω_0 and a hyperbolic saddle σ_0 such that the unstable separatrix γ_{φ_0} of the saddle σ_0 lies in the basin of the sink $W^s_{\omega_0}$ and has the same period m as the sink ω_0. Let (U_0, ψ_0) be a local chart of the manifold M^2 such that $\omega_0 \in U_0$, $\psi_0(\omega_0) = O$, and $\varphi_0^m(U_0) \subset U_0$. Then there exist neighborhoods*

V_1 and V_2 of the point ω_0 such that $V_2 \subset V_1 \subset U_0$ and an arc $\varphi_t : M^2 \to M^2$, $t \in [0,1]$, without bifurcations such that

(1) the diffeomorphism φ_t, $t \in [0,1]$, coincides with the diffeomorphism φ_0 outside $\bigcup\limits_{k=0}^{m-1} \varphi_0^k(V_1)$, and $\bigcup\limits_{k=0}^{m-1} \varphi_0^k(\omega_0)$ is the hyperbolic sink orbit of period m for all φ_t,

(2) $\psi_0(\gamma_{\varphi_1} \cap V_2) \subset OX_1$, where γ_{φ_1} is an unstable separatrix of the saddle σ_0 relative to the diffeomorphism φ_1 (cf. Figure 11).

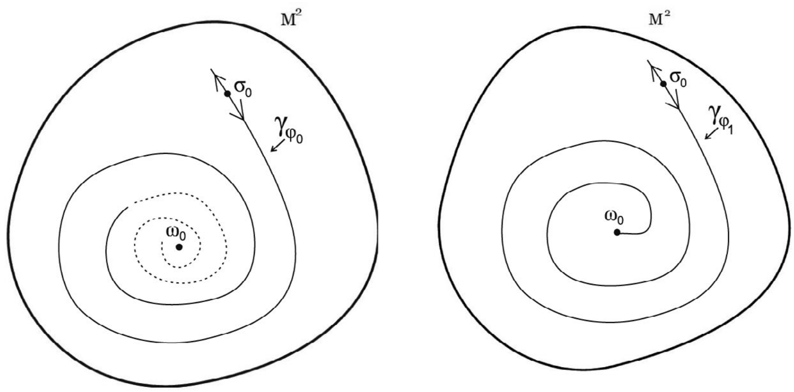

Figure 11. Rectification of separatrix.

Proof. Let $\varphi_0 = \varphi_0^m, \overline{\varphi}_0 = \psi_0\varphi_0\psi_0^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$. By Proposition 4.3, without loss of generality we can assume that $\overline{\varphi}_0 = g$ on the disk B_{2r_0} for some $r_0 > 0$. We set $K_0 = B_{2r_0} \setminus B_{r_0}$ and $\gamma_{\overline{\varphi}_0} = \psi_0(\gamma_{\varphi_0})$.

We denote by E_g the set of contractions $\overline{\varphi} : \mathbb{R}^2 \to \mathbb{R}^2$ coinciding with $\overline{\varphi}_0$ outside B_{2r_0} and with g on $B_{r_{\overline{\varphi}}}$, where $r_{\overline{\varphi}} \leqslant 2r_0$. For any $\overline{\varphi} \in E_g$ we set $\gamma_{\overline{\varphi}} = \bigcup\limits_{k\in\mathbb{Z}} \overline{\varphi}^k(\gamma_{\overline{\varphi}_0} \cap K_0)$. By construction, the $\overline{\varphi}$-invariant curve $\gamma_{\overline{\varphi}}$ coincides with the $\overline{\varphi}_0$-invariant curve $\gamma_{\overline{\varphi}_0}$ outside B_{r_0}. Then it suffices to construct an arc from the contractions $\overline{\varphi}_t : \mathbb{R}^2 \to \mathbb{R}^2$, $t \in [0,1]$, such that

$\overline{1}$) the diffeomorphism $\overline{\varphi}_t$, $t \in [0,1]$, coincides with the diffeomorphism $\overline{\varphi}_0$ outside B_{r_0},

$\overline{2}$) $(\gamma_{\overline{\varphi}_1} \cap B_{r_{\overline{\varphi}_1}}) \subset OX_1$.

Then the arc $\varphi_t : M^2 \to M^2$ is obtained from the arc $\overline{\varphi}_t$ as follows. Assume that $V_1 = h^{-1}(B_{r_0})$, $V_2 = h^{-1}(B_{r_{\overline{\varphi}_1}})$, and $\varphi_t = h^{-1}\overline{\varphi}_t h$ on V_1. Then φ_t, $t \in [0,1]$, coincides with φ_0 outside $\bigcup\limits_{k=0}^{m-1} \varphi_0^k(V_1)$, $\varphi_t(z) = \varphi_0(z)$ for $z \in \varphi_0^k(V_2)$, $k \in \{0, \dots, m-2\}$, and $\varphi_t(z) = \varphi_t(\varphi_0^{1-m}(z))$ for $z \in \varphi_0^{m-1}(V_2)$.

To construct the arc $\overline{\varphi}_t$, for an arbitrary diffeomorphism $\overline{\varphi} \in E_g$ we introduce the notation. We represent the two-dimensional torus $\mathbb{T}^2$ as the space of orbits of the action of the diffeomorphism g on $\mathbb{R}^2 \setminus O$ and denote by $p : \mathbb{R}^2 \setminus O \to \mathbb{T}^2$ the natural projection. We fix generators $\widehat{a} = p(OX_1)$ and $\widehat{b} = p(\mathbb{S}^1)$ on $\mathbb{T}^2$. We set $K_{\overline{\varphi}} = B_{r_{\overline{\varphi}}} \setminus B_{r_{\overline{\varphi}}/2}$ and $\widehat{\gamma}_{\overline{\varphi}} = p(\gamma_{\overline{\varphi}} \cap K_{\overline{\varphi}})$. Then the curve $\widehat{\gamma}_{\overline{\varphi}}$ is a node on the torus $\mathbb{T}^2$ admitting the decomposition $\langle 1, -n_{\overline{\varphi}}\rangle$, $n_{\overline{\varphi}} \in \mathbb{Z}$, in the basis $\widehat{a}$, $\widehat{b}$ (cf., for example, [9]).

The arc $\overline{\varphi}_t$ is the smooth product of arcs η_t and ζ_t, where

(I) the arc η_t, $t \in [0,1]$, consists of contractions coinciding with the diffeomorphism $\overline{\varphi}_0$ outside B_{r_0} and connects the diffeomorphism $\eta_0 = \overline{\varphi}_0$ with some diffeomorphism $\eta_1 \in E_g$ such that the knot $\widehat{\gamma}_{\eta_1}$ admits the decomposition $\langle 1, 0\rangle$ in the basis $\widehat{a}$, $\widehat{b}$,

(II) the arc $\zeta_t \in E_g$, $t \in [0,1]$, connects the diffeormorphism $\zeta_0 = \eta_1$ with a diffeomorphism ζ_1 such that $\widehat{\gamma}_{\zeta_1} = \widehat{a}$.

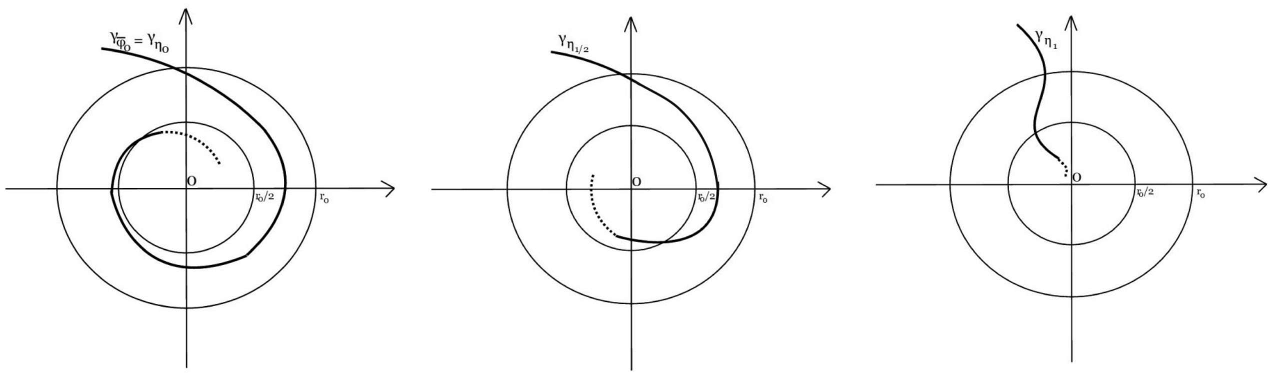

Figure 12. Illustration for Lemma 4.1 (1).

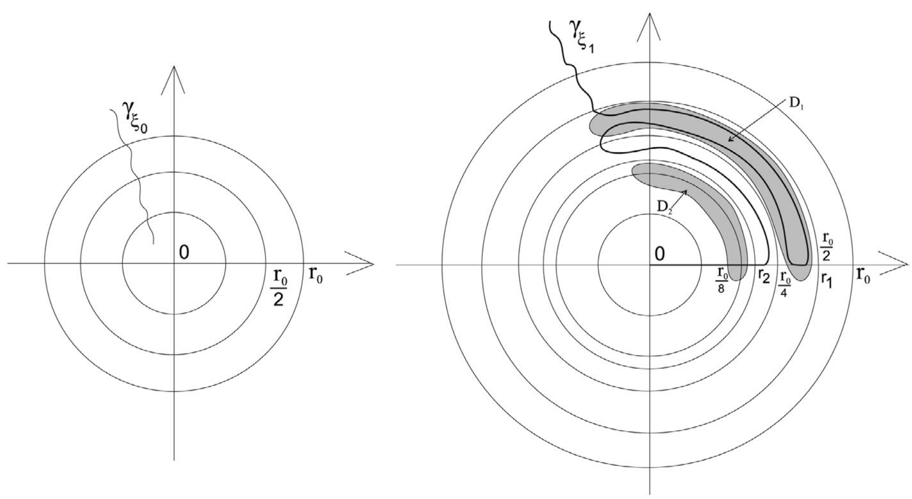

Figure 13. Illustration for Lemma 4.1 (2).

(I) If $n_{\overline{\varphi}} = 0$, then we set $\eta_t = \overline{\varphi}_0$ for all $t \in [0,1]$. Otherwise, we define the diffeomorphism $\theta_t : \mathbb{R}^2 \to \mathbb{R}^2$, $t \in [0,1]$, such that $\theta_t(O) = O$ and

$$\theta_t(\rho e^{i\varphi}) = \begin{cases} \rho e^{i\varphi}, & \rho > r_0, \\ \rho e^{i(\varphi + 4n_{\overline{\varphi}}\pi t(1-\rho/r_0))}, & r_0/2 \leqslant \rho \leqslant r_0, \\ \rho e^{i(\varphi+2n_{\overline{\varphi}}\pi t)}, & \rho < r_0/2. \end{cases}$$

Then $\eta_t = \theta_t \overline{\varphi}_0 : \mathbb{R}^2 \to \mathbb{R}^2$ is the desired arc (cf. Figure 12).

(II) By construction, $\eta_1 \in E_g$ and the node $\widehat{\gamma}_{\eta_1}$ admits the decomposition $\langle 1, 0\rangle$ in the basis $\widehat{a}$, $\widehat{b}$. There exists a diffeomorphism $\widehat{h} : \mathbb{T}^2 \to \mathbb{T}^2$ that is smoothly isotopic to the identity and such that $\widehat{h}(\widehat{\gamma}_{\eta_1}) = \widehat{a}$. For $r > 0$ we set $K_r = B_r \backslash B_{r/2}$. We choose an open covering $D = \{D_1, \ldots, D_q\}$ of the torus $\mathbb{T}^2$ such that the connected component $\overline{D}_i$ of the set $p^{-1}(D_i)$ is a subset of K_{r_i} with some $r_i < r_{i-1}/2$ and $r_1 \leqslant r_0/2$. By [14], there exist diffeomorphisms $\widehat{w}_1, \dots, \widehat{w}_q : \mathbb{T}^2 \to \mathbb{T}^2$ that are smoothly isotopic to the identity and possess the following properties:

(i) for every $i \in \{1, \dots, q\}$ there exists a smooth isotopy $\{\widehat{w}_{i,t}\}$ that is equal to the identity outside D_i and connects the identity with $\widehat{w}_i$,

(ii) $\widehat{h} = \widehat{w}_1 \dots \widehat{w}_q$.

Let $w_{i,t} : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism coinciding with $(p|_{K_{r_i}})^{-1}\widehat{w}_{i,t}p$ on K_{r_i} and the identity outside K_{r_i} (cf. Figure 13). Then the desired arc is defined by $\zeta_t = w_{1,t} \dots w_{q,t}$. The lemma is proved. □

Lemma 4.2. *For any diffeomorphisms $f_{\lambda,\mu} \in G_{\lambda,\mu}, \lambda > 1$, $f_{\lambda-1,\mu} \in G_{\lambda-1,\mu}$ there exists a connecting stable arc $H_{f_{\lambda,\mu}, f_{\lambda-1,\mu},t}$.*

Proof. Let $f = f_{\lambda,\mu}$. By Lemma 3.2, there exist points $\sigma, \omega \in A_f$ such that $q_\omega = 0$, $q_\sigma = 1$, $\omega \in \mathrm{cl}\, W^u_\sigma$, and the intersection $W^s_\omega \cap A_f$ consists of exactly one unstable separatrix γ of the saddle σ and the sink ω; moreover, the period of this separatrix is equal to the sink period m. By Proposition 4.3 and Lemma 4.1, we can assume that there exists a local chart (U, ψ) of the manifold $\mathbb{S}^2$ such that $\omega \in U$, $\psi(\omega) = O$, $f^m(U) \subset U$, and $\psi(\gamma \cap U) \subset OX_1$. By Lemma 3.2, for the diffeomorphism f two cases are possible: 1) $\nu_\sigma = -1$ and 2) $\nu_\sigma = 1$. We construct the required arc separately for each case.

Case 1: $\nu_\sigma = -1$. In this case, $A_f = W^u_\sigma \cup \omega \cup f(\omega)$ and $m = 2$. We set $l = W^u_\sigma \cup \psi^{-1}(OX_1) \cup f(\psi^{-1}(OX_1))$. Then l is a smooth curve containing A_f and such that the points $\omega, f(\omega)$ are interior; moreover, $f(l) \subset l$. We set $\Pi_1 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| < 1/2\}$. On $[-1/2; 1/2] \subset OX_1$, we introduce the family of diffeomorphisms $\varphi_t : [-1/2, 1/2] \to \mathbb{R}^2$ by the formula

$$\varphi_t(x_1) = \Big(-x_1\Big(1 + \frac{1}{10}(1-2t)\Big) + x_1^3\Big).$$

We define the diffeomorphism $\widetilde{\varphi}_t : \widetilde{\Pi}_1 \to \mathbb{R}^2$ by the formula $\widetilde{\varphi}_t(x_1, x_2) = (\varphi_t(x_1), -x_2/2,)$. By construction, the diffeomorphism $\widetilde{\varphi}_0$ has three periodic points of the period; namely, two sinks $P_1(-1/\sqrt{10}, 0)$, $P_2(1/\sqrt{10}, 0)$ of period 2 and the fixed source point $P_3(0,0)$. For $\varepsilon > 0$, $\delta > 0$ we set

$$J_\varepsilon = \Big[-\frac{1}{\sqrt{10}} - \varepsilon, \frac{1}{\sqrt{10}} + \varepsilon \Big] \subset OX_1,$$

$$V_{\varepsilon,\delta} = \big\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < 1/\sqrt{10} + \varepsilon, |x_2| < \delta\big\}.$$

We choose a neighborhood Π_1 of the arc A_f and a diffeomorphism $\beta : \Pi_1 \to \widetilde{\Pi}_1$ such that $\beta(\omega) = P_1$, $\beta(f(\omega)) = P_2$, $\beta(\sigma) = P_3$, and $\beta(l \cap \Pi_1) = Ox_1 \cap \widetilde{\Pi}_1$. Then, in some neighborhood $V_{\varepsilon_1,\delta_1}$, the diffeomorphism $\widetilde{f} = \beta f \beta^{-1}$ is well defined. By Proposition 4.3, we can assume that, in the neighborhoods

$$V_{P_1} = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 + 1/\sqrt{10}| < \varepsilon_1, |x_2| < \delta_1\},$$

$$V_{P_2} = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - 1/\sqrt{10}| < \varepsilon_1, |x_2| < \delta_1\},$$

$$V_{P_3} = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| < \delta_1\}$$

of the points P_1, P_2, P_3, the diffeomorphism $\widetilde{f}$ coincides with $(D\widetilde{\varphi}_0)_{P_1}$, $(D\widetilde{\varphi}_0)_{P_2}$, $(D\widetilde{\varphi}_0)_{P_3}$ respectively. We denote by φ the restriction of $\widetilde{f}$ onto J_{ε_1}. On the cylinder $V_{\varepsilon_1,\delta_1}$, we define the diffeomorphism $\widetilde{\varphi}$ by the formula $\widetilde{\varphi}(x_1, x_2) = (\varphi(x_1), -x_2/2)$. For $\delta > 0$ we define the bump function $\rho_\delta(r)$, $r \geqslant 0$, equal to 1 for $r \in [0, \delta]$ and 0 for $r \geqslant 2\delta$. For $\delta_2 = \delta_1/2$ we define the family of diffeomorphisms $\widetilde{a}_t$, $t \in [0, 1]$, on the cylinder $V_{\varepsilon_1,\delta_1}$ by the formula

$$\widetilde{a}_t(x_1, x_2) = t\rho_{\delta_2}(|x_2|)\widetilde{\varphi}(x_1, x_2) + (1 - t\rho_{\delta_2}(|x_2|))\widetilde{f}(x_1, x_2).$$

By construction, $\widetilde{a}_t$, $t \in [0, 1]$, coincides with $\widetilde{f}$ on $\partial V_{\varepsilon_1,\delta_1}$, $\widetilde{a}_0 = \widetilde{f}$ on $V_{\varepsilon_1,\delta_1}$ and $\widetilde{a}_1 = \widetilde{\varphi}$ on $V_{\varepsilon_1,\delta_2}$. Since the diffeomorphisms $\widetilde{f}$ and $\widetilde{\varphi}$ coincide on the segment J_{ε_1} and rectangles V_{P_1}, V_{P_2}, V_{P_3}, we can assume without loss of generality that δ_2 is chosen in such a way that the diffeomorphism $\widetilde{a}_t$ has no nonwandering points different from P_1, P_2, P_3.

For $\varepsilon_2 = \varepsilon_1/4$ and $t \in [0, 1]$ we set

$$v_t(x_1) = \rho_{\varepsilon_2}(|x_1|)\varphi_t(x_1) + (1 - \rho_{\varepsilon_2}(|x_1|))\varphi(x_1), \quad x_1 \in J_{\varepsilon_1}.$$

By construction, the diffeomorphism v_t coincides with φ_t on J_{ε_2} and φ on $J_{\varepsilon_1} \setminus J_{2\varepsilon_2}$. For $t \in [0, 1]$ we set $\nu_t(x_1) = tv_0(x_1) + (1 - t)\varphi(x_1)$. By construction, ν_0 coincides with φ and ν_1 coincides with v_0. We set $w_t = \nu_t * v_t$ and $\widetilde{w}_t(x_1, x_2) = (w_t(x_1), -x_2/2)$ for $(x_1, x_2) \in V_{\varepsilon_1,\delta_1}$.

For $\delta_3 = \delta_2/2$ we define the family of diffeomorphisms $\widetilde{b}_t$, $t \in [0, 1]$, on the cylinder $V_{\varepsilon_1,\delta_1}$ by the formula

$$\widetilde{b}_t(x_1, x_2) = \rho_{\delta_3}(|x_2|)\widetilde{w}_t(x_1, x_2) + (1 - \rho_{\delta_3}(|x_2|))\widetilde{a}_1(x_1, x_2).$$

By construction, $\widetilde{b}_t$, $t \in [0, 1]$, coincides with $\widetilde{f}$ on $\partial V_{\varepsilon_1,\delta_1}$, $\widetilde{b}_0 = \widetilde{a}_1$ on $V_{\varepsilon_1,\delta_1}$, $\widetilde{b}_t = \widetilde{w}_t$ on $V_{\varepsilon_1,\delta_3}$, and $\widetilde{b}_1 = \widetilde{v}_1$ on $V_{\varepsilon_1,\delta_3}$.

We set $\widetilde{c}_t = \widetilde{a}_t * \widetilde{b}_t$ and $U_2 = \beta^{-1}(V_{\varepsilon_1,\delta_1})$. Then the desired arc $\Gamma_{f_\mu, f_{\mu-1}}$ coincides with f outside $\bigcup\limits_{k=0}^{m-1} f^k(U_2)$, $f_t(z) = f(z)$ for $z \in f^k(U_2)$, $k \in \{0, \dots, m-2\}$, and $f_t(z) = \beta^{-1}(\widetilde{c}_t(\beta(f^{1-m}(z))))$ for $z \in f^{m-1}(U_2)$.

Case 2: $\nu_\sigma = 1$. In this case, the saddle σ and sink ω have the same period m. We set $l = W^u_\sigma \cup \psi^{-1}(OX_1)$. Then l is a smooth curve containing γ such that ω and σ are interior points. We set $\Pi_1 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| < 1/2\}$. On $[-1/2; 1/2] \subset OX_1$, we define the family of diffeomorphisms $\varphi_t : [-1/2, 1/2] \to \mathbb{R}^2$ by the formula

$$\varphi_t(x_1) = x_1 + \frac{x_1^2}{2} + \frac{1}{10}(2t - 1).$$

We define the diffeomorphism $\widetilde{\varphi}_t : \widetilde{\Pi}_1 \to \mathbb{R}^2$ by the formula $\widetilde{\varphi}_t(x_1, x_2) = (\varphi_t(x_1), x_2/2)$. By construction, the diffeomorphism $\widetilde{\varphi}_0$ has the sink point $P_1(-1/\sqrt{5}, 0)$ and saddle point $P_2(1/\sqrt{5}, 0)$. For $\varepsilon > 0$, $\delta > 0$ we set $I_\varepsilon = [-1/\sqrt{5} - \varepsilon, 1/\sqrt{5} + \varepsilon] \subset OX_1$ and

$$V_{\varepsilon,\delta} = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < 1/\sqrt{5} + \varepsilon, |x_2| < \delta\}.$$

We choose a neighborhood Π_1 of the arc γ and a diffeomorphism $\beta : \Pi_1 \to \widetilde{\Pi}_1$ such that $\beta(\omega) = P_1, \beta(\sigma) = P_2$ and $\beta(l \cap \Pi_1) = Ox_1 \cap \widetilde{\Pi}_1$. Then, in some neighborhood $V_{\varepsilon_1,\delta_1}$, we have a well-defined diffeomorphism $\widetilde{f} = \beta f^m \beta^{-1}$. By Proposition 4.3, we can assume that, in the neighborhood

$$V_{P_1} = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 + 1/\sqrt{5}| < \varepsilon_1, |x_2| < \delta_1\},$$

$$V_{P_2} = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - 1/\sqrt{5}| < \varepsilon_1, |x_2| < \delta_1\}$$

of the points P_1 and P_2, the diffeomorphism $\widetilde{f}$ coincides with $(D\widetilde{\varphi}_0)_{P_1}$ and $(D\widetilde{\varphi}_0)_{P_2}$ respectively. We denote by φ the restriction of $\widetilde{f}$ on I_{ε_1}. On the cylinder $V_{\varepsilon_1,\delta_1}$, we define the diffeomorphism $\widetilde{\varphi}$ by $\widetilde{\varphi}(x_1,x_2)=(\varphi(x_1),x_2/2)$. For $\delta>0$ we introduce the bump function $\rho_\delta(r)$, $r\geqslant 0$, equal to 1 for $r\in[0,\delta]$ and 0 for $r\geqslant 2\delta$. For $\delta_2=\delta_1/2$ we define the family of diffeomorphisms $\widetilde{a}_t$, $t\in[0,1]$, on the cylinder $V_{\varepsilon_1,\delta_1}$ by the formula

$$\widetilde{a}_t(x_1,x_2)=t\rho_{\delta_2}(|x_2|)\widetilde{\varphi}(x_1,x_2)+(1-t\rho_{\delta_2}(|x_2|))\widetilde{f}(x_1,x_2).$$

By construction, $\widetilde{a}_t$, $t\in[0,1]$, coincides with $\widetilde{f}$ on $\partial V_{\varepsilon_1,\delta_1}$, $\widetilde{a}_0=\widetilde{f}$ on $V_{\varepsilon_1,\delta_1}$ and $\widetilde{a}_1=\widetilde{\varphi}$ on $V_{\varepsilon_1,\delta_2}$. Since the diffeomorphisms $\widetilde{f}$ and $\widetilde{\varphi}$ coincide on I_{ε_1} and rectangles V_{P_1}, V_{P_2}, without loss of generality we can assume that δ_2 is chosen in such a way that the diffeomorphism $\widetilde{a}_t$ has no nonwandering points different from P_1 and P_2.

For $\varepsilon_2=\varepsilon_1/4$ and $t\in[0,1]$ we set

$$v_t(x_1)=\rho_{\varepsilon_2}(|x_1|)\varphi_t(x_1)+(1-\rho_{\varepsilon_2}(|x_1|))\varphi(x_1), x_1\in I_{\varepsilon_1}.$$

By construction, the diffeomorphism v_t coincides with φ_t on I_{ε_2} and φ on $I_{\varepsilon_1}\setminus I_{2\varepsilon_2}$. For $t\in[0,1]$ we set $\nu_t(x_1)=tv_0(x_1)+(1-t)\varphi(x_1)$. By construction, ν_0 coincides with φ and ν_1 coincides with v_0. We set $w_t=\nu_t * v_t$ and $\widetilde{w}_t(x_1,x_2)=(w_t(x_1),x_2/2)$ for $(x_1,x_2)\in V_{\varepsilon_1,\delta_1}$.

For $\delta_3=\delta_2/2$ we define the family of diffeomorphisms $\widetilde{b}_t$, $t\in[0,1]$, on the cylinder $V_{\varepsilon_1,\delta_1}$ by the formula

$$\widetilde{b}_t(x_1,x_2)=\rho_{\delta_3}(|x_2|)\widetilde{w}_t(x_1,x_2)+(1-\rho_{\delta_3}(|x_2|))\widetilde{a}_1(x_1,x_2).$$

By construction, $\widetilde{b}_t$, $t\in[0,1]$, coincides with $\widetilde{f}$ on $\partial V_{\varepsilon_1,\delta_1}$, $\widetilde{b}_0=\widetilde{a}_1$ on $V_{\varepsilon_1,\delta_1}$, $\widetilde{b}_t=\widetilde{w}_t$ on $V_{\varepsilon_1,\delta_3}$ and $\widetilde{b}_1=\widetilde{v}_1$ on $V_{\varepsilon_1,\delta_3}$.

We set $\widetilde{c}_t=\widetilde{a}_t * \widetilde{b}_t$ and $U_2=\beta^{-1}(V_{\varepsilon_1,\delta_1})$. Then the desired arc $\Gamma_{f_\mu,f_{\mu-1}}$ coincides with f outside the set $\bigcup\limits_{k=0}^{m-1} f^k(U_2)$, $f_t(z)=f(z)$ for $z\in f^k(U_2), k\in\{0,\dots,m-2\}$ and $f_t(z)=\beta^{-1}(\widetilde{c}_t(\beta(f^{1-m}(z))))$ for $z\in f^{m-1}(U_2)$. $\square$

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