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# On existence of global attractors of foliations with transverse linear connections

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The existence problem for attractors of foliations with transverse linear connection is investigated. In general foliations with transverse linear connection do not admit attractors. Sufficient conditions are found for the existence of a global attractor that is a minimal set. An application to transversely similar pseudo-Riemannian foliations is obtained. The global structure of transversely similar Riemannian foliations is described. Different examples are constructed.



## 1. Introduction and main results

Investigation of dynamical properties of foliations is an actual area, see, e.g., the book [2].

There are several nonequivalent notions of an attractor in the theory of dynamical systems (e.g., see [9]). Some of these notions are equivalent [6]. For “typical” dynamical systems in the metric sense different notions of an attractor should coincide according to Palis’s hypothesis [14]. We use the most general notion of an attractor for a foliation that generalizes the notion of an attractor from [15]. Note that the attractor of a foliation may be disconnected and it may contain other attractors. This is not the case for a transitive attractor that contains a dense leaf. Attractors which are minimal sets are examples of transitive attractors.

We investigate foliations with transverse linear connections. They include, in particular, Weyl foliations [21] and transversely similar pseudo-Riemannian foliations, which are of special interest. Among them

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are transversely similar Loretzian and transversely similar Riemannian foliations. There are foliations from each of the indicated classes admitting attractors.

Deroin and Kleptsyn [6] investigated attractors of foliations with conformal transversal structures on compact manifolds. The Main theorem of [6] states that for every conformal foliation on a compact manifold either there exists a transverse invariant measure, or there exists a finite number of minimal sets equipped with probability measures, which are attractors satisfying some properties.

The case of transversely similar foliations is considered by the author in [18, Sec. 9]. In [20], it is shown that every non-Riemannian conformal foliation $(M,F)$ of codimension $q \geq 3$ admits an attractor which is a minimal set, and the restriction of the foliation to the basin of the attractor is a transversely conformally flat foliation. Moreover, if the foliated manifold $M$ is compact, then $(M,F)$ is a $(\mathrm{Conf}(\mathbb{S}^q), \mathbb{S}^q)$-foliation [19, Th. 4]. Every complete non-Riemannian conformal foliation $(M,F)$ of codimension $q \geq 3$ admits a global attractor $\mathcal{M}$ such that one of the following two conditions holds: 1) $\mathcal{M}$ is transitive; 2) $\mathcal{M}$ is either a closed leaf or a union of two leaves, and $(M,F)$ is covered by a locally trivial bundle over the standard $q$-dimensional sphere $\mathbb{S}^q$ or the Euclidean space $\mathbb{E}^q$ [20, Th. 5].

Note that in [19,20] as well as in the present work we use the methods of local and global differential geometry, while Deroin and Kleptsyn [6] used methods of random dynamical systems. In [6] they considered a Laplace operator along leaves and investigated the existence of a transverse invariant measure.

We give the following general definition of an attractor of a foliation.

**Definition 1.** Let $(M,F)$ be a foliation. A subset of a manifold $M$ is called saturated if it is a union of leaves of this foliation. A nonempty closed saturated subset $\mathcal{M}$ of $M$ is called *an attractor* of $(M,F)$ if there exists an open saturated neighbourhood $\mathcal{U} = \mathcal{U}(\mathcal{M})$ of the set $\mathcal{M}$ such that the closure of every leaf from $\mathcal{U} \setminus \mathcal{M}$ contains the set $\mathcal{M}$. The neighbourhood $\mathcal{U}$ is uniquely determined by this condition and it is called *the basin of this attractor*; we denote it by $\mathcal{A}ttr(\mathcal{M})$. If in addition $\mathcal{A}ttr(\mathcal{M}) = M$, then the attractor $\mathcal{M}$ is called *global*.

An attractor $\mathcal{M}$ of a foliation $(M,F)$ is said to be *transitive* if there exists a leaf $L$ which is dense in $\mathcal{M}$, i.e., if $\overline{L} = \mathcal{M}$.

Recall that *a minimal set* of a foliation on a manifold $M$ is a nonempty closed subset in $M$ that consists of a union of leaves and has no proper subset satisfying this condition. A minimal set is said to be *trivial* if it is a leaf of a foliation. Minimal sets for transformation groups are defined in a similar way.

At first we give different approaches to the concept of holonomy groups of foliations with transverse linear connection which used further. Denote by $\Gamma(L,x)$ the germ holonomy group of a leaf $L = L(x)$ of a foliation $(M,F)$ usually used in the foliation theory [4]. In Section 2.5 we recall the notion of a foliated bundle over a foliation with transverse linear connection. Applying Proposition 5 from [20] about different interpretations of the notion of holonomy groups of Cartan foliations to foliations with transverse linear connection and taking into account that a linear connection is a structure of the first order, we obtain the following statement.

**Theorem 1.** *Let $(M,F)$ be a foliation with transverse linear connection and $\pi : \mathcal{R} \to M$ be its foliated $H$-bundle where $H := GL(q,\mathbb{R})$, $x \in M$, $u \in \pi^{-1}(x)$. Let $\mathcal{L} = \mathcal{L}(u)$ be the leaf through $u$ of the lifted foliation $(\mathcal{R},\mathcal{F})$. Then the germ holonomy group $\Gamma(L,x)$ of the leaf $L = L(x)$ through $x$ is isomorphic to each of the following three groups:*

1) *the subgroup $H(u) := \{a \in H | R_a(\mathcal{L}) = \mathcal{L}\}$ of $H$ which preserves the leaf $\mathcal{L} = \mathcal{L}(u)$ of $(\mathcal{R},\mathcal{F})$;*
2) *the group of deck transformations of the regular covering map $\pi|_{\mathcal{L}} : \mathcal{L} \to L$;*
3) *the linear holonomy group $D\Gamma(L,x)$ formed by the differentials of the local holonomy diffeomorphisms along leaf loops of a transversal $q$-dimensional disc at $x$.*

**Remark 1.** If $\hat{u}$ is an other point from $\pi^{-1}(x)$, then there exists $b \in H$ such that $\hat{u} = u \cdot b$ and $H(\hat{u}) = b^{-1} \cdot H(u) \cdot b$.

**Remark 2.** Since the isomorphism of the subgroups $D\Gamma(L, x)$ and $H(u)$, $x = \pi(u)$, of the Lie group $H$ is induced by the action of $H$ on the space of the foliated bundle $\mathcal{R}$, it is an isomorphism of the topological subgroups $D\Gamma(L, x)$ and $H(u)$ of $H$.

A leaf $L$ is referred to as proper if $L$ is an embedded submanifold of $M$. A foliation having only proper leaves is said to be proper. A leaf $L$ is called closed if it is a closed subset of $M$. As known, every closed leaf is proper, the converse is not true in general.

Recall that a subgroup of a Lie group $G$ is called relatively compact if its closure in $G$ is compact. In Section 3.1 we recall the notion of an Ehresmann connection for a foliation introduced in [3].

We denote by $K =< A >$ the group generated by $A$. The next theorem contains conditions for a foliation $(M, F)$ that guarantee the existence of a global attractor which is a minimal set of $(M, F)$.

**Theorem 2.** *Let $(M, F)$ be a foliation with transverse linear connection of codimension $q$. Assume that $(M, F)$ admits an Ehresmann connection. If there exists a leaf $L$ such that its linear holonomy group contains an element defined by a matrix of the form $A \cdot D$, where $K = < A >$ is a relatively compact subgroup in the linear group $GL(q, \mathbb{R})$ and $D = \operatorname{diag}(d_1, ..., d_q)$ with $0 < |d_i| < 1$ for $1 \leq i \leq q$, then $(M, F)$ has a global attractor $\mathcal{M} = \overline{L}$ which is a minimal set.*

*If the leaf $L$ is proper, then $L$ is a unique closed leaf of $(M, F)$.*

In Section 2.3 we recall the notions of transversely similar pseudo-Riemannian and Riemannian foliations. Applying Theorems 1 and 2 we get the following known statement [21, Th. 5].

**Corollary 1.** *Let $(M, F)$ be a transversely similar pseudo-Riemannian foliation of codimension $q$ on an $n$-dimensional manifold $M$ modelled on a transverse pseudo-Riemannian manifold $(N, g^N)$ of signature $(k, s)$, where $k + s = q$. Assume that $(M, F)$ has an Ehresmann connection. If there exists a leaf $L$ such that its linear holonomy group contains an element defined by a matrix of the form $\lambda \cdot A$, where $0 < \lambda < 1$ and $A$ belongs to a compact subgroup of the pseudo-orthogonal group $O(k, s)$, then the closure $\mathcal{M} := \overline{L}$ of the leaf $L$ is an attractor and a minimal set.*

*If, moreover, the leaf $L$ is proper, then $\mathcal{M}$ is a global attractor and a unique closed leaf of $(M, F)$.*

Denote by $Sim(\mathbb{E}^q)$ the Lie group of all the similarities of the $q$-dimensional Euclidean space $\mathbb{E}^q$. As is well known [8], the topology in the Lie group $Sim(\mathbb{E}^q)$ coincides with the $C^\infty$ compact-open topology in $Sim(\mathbb{E}^q)$ considered as a transformation group of $\mathbb{E}^q$.

**Definition 2.** If the linear holonomy group $D\Gamma(L, x)$ of a leaf $L$ of a transversely similar Riemannian foliation is relatively compact, then we say that $L$ has an *inessential* holonomy group. Otherwise, the holonomy group of a leaf $L$ is called *essential*.

A regular covering map $L_0 \to L_\alpha$ onto a leaf $L_\alpha$ of a foliation is called holonomic if its deck transformation group is isomorphic to the holonomy group of $L_\alpha$.

**Theorem 3.** *Let $(M, F)$ be a transversely similar Riemannian foliation of codimension $q$, $q \geq 1$, with an Ehresmann connection, and $(M, F)$ is not Riemannian. Then:*

$(i)$ *there exists a leaf $L$ with an essential holonomy group, and $\mathcal{M} = \overline{L}$ is a unique global attractor and a minimal set;*

(*ii*) *$(M,F)$ is a complete $(Sim(\mathbb{E}^q),\mathbb{E}^q)$-foliation;*
(*iii*) *there exists a regular covering map $\kappa : \widetilde{M} \to M$ such that $\widetilde{M}$ may be identified with the product of manifolds $L_0 \times \mathbb{E}^q$, at that the induced foliation $\widetilde{F} = \kappa^* F$ is formed by the fibres of the canonical projection $r : L_0 \times \mathbb{E}^q \to \mathbb{E}^q$ and the restriction $\kappa|_{L_0\times\{b\}}$, $b \in \mathbb{E}^q$, is a holonomic covering map onto the corresponding leaf of $(M,F)$;*
(*iv*) *there exists an epimorphism $\chi : \pi_1(M,x_0) \to Sim(\mathbb{E}^q)$ of the fundamental group $\pi_1(M,x_0)$ of $M$ onto some subgroup $\Psi = \chi(\pi_1(M,x_0))$ of $Sim(\mathbb{E}^q)$;*
(*v*) *the holonomy group of any leaf $L_\alpha = L_\alpha(x)$, $x \in M$, is isomorphic to the isotropy subgroup $\Psi_z$ of $\Psi$ at $z \in pr(\kappa^{-1}(L_\alpha)) \subset \mathbb{E}^q$;*
(*vi*) *there exists a global attractor $\mathcal{K}$ of the group $\Psi$, and $\mathcal{M} = \kappa(r^{-1}(\mathcal{K}))$;*
(*vii*) *the closure $\overline{L_\alpha}$ of every leaf $L_\alpha$, belonging to $M_0 := M \setminus \mathcal{M}$, is equal to $\mathfrak{L}_\alpha \cup \mathcal{M}$, where $\mathfrak{L}_\alpha$ is an embedded submanifold of $M$ which is the closure of $L_\alpha$ in $M_0$.*

*In the case when $(M,F)$ is a proper foliation, the global attractor $\mathcal{M}$ is a unique closed leaf of $(M,F)$.*

**Definition 3.** The subgroup $\Psi := \chi(\pi_1(M,x_0))$ of $Sim(\mathbb{E}^q)$ indicated in Theorem 3 is referred to as a *global holonomy group* of the transversely similar Riemannian foliation $(M,F)$.

The following statement is proved in the constructive way analogous to [20, Th. 7].

**Theorem 4.** *Every countable subgroup $\Psi$ of the group $Sim(\mathbb{E}^q)$ is realized as the global holonomy group of some transversely similar Riemannian foliation $(M,F)$ of codimension $q$.*

Denote by $\mathfrak{Fol}$ the category of foliations, where every morphism transforms each leaf of one foliation to a leaf of the other foliation.

It is said that a leaf $L$ of a foliation $(M,F)$ is without holonomy if the holonomy group of $L$ vanishes. If every leaf of $(M,F)$ is without holonomy, then $(M,F)$ is referred to as a foliation without holonomy.

**Theorem 5.** *If $(M,F)$ is a transversely similar Riemannian foliation of codimension one, and $(M,F)$ is not Riemannian and admits an Ehresmann connection $\mathfrak{M}$, then $(M,F)$ is a complete $(Sim(\mathbb{E}^1),\mathbb{E}^1)$-foliation satisfying one of the following two statements:*

*I. Every leaf is dense in $M$, i.e. $M$ is a minimal set of $(M,F)$.*
*II. There exists a unique closed leaf $L$ of $(M,F)$ which is a global attractor homotopy equivalent to $M$, and $M$ is not compact. The distribution $\mathfrak{M}$ is formed by the tangent vector spaces to fibres of a locally trivial bundle $p : M \to L$ with the standard fibre $\mathbb{R}^1$. The foliation $(M,F)$ is defined by the suspension of a group homomorphism*

$$\rho : \pi_1(L,x_0) \to Sim(\mathbb{E}^1)$$

*of the fundamental group $\pi_1(L,x_0)$ of $L$ to $Sim(\mathbb{E}^1)$, and $\rho(\pi_1(L,x_0)) = \Psi$, where $\Psi$ is the global holonomy group of $(M,F)$. The induced foliation $(M_0,F_0)$ on the saturated open dense subset $M_0 = M \setminus L$ is without holonomy and $(M_0,F_0)$ has the following structure:*
(1) *if $(M,F)$ is proper and transversally non-orientable, then $(M_0,F_0)$ is isomorphic in the category $\mathfrak{Fol}$ to the trivial foliation $(L_0 \times \mathbb{S}^1, F_{tr})$ where $\mathbb{S}^1$ is the circle and $F_{tr} = \{L_0 \times \{t\} \,|\, t \in \mathbb{S}^1\}$. The closure $\overline{L_\alpha}$ of every leaf $L_\alpha \subset M_0$ satisfies the equality $\overline{L_\alpha} = L_\alpha \cup L$;*
(2) *if $(M,F)$ is proper and transversely orientable, then $M_0$ has two connected components $M_0^{(i)}$, $i = 1,2$, and each induced foliation $(M_0^{(i)},F_0^{(i)})$ is isomorphic in $\mathfrak{Fol}$ to the trivial foliation $(L_0 \times \mathbb{S}^1, F_{tr})$ as in (1). The closure of every leaf $L_\alpha \subset M_0$ satisfies the equality $\overline{L_\alpha} = L_\alpha \cup L$;*

(3) *if* $(M, F)$ *is improper and transversally non-orientable, then* $M_0$ *is connected and every leaf* $L_\alpha \subset M_0$ *is dense in* $M$*;*
(4) *if* $(M, F)$ *is improper and transversely orientable, then* $M_0$ *has two connected components* $M_0^{(i)}$*,* $i = 1, 2$*, and the closure of each leaf* $L_\alpha \subset M_0^{(i)}$ *is equal to* $\overline{L_\alpha} = M_0^{(i)} \cup L$*.*

**Corollary 2.** *If* $(M, F)$ *is a complete non-Riemannian affine foliation of codimension one, then* $(M, F)$ *is a complete* $(Sim(\mathbb{E}^1), \mathbb{E}^1)$*-foliation satisfying Theorem 5.*

Next theorem is proved without assumption of existence of an Ehresmann connection for a transversally similar Riemannian foliation $(M, F)$ and it contains conditions that guarantee the existence of a global attractor which is a minimal set of $(M, F)$. In Section 6.3 we recall the concept of the set of ends of a manifold $L$ following [10].

**Theorem 6.** *Let* $(M, F)$ *be a transversally similar Riemannian foliation of codimension* $q$ *on* $n$*-dimensional compact manifold* $M$*, and* $(M, F)$ *is non-Riemannian. Assume that every leaf* $L_\alpha$ *without holonomy has only one end. Then:*

1. $(M, F)$ *is* $(Sim(\mathbb{E}^q), \mathbb{E}^q)$*-foliation.*
2. *There exists a unique global attractor* $\mathcal{M}$ *such that* $\mathcal{M}$ *is a minimal set of this foliation, and* $\mathcal{M}$ *is the closure of a leaf* $L$ *with an essential holonomy group.*

*If the leaf* $L$ *is proper, then* $L$ *is a unique closed leaf of* $(M, F)$*.*

**Corollary 3.** *I. Let* $(M, F)$ *be a transversally similar Riemannian foliation of codimension* $q$*,* $0 < q < n$*, on* $n$*-dimensional compact manifold* $M$*, and* $(M, F)$ *is non-Riemannian. Assume that every leaf* $L_\alpha$ *without holonomy is diffeomorphic to* $\mathbb{R}^{n-q}$*. Then there exists a unique global attractor* $\mathcal{M}$ *which is a minimal set of this foliation, and* $\mathcal{M}$ *is the closure* $\overline{L}$ *of a leaf* $L$ *with an essential holonomy group.*

*If the leaf* $L$ *is proper,* $\mathcal{M}$ *is a global attractor and a unique closed leaf of* $(M, F)$*.*

**Remark 3.** In Example 1, foliations satisfying Theorem 6 and Corollary 3 are constructed.

Denote by $E_q$ the unit $q$-dimensional matrix. Recall that an $((\mathbb{R}^+ \cdot \{\mathbb{E}_q\}) \ltimes \mathbb{R}^q, \mathbb{R}^q)$-foliation is referred to as a transversely homothety foliation [18]. Some examples of complete transversely homothety foliations which are not Riemannian were constructed in [18, Sec. 9].

The text is structured as follows:

- Section 2 contains well known fundamental facts about foliations with transverse linear connection and the associated constructions.
- In Section 3 we find condition for the existence of a global attractor which is a minimal set of a foliation with transverse linear connection admitting an Ehresmann connection.
- In Section 4 we describe the structure of transversely similar Riemannian foliations admitting an Ehresmann.
- Section 5 is devoted to a detailed description of the structure of codimension one transversely similar Riemannian foliations which are not Riemannian.
- In Section 6 we obtain sufficient conditions for the existence of global attractors of transversely similar foliations on compact manifolds.
- In Section 7 we construct examples.

**Assumptions.** Throughout this paper we assume for simplicity that all manifolds and maps are smooth of the class $C^\infty$; in fact, the main results of the paper are valid for foliations of the class $C^2$. All neighbourhoods are assumed to be open and all manifolds are assumed to be Hausdorff.

**Notations.** The algebra of smooth functions on a manifold $M$ will be denoted by $\mathfrak{F}(M)$. Let $\mathfrak{X}(N)$ denote the Lie algebra of smooth vector fields on a manifold $N$. If $\mathfrak{M}$ is a smooth distribution of a constant rank on $M$ and $f : K \to M$ is a submersion, then let $f^*\mathfrak{M}$ be the distribution on the manifold $K$ such that $(f^*\mathfrak{M})_z = \{X \in T_zK \,|\, f_{*z}(X) \in \mathfrak{M}_{f(z)}\}$, where $z \in K$. Let $\mathfrak{X}_{\mathfrak{M}}(M) = \{X \in \mathfrak{X}(M) \mid X_u \in \mathfrak{M}_u \quad \forall u \in M\}$. As usually we denote by $P(N, H)$ the principal $H$-bundle $P$ over the manifold $N$. The symbol $\cong$ will denote the isomorphism of objects in the corresponding category.

## 2. Foliations with transverse linear connection and the associated constructions

### 2.1. $(G, N)$-manifolds and $(G, N)$-foliations

Recall the following notions usually used in the geometric foliation theory.

Let $N$ be a $q$-dimensional manifold and $M$ be a smooth $n$-dimensional manifold, where $0 < q < n$. Unlike $M$, the connectivity of the topological space $N$ is not assumed. An $N$-cocycle is the set $\{U_i, f_i, \{k_{ij}\}\}_{i,j\in J}$ such that:

1. The family $\{U_i \,|\, i \in J\}$ forms an open cover of $M$.
2. The mappings $f_i : U_i \to N$ are submersions into $N$ with connected fibres, and $\{V_i := f_i(U_i) \,|\, i \in J\}$ is a cover of $N$.
3. If $U_i \cap U_j \neq \varnothing$, $i, j \in J$, then a diffeomorphism $k_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ is well-defined and satisfies the equality $f_i = k_{ij} \circ f_j$.

Let $N$ be a connected manifold and $G$ be a Lie group of diffeomorphisms of $N$. It is customary to say that the group $G$ acts quasi-analytically on $X$ if, for any open subset $U$ in $N$ and an element $g \in G$, the condition $g|_U = id_U$ implies that $g$ is the identity transformation of $N$. We assume that the group $G$ of diffeomorphisms of a manifold $N$ acts on $N$ quasi-analytically.

**Definition 4.** A foliation $(M, F)$ determined by an $N$-cocycle $\{U_i, f_i, \{k_{ij}\}\}_{i,j\in J}$ is called a $(G, N)$-foliation if for any $U_i \cap U_j \neq \varnothing$, $i, j \in J$, there exists an element $g \in G$ such that $k_{ij} = g|_{f_j(U_i\cap U_j)}$.

**Definition 5.** A manifold $B$ is called a $(G, N)$-manifold if its foliation by points is a $(G, N)$-foliation.

Thus, the concept of $(G, N)$-foliation generalizes the concept of $(G, N)$-manifold introduced by Thurston [16].

### 2.2. Foliations with transverse linear connection

Let $(N^{(1)}, \nabla^{(1)})$ and $(N^{(2)}, \nabla^{(2)})$ be manifolds with linear connections $\nabla^{(1)}$ and $\nabla^{(2)}$ respectively. A smooth map $f : N^{(1)} \to N^{(2)}$ is said to be a morphism from $(N^{(1)}, \nabla^{(1)})$ to $(N^{(2)}, \nabla^{(2)})$ if

$$f_*(\nabla^{(1)}_X Y) = \nabla^{(2)}_{f_*X} f_*Y$$

for all vector fields $X, Y \in \mathfrak{X}(N^{(1)})$, where $f_*$ is the differential of $f$.

**Definition 6.** Let a foliation $(M, F)$ be given by an $N$-cocycle $\{U_i, f_i, \{k_{ij}\}\}_{i,j\in J}$.

If the manifold $N$ admits a linear connection $\nabla$ such that every local diffeomorphism $k_{ij}$ is an isomorphism of the linear connections induced by $\nabla$ on open subsets $f_i(U_i \cap U_j)$ and $f_j(U_i \cap U_j)$, then $(M, F)$ is referred to as *a foliation with transverse linear connection* defined by the $(N, \nabla)$-cocycle $\{U_i, f_i, \{k_{ij}\}\}_{i,j\in J}$. It is also said that $(M, F)$ is modelled on $(N, \nabla)$.

Emphasize that the vanishing of the torsion tensor of a linear connection $\nabla$ on $N$ is not assumed. If $(N, \nabla)$ is torsion free and it has the vanishing curvature, then $(M, F)$ is referred to as a *transversely affine* foliation.

**Remark 4.** Every transversely affine foliation of codimension $q$ is a $(Aff(A^q), A^q)$-foliation where $Aff(A^q)$ is the affine group of the affine space $A^q$. A transversely similar foliation of codimension $q$ is $(Sim(\mathbb{E}^q), \mathbb{E}^q)$-foliation where $Sim(\mathbb{E}^q)$ is the similarity group of the Euclidean space $\mathbb{E}^q$.

*2.3. Transversely similar pseudo-Riemannian and Riemannian foliations*

Let $(N_i, g_i)$, $i = 1, 2$, be pseudo-Riemannian manifolds. Recall that a diffeomorphism $f : N_1 \to N_2$ is said to be similar, if $f^*g_2 = cg_1$ for some constant $c > 0$. In the case when $(N_1, g_1) = (N_2, g_2)$ such diffeomorphism is called a similar transformation or a similarity of the pseudo-Riemannian manifold $(N_1, g_1)$.

**Definition 7.** Let a foliation $(M, F)$ be given by an $N$-cocycle $\{U_i, f_i, \{k_{ij}\}\}_{i,j\in J}$. If the manifold $N$ admits a pseudo-Riemannian (or Riemannian) metric $g^N$ such that every local diffeomorphism $k_{ij}$ is a similar diffeomorphism of the metrics induced by $g^N$ on open subsets $f_i(U_i \cap U_j)$ and $f_j(U_i \cap U_j)$, then we refer to $(M, F)$ as *a transversely similar pseudo-Riemannian (or a transversely similar Riemannian) foliation*. If the curvature of the Levi-Civita connection $\nabla^N$ of a Riemannian metric $g^N$ is zero, then $(M, F)$ is called *transversely similar foliation*.

If every local diffeomorphism $k_{ij}$ is an isometry of the metrics induced by $g^N$ on open subsets $f_i(U_i \cap U_j)$ and $f_j(U_i \cap U_j)$, then we refer to $(M, F)$ as *a pseudo-Riemannian (or Riemannian) foliation*. It is said also that $(M, F)$ is modelled on a pseudo-Riemannian (respectively, Riemannian) transverse geometry $(N, g^N)$.

Emphasize that transversely similar pseudo-Riemannian and transversely similar Riemannian foliations as well as pseudo-Riemannian and Riemannian foliations belong to the class of foliations with transverse linear connection.

*2.4. The associated connection in the frame bundle*

Let $H = GL(q, \mathbb{R})$ be the general linear group and let $G = H \ltimes \mathbb{R}^q$ be the semidirect product of $H$ and the Abelian group $\mathbb{R}^q$, where $\mathbb{R}^q$ is the normal subgroup of $G$. We will consider $G$ as the affine group $Aff(A^q)$ of all affine transformations of the $q$-dimensional affine space $A^q$. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of the Lie groups $G$ and $H$, respectively. Let $N$ be a smooth (not necessary connected) manifold. Consider the frame bundle $\mathcal{T}(N, H)$ over $N$ with the projection $p : \mathcal{T} \to N$. Denote by $\mathfrak{p}$ the Lie algebra of the Lie group $\mathbb{R}^q$.

It is well known that setting the linear connection $\nabla$ on $N$ is equivalent to setting an $H$-connection in $\mathcal{T}$, i.e. the $H$-invariant $q$-dimensional distribution $Q$ on $\mathcal{T}$. We call $Q$ the *associated H-connection with* $\nabla$. Since $Q$ is an $H$-connection in $\mathcal{T}$, we have the $\mathfrak{h}$-valued 1-form $\alpha$ and the canonical $\mathfrak{p}$-valued 1-form $\theta$ of the connection $Q$ on $\mathcal{T}$ [12]. Then the equality $\beta(X) := \alpha(X) + \theta(X) \in \mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{p}$, $X \in \mathfrak{X}(\mathcal{T})$, defines the $\mathfrak{g}$-valued 1-form on $\mathcal{T}$ satisfying the following conditions:

$(c_1)$ the map $\beta_w : T_w\mathcal{T} \to \mathfrak{g}$ is an isomorphism of the vector spaces for every $w \in \mathcal{T}$;

($c_2$) $R_h^*\beta = \mathrm{Ad}_G(h^{-1})\beta$ for all $h \in H$, where $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ is the adjoint representation of the Lie group $G$ on its Lie algebra $\mathfrak{g}$;
($c_3$) $\beta(A^*) = A$ for any $A \in \mathfrak{h}$, where $A^*$ is the fundamental vector field defined by the element $A$.

The pair $(N, \nabla)$ is called a *manifold of a linear connection* $\nabla$. We emphasize that the setting of a linear connection $\nabla$ on $N$ is equivalent to the setting the Cartan geometry $\xi = (\mathcal{T}(N, H), \beta)$ of type $(G, H)$ [5].

### 2.5. The foliated bundle over $(M, F)$

We will use the following well known construction of the lifted foliation $(\mathcal{R}, \mathcal{F})$ for a foliation $(M, F)$ with transverse linear connection. Let $(M, F)$ be a given foliation with transverse linear connection modelled on a $q$-dimensional manifold of a linear connection $(N, \nabla)$. One may construct a principle $H$-bundle $\mathcal{R}(M, H)$ (called a *foliated bundle*) with the projection $\pi : \mathcal{R} \to M$, an $H$-invariant transversely parallelizable foliation $(\mathcal{R}, \mathcal{F})$ such that $\pi$ is a morphism of $(\mathcal{R}, \mathcal{F})$ into $(M, F)$ in the category of foliations $\mathfrak{Fol}$; moreover, there exists a $\mathfrak{g}$-valued 1-form $\omega$ on $\mathcal{R}$ having the following properties:

(i) $\omega(A^*) = A$ for any $A \in \mathfrak{h}$, where $A^*$ is the fundamental vector field corresponding to $A$;
(ii) $R_a^*\omega = \mathrm{Ad}_G(a^{-1})\omega \ \forall a \in H$;
(iii) for any $u \in \mathcal{R}$, the map $\omega_u : T_u\mathcal{R} \to \mathfrak{g}$ is surjective with the kernel $\ker\omega = T\mathcal{F}$, where $T\mathcal{F}$ is the tangent distribution to the foliation $(\mathcal{R}, \mathcal{F})$;
(iv) the Lie derivative $L_X\omega$ is zero for any vector field $X$ tangent to the leaves of $(\mathcal{R}, \mathcal{F})$;
(v) there exists the $H$-connection $\mathcal{Q}$ on $\mathcal{R}$ transversely projectable with respect to $(\mathcal{R}, \mathcal{F})$ in sense of [13], i.e. its $\mathfrak{h}$-valued 1-form $pr \circ \omega$ satisfies the following conditions $i_X(pr \circ \omega) = 0$ and $i_X d(pr \circ \omega) = 0$ for all $X \in \mathfrak{X}_{\mathcal{F}}(\mathcal{R})$, where $pr : \mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{p} \to \mathfrak{h}$ is the canonical projection.

The foliation $(\mathcal{R}, \mathcal{F})$ is called the *lifted foliation*. The restriction $\pi_{\mathcal{L}} : \mathcal{L} \to L$ of $\pi$ to a leaf $\mathcal{L}$ of $(\mathcal{R}, \mathcal{F})$ is a holonomy covering map onto the corresponding leaf $L$ of $(M, F)$.

If $\mathcal{R}$ is disconnected, then we consider a connected component of $\mathcal{R}$.

## 3. Existence of a global attractor

### 3.1. Ehresmann connection for foliations

The notion of an Ehresmann connection for a foliation was introduced by Blumenthal and Hebda [3]. Just like in [18] we use the terminology suggested earlier by Hermann. Let $(M, F)$ be a smooth foliation of codimension $q \geq 1$ and $\mathfrak{M}$ be a $q$-dimensional distribution transverse to $(M, F)$, i.e. $T_xM = \mathfrak{M}_x \oplus T_xF$ for any point $x \in M$. All maps considered here are assumed to be piecewise smooth. The curves in the leaves of the foliation are called vertical; the distribution $\mathfrak{M}$ and its integral curves are called horizontal.

A map $H : I_1 \times I_2 \to M$, where $I_1 = I_2 = [0, 1]$, is called *a vertical-horizontal homotopy* if for each fixed $t \in I_2$, the curve $H_{|I_1 \times \{t\}}$ is horizontal, and for each fixed $s \in I_1$, the curve $H_{|\{s\} \times I_2}$ is vertical, see Fig. 1. The pair of curves $(H_{|I_1 \times \{0\}}, H_{|\{0\} \times I_2})$ is called the base of $H$.

A pair of curves $(\sigma, h)$ with a common starting point $\sigma(0) = h(0)$, where $\sigma : I_1 \to M$ is a horizontal curve, and $h : I_2 \to M$ is a vertical curve, is called admissible. If for each admissible pair of curves $(\sigma, h)$ there exists a vertical-horizontal homotopy with the base $(\sigma, h)$, then the distribution $\mathfrak{M}$ is called an Ehresmann connection for the foliation $(M, F)$. Note that there exists at most one vertical-horizontal homotopy with a given base. Let $H$ be a vertical-horizontal homotopy with the base $(\sigma, h)$. We say that $\widetilde{\sigma} = H|_{I_1 \times \{1\}}$ is the result of the translation of the horizontal curve $\sigma$ along the vertical curve $h$ with respect to the Ehresmann

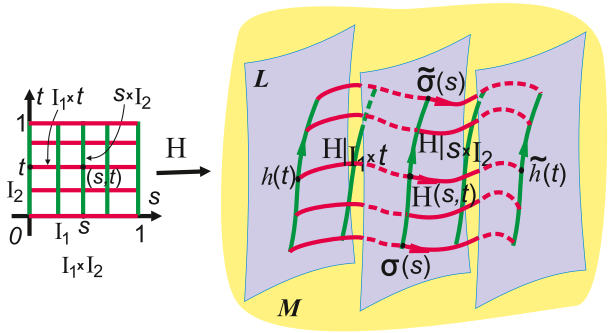


**Fig. 1.** A vertical-horizontal homotopy $H$.

connection $\mathfrak{M}$. Similarly the curve $\widetilde{h} = H_{|\{1\}\times I_2}$ is called the translation of the curve $h$ along $\sigma$ with respect to $\mathfrak{M}$.

Let $q : M \to B$ be a submersion of manifolds with connected fibres. Let $\mathfrak{M}$ be a distribution on $M$ complementary to the vertical distribution tangent to the fibres of $q$. Then $\mathfrak{M}$ is said to be *an Ehresmann connection for the submersion* $q$ if for any curve $h : [0, 1] \to B$ and every $x \in q^{-1}(h(0))$, there exists an integral curve $\widetilde{h} \to M$ of $\mathfrak{M}$ with the origin $\widetilde{h}(0) = x$ such that $q \circ \widetilde{h} = h$. The curve $\widetilde{h}$ is called the $\mathfrak{M}$-lift of $h$. It is known [3, Prop. 1.3] that a distribution $\mathfrak{M}$ is an Ehresmann connection for the submersion $q : M \to B$ with connected fibres, if and only if $\mathfrak{M}$ is an Ehresmann connection for the foliation $(M, F)$ where $F = \{q^{-1}(b) \,|\, b \in B\}$.

*3.2. A Riemannian metric and a connection adapted to the lifted foliation*

Let $(M, F)$ be a foliation of codimension $q$ with transverse linear connection. Let $\mathcal{R}(M, H)$ be its foliated bundle with the lifted foliation $(\mathcal{R}, \mathcal{F})$ and the projection $\pi : \mathcal{R} \to M$ having properties indicated in Section 2.5. Consider a smooth $q$-dimensional distribution $\mathfrak{M}$ transverse to $(M, F)$. Let $\widetilde{\mathfrak{M}} := \pi^*\mathfrak{M}$, i.e. $\widetilde{\mathfrak{M}}_u := \{X \in T_u\mathcal{R} \,|\, \pi_{*u}X \in \mathfrak{M}_x, x = \pi(u)\}$ for all $u \in \mathcal{R}$. Denote by $\mathfrak{P}$ the smooth $q$-dimensional distribution on $\mathcal{R}$ which is equal to the intersection $\mathcal{Q}$ and $\widetilde{\mathfrak{M}}$, i.e. $\mathfrak{P}_u = \{X \in \widetilde{\mathfrak{M}}_u \mid \omega(X) \in \mathfrak{p}\}$ for all $u \in \mathcal{R}$. The $H$-invariance of the distributions $\mathcal{Q}$ and $\widetilde{\mathfrak{M}}$ implies the $H$-invariance of the distribution $\mathfrak{P}$.

**Definition 8.** A smooth vector field $X \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$, for which $\omega(X) = c = const \in \mathfrak{g}$ is said to be a $\mathfrak{g}$-*field*. If moreover $c \in \mathfrak{p}$, then $X$ is said to be a $\mathfrak{p}$-*field*.

A piecewise smooth curve in $\mathcal{R}$ is called a $\mathfrak{g}$-*curve* (respectively a $\mathfrak{p}$-*curve*), if each its smooth piece is an integral curve of some vector $\mathfrak{g}$-field (respectively a $\mathfrak{p}$-*field*).

Remark, that locally each smooth $\mathfrak{g}$-curve $\sigma$ can be represented as $\sigma(t) = \varphi_t^X(v), t \in (-\varepsilon, \varepsilon)$, where $\varepsilon > 0$, $\varphi_t^X$ is the 1-parameter group of local diffeomorphisms of the manifold $\mathcal{R}$ generated by the $\mathfrak{g}$-field $X$ for which $\sigma(t)$ is an integral curve, and $v = \sigma(0) = \varphi_0^X(v)$.

**Lemma 1.** *Let $H = GL(q, \mathbb{R})$ and $O(q)$ be the orthogonal subgroup of $H$. Denote by $g_{\mathcal{R}}$ a Riemannian metric on the space of the foliated bundle $\mathcal{R}$. Let $d_0$ be the Euclidean metric on the vector space of $\mathfrak{g}$ invariant with respect to the action of the compact group $Ad_G(O(q))$. Let $Z = Z_{\mathcal{F}} \oplus Z_{\widetilde{\mathfrak{M}}}$ be the decomposition of a vector field $Z \in \mathfrak{X}(\mathcal{R})$ corresponding to the decomposition of the tangent vector space to $\mathcal{R}$ in the direct sum of the vector subspaces $T_u\mathcal{R} = T_u\mathcal{F} \oplus \widetilde{\mathfrak{M}}_u, u \in \mathcal{R}$.*

*Then the equality*

$$d(X, Y) := g_{\mathcal{R}}(X_{\mathcal{F}}, Y_{\mathcal{F}}) + d_0(\omega(X), \omega(Y)) \ \ \forall X, Y \in \mathfrak{X}(\mathcal{R}),$$

*defines a Riemannian metric $d$ on $\mathcal{R}$ transversally projectable with respect to the foliation $(\mathcal{R}, \mathcal{F})$ satisfying the following properties:*

1) *the length $l(\sigma)$ of any smooth $\mathfrak{g}$-curve $\sigma$, $\sigma(t) = \varphi_t^X(v)$, where $t \in [0, t_1], v = \sigma(0)$, is equal to $\parallel \omega(X) \parallel_{d_0} \cdot t_1$, where $\parallel \omega(X) \parallel_{d_0}^2 = d_0(\omega(X), \omega(X))$;*
2) *$l(\varphi_t^X(v)) = l(\varphi_t^X(v'))$, where $X$ is a $\mathfrak{g}$-field, $t \in [0, t_1]$, for all $v, v' \in \mathcal{R}$, if $\varphi_t^X(v)$ and $\varphi_t^X(v')$ are defined at $t \in [0, t_1]$;*
3) *for any element $a = (D \cdot A)^{-1}$, where $A \in O(q)$, $D = \mathrm{diag}(d_1, ..., d_q)$ with $0 < |d_i| < 1$ for $1 \leq i \leq q$, and $\lambda = max\{|d_1|, ...|d_q|\}$ and for any $\mathfrak{p}$-curve $\sigma$ the curve $\widetilde{\sigma} := R_a \circ \sigma$ is a $\mathfrak{p}$-curve, with $l(\widetilde{\sigma}) \leq \lambda \cdot l(\sigma)$.*

**Proof.** Since $\mathfrak{g}$-valued 1-form $\omega$ is projectable with respect to $(\mathcal{R}, \mathcal{F})$, the Riemannian metric $d$ is also projectable with respect to this foliation. As $\sigma(t) = \varphi_t^X(v)$, where $t \in [0, t_1]$, is an integral curve of some $\mathfrak{g}$-field $X$ such that $d(X, X) = \parallel \omega(X) \parallel_{d_0}^2$ and $d(\sigma)/dt = X_{\sigma(t)}$, the length $l(\sigma)$ is calculated by the formula specified in 1).

The relation 2) follows from 1).

Let us verify 3). By the condition $a = (D \cdot A)^{-1}$, where $A \in O(q)$, $D = \mathrm{diag}(d_1, ..., d_q)$ with $0 < |d_i| < 1$ for $1 \leq i \leq q$ and $\lambda = max\{|d_1|, ...|d_q|\}$.

*Case I:* $\sigma(t) = \varphi_t^X(v), t \in [0, t_1]$, is a smooth $\mathfrak{p}$-curve with the origin $\sigma(0) = v$. Consider $\widetilde{\sigma} := R_a \circ \sigma$. As $\widetilde{\sigma}(t) = \varphi_t^Y(v \cdot a)$, where $Y = R_{a*}(X)$, the $Ad_G(H)$-invariance of $\mathfrak{p}$ implies that $Y$ is a $\mathfrak{p}$-field. Hence $\widetilde{\sigma}$ is a $\mathfrak{p}$-curve.

According to 1), its length is calculated by the formula $l(\widetilde{\sigma}) = \parallel \omega(X) \parallel_{d_0} \cdot t_1$. The $H$-equivariance of the form $\omega$ implies the equality $\omega(Y) = \omega(R_{a*}(X)) = Ad_G(a^{-1})\omega(X) = a^{-1}\omega(X)$. Therefore, $l(\widetilde{\sigma}) = \parallel Ad_G(a^{-1})\omega(X) \parallel_{d_0} \cdot t_1$. Since $a^{-1} = D \cdot A$ and $A \in O(q)$, using $Ad(O(q))$-invariance of $d_0$, we get the following relations: $\parallel Ad_G(a^{-1})\omega(X) \parallel_{d_0} = \parallel Ad_G(D \cdot A)\omega(X) \parallel_{d_0} = \parallel Ad_G(D) \circ Ad_G(A)\omega(X) \parallel_{d_0} \leq \lambda \cdot \parallel Ad_G(A)\omega(X \parallel_{d_0} = \lambda \cdot \parallel \omega(X) \parallel_{d_0}$, hence $l(\widetilde{\sigma}) \leq \lambda \cdot l(\sigma)$.

*Case II:* $\sigma$ is a piecewise smooth $\mathfrak{p}$-curve. Then it is divided into finite number of smooth pieces $\sigma|_{I_i}, i = 1, ..., m$, for each of which, as it was proved above, there is the inequality $l(\widetilde{\sigma}|_{I_i}) \leq \lambda \cdot l(\sigma|_{I_i})$, hence $l(\widetilde{\sigma}) \leq \lambda \cdot l(\sigma)$. □

The following easily proved lemma takes place.

**Lemma 2.** *Let $E_i, i = \overline{1, dim(\mathfrak{g})}$ be a basis of the Lie algebra $\mathfrak{g}$. Let $X_i$ be a $\mathfrak{g}$-field such that $\omega(X_i) = E_i$. We shall denote by $\widetilde{\nabla}$ the Levi-Civita connection of the Riemannian manifold $(\mathcal{R}, d)$. Then the equality*

$$\nabla_Y^0 Z := Y(Z^i)X_i + \widetilde{\nabla}_Y Z_F, \tag{*}$$

*where $Z = Z_F \oplus Z_{\widetilde{\mathfrak{M}}}$, $Z_{\widetilde{\mathfrak{M}}} = Z^i X_i \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$, $Z_F \in \mathfrak{X}_{T\mathcal{F}}(\mathcal{R})$, $Y \in \mathfrak{X}(\mathcal{R})$, defines a linear connection $\nabla^0$ in $\mathcal{R}$ (generally speaking with torsion) with respect to which all $\mathfrak{g}$-fields are parallel. Besides the parallel transfer keeps the scalar product of $\mathfrak{g}$-fields inducted by $d$, and integrated curves of $\mathfrak{g}$-fields are geodetic lines of the connection $\nabla^0$.*

Recall that a smooth vector field $X$ on $\mathcal{R}$ is *complete*, if $X$ generates a global 1-parameter group of diffeomorphisms of $\mathcal{R}$.

**Definition 9.** A foliation $(M, F)$ of codimension $q$ with transverse linear connection is called *complete,* if there exists a $q$-dimensional smooth distribution $\mathfrak{M}$ on $M$ transverse to $TF$ such that every $\mathfrak{g}$-field $X \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$, where $\widetilde{\mathfrak{M}} := \pi^*\mathfrak{M}$, is complete.

Thus, by the definition, the completeness of a foliation $(M, F)$ with transverse linear connection is equivalent to the completeness of $(M, F)$ considered as a Cartan foliation [18].

**Remark 5.** It is not difficult to show that a foliation $(M, F)$ of codimension $q$ with transverse linear connection is *complete*, if and only if there exists a $q$-dimensional smooth distribution $\mathfrak{M}$ on $M$ transverse to $TF$ such that every $\mathfrak{p}$-field $X \in \mathfrak{X}_{\mathfrak{P}}(\mathcal{R})$ is complete.

### *3.3. Lemma*

We use the notations from the previous section.

The following lemma will be used in the proof of the existence of a global attractor.

**Lemma 3.** *Let $(M, F)$ be a foliation with transverse linear connection admitting an Ehresmann connection $\mathfrak{M}$. Let $L$ and $L'$ be two arbitrary leaves of $(M, F)$. Then the sets $\pi^{-1}(L)$ and $\pi^{-1}(L')$ of $\mathcal{R}$ may be connected by some piecewise smooth $\mathfrak{p}$-curve.*

**Proof.** Let $f : M \to M/F$ be the quotient map onto the leaf space. We denote by $[L]$ the leaf $L$ of $(M, F)$ considered as a point of $M/F$. We say that $[L]$ is equivalent to $[L']$ if there exists a piecewise smooth $\mathfrak{p}$-curve $\sigma : [0, 1] \to \mathcal{R}$ connecting $\pi^{-1}(L)$ with $\pi^{-1}(L')$, i.e., $\sigma(0) \in \pi^{-1}(L)$ and $\sigma(1) \in \pi^{-1}(L')$.

Let us show that this relation is indeed an equivalence relation on $M/F$. The reflexivity and symmetry are obvious. Let us check the transitivity of the introduced relation. Let $[L_0] \sim [L_1]$ and $[L_1] \sim [L_2]$, and a $\mathfrak{p}$-curve $\sigma$ connects $\pi^{-1}(L_0)$ with $\pi^{-1}(L_1)$, a $\mathfrak{p}$-curve $\sigma_1$ connects $\pi^{-1}(L_1)$ with $\pi^{-1}(L_2)$. Let $v_0 = \sigma(0) \in \pi^{-1}(L_0)$, $v_1 = \sigma(1) \in \pi^{-1}(L_1)$, $v_2 = \sigma_1(0) \in \pi^{-1}(L_1)$, $v_3 = \sigma_1(1) \in \pi^{-1}(L_2)$ and $x_i = \pi(v_i), i = 0, ..., 3$, and $x_1, x_2 \in L_1$. Hence, there exists a path $h : [0, 1] \to L_1$ connecting $x_2 = h(0)$ with $x_1 = h(1)$.

Use the following notation: $\gamma_1 = \pi \circ \sigma_1$. Emphasize that $\gamma_1$ is $\mathfrak{M}$-horizontal curves. As $\mathfrak{M}$ is an Ehresmann connection for $(M, F)$, there exists $\gamma_2$, the result of the translation of $\gamma_1$ along $h$ with respect to $\mathfrak{M}$, and $\gamma_2$ is $\mathfrak{M}$-horizontal curve with the end points $\gamma_2(0) = x_1 \in L_1$ and $\gamma_2(1) = x_4 \in L_2$. Note that the defined in Section 2.5 $H$-invariant distribution $\mathcal{Q}$ is a connection in the principal $H$-bundle with the projection $\pi : \mathcal{R} \to M$. Therefore there exists the $\mathcal{Q}$-lift $\sigma_2$ of $\gamma_2$ to the point $v_1$. Since $\pi \circ \sigma_2 = \gamma_2$, then $v_4 := \sigma_2(1) \in \pi^{-1}(x_4) \subset \pi^{-1}(L_2)$. Thus, the product of paths $\sigma \cdot \sigma_2$ is a $\mathfrak{p}$-curve connecting $\pi^{-1}(L_0)$ with $\pi^{-1}(L_2)$. This means that $[L_0] \sim [L_2]$, i.e. the relation $\sim$ is transitive, hence the introduced relation is an equivalence relation.

We now show that each equivalence class is an open subset of $M/F$. Consider a point $[L] \in M/F$, where $L = L(x_0)$, $x_0 \in M$. Let $A([L])$ be the equivalence class containing $[L]$. Using the exponential map of the linear connection $\nabla^0$ on $\mathcal{R}$ introduced in Lemma 2, we see that for every point $u_0 \in \pi^{-1}(x_0)$ there exists a normal neighbourhood $\mathcal{V}$ in $\mathcal{R}$ such that $U = \pi(\mathcal{V})$ is a neighbourhood adapted to $(M, F)$ and projections of $\mathfrak{p}$-geodesics with the origin in $u_0$ to $U$ intersect every leaf of the induced foliation $(U, F_U)$. This means that the open subset $U$ in $M$ satisfies the following inclusion $f(U) \subset A([L])$. Therefore, $A([L])$ is an open subset in $M/F$.

Since the complement of $A([L])$ is formed by the union of the remaining equivalence classes each of which is open, then $A([L])$ is a closed subset of $M/F$. Due to the connectivity of the topological space of $M$, the leaf space $M/F$ is also connected. Hence a non-empty open-closed subset $A([L])$ coincides with $M/F$. $\square$

### *3.4. Proof of Theorem 2*

Let $(M, F)$ be a foliation with transverse linear connection admitting an Ehresmann connection $\mathfrak{M}$ and $q = codim(M, F)$. Let $f : U \to V$ be a submersion from $N$-cocycle defining $(M, F)$. Assume that for $x \in U$ and $z = f(x) \in V$ the linear holonomy group $D\Gamma(L, x) \cong \mathcal{H}_{z*}$ of a leaf $L = L(x)$ contains an element $\phi_{*z}$

defined by a matrix of the form $A \cdot D$, where $K = < A >$ is a relatively compact subgroup of the linear group $H = GL(q, \mathbb{R})$ and $D = \mathrm{diag}(d_1, ..., d_q)$ with $0 < |d_i| < 1$ for $1 \le i \le q$ in the orthonormal basis of the Euclidean tangent vector space $(T_z N, g_0)$, and $g_0$ is invariant with respect to the compact group $K$. Put $\lambda = max\{|d_1|, ..., |d_q|\}$, then $\lambda \in (0, 1)$.

Let $\pi : \mathcal{R} \to M$ be the projection of the foliated $H$-bundle, $H = GL(q, \mathbb{R})$, over $(M, F)$. Recall that $H(u) = \{a \in H \,|\, R_a(\mathcal{L}) = \mathcal{L}\}$, where $u \in \pi^{-1}(x) \subset \mathcal{R}$ and $\mathcal{L} = \mathcal{L}(u)$ is a leaf of the lifted foliation $(\mathcal{R}, \mathcal{F})$. By Theorem 1, there exists an isomorphism $h$ of the subgroups $D\Gamma(L, x)$ and $H(u)$ of the linear group $H$, and $h$ is a homeomorphism of $D\Gamma(L, x)$ and $H(u)$ with respect to the induced topologies. Therefore the group $H(u)$ contains an element $\widehat{a} = (D \cdot A)^{-1}$, where $D$ and $A$ are indicated above.

Let $K_0$ be the maximal compact subgroup of $H = GL(q, \mathbb{R})$ containing $K$. As $K_0$ is conjugate to the maximal compact subgroup $O(q)$ in $H$, there exists $C \in H$ such that $K_0 = C \cdot O(q) \cdot C^{-1}$. According to Remark 1, at point $v = uC$, we get that $H(v) = C^{-1} \cdot H(u) \cdot C$. Therefore, $H(v)$ contains the element $b := C^{-1} \cdot (D \cdot A) \cdot C = D \cdot C^{-1} \cdot A \cdot C = D \cdot \widetilde{A}$ where $\widetilde{A} \in O(q)$.

We use the notations of the previous lemmas. Let $L'$ be any other leaf of this foliation. By Lemma 3, there exists a $\mathfrak{p}$-curve $\sigma$ connecting $\pi^{-1}(L)$ and $\pi^{-1}(L')$. Let $v = \sigma(0) \in \pi^{-1}(x)$, $x \in L$, $v_0 := \sigma(1) \in \pi^{-1}(x_0)$, $x_0 \in L'$. Due to the continuity of $\pi$, for any however small $\varepsilon > 0$ there is the ball $B_\varepsilon$ of a radius $\varepsilon$ with the centre at $v$ of the Riemannian manifold $(\mathcal{R}, d)$ such that $\pi(B_\varepsilon) \subset U$.

Consider the curve $\widetilde{\sigma} := R_a(\sigma)$ where $a := b^k$, $b$ is as above. Then $w := va = \widetilde{\sigma}(0)$ and $w_0 := v_0 a = \widetilde{\sigma}(1)$. As $\lambda \in (0, 1)$, for any however small $\varepsilon > 0$ there is a natural number $k$ for which $\lambda^k \cdot l(\sigma) < \varepsilon$. According to the statement 3) of Lemma 1, the length of the curve $\widetilde{\sigma} := R_a \circ \sigma$, where $a = b^k \in H(\mathcal{L}, v) \subset H$, satisfies the relation $l(\widetilde{\sigma}) = \lambda^k \cdot l(\sigma) < \varepsilon$. Since $a \in H(v)$, we can connect the points $w := R_a(v)$ and $v$ by a smooth path $h$ in the leaf $\mathcal{L} = \mathcal{L}(v)$, $h(0) = w$, $h(1) = v$. By the assumption, $\mathfrak{M}$ is an Ehresmann connection for the foliation $(M, F)$. Therefore the induced distribution $\widetilde{\mathfrak{M}} = \pi^* \mathfrak{M}$ is an Ehresmann connection for the lifted foliation $(\mathcal{R}, \mathcal{F})$. Hence there exists the translation of the $\mathfrak{p}$-curve $\widetilde{\sigma}$ along the leaf path $h$ with respect to the Ehresmann connection $\widetilde{\mathfrak{M}}$. Let $\widehat{\sigma}$ be the result of the translation. We observe that $\widehat{\sigma}$ is a $\mathfrak{p}$-curve and $\widehat{\sigma}(0) = v$. Recall that according to Lemma 1, $d$ is transversely projectable Riemannian metric with respect to $(\mathcal{R}, \mathcal{F})$. Therefore $(\mathcal{R}, \mathcal{F})$ is a Riemannian foliation, and $l(\widehat{\sigma}) = l(\widetilde{\sigma}) < \varepsilon$ in $(\mathcal{R}, d)$, hence $\widehat{\sigma}(1) = v_1 \in B_\varepsilon(v)$. As points $\pi(v_1)$ and $\pi(v_0)$ belong to the same leaf $\pi(\mathcal{L}(v_0)) = L' = \pi(\mathcal{L}(v_1))$, so $\pi(v_1) \in L' \cap \pi(B_\varepsilon)$. Hence the closure $\overline{L'}$ of the leaf $L'$ satisfies the inclusion $\overline{L'} \supset L$. By property of closure it implies $\overline{L'} \supset \overline{L} = \mathcal{M}$. Thus, we proved that

$$\mathcal{M} \subset \overline{L'} \quad \forall L' \in F. \tag{1}$$

Let $L_\alpha \subset \mathcal{M}$, then $\overline{L_\alpha} \subset \mathcal{M}$. According to (1), it is necessary $\mathcal{M} \subset \overline{L_\alpha}$. Therefore, $\overline{L_\alpha} = \mathcal{M}$ for all $L_\alpha \subset \mathcal{M}$. This means that $\mathcal{M}$ is a minimal set. According to (1), $\mathcal{M}$ is a global attractor.

If $L$ is a proper leaf, then the minimal set $\mathcal{M}$ is trivial, and $\mathcal{M} = L$. Since $L$ is a global attractor, it is necessary that $L$ is a unique closed leaf of $(M, F)$. This completes the proof of Theorem 2. □

### *3.5. Proof of Corollary 1*

Let $(M, F)$ be a transversely similar pseudo-Riemannian foliation of codimension $q$ on an $n$-dimensional manifold $M$, and $(M, F)$ admits an Ehresmann connection. Assume that $(M, F)$ is modelled on a pseudo-Riemannian manifold $(N, g^N)$ of signature $(k, s)$, where $q = k + s$. Denote by $L = L(x)$ the leaf having an element $\varphi_{*x}$ in the linear holonomy group $D\Gamma(L, x)$ of the form $\lambda \cdot A$, where $\lambda \in (0, 1)$ and $A$ belongs to a compact subgroup of the pseudo-orthogonal group $O(k, s)$. Note that every local similarity $k_{ij}$ belonging to the holonomy pseudogroup of $(M, F)$ is an isomorphism of the Levi-Civita connection $(N, \nabla)$ of the pseudo-Riemannian manifold $(N, g^N)$. Therefore $(M, F)$ satisfies the assumptions of Theorem 2 and the statements of Corollary 1 follow from Theorem 2. □

## 4. The structure of transversely similar Riemannian foliations

### 4.1. A criterion of being Riemannian for a transversely similar Riemannian foliation

Let $N$ be a $q$-dimensional not necessary connected manifold and $g$ be a Riemannian metric on $N$. A similar Riemannian structure on $N$ is the class $[|g|]$ of Riemannian metrics similar to $g$.

Denote by $\mathbb{R}^+$ the multiplicative group of positive numbers and $G = (\mathbb{R}^+ \cdot O(q)) \ltimes \mathbb{R}^q$ the Lie group of all similarities of the $q$-dimensional Euclidean space $\mathbb{E}^q$. Put $H = \mathbb{R}^+ \cdot O(q)$. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of the Lie groups $G$ and $H$ respectively. A similar Riemannian structure $[|g|]$ on $N$ defines the Levi-Civita connection $\nabla$ which is same for every metric $h \in [|g|]$. Emphasize that the setting a similar Riemannian structure $[|g|]$ on $N$ is equivalent to the setting an $H$-invariant connection $Q$ in the $H$-bundle $P(N, H)$. Let $p : P \to N$ be the projection. Denote by $\xi = (P(N, H), \beta)$ the associated Cartan geometry on $N$, where $\beta$ is the associated $\mathfrak{g}$-valued 1-form on $P$ (details see in Section 2.4).

Consider a transversely similar Riemannian foliation $(M, F)$. Then there are the following objects: an $H$-bundle $\mathcal{R}(M, H)$ over $(M, F)$ with the projection $\pi : \mathcal{R} \to M$, a $\mathfrak{g}$-valued 1-form $\omega$ on $\mathcal{R}$ and the lifted foliation $(\mathcal{R}, \mathcal{F})$. Note that the Lie group $H$ acts freely on the right on $\mathcal{R}$, and the action is smooth.

**Proposition 1.** *A transversely similar Riemannian foliation is Riemannian if and only if all its holonomy groups are relatively compact.*

**Proof.** Let $E_q$ be the $q$-dimensional unit matrix, $E_q \in O(q)$. For short we denote by $\mathbb{R}^+$ the Lie subgroup $\mathbb{R}^+ \cdot \{E_q\}$ of $H$. Since $H$ acts freely on $\mathcal{R}$, a free action of its normal subgroup $\mathbb{R}^+$ is also defined on $\mathcal{R}$, and the quotient space $\mathcal{R}/\mathbb{R}^+$ is a smooth manifold $\widehat{\mathcal{R}}$. Moreover, a free action of the quotient group $O(q) = H/\mathbb{R}^+$ on $\widehat{\mathcal{R}}$ is defined, and $M = \widehat{\mathcal{R}}/O(q)$. The projections to the orbit spaces $\alpha : \mathcal{R} \to \hat{\mathcal{R}}$ and $\hat{\pi} : \hat{\mathcal{R}} \to M$ satisfy the equation $\pi = \hat{\pi} \circ \alpha$. Since the lifted foliation $(\mathcal{R}, \mathcal{F})$ is $H$-invariant, it is $\mathbb{R}^+$-invariant, hence there exists a foliation $(\widehat{\mathcal{R}}, \widehat{F})$ such that $\alpha : \mathcal{R} \to \widehat{\mathcal{R}}$ is a morphism in the category of foliations $\mathfrak{Fol}$.

By the assumption, all linear holonomy groups of $(M, F)$ are relatively compact. According to Theorem 1, this is equivalent to relative compactness of every subgroup $H(u)$, $u \in \mathcal{R}$, where $H(u) = \{a \in H \mid R_a(\mathcal{L}) = \mathcal{L}\}$ and $\mathcal{L} = \mathcal{L}(u)$ is a leaf of $(\mathcal{R}, \mathcal{F})$.

Since the Lie group $\mathbb{R}^+$ does not admit nontrivial relatively compact subgroups, it is necessary that $\mathcal{L}(u) \cap \alpha^{-1}(\alpha(u)) = \{u\}$ for every $u \in \mathcal{R}$. Therefore, $H(u) \cap R^+ = \{1\}$. This implies that $\alpha|_{\mathcal{L}} : \mathcal{L} \to \widehat{L}$ is a diffeomorphism of $\mathcal{L}$ onto the appropriate leaf $\widehat{L}$ of the foliation $(\widehat{\mathcal{R}}, \widehat{F})$. According to [20, Prop. 4], there exists a section $\sigma : \widehat{\mathcal{R}} \to \mathcal{R}$ such that the image $\sigma(\widehat{L})$ of a leaf $\widehat{L} \in \widehat{F}$ is a leaf of $(\mathcal{R}, \mathcal{F})$. This means that $\sigma : \widehat{\mathcal{R}} \to \mathcal{R}$ is an isomorphism of foliations $(\widehat{\mathcal{R}}, \widehat{F})$ and $(\widetilde{\mathcal{R}}, \widetilde{F})$, where $\widetilde{\mathcal{R}} = \sigma(\widehat{\mathcal{R}})$ and $\widetilde{F} = \mathcal{F}|_{\widetilde{\mathcal{R}}}$. This implies the triviality of the $\mathbb{R}^+$-bundle $\mathcal{R}(\widehat{\mathcal{R}}, \mathbb{R}^+)$. Hence, without loss of generality, we can identify $\mathcal{R}$ with the product of manifolds $\widehat{\mathcal{R}} \times \mathbb{R}^+$ and assume that $\widetilde{\mathcal{R}} = \sigma(\widehat{\mathcal{R}}) = \widehat{\mathcal{R}} \times \{1\}$. Thus, $\widehat{\mathcal{R}}(M, O(q))$ is a foliated reduction of the $H$-bundle $\mathcal{R}(M, H)$ to the closed subgroup $O(q)$.

Note that $\mathfrak{h} = \mathfrak{R}^1 \oplus \mathfrak{so}(q)$, where $\mathfrak{h}$, $\mathfrak{R}^1$ and $\mathfrak{so}(q)$ are the Lie algebras of the Lie groups $H$, $\mathbb{R}^+$ and $O(q)$, respectively. Let $pr : \mathfrak{R}^1 \oplus \mathfrak{so}(q) \to \mathfrak{so}(q)$ be the canonical projection. As $T_v\mathcal{R} = \mathbb{R}^1 \oplus T_v\widetilde{\mathcal{R}}$, $v \in \widetilde{\mathcal{R}}$, $\omega(X) \in \{\mathfrak{o}\} \oplus \mathfrak{so}(q)$ for any $X \in T_v\widetilde{\mathcal{R}}$. The equality $\widetilde{\omega}(X) = pr \circ \omega(X)$, $X \in T_v\widetilde{\mathcal{R}}$ defines a $\mathfrak{so}(q)$-valued 1-form $\widetilde{\omega}$ on $\widetilde{\mathcal{R}}$.

It is easy to show that $H$-equivariance of $\omega$ implies the $\mathfrak{so}(q)$-equivariance of $\widetilde{\omega}$, and the transversal projectability of $\omega$ with respect to $(\mathcal{R}, \mathcal{F})$ implies the transversal projectability of $\widetilde{\omega}$ with respect to $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{F}})$.

Thus, $(M, F)$ is a Riemannian foliation and $\widetilde{\mathcal{R}}(M, O(q))$ is its foliated bundle.

The converse may be proved in an obvious way. □

**Corollary 4.** *If a transversely similar Riemannian foliation $(M, F)$ is not Riemannian, then there exists a leaf $L$ with essential holonomy group.*

*4.2. Proof of Theorem 3*

$(i)$. Let $(M, F)$ be a non-Riemannian transversely similar Riemannian foliation of codimension $q$, $q \geq 1$. According to Corollary 4, there is a leaf $L = L(x)$, $x \in M$, with an essential holonomy group. This is equivalent to the existence of an element of the form $\lambda \cdot A$ in the linear holonomy group $D\Gamma(L, x)$, where $\lambda \in (0, 1)$ and $A \in O(q)$. Besides by the assumption, $(M, F)$ admits an Ehresmann connection. Hence, according to Theorem 2, the foliation $(M, F)$ has a global attractor $\mathcal{M}$. Thus the statement $(i)$ is proved.

$(ii)$, $(iv)$. Assume that $(M, F)$ is modelled on a similar Riemannian geometry $(N, [|g|])$, where $(N, g)$ is a not necessary connected Riemannian manifold. There exists a unique class of conformal metrics $[g]$ on $N$ containing $[|g|]$. Therefore $(M, F)$ may be considered as a conformal foliation modelled on a conformal geometry $(N, [g])$.

Let $q \geq 4$ and $W$ be the Weyl tensor of type $(1, 3)$ of the conformal curvature for the Riemannian manifold $(N, g)$. Considering $W$ as a multilinear map $W : \mathfrak{X}N \times \mathfrak{X}N \times \mathfrak{X}N \to \mathfrak{X}N$, we define the norm $\parallel W \parallel (x)$, $x \in N$ by the following way. Let $\parallel X \parallel (x) := \sqrt{g_x(X, X)}$ for any vector field $X \in \mathfrak{X}(N)$. We put $\parallel W \parallel (x) := \sup\limits_{\parallel X_i \parallel (x) \leq 1} \parallel W(X_1, X_2, X_3) \parallel (x)$ $\forall x \in N$, $i = 1, 2, 3$.

Recall that the holonomy pseudogroup $\mathcal{H} = \mathcal{H}(M, F)$ of $(M, F)$ consists of local conformal diffeomorphisms of the Riemannian manifold $(N, g)$. It is well known that the Weyl tensor $W$ is a conformal invariant. Let $\mathfrak{M}$ be a $q$-dimensional distribution on $M$ transversal to $(M, F)$, i.e. $T_xM = \mathfrak{M}_x \oplus T_xF$ for every $x \in M$. Therefore, on the distribution $\mathfrak{M}$ the transversely projectable Weyl tensor $\widetilde{W}$ is induced. Thus $\widetilde{f}(x) :=\parallel \widetilde{W} \parallel (x)$, $x \in M$, is the base function with respect to $(M, F)$, i.e. a function which is constant on leaves of this foliation. The existence of a global attractor and the continuity of this function imply $\parallel \widetilde{W} \parallel = const$, i.e. the constancy of the function $f(z) :=\parallel W \parallel (z)$, $z \in N$. Let $c =\parallel W \parallel$.

For every $v \in N$ there exists a submersion $f_i : U_i \to V_i$ from $(N, [|g|])$-cocycle defining $(M, F)$ such that $v \in f_i(U_i) = V_i$. The equality $\hat{f}(v) := f(z)$ where $z \in f_i^{-1}(v)$ defines a smooth function $\hat{f}$ on $N$, and $\hat{f}(v) \equiv c$. Assume, that $c \neq 0$. As all transformations from the holonomy pseudogroup $\mathcal{H}$ of $(M, F)$ are local conformal diffeomorphisms, it is not difficult to check up that each transformation from $\mathcal{H}$ preserves the Riemannian metric $cg$ on $N$, i.e. $\mathcal{H}$ is a pseudogroup of local isometries of the Riemannian manifold $(N, cg)$. Hence $(M, F)$ is a Riemannian foliation. This contradicts the assumption of Theorem 3.

Therefore it is necessary that $W \equiv 0$.

If $q = 3$, then $W \equiv 0$ and the replacing $W$ by the Schouten tensor $V$ of type $(1, 2)$ [17, Th. P.5.1], similarly to the previous case we show the invariance of the Riemannian metric $\parallel V \parallel^{\frac{2}{3}} \cdot g$ with respect to the holonomy pseudogroup $\mathcal{H}$ of $(M, F)$ and get the contradiction. Hence, $V \equiv 0$.

The Weyl–Schouten theorem [17, Th. P.5.1] contains the following criteria for a $q$-dimensional Riemannian manifold $(N, g)$, $q \geq 3$, to be locally conformally flat, i.e. to be locally conformally equivalent to the Euclidean space $\mathbb{E}^q$ (or the standard sphere $\mathbb{S}^q$). For $q \geq 4$, a $q$-dimensional Riemannian manifold $(N, g)$ is locally conformally flat if and only if, the Weyl conformal curvature tensor $W$ is equal to zero. A 3-dimensional Riemannian manifold $(N, g)$ is locally conformally flat if and only if, the Schouten tensor $V$ is equal to zero.

By the Lichtenstein theorem [17, Th. P.4.6], any 2-dimensional Riemannian manifold is locally conformally flat. Every 1-dimensional Riemannian manifold is also conformally flat.

Thus for $q \geq 1$ the transverse Riemannian manifold $(N, g)$ is locally conformally flat, hence every open subset $U \subset N$ may be considered as a subset of $\mathbb{S}^q$.

According to the Liouville theorem, for any connected open subsets $U$, $V$ of $\mathbb{S}^q$ and for each similar transformation $\gamma : U \to V$ there exists a unique element $f \in Conf(\mathbb{S}^q)$ such that $\gamma = f|_U$. Therefore, $(M, F)$ is a $(Conf(\mathbb{S}^q), \mathbb{S}^q)$-foliation.

By the assumption, there exists an Ehresmann connection for $(M, F)$, then we may apply [20, Th. 2]. According to this theorem, there exists a regular covering map $\kappa : \widetilde{M} \to M$ and a simply connected $(Conf(\mathbb{S}^q), \mathbb{S}^q)$-manifold $B$ such that the induced foliation $\widetilde{F} := \kappa^* F$ is formed by fibres of a submersion

$r : \widetilde{M} \to M$. Moreover, there exists a Riemannian metric $g^B$ on $B$ and a group homomorphism $\chi : \pi_1(M,x) \to Conf(B, g^B)$ such that the global holonomy group $\Psi := \chi(\pi_1(M,x))$ is isomorphic to the group of covering transformations of the map $\kappa : \widetilde{M} \to M$.

Without loss of generality we can assume that the holonomy pseudogroup of $(M,F)$ is generated by the group $\Psi$. Since $(M,F)$ is a transversally similar foliation, $\Psi$ is a group of similarities of $(B, g^B)$. Therefore $\Psi$ is both an essential similarity group and an essential conformal transformation group of the Riemannian manifold $(B, g^B)$. Therefore, by Ferrand results [8] (see also [1]), for $q \geq 3$ the Riemannian manifold $(B, g^B)$ has to be conformal either to the standard sphere $\mathbb{S}^q$ or to the Euclidean space $\mathbb{E}^q$. The same statement holds for $q = 2$ [1].

Assume that $(B, g^B)$ is conformal to $\mathbb{S}^q$, hence $B$ is compact. In this case, according to theorem of Hopf and Rinow, $(B, g^B)$ is complete. Thus the global holonomy group $\Psi$ has an essential similarity $\psi$ of a complete Riemannian manifold $B$ with the fixed point $b = r(\kappa^{-1})(x)$. According to [12, Lem. 2, Chap VI] this is only possible when $(B, g^B)$ is the Euclidean space $\mathbb{E}^q$. The contradiction with the assumption shows that $B = \mathbb{E}^q$ and $\Psi \subset Sim(\mathbb{E}^q)$. It is easy to see that the completeness of $B = \mathbb{E}^q$ implies the completeness of both transversely similar foliations $(\widetilde{M}, \widetilde{F})$ and $(M,F)$.

Therefore $(M,F)$ is a complete $(Sim(\mathbb{E}^q), \mathbb{E}^q)$-foliation and the statements $(ii)$ and $(iv)$ are proved.

$(v)$. Since the holonomy pseudogroup $\mathcal{H} = \mathcal{H}(M,F)$ is generated by the group $\Psi$, then $\mathcal{H}$ is analytical. Therefore the statements proved above imply that the holonomy group $\Gamma(L,x)$ of the leaf $L$ is isomorphic to the isotropy subgroup $\Psi_z$ of $\Psi$ at point $z \in pr(\kappa^{-1}(L))$. Hence $(v)$ holds true.

$(iii)$. Since $\mathbb{E}^q$ is contractible, the locally trivial bundle $r : \widetilde{M} \to \mathbb{E}^q$ is trivial, hence $\widetilde{M} \cong L_0 \times \mathbb{E}^q$ and $r = pr : L_0 \times \mathbb{E}^q \to \mathbb{E}^q$ is the canonical projection.

It is easy to see that the restriction $\kappa|_{L_0 \times \{b\}}$, $b \in \mathbb{E}^q$ is a regular covering map onto the corresponding leaf $L$ of $(M,F)$, and the deck transformation group is isomorphic to the isotropy subgroup $\Psi_b$ of $\Psi$ at the point $b$. According to the statement $(v)$ proved above, the group $\Psi_b$ is isomorphic to the holonomy group of $L$. This completes the proof of $(iii)$.

For $q \geq 1$, statements $(vi)$ and $(vii)$ are proved similarly to the corresponding statements in [20, Th. 5]. □

## 5. Proof of Theorem 5

Denote by $< A, a >$, where $A \in GL(q,\mathbb{R})$, $a \in \mathbb{R}^q$, $q \geq 1$, an element of the affine group $Aff(A^q)$. This element acts on the $q$-dimensional affine space $A^q$ by the rule $< A, a > x = Ax + a \ \forall x \in A^q$. Therefore the group operation in $Aff(A^q)$ is defined by the following equality:

$$< A, a >< B, b >=< AB, Ab + a > \quad \forall < A, a >, < B, b >\in Aff(A^q).$$

Since $Sim(\mathbb{E}^q)$ is a subgroup of $Aff(A^q)$, it has the same group operation.

Let $(M,F)$ be a non-Riemannian transversely similar foliation of codimension one. Assume that $(M,F)$ admits an Ehresmann connection $\mathfrak{M}$. Since $\mathfrak{M}$ is a 1-dimensional distribution, it is integrable, i.e. there exists an 1-dimensional foliation $(M, F^t)$ for which $TF^t = \mathfrak{M}$. Consider the universal covering map $\hat{\kappa} : \hat{M} \to M$ with two induced foliations $(\hat{M}, \hat{F})$, $\hat{F} := \hat{\kappa}^* F$ and $(\hat{M}, \hat{F}^t)$, $\hat{F}^t = \hat{\kappa}^* F^t$. Note that the distribution $\hat{\mathfrak{M}} := \kappa^* \mathfrak{M}$ is equal to $T\hat{F}^t$. Since $(M,F)$ admits an Ehresmann connection, then there exists a manifold $\hat{L_0}$ diffeomorphic to the common universal covering space for all leaves of $(M,F)$. According to Kashiwabara's theorem [11, Th. 2], we may identify $\hat{M}$ with the product $\hat{L_0} \times \mathbb{R}^1$, and $\hat{F} = \{\hat{L_0} \times \{y\} \,|\, y \in \mathbb{R}^1\}$, $\hat{F}^t = \{\{z\} \times \mathbb{R}^1 \,|\, z \in \hat{L_0}\}$. Therefore, every two leaves of different foliations $(M,F)$ and $(M,F^t)$ have a nonempty intersection.

For a fixed point $\hat{x} \in \hat{\kappa}^{-1}(x)$, the fundamental group $\pi_1(M,x)$ acts on the product $\hat{L_0} \times \mathbb{R}^1$ as the deck transformation group $G$. Since $G$ conserves both induced trivial foliations of $\hat{L_0} \times \mathbb{R}^1$, the projection

$\hat{L}_0 \times \mathbb{R}^1 \to \mathbb{R}^1$ induces a group $\Psi$ on $\mathbb{R}^1$, and $\Psi$ is a subgroup of the Lie group $Sim(\mathbb{E}^1) = (\mathbb{R}^+ \cdot O(1)) \ltimes \mathbb{R}^1$. Emphasize that $\Psi$ is the global holonomy group of $(M, F)$. By Theorem 3, a group homomorphism $\chi : \pi_1(M, x_0) \cong G \to Sim(\mathbb{E}^1)$ is defined. Let $\hat{G} = Ker(\chi)$ be the kernel of $\chi$, then $\hat{G}$ acts freely and proper discontinuously on $\hat{M}$, and the orbit space $\widetilde{M} = \hat{M}/\hat{G}$ is a manifold on which the quotient group $\widetilde{\Psi} := G/\hat{G}$ acts freely and proper discontinuously, and the orbit space $\widetilde{M}/\widetilde{\Psi}$ is diffeomorphic to $M$. The quotient map $\kappa : \widetilde{M} \to M$ is a regular covering map with the deck transformation group $\widetilde{\Psi}$ isomorphic to $\Psi$ satisfying the equality $\hat{\kappa} = \kappa \circ \widetilde{\kappa}$, where $\widetilde{\kappa} : \hat{M} \to \hat{M}/\widetilde{\Psi} = \widetilde{M}$ is the quotient map. Note that $\kappa : \widetilde{M} \to M$ and $\Psi$ satisfy Theorem 3.

Let $< E_1, 0 >$ be the unit element in $\Psi$. Every similarity of $\mathbb{E}^q$ $\forall q \geq 1$, different from an isometry, has a unique fixed point. Therefore, since $\Psi$ is essential, there exists an essential stationary subgroup $\Psi_a$ at some point $a \in \mathbb{E}^1$. Hence, there exists a homothety $\psi = < \lambda E_1, a > \in \Psi$ where $\lambda \neq 1$ and $\lambda > 0$. If $\lambda < 0$, then we replace $\psi$ with $\psi^2$. Consider a coordinate system on $\mathbb{E}^1$ with the origin at $a$. Then $a = 0$ and $\psi =< \lambda E_1, 0 >\in \Psi$.

*Case I:* there exists $\psi' \in \Psi$ with a fixed point different from 0. Then there exists $\psi'' = < \mu E_1, d > \in \Psi$, where $\mu > 1$ and $d \neq 0$. In this case by [18, Lem. 2], the orbit $\Psi.b$ $\forall b \in \mathbb{E}^1$ is dense in $\mathbb{E}^1$. This means that every leaf of $(M, F)$ is dense in $M$, i.e. $M$ is a minimal set.

*Case II:* there exists a unique essential stationary subgroup $\Psi_{b_0}$, $b_0 \in \mathbb{E}^1$, of $\Psi$. Without loss generality we put $b_0 = 0$, hence there exists $\psi =< \lambda E_1, 0 >\in \Psi_0$, $\lambda \in (0, 1)$. Let $\kappa : L_0 \times \mathbb{E}^1 \to M$ be the regular covering map satisfying Theorem 3. Therefore $L := \kappa(L_0 \times \{0\})$ is the unique closed leaf of $(M, F)$. It is not difficult to check that for every $x \in L$, the leaf $L^t(x)$ of $(M, F^t)$ containing $x$ intersects $L$ only at $x$, i.e. $L \cap L^t(x) = \{x\}$ $\forall x \in L$. Moreover, the map $p : M \to L$ such that $p(y) := L^t(y) \cap L$ $\forall y \in M$, is defined, and it forms a locally trivial bundle over $L$ with the standard fibre $\mathbb{R}^1$. This implies that $M$ and $L$ are homotopy equivalent. Hence, in particular, the fundamental groups $\pi_1(L, x_0)$ and $\pi_1(M, x_0)$ are isomorphic. Moreover, $(M, F)$ is the suspension foliation defined by a group homomorphism $\rho : \pi_1(L, x_0) \to Sim(\mathbb{E}^1)$. Identifying $\pi_1(L, x_0)$ and $\pi_1(M, x_0)$ by the mentioned above isomorphism we get $\rho = \chi$, where $\chi$ satisfies Theorem 3. Therefore, $\rho(\pi_1(L, x_0)) = \Psi$, where $\Psi$ is the global holonomy group of $(M, F)$.

Since $p : M \to L$ is a locally trivial bundle with the non-compact fibre $\mathbb{R}^1$, then $M$ must be non-compact in Case II.

As $L$ is a closed leaf, $M_0 := M \setminus L$ is an open dense saturated subset in $M$. Let $F_0 := F|_{M_0}$ be the induced foliation. Since all holonomy groups of $(M_0, F_0)$ are inessential, $(M_0, F_0)$ is a Riemannian foliation. The restriction $\mathfrak{M}|_{M_0}$ of an Ehresmann connection $\mathfrak{M}$ for $(M, F)$ is an Ehresmann connection for $(M_0, F_0)$.

Note that there exists a homeomorphism $h : M/F \to \mathbb{E}^1/\Psi$ of the leaf space $M/F$ of $(M, F)$ onto the orbit space $\mathbb{E}^1/\Psi$ satisfying the equality $h \circ r \circ \kappa = \tau \circ pr$, where $pr : \widetilde{M} \cong L_0 \times \mathbb{E}^1 \to \mathbb{E}^1$ is the projection, $r : M \to M/F$ and $\tau : \mathbb{E}^1 \to \mathbb{E}^1/\Psi$ are the respective quotient maps. Then the restriction $h|_{M_0/F_0} : M_0/F_0 \to (\mathbb{E}^1 \setminus \{0\})/\Psi$ is also a homeomorphism. If $\Psi$ contains a translation $\psi' =< E_1, d >$, where $d \neq 0$, then the composition $\psi' \circ \psi =< \lambda E_1, d >$ belongs to the stationary subgroup $\Psi_b$ at $b = d/(1-\lambda)$. As $d \neq 0$, it is necessary $b \neq 0$, i.e. $\Psi_b$ is essential at $b \neq 0$. That contradicts with the assumption of Case II. Thus $\Psi = \Psi_0$.

Consider a leaf $L_\alpha \subset M_0$. Let $b \in pr \circ \kappa^{-1}(L_\alpha) \in \mathbb{E}^1$, then $b \neq 0$. By Theorem 3, $\Gamma(L_\alpha, x)$ is isomorphic to the stationary subgroup $\Psi_b$. Since $\Psi = \Psi_0$, every leaf $L_\alpha \subset M_0$ is without holonomy.

$(a)$ Assume that $(M, F)$ is a proper foliation. In this case the global holonomy group $\Psi$ is a discrete subgroup of $Sim(\mathbb{E}^1)$. As $\Psi_0 \subset \mathbb{R}^+ \cdot O(1)$ where $O(1) = \{< \pm E_1, 0 >\}$, according to the statements proved above, $\Psi$ is either isomorphic to the group of integers $\mathbb{Z}$, when $\Psi$ preserves the orientation of $\mathbb{E}^1$, or to the group $\mathbb{Z}_2 \times \mathbb{Z}$ otherwise. In the both cases the closure $\overline{L_\alpha}$ of every leaf $L_\alpha \subset M_0$ is equal to $L_\alpha \cup L$.

(1) Let $(M, F)$ be proper transversely non-orientable, then $M_0/F_0$ is homeomorphic to $(\mathbb{E}^1 \setminus \{0\})/\Psi$ which is homeomorphic to $\mathbb{S}^1$. Therefore $M_0/F_0$ is also homeomorphic to $\mathbb{S}^1$. Since $(M_0, F_0)$ is without holonomy, we have the chain of diffeomorphisms $M_0 = (\widetilde{M} \setminus L_0 \times \{0\})/\widetilde{\Psi} \cong L_0 \times ((\mathbb{E}^1 \setminus \{0\})/\Psi) \cong L_0 \times \mathbb{S}^1$.

Thus, the manifold $M_0$ is diffeomorphic to the product of manifolds $L_0 \times \mathbb{S}^1$ and $(M_0, F_0)$ is isomorphic to the trivial foliation $(L_0 \times \mathbb{S}^1, F_{tr})$ where $F_{tr} = \{L_0 \times \{t\} \,|\, t \in \mathbb{S}^1\}$, in the category of foliations $\mathfrak{Fol}$.

(2) Let $(M, F)$ be proper transversely orientable, then $M_0$ has two connected components $M_0^{(i)}$, $(i = 1, 2)$, and the leaf space of each induced foliation $(M_0^{(i)}, F_0^{(i)})$ is homeomorphic to $\mathbb{R}^1$. Therefore, as above, each induced foliation $(M_0^{(i)}, F_0^{(i)})$ is isomorphic in the category $\mathfrak{Fol}$ to the trivial foliation $(L_0 \times \mathbb{S}^1, F_{tr})$ where $F_{tr} = \{L_0 \times \{t\} \,|\, t \in \mathbb{S}^1\}$.

$(b)$ Assume that $(M, F)$ is an improper foliation. In this case the closure $\overline{\Psi}$ of $\Psi$ in $Sim(\mathbb{E}^1)$ is isomorphic to $\mathbb{R}^+ \cdot \mathbb{Z}_2$ when $(M, F)$ is transversely non-orientable or isomorphic to $\mathbb{R}^+$ otherwise. Since the closure $\overline{L_\alpha}$ of a leaf $L_\alpha$ corresponds to the closure $\overline{\Psi.z}$ of the orbit of $z \in pr(\kappa^{-1}(L_\alpha))$ with respect to $\Psi$, we get the following.

(3) Let $(M, F)$ be improper transversely non-orientable, then $M_0$ is connected, and every leaf $L_\alpha \subset M_0$ is dense in $M$.

(4) Let $(M, F)$ be improper transversely orientable, then $M_0$ has two connected components $M_0^{(i)}$, $i = 1, 2$, and closure $\overline{L_\alpha}$ of every leaf $L_\alpha \subset M_0^{(i)}$ is equal to $M_0^{(i)} \cup L$.

This completes the proof of Theorem 5. □

## 6. Existence of global attractors of transversely similar foliations on compact manifolds

### 6.1. The existence of an attractor which is a minimal set

We give here a complete proof of the Theorem 6, some of whose arguments were used by us in [19], since there was a restriction on the codimension $q \geq 3$ and the proofs were found in a more general class of conformal foliations.

**Proposition 2.** *Let $(M, F)$ be a transversely similar Riemannian foliation, and it is not Riemannian. Let $q$ be the codimension of $(M, F)$, $0 < q \leq n = dim(M)$. Then:*

1) *there exists a leaf with essential holonomy group;*
2) *if $L$ is a leaf with essential holonomy group $\Gamma(L, x)$, then the closure $\mathcal{M} = \overline{L}$ in $M$ is an attractor and a minimal set, and the restriction $(Attr(\mathcal{M}), F|_{Attr(\mathcal{M})})$ is $(Sim(\mathbb{E}^q), \mathbb{E}^q)$-foliation.*

**Proof.** The statement 1) follows from Corollary 4.

2) Assume that a leaf $L = L(x)$ has an essential holonomy group $\Gamma(L, x)$. As above, $\Gamma(L, x)$ is the germ holonomy group of a leaf $L = L(x)$ consisting of germs of certain holonomic diffeomorphisms $\psi$ of a transverse $q$-dimensional disk $D_x^q$ at the point $x$ [4]. Let $\mathfrak{M}_x = T_x D_x^q$ and $D\Gamma(L, x)$ be the linear holonomy group consisting of the differentials $\psi_{*x} : \mathfrak{M}_x \to \mathfrak{M}_x$.

There exists a submersion $f : U \to V$ from an $(N, g^N)$-cocycle $\{U_i, f_i, \{k_{ij}\}\}_{i,j \in J}$ defining the foliation $(M, F)$ such that $x \in U$. Let $v = f(x) \in V$. Denote by $\mathcal{H}$ the holonomy pseudogroup generated by the local similarities $k_{ij}$, $i, j \in J$ of the transversal Riemannian manifold $(N, g^N)$. Let $\mathcal{H}_v = \{\phi \in \mathcal{H} \,|\, \phi(v) = v\}$. Denote by $[\phi]_v$ the germ at $v$ of $\phi \in \mathcal{H}_v$. Let $\widehat{\mathcal{H}_v} = \{[\phi]_v \,|\, \phi \in \mathcal{H}_v\}$ be the group of germs at $v$ of local transformations from $\mathcal{H}_v$. There exists a group homomorphism $\nu : \Gamma(L, x) \to \widehat{\mathcal{H}_v}$, $\nu([\varphi]_x) = [\phi]_v$, where $f \circ \varphi = \phi \circ f$ in some neighbourhood of the point $x$ belonging to $D_x^q$. Note that $\mathcal{H}_{v*} = \{\phi_{*v} \,|\, \phi \in \mathcal{H}_v\}$ is a subgroup of $GL(T_v N)$. The differential $f_{*x} : \mathfrak{M}_x \to T_v N$ of $f$ at $x$ induces a group isomorphism

$$\mu : D\Gamma(L, x) \to \mathcal{H}_{v*}, \;\; \mu(\varphi_{*x}) = \phi_{*v},$$

where $\nu([\varphi]_x) = [\phi]_v$. For simplicity we identify $\Gamma(L, x)$ with $\widehat{\mathcal{H}_v}$ and $D\Gamma(L, x)$ with $\mathcal{H}_{v*}$ using the group isomorphisms of $\nu$ and $\mu$ respectively. Since the holonomy group $\Gamma(L, x)$ is essential, the linear holonomy group $D\Gamma(L,x) \cong \mathcal{H}_{v*}$ of a leaf $L = L(x)$ contains an element $\phi_{*v}$ defined by a matrix of the form $\lambda \cdot A$, where $A \in O(q)$ is an element of the orthogonal subgroup $O(q)$ in the linear group $GL(q,\mathbb{R})$ and $\lambda \in (0,1)$.

Let $W_r$ be a normal neighbourhood of the origin of radius $r > 0$ in the Euclidean tangent vector space $(T_v N, g_v^N$. Denote by $\nabla^N$ the Levi-Civita of the Riemannian manifold $(N, g^N)$. The exponential map

$$\mathrm{Exp}_v : W_r \to W : X \mapsto \gamma_X(1)$$

defined by the connection $\nabla^N$ is a diffeomorphism onto an open neighbourhood $W$ of $v$ in $N$.

We assume that $\phi : D(\phi) \to R(\phi) = \phi(D(\phi))$, and $D(\phi) \subset W$, otherwise we will achieve this by reducing the neighbourhood $D(\phi)$. Let $W_0 := (\mathrm{Exp}_v)^{-1} D(\phi)$. Since $\phi_{*v}(W_r) \subset W_{\lambda r}$ for $\lambda \in (0,1)$, then $\phi_{*v}^k(W_r) \to 0$ as $k \to +\infty$. Therefore, there is $k_0 \in \mathbb{N}$ such that for all $k > k_0$, the inclusion

$$\phi^k(D(\phi)) \subset D(\phi) \subset W \tag{2}$$

is performed. By the property of the exponential map of the manifold of linear connection $(N, \nabla^N)$, the local similarity $\phi^k \in \mathcal{H}_v$ satisfies the following equality

$$\phi^k \circ \mathrm{Exp}_v = \mathrm{Exp}_v \circ \phi_{*v}^k, \quad X \in W_0, \quad k > k_0. \tag{3}$$

The relations (2) and (3) imply

$$\phi^k(D(\phi)) \to v \quad as \quad k \to +\infty. \tag{4}$$

Therefore, similar to the proof of Theorem 3, we get that the Riemannian manifold $(D(\phi), g^N)$ is conformally flat for every $q \geq 1$. Hence, we can assume that $D(\phi)$ is an open subset of the sphere $\mathbb{S}^q$ and $\phi$ is a restriction of a transformation $h \in Conf(\mathbb{S}^q)$. So $h|_W$ is a continuation of $\phi$ on $W$. Thus, without loss generality, we can assume that the similarity $\phi$ is defined on the entire normal neighbourhood of $W$ and satisfies the following equality

$$\phi \circ \mathrm{Exp}_v = \mathrm{Exp}_v \circ \phi_{*v}, \quad X \in W_0. \tag{5}$$

Introduce the notation $U_0 = f^{-1}(W) \subset U$ and

$$\mathcal{U} = \bigcup_{L_\alpha \in F,\, L_\alpha \cap U_0 \neq \varnothing} L_\alpha.$$

Consider any leaf $L_\alpha \subset \mathcal{U}$. Let $x_\alpha \in L_\alpha \cap U_0$. Then $y = f(x_\alpha) \in W$. Since the map $\mathrm{Exp}_v|_{W_r} : W_r \to W$ is a diffeomorphism, for the point $y \in W$ there exists a vector $Y = \mathrm{Exp}_v^{-1}(y) \in W_r$. From (2) it follows that $\phi^k(y) \to v$ as $k \to +\infty$. We emphasize that the set $f^{-1}(\{\phi^k(y) \,|\, k \in \mathbb{N}\})$ is contained in $L_\alpha \cap U_0$. Consequently, $L \subset \overline{L_\alpha}$. Thus it holds

$$\mathcal{M} = \overline{L} \subset \overline{L_\alpha} \quad \forall L_\alpha \subset \mathcal{U}. \tag{6}$$

In order to show that $\mathcal{M}$ is an attractor with the basin $\mathcal{U}$, it is sufficient to check that $\mathcal{U}$ is a neighbourhood of $\mathcal{M}$. Let $L_\beta$ be an arbitrary leaf of the foliation contained in $\mathcal{M}$, i.e., $L_\beta \subset \overline{L} = \mathcal{M}$. There is a submersion $f_i : U_i \to V_i$ from $(N, g^N)$-cocycle defining the foliation $(M, F)$ such that $z \in U_i \cap L_\beta$. Let $w := f_i(z) \in V_i$. It is well known that there exists a normal convex neighbourhood $W_w$ of a point $w$ in $(N, g^N)$ belonging to some normal neighbourhood of any of its points. Without loss generality we assume that $V_i \subset W_w$. Since $L_\beta \subset \overline{L}$, then $L \cap U_i \neq \varnothing$ and there exists a point $\hat{w} \in f_i(L \cap U_i)$. Hence, there exists a normal neighbourhood $W_{\hat{w}}$ such that $W_w \subset W_{\hat{w}}$. Analogously to the previous one, we get that the closure of each leaf that intersects $U_i$ contains the leaf $L$. Therefore $L \subset \overline{L_\beta}$. This implies that $L_\beta \cap U_0 \neq \varnothing$, hence $L_\beta \subset \mathcal{U}$ and $\mathcal{M} \subset \mathcal{U}$. According to the definition, $\mathcal{U}$ is a open saturated subset, then $\mathcal{U} = Attr(\mathcal{M})$, and the restriction $(Attr(\mathcal{M}), F|_{Attr(\mathcal{M})})$ is $(Sim(\mathbb{E}^q), \mathbb{E}^q)$-foliation.

Thus we have two inclusions $L_\beta \subset \overline{L}$ and $L \subset \overline{L_\beta}$. This implies the equality $\overline{L_\beta} = \overline{L}$, i.e., $\overline{L_\beta} = \mathcal{M}$ for every $L_\beta \subset \mathcal{M}$. This means that $\mathcal{M}$ is a minimal set of the foliation $(M, F)$.

Assume now that $L$ is a proper leaf. Since any nontrivial minimal set contains only improper leaves, it is necessary that $L$ is a closed leaf.

Thus Proposition 2 is proved. □

### 6.2. Transversely similar foliations on compact manifolds

**Theorem 7.** *Let $(M, F)$ be a transversely similar Riemannian foliation on a compact manifold $M$, and $(M, F)$ is not Riemannian. Let $q$ be the codimension of $(M, F)$, $0 < q \leq n = dim(M)$. Then $(M, F)$ is a $(Sim(\mathbb{E}^q), \mathbb{E}^q)$-foliation. It has a finite number of attractors, and every leaf belongs to the basin of at least one of them. Moreover, each attractor is a minimal set and equal to the closure of a leaf with essential holonomy group.*

**Proof.** According to Corollary 4, there exists a leaf with an essential holonomy group. Therefore, the union $K := \bigcup_{\beta \in B} \mathcal{M}_\beta$ of closures $\mathcal{M}_\beta$ in $M$ of all leaves with essential holonomy groups is not empty. By Proposition 2, every $\mathcal{M}_\beta$ is an attractor and a minimal set of $(M, F)$. Let $\mathcal{U}_\beta := Attr(\mathcal{M}_\beta)$. Then the union

$$\mathfrak{U} := \bigcup_{\beta \in B} \mathcal{U}_\beta$$

is an open saturated neighbourhood of $K$, and by Proposition 2, $(\mathfrak{U}, F_{\mathfrak{U}})$ is a $(Sim(\mathbb{E}^q), \mathbb{E}^q)$-foliation.

Show that the set $K$ is closed in $M$. Otherwise, there is $y \in \overline{K} \setminus K$ and a sequence $\{y_n\} \subset K$ converging to $y$ as $n \to +\infty$. For simplicity, we denote by $\mathcal{M}_n$ the minimal set containing $y_n$ and $L_n := L_n(y_n)$. According to Proposition 2, $\overline{L_n} = \mathcal{M}_n$ for each leaf $L_n \subset \mathcal{M}_n$, and there exists a leaf with essential holonomy group in $\mathcal{M}_n$. Therefore, without loss generality, we assume that the holonomy group of $L_n$ is essential. Let $L_\delta = L_\delta(y)$. There exists a submersion $f_i : U_i \to V_i$ from $(N, g^N)$-cocycle defining the foliation $(M, F)$ such that $y \in U_i$. Let $v := f_i(y) \in V_i$. It is well known that at $v$ there is a normal convex neighbourhood $W_v$ in $(N, g)$ belonging to some normal convex neighbourhood of every of its points. Without loss generality, we assume that $V_i \subset W_v$. There exists a natural number $n_0$ such that $y_n \in U_i$ and $v_n := f_i(y_n) \in V_i$ for all $n \geq n_0$. The continuity of $f_i$ implies the convergence of the sequence $v_n$ to $v$. It follows from the proof of Proposition 2 that the closure of each leaf $L_\alpha$ that intersects $U_i$, contains the closure $\overline{L_n} = \mathcal{M}_n$ for the leaf $L_n$ with an essential holonomy group. Therefore $\overline{L_n} \subset \overline{L_{n_0}}$ and $\overline{L_{n_0}} \subset \overline{L_n}$ for all $n \geq n_0$. This implies that $\mathcal{M}_n = \mathcal{M}_{n_0}$ for all $n \geq n_0$. Therefore, $L_\delta \subset \mathcal{M}_{n_0} \subset K$, hence $K$ is closed and compact in $M$.

Emphasize that according to Proposition 1, the induced foliation $(M', F_{M'})$ on the open subset $M' := M \setminus K$ is Riemannian. Using this fact, show that $\mathfrak{U} = M$. Since $M$ is connected, it sufficient to verify that $\mathfrak{U}$ is closed in $M$. Assume on the contrary, then $\partial\mathfrak{U} \neq \varnothing$. As $\partial\mathfrak{U}$ is a nonempty saturated compact subset of $M$, according to [4], there exists a minimal set $\mathcal{M} \subset \partial\mathfrak{U}$. Since $\partial\mathfrak{U} \subset M'$, every leaf $L$ belonging to $\mathcal{M}$ has inessential holonomy group. Therefore we may apply [19, Lem. 8] according to which, each open neighbourhood $\mathcal{W}$ of $\mathcal{M}$ contains an open saturated neighbourhood $\mathcal{V}$ of $\mathcal{M}$ consisting of leaves with inessential holonomy groups. The disjoint compact subsets $K$ and $\mathcal{M}$ have open disjoint neighbourhoods $U_K$ and $U_{\mathcal{M}}$. Let $U_K \subset \mathfrak{U}$, otherwise replace $U_K$ by $\mathfrak{U} \cap U_K$. According to [19, Lem. 8], there exists an open saturated neighbourhood $\mathcal{V}$ of $\mathcal{M}$ such that $\mathcal{V} \subset U_{\mathcal{M}}$. Note that $\mathcal{V} \cap U_K = \varnothing$. The inclusion $\mathcal{M} \subset \partial\mathfrak{U}$ implies existence of a point $x \in \mathfrak{U} \cap \mathcal{V}$. Let $L_\gamma := L_\gamma(x)$. As $\mathcal{V}$ is a saturated subset of the foliated manifold $M$, it is necessary that $L_\gamma \subset \mathcal{V}$. Since $x \in \mathfrak{U}$, then there is an attractor $\mathcal{M}_j \subset K$ for which $L_\gamma \subset \mathcal{U}_j = Attr(\mathcal{M}_j)$

and $L_\gamma \cap U_K \neq \emptyset$, hence $U_K \cap \mathcal{V} \neq \varnothing$. The contradiction implies that $\partial\mathfrak{U} = \varnothing$. Due to the connectivity of the topological space $M$, it is necessary that $M = \mathfrak{U}$. Thus, $(M, F)$ is a $(Sim(\mathbb{E}^q), \mathbb{E}^q)$-foliation.

The family $\xi := \{\mathcal{U}_\beta \,|\, \beta \in B\}$ is an open covering of the compact manifold $M$, and it has not subcoverings. Compactness of $M$ implies that $B$ is finite, hence there exists a finite number $k$, $k \geq 1$, of attractors $\{\mathcal{M}_m \,|\, m = 1, ..., k\}$, and $\mathcal{M}_m$ is also a minimal set of $(M, F)$. Moreover, every leaf belongs to the basin of at least one of them. □

### *6.3. The proof of Theorem 6*

*The concept of the set of ends of a manifold [10].* Let $\mathcal{K}(L)$ be the set of subsets $K$ of $L$, satisfying the following conditions:

1) a subset of $K$ is a connected compact submanifold with a boundary $\partial K$ and has a codimension of 0 in $L$;
2) the complement $L \backslash K$ does not have relatively compact connectivity components;
3) for any two distinct components $A_0$ and $A_1$ of the boundary $\partial K$ there is no a path $h$ in $L$ such that $h(0) \in A_0$, $h(1) \in A_1$ and $h(]0, 1[) \cap K = \varnothing$.

Let $\mathcal{P}(L)$ be the set of closures $P_i$ of all complements $L \backslash K$, $K \in \mathcal{K}(L)$. *An escaping sequence* is a decreasing sequence $\{P_n\}_{n\in\mathbb{N}}$ of elements $P_n \in \mathcal{P}(L)$, $P_1 \supset \ldots \supset P_n \supset P_{n+1} \supset \ldots$, such that $\bigcap_n P_n = \varnothing$. Two escaping sequences $\{P_n\}$ and $\{P'_m\}$ are called *equivalent* if for each $n$, there exists $m$ such that $P'_m \subset P_n$. The equivalent class $e = [\{P_n\}]$ is said to be an *end* of $L$. The set of all ends of $L$ is denoted by $\mathcal{E}(L)$.

*The proof of Theorem 6.* Let $(M, F)$ be a transversely similar Riemannian foliation, and it is not Riemannian. According to Theorem 7, there exists a finite number of attractors $\{\mathcal{M}_m \,|\, m = 1, ..., k\}$, where $k \in \mathbb{N}$. Let $\mathcal{U}_m := Attr(\mathcal{M}_m)$.

Assume that boundary $\partial\mathcal{U}_1$ is not empty. Since $\partial\mathcal{U}_1$ is a compact saturated subset in $M$, there exists a minimal set $\mathcal{M} \subset \partial\mathcal{U}_1$. Theorem 7 implies that $\mathcal{M} = \mathcal{M}_j$ for some $j \in \{2, ..., k\}$. As $\mathcal{M}_j \subset \partial\mathcal{U}_1$, it is necessary that $\mathcal{U}_j \cap \mathcal{U}_1 \neq \varnothing$. Since $\mathcal{M}_j \cap \mathcal{M}_1 = \varnothing$, then

$$\mathcal{V} := (\mathcal{U}_j \setminus \mathcal{M}_j) \cap (\mathcal{U}_1 \setminus \mathcal{M}_1) \neq \varnothing.$$

According to [7], for a Hausdorff paracompact manifold $M$, the union of all leaves without holonomy of a foliation $(M, F)$ is a dense $G_\delta$ subset in $M$. Therefore, there exists a leaf $L_\alpha$ without holonomy belonging to the open saturated subset $\mathcal{V}$, hence $\mathcal{M}_j \cup \mathcal{M}_1 \subset \overline{L_\alpha}$. Since $\mathcal{M}_j$ and $\mathcal{M}_1$ are disjoint compact subsets in $M$, it implies that the leaf $L_\alpha$ has at least two ends. This contradicts the condition of the proved theorem. Thus, $\partial\mathcal{U}_1 = \varnothing$. Due to the connectivity of the topological space $M$, this implies $\mathcal{U}_1 = M$, i.e. $\mathcal{M}_1$ is a global attractor. □

## 7. Examples

**Example 1.** Consider the submanifold $\widehat{M} = \mathbb{R}^n \setminus \{0_n\}$, $n \geq 3$, of $\mathbb{R}^n$, where $0_n$ is zero in $\mathbb{R}^n$. Let $(\widehat{M}, \widehat{F})$ be the simple foliation defined by the submersion

$$r : \widehat{M} \to \mathbb{R}^1, \quad (x_1, ..., x_n) \mapsto x_1.$$

Consider the homothety $\gamma = \langle \lambda E_n, 0 \rangle$ with the coefficient $\lambda > 1$, where $E_n$ is the identity matrix of the order $n$. We obtain the affine Hopf $n$-manifold $M = \widehat{M}/\Gamma$, where $\Gamma =< \gamma >$ is a group of similarity transformations

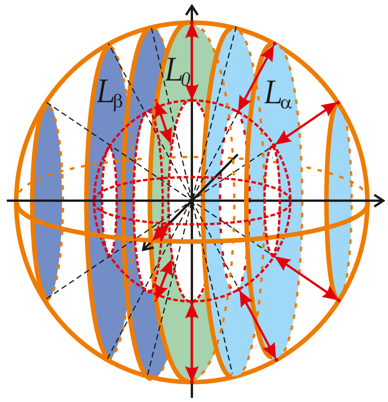


**Fig. 2.** $n = 3$.

of $\widehat{M}$. Note that $M \cong \mathbb{S}^{n-1} \times \mathbb{S}^1$ is compact. Since $\gamma(r^{-1}(x_1)) = r^{-1}(\lambda x_1)$ for every $x_1 \in \mathbb{R}^1$, the group $\Gamma$ preserves the foliation $(\widehat{M}, \widehat{F})$ formed by connected components of fibres of $r : \widehat{M} \to \mathbb{R}^1$. Therefore, on the manifold $M = \widehat{M}/\Gamma$, the foliation $(M, F)$ is induced such that its leaves are images of the leaves of the foliation $(\widehat{M}, \widehat{F})$ under the universal covering $\kappa : \widehat{M} \to M = \widehat{M}/\Gamma$. Consequently the foliation $(M, F)$ is covered by the simple foliation $(\widehat{M}, \widehat{F})$, and the both these foliations are transversally affine. More precisely, $(M, F)$ is a transversely similar foliation. Note that the leaf $L_0 = \kappa(r^{-1}(0))$ is a unique compact leaf. If $n = 3$, it is diffeomorphic to the 2-dimensional torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, all the other leaves of $(M, F)$ are diffeomorphic to the plane $\mathbb{R}^2$, see Fig. 2. For $n > 3$, the leaf $L_0$ is diffeomorphic to $\mathbb{S}^{n-2} \times \mathbb{S}^1$, and all the other leaves are diffeomorphic to $\mathbb{R}^{n-1}$. Emphasize that $(M, F)$ is a proper foliation, and the closed leaf $L_0$ is its global attractor. Note that every leaf without holonomy has only one end.

Suppose that the foliation $(M, F)$ has an Ehresmann connection. Then by Theorem 3, it is covered by a locally trivial bundle $p : \widehat{M} \to \mathbb{R}^1$, whose fibres are all diffeomorphic to each other. The fibre $p^{-1}(0)$ is diffeomorphic to $\mathbb{R}^{n-1} \setminus \{0_n\}$ and any other fibre is diffeomorphic to $\mathbb{R}^{n-1}$. The contradiction shows that $(M, F)$ does not admit an Ehresmann connection. Therefore, the existence of an Ehresmann connection is not a necessary condition for the existence of a global attractor for transversely affine foliation.

Emphasize that $(M, F)$ satisfies Theorem 6.

**Example 2.** Consider a plane without zero $\widehat{M} = \mathbb{R}^2 \setminus \{0_2\}$. Let $(\widehat{M}, \widehat{F})$ be the simple foliation defined by the submersion

$$r : \widehat{M} \to \mathbb{R}^1, \quad (x_1, x_2) \mapsto x_1.$$

Consider the homothety $\gamma = \langle \lambda E_2, 0 \rangle$ with the coefficient $\lambda > 1$ of $\widehat{M}$. We obtain the affine Hopf 2-manifold $M = \widehat{M}/\Gamma$, where $\Gamma =< \gamma >$ is a group of similarity transformations of $\widehat{M}$. Note that $M \cong \mathbb{S}^1 \times \mathbb{S}^1$ is diffeomorphic to the torus $\mathbb{T}^2$. Since the group $\Gamma$ preserves the foliation $(\widehat{M}, \widehat{F})$ formed by connected components of fibres of $r : \widehat{M} \to \mathbb{R}^1$, then on the manifold $M = \widehat{M}/\Gamma$, the foliation $(M, F)$ is induced, and its leaves are images of the leaves of the foliation $(\widehat{M}, \widehat{F})$ under the regular covering $\kappa : \widehat{M} \to M = \widehat{M}/\Gamma$. Consequently the foliation $(M, F)$ is covered by the regular foliation $(\widehat{M}, \widehat{F})$, and the both these foliations are transversely affine. More precisely, $(M, F)$ is a transversely similar foliation. As $\kappa(r^{-1}(0))$ has two components, it is the union of two compact leaves $L_1$ and $L_2$ diffeomorphic to the circle $\mathbb{S}^1$. Emphasize that $(M, F)$ is a proper transversely similar foliation having two closed leaves $L_1$ and $L_2$ which are attractors. Every other leaf $L_\alpha \cong \mathbb{R}^1$ is without holonomy and has two ends, see Fig. 3.

Note that we may consider the union $L_1 \cup L_2$ as a global attractor of $(M, F)$, but it is not a minimal set.

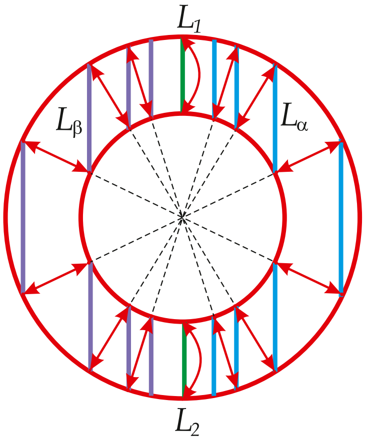


**Fig. 3.** $n = 2$.

**Example 3.** Let us construct an example using a suspension of a group homomorphism. The construction of suspension foliations was represented by Haefliger (see, for example, [18]).

Let $\mathbb{E}^q$ be the $q$-dimensional Euclidean space and let $\mathbb{E}^k$, $2 \le k < q$ be its $k$-dimensional subspace. Let $e_i$, $i = \overline{1,k}$, be a basis of vector space $\mathbb{R}^k$ corresponding to $\mathbb{E}^k$. Consider a subgroup $\Psi_k$ of the similarity group $Sim(\mathbb{E}^q)$ generated by transformations $\psi_j$, $j = \overline{1,k+1}$, where $\psi_j(z) := z + e_j$ for $j = \overline{1,k}$ and $\psi_{k+1}(z) := \lambda \cdot z$ for some number $\lambda \in (0,1)$ and all $z \in \mathbb{E}^q$. Note that $\Psi_k$ is essential group of homotheties, and, according to [18, Prop. 16], $\mathcal{M} := \mathbb{E}^k$ is a minimal set and a global attractor of $\Psi_k$.

Denote by $B_{k+1}$ the smooth two-dimensional sphere with $k+1$ handles. The fundamental group of $B_{k+1}$ is equal to

$$G_k := \pi_1(B_{k+1}, b) = < a_i, b_i \,|\, i = \overline{1,k+1};\ a_1 b_1 a_1^{-1} b_1^{-1} ... a_{k+1} b_{k+1} a_{k+1}^{-1} b_{k+1}^{-1} > .$$

Define the group homomorphism $\rho_k : G_k \to Sim(\mathbb{E}^q)$ by setting it on generators: $\rho_k(a_i) := \psi_i, i = \overline{1,k+1}$, $\rho_k(b_i) := Id_{\mathbb{E}^q}$ where $Id_{\mathbb{E}^q}$ is the identical map of $\mathbb{E}^q$.

Let $h_k : \mathbb{R}^2 \to B_{k+1}$ be the universal covering map. For a fixed point $\hat{b} \in h_k^{-1}(b)$, the right action of the group $G_k$ on $\mathbb{R}^2$ as a group of deck transformations of $h_k$ is defined. Denote it by $(y,g) \mapsto y.g$, $(y,g) \in \mathbb{R}^2 \times G_k$. Then the map

$$\Phi_k : G_k \times \mathbb{R}^2 \times \mathbb{E}^q, \ \ \Phi(g,y,z) := (y.g, \rho_k(g^{-1})) \ \ \forall (g,y,z) \in G_k \times \mathbb{R}^2 \times \mathbb{E}^q,$$

defines a free proper discontinuous action of $G_k$ on $\mathbb{R}^2 \times \mathbb{E}^q$. Therefore, the quotient manifold $M_k := \mathbb{R}^2 \times_{G_k} \mathbb{E}^q$ and the suspension foliation $(M_k, F_k) := Sus(\mathbb{E}^q, B_{k+1}, \rho_k)$ are defined. The foliation $(M_k, F_k)$ is covered by the trivial bundle $r : \mathbb{R}^2 \times \mathbb{E}^q \to \mathbb{E}^q$ and has the global holonomy group $\Psi_k$. By properties of a suspension foliation, the $(q+2)$-dimensional manifold $M_k$ is the space of a locally trivial bundle $p_k : M_k \to B_{k+1}$ with a standard fibre $\mathbb{E}^q$ over the base $B_{k+1}$. Therefore $M_k$ is not compact.

Thus, we get a transversely similar foliation $(M_k, F_k)$ of codimension $q$, $q \ge 2$, with a regular minimal set $\mathcal{M}_k := f_k(r^{-1}(\mathbb{E}^k))$, and $\mathcal{M}_k$ is a global attractor and a minimal set of $(M_k, F_k)$. According to [18, Th. 9], $\mathcal{M}_k$ and $M_k$ are homotopy equivalent.

Let $M_0^{(k)} := M_k \setminus \mathcal{M}_k$ and $L_\alpha$ be an arbitrary leaf in $M_0^{(k)}$. Emphasize that the induced foliation $(M_0^{(k)}, F_{M_0^{(k)}})$ is an improper Riemannian foliation without holonomy, admitting an Ehresmann connection. According to Theorem 3, the closure $\overline{L_\alpha}$ is equal to $\mathfrak{L}_\alpha \cup \mathcal{M}_k$, where $\mathfrak{L}_\alpha$ is the closure of $L_\alpha$ in $M_0^{(k)}$. Pick $z_\alpha \in r(f_k^{-1}(L_\alpha)) \subset \mathbb{E}^q$. Applying [18, Lem. 5] it is easy to show that $\mathfrak{L}_\alpha = f_k(\mathbb{R}^2 \times \overline{\Psi}.z_\alpha)$ where $\overline{\Psi}$ is the closure of $\Psi$ in the Lie group $Sim(\mathbb{E}^q)$, hence $\overline{\Psi} = (< \psi_{k+1} > \cdot \{E_q\}) \ltimes \mathbb{E}^k$. Therefore, $\mathfrak{L}_\alpha$ is a smooth $(k+2)$-dimensional embedding submanifold in $M_k$. Thus, $\overline{L_\alpha}$ is the union of two $(k+2)$-dimensional embedding submanifolds $\mathfrak{L}_\alpha$ and $\mathcal{M}_k$ in $M_k$.

Foliations $(M_k, F_k)$, $k \in \mathbb{N}$, satisfy Theorems 2 and 3.

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