On topological classification of Morse–Smale diffeomorphisms on the sphere S^n ($n > 3$)

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Abstract

We consider the class $G(S^n)$ of orientation preserving Morse–Smale diffeomorphisms of the sphere S^n of dimension $n > 3$, assuming that invariant manifolds of different saddle periodic points have no intersection. For any diffeomorphism $f \in G(S^n)$, we define a coloured graph Γ_f that describes a mutual arrangement of invariant manifolds of saddle periodic points of the diffeomorphism f. We enrich the graph Γ_f by an automorphism P_f induced by dynamics of f and define the isomorphism notion between two coloured graphs. The aim of the paper is to show that two diffeomorphisms $f, f' \in G(S^n)$ are topologically conjugated if and only if the graphs Γ_f, Γ'_f are isomorphic. Moreover, we establish the existence of a linear-time algorithm to distinguish coloured graphs of diffeomorphisms from the class $G(S^n)$.

(Some figures may appear in colour only in the online journal)

1. Introduction and the statement of results

The problem of a topological classification of dynamical systems takes its origin in papers of A Andronov, L Pontryagin, E Leontovich, A Mayer, and M Peixoto. In 1937, A Andronov and L Pontryagin introduced the notion of a roughness of dynamical systems and showed that necessary and sufficient conditions of the roughness of a flow on the two-dimensional sphere are finiteness of the non-wandering orbits set, its hyperbolicity, and the absence of trajectories joining two saddle equilibria or going from a saddle to the same saddle. In 1960, S Smale

Figure 1. Morse−Smale diffeomorphisms on S^3 with wild closures of separatrices

introduced a similar class of dynamical systems on manifolds of an arbitrary dimension and transform the condition of the absence of trajectories between two saddle equilibria into more general conditions of transversality of the intersection of equilibria invariant manifolds and periodic orbits. Later, such systems were called *Morse−Smale systems*.

The finiteness of the set of non-wandering orbits leads to an idea of reducing the problem of topological classification of Morse−Smale systems to a combinatorial problem of a description of the mutual arrangement of such orbits and their invariant manifolds in the ambient manifolds. First time, this approach was applied by E Leontovich and A Mayer for classification of flows on the one-dimensional sphere with a finite set of singular orbits. It was developed in papers of M Peixoto, A Oshemkov, V Sharko, Y Umanskii, S Pilyugin, where a similar problem was solved for Morse−Smale flows on closed manifolds with dimension 2, 3, and higher. It was also developed by V Grines and A Bezdenezhnych for Morse−Smale diffeomorphisms with a finite number of heteroclinic orbits on surfaces[1].

It turned out that this idea in general does not work in the case of diffeomorphisms on three-dimensional manifolds. It is related to the fact that the closures of the invariant manifolds of hyperbolic saddle points may be wild at one point[2]. Figure 1(a) illustrates the phase portrait of a Morse−Smale cascade with the non-wandering set consisting of four fixed points $\alpha, \sigma, \omega_1, \omega_2$, the closure of the stable manifold, and the closure of one of the unstable separatricies of the saddle point σ being the wild sphere and the wild arc, respectively. Figure 1(b) illustrates the phase portrait of a Morse−Smale cascade, such that all closures of the separatricies of its saddle points are locally flat, but the union of the one-dimensional separatrices of the points σ_1, σ_2 together with the sink ω (the union of the two-dimensional separatrices of σ_1, σ_2 together with the source α) forms a wild frame of arcs (a wild bouquet of the 2-spheres).

First examples of wild spheres and arcs were constructed by J W Alexander (1924), E Artin and R Fox (1948). D Pixtion (1977), C Bonatti and V Grines (2000) constructed Morse−Smale cascades with the wild object being the closures of saddle invariant sets, see [1, 27]. This fact called for a new language for construction of topological invariants. A complete topological classification of Morse−Smale diffeomorphisms on three-dimensional manifolds was obtained in a series of papers by C Bonatti, V Grines, F Laudenbach, O Pochinka, E Pecou, and V Medvedev (see the reviews [3, 11] for references). A new invariant called *the scheme of a*

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[1] A non-empty intersection of invariant manifolds of different saddle periodic points is called *a heteroclinic intersection*, an isolated point of such intersection is called a *heteroclinic point*, and its orbit is called a *heteroclinic orbit*.

[2] A manifold $N^k \subset M^n$ of dimension k without boundary is *locally flat at a point* $x \in N^k$ if there exists a neighbourhood $U(x) \subset M^n$ of x and a homeomorphism $\varphi : U(x) \to \mathbb{R}^n$, such that $\varphi(N^k \cap U(x)) = \mathbb{R}^k$, where $\mathbb{R}^k = \{(x_1, \ldots, x_n) \in \mathbb{R}^n |\ x_{k+1} = x_{k+2} = \cdots = x_n = 0\}$. If the condition of local flatness fails at a point $x \in N^k$, then the manifold N^k is called *wild* and x is called *a point of wildness*.

diffeomorphism includes topological structures of the orbit space of action of the diffeomorphism on some wandering set and embedding of projections of the invariant manifolds in this orbit space.

Surprisingly, it turned out that for some classes of Morse–Smale cascades on manifolds with dimensions greater than three, a complete invariant can be borrowed from the two-dimensional case. Due to papers of J Cantrell and C Edwards [6–8] (see also [10], proposition 3A.6), arcs and spheres of the codimension one in a manifold M^n of dimension $n > 3$ either are locally flat at each point or the set of their points of wildness is more than countable. So, if we consider the class of Morse–Smale diffeomorphisms on a manifold M^n of dimension $n > 3$ under the assumption that all saddle periodic points have invariant manifolds of dimension either one or $(n - 1)$ and do not contain heteroclinic intersections, then we can state that all closures of such invariant manifolds are locally flat arcs and spheres. Moreover, we can say that the unions of the closures of one-dimensional separatrices form tame (not wild) frames. These facts give us a hope that a topological invariant for such diffeomorphisms must be more simple than for three-dimensional case. Indeed, topological techniques developed in the papers of Cantrell and Edwards allowed to V Grines, E Gurevich and V Medvedev to obtain a topological classification of Morse–Smale diffeomorphisms on manifolds of dimension $n \geqslant 4$ in combinatorial terms under the assumption that all unstable manifolds of saddle periodic points of such diffeomorphisms have the dimension one and the wandering set does not contain heteroclinic orbits ([12, 13]). The key fact, which shows the difference between the dimension three and greater than three, is proposition 2.10. Roughly speaking, it states that for any two homotopic simple closed curves in a compact manifold M^n of dimension $n \geqslant 4$ there exists a homeomorphism of M^n that sends one of the curves to the other one. The trefoil knot is a simple example that this fact is not true in the dimension three.

In this paper, we consider the class $G(S^n)$ of preserving orientation Morse–Smale diffeomorphisms on the sphere S^n ($n \geqslant 4$), such that the stable and unstable manifolds of different saddle periodic points of any diffeomorphism from $G(S^n)$ have no intersections. We provide a combinatorial invariant called *a coloured graph*, which we describe below after an exposition of some necessary facts. Let us remark that an attempt to generalize the results of [12, 13] leads to new difficulties. In fact, we know everything about one-dimensional separatricies but cannot see that codimension one separatrices also are trivially embedded, in particular, that they do not form tame bouquets of spheres. In section 4.2, we show that there is a duality between embedding of one-dimensional and codimension one separaticies that finally allows to obtain a topological classification of diffeomorphisms of $G(S^n)$ in combinatorial terms.

Let Ω_f be the non-wandering set of a diffeomorphism $f \in G(S^n)$ and $\Omega_f^i = \{p \in \Omega_f \mid \dim W_p^u = i\}$, $i \in \{0, 1, \ldots, n\}$. We prove in proposition 2.1 that for $f \in G(S^n)$ the sets Ω_f^i are empty, for $i \in \{2, \ldots, n-2\}$.

Let $p \in \Omega_f^1$. It follows from [29, theorem 2.3] that the closure $\mathrm{cl}\, W_p^s$ of its invariant manifold W_p^s contains, apart the W_p^s itself, exactly one periodic point and this point is a source α. Then, the set $\mathrm{cl}\, W_p^s$ is homeomorphic to the sphere of dimension $(n - 1)$ and W_p^s is smoothly embedded at all points, except, possibly, the point α. J Cantrell proved in [6] that the $(n - 1)$-dimensional sphere $S^{n-1} \subset S^n$ cannot have one point of wildness (that contrasts with the case $n = 3$). Hence, the sphere $\mathrm{cl}\, W_p^s$ is locally flat at every point, and, due to the generalized Schoenflies theorem (see [5, theorem 4] and [4, theorem 5]), cuts the ambient sphere S^n into two connected components, each closure of which is a ball.

Denote by $\mathcal{L}_f$ the union of all the spheres $\{\mathrm{cl}\, W_p^s, p \in \Omega_f^1\}$ and the spheres $\{\mathrm{cl}\, W_q^u, q \in \Omega_f^{n-1}\}$, and put $k_f = |\Omega_f^1 \cup \Omega_f^{n-1}|$ (here $|X|$ means the cardinality of a set X). Since each $(n - 1)$-sphere from the set $\mathcal{L}_f$ cuts the sphere S^n into two connected components, the set
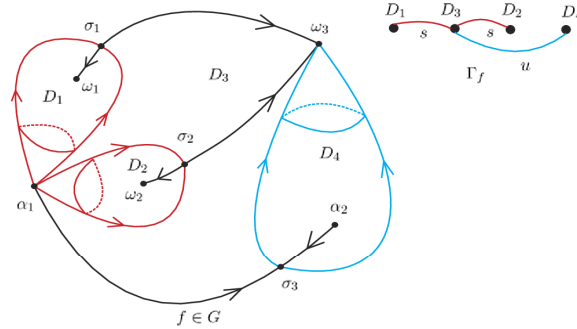
Figure 2. The phase portrait of a diffeomorphism $f \in G(S^n)$ and its coloured graph Γ_f

$S^n \backslash (\bigcup_{p \in \Omega^1_f} W^s_p \cup \bigcup_{q \in \Omega^{n-1}_f} W^u_q)$ consists of $k_f + 1$ connected components $D_1, \ldots, D_{k_f+1}$. Denote by $\mathcal{D}_f$ the set of all these components.

Definition 1.1. The coloured graph of a diffeomorphism $f \in G(S^n)$ is a graph Γ_f, defined as follows:

(a) the set $V(\Gamma_f)$ of vertices of the graph Γ_f is isomorphic to the set $\mathcal{D}_f$, the set $E(\Gamma_f)$ of edges of the graph Γ_f is isomorphic to the set $\mathcal{L}_f$;

(b) the vertices v_i and v_j are incident to an edge $e_{i,j}$ if and only if the corresponding domains D_i and D_j have a common boundary;

(c) an edge $e_{i,j}$ has the colour s (respectively, u) if it corresponds to a manifold cl $W^s_p \subset \mathcal{L}_f$ (cl $W^u_q \subset \mathcal{L}_f$).

Figure 2 illustrates the phase portrait of a diffeomorphism $f \in G(S^n)$ with the non-wandering set, consisting of eight points: the sources α_1, α_2, the saddles $\sigma_1, \sigma_2, \sigma_3$, the sinks $\omega_1, \omega_2, \omega_3$, and the coloured graph Γ_f of the diffeomorphism f.

Denote by $\xi : V(\Gamma_f) \rightarrow \mathcal{D}_f$ an arbitrary isomorphism and define an automorphism $P_f : V(\Gamma_f) \rightarrow V(\Gamma_f)$ by $P_f = \xi^{-1} f \xi|_{V(\Gamma_f)}$.

Definition 1.2. The graphs Γ_f and $\Gamma_{f'}$ of diffeomorphisms $f \in G(S^n)$ and $f' \in G(S^n)$ are isomorphic if there exists an isomorphism $\zeta : V(\Gamma_f) \rightarrow V(\Gamma_{f'})$, preserving colours of edges, such that $P_{f'} = \zeta P_f \zeta^{-1}$.

Theorem 1. *Diffeomorphisms $f \in G(S^n)$ and $f' \in G(S^n)$ are topologically conjugated if and only if their graphs Γ_f and $\Gamma_{f'}$ are isomorphic.*

We show in proposition 2.4 that for any diffeomorphism $f \in G(S^n)$ the coloured graph Γ_f is a tree, that will play the key role in the proof of theorem 1. In [14], it is shown that for any tree Γ, arbitrary coloured in two colours and enriched with arbitrary colours preserving automorphism P, there exists a diffeomorphism $f \in G(S^n)$, whose graph Γ_f is isomorphic to Γ by means of an isomorphism $\theta : \Gamma \rightarrow \Gamma_f$, such that $\theta P = P_f \theta$.

The following result shows that the coloured graph is the most effective invariant for classification of diffeomorphisms from the class $G(S^n)$, because there exists an optimal linear-time algorithm for recognizing such graphs up to isomorphism.

Theorem 2.   *Let Γ_f and $\Gamma_{f'}$ be the graphs of diffeomorphisms $f, f' \in G(S^n)$ with the same number k of vertices. Then, there exists an algorithm for checking the existence of an isomorphism between Γ_f and $\Gamma_{f'}$ in $O(k)$ time[3], provided Γ_f and $\Gamma_{f'}$ are stored by the adjacency lists, i.e. all neighbours are listed, for any vertex of the graphs.*

The structure of the paper is the following. In section 2, we list all auxiliary results, necessary for proof of theorem 1, next we provide a proof of theorem 1 in section 2.4. Proofs of the auxiliary results, required special techniques, are given in sections 4.1 and 4.2. The proof of theorem 2, presented in section 5, can be read independently from the other sections.

2. Auxiliary results

2.1. The structure of the non-wandering set of a diffeomorphism $f \in G(S^n)$

In the paper [15], the following proposition was obtained. We outline its proof for the possibility of an independent reading of the present paper.

Let us recall that *the Morse index* of a hyperbolic periodic point p is the number, equal to the dimension of the manifold W_p^u.

Proposition 2.1.   *Let $f \in G(S^n)$. Then, the set of its saddle periodic points consists only on points with the Morse indices, equal to 1 and $(n-1)$.*

Proof.   Suppose that there exists a point $\sigma \in \Omega_f$, such that dim $W_\sigma^u = j, 1 < j < n-1$. Since W_σ^u, W_σ^s does not intersect invariant manifolds of any saddle periodic point p different from σ, the closures cl W_σ^u and W_σ^s of the stable and unstable manifolds of σ are spheres of dimensions j and $n-j$, correspondingly. The spheres $S^j = W_\sigma^u$ and $S^{n-j} = W_\sigma^s$ intersect transversely at the single point σ. Therefore, their intersection index equals either 1 or -1, depending on the choice of orientation of S^j, S^{n-j} and S^n. Since the homology groups $H_j(S^n)$ and $H_{n-j}(S^n)$ are trivial, then there is a sphere $\tilde{S}^j$, homological to the sphere S^j and having the empty intersection with the sphere S^{n-j}. Hence, the intersection index of spheres $\tilde{S}^j, S^{n-j}$ is zero. As the intersection index is the homology invariant, the intersection index of the spheres S^j and S^{n-j} must be also equal to zero (see, for example, [32], section 69). This contradiction proves the statement. $\diamond$

2.2. Linearizing neighbourhood and canonical manifolds connected, with hyperbolic periodic points

Define a homeomorphism $b_\nu : \mathbb{R}^n \to \mathbb{R}^n, \nu \in \{+1, -1\}$ by

$$ b_\nu(x_1, x_2, \ldots, x_n) = \left(\nu 2 x_1, \frac{1}{2} x_2, \ldots, \frac{1}{2} x_{n-1}, \nu \frac{1}{2} x_n \right). $$

The origin O is the unique fixed point of b_ν, and it is a hyperbolic saddle point. The stable manifold W_O^s coincides with the hyperplane $x_1 = 0$, and the unstable manifold W_O^u coincides with the Ox_1-axe.

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[3] That is, the computation complexity of the algorithm is a linear function in k.

Put

$$\mathbb{U} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2(x_2^2 + \cdots + x_n^2) \leqslant 1\}, \quad \mathbb{U}_0 = \{(x_1, \ldots, x_n)| \ x_1 = 0\},$$

$$\mathbb{N}^u = \mathbb{U} \backslash \mathbb{U}_0, \mathbb{N}^s = \mathbb{U} \backslash Ox_1, \mathbb{K}_\nu^{n-1} = \mathbb{U}_0/_{b_\nu}, \widehat{\mathbb{N}}_\nu^s = \mathbb{N}^s/_{b_\nu}, \widehat{\mathbb{N}}_\nu^u = \mathbb{N}^u/_{b_\nu},$$

and denote by $p_{b_\nu}^s : \mathbb{N}_\nu^s \to \widehat{\mathbb{N}}_\nu^s$, $p_{b_\nu}^u : \mathbb{N}_\nu^u \to \widehat{\mathbb{N}}_\nu^u$ the canonical projections.

Recall that *an n-ball* (*n-disk*) is a manifold B^n homeomorphic to the unit ball

$$\mathbb{B}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + \cdots + x_n^2 \leqslant 1\}, \quad n \geqslant 1.$$

An open n-ball (*n-disk*) is a manifold homeomorphic to the interior of $\mathbb{B}^n$, and *a sphere* S^{n-1} is a manifold homeomorphic to the boundary $\mathbb{S}^{n-1}$ of the ball $\mathbb{B}^n$. The sphere S^1 is also called *the knot*.

It is easy to show that the manifold $\mathbb{K}_{+1}^{n-1}$ is homeomorphic to $\mathbb{S}^{n-2} \times \mathbb{S}^1$. We call the manifold $\mathbb{K}_{-1}^{n-1}$ *the standard generalized* $(n-1)$-*dimensional Klein Bottle* and a manifold homeomorphic to $\mathbb{K}_{-1}^{n-1}$ *the generalized Klein Bottle*. A canonical projection $p_{b_{-1}}^s|_{\mathbb{U}_0} : \mathbb{U}_0 \to \mathbb{K}_{-1}^{n-1}$ induces on $\mathbb{K}_{-1}^{n-1}$ a structure of non-oriented fibre bundle over $\mathbb{S}^1$ with a fibre $\mathbb{S}^{n-2}$. Hence, K_{-1}^{n-1} is a non-oriented manifold. Since $\mathbb{U}_0$ is the universal cover for $\mathbb{K}_\nu^{n-1}$, the fundamental group $\pi_1(\mathbb{K}_\nu^{n-1})$ is isomorphic to $\mathbb{Z}$ (see [24, corollary 19.4]).

We call the manifold $\widehat{\mathbb{N}}_{-1}^s$ *the canonical neighbourhood of the generalized Klein Bottle* $\mathbb{K}_{-1}^{n-1}$.

The proposition below follows directly from the definition of $\widehat{\mathbb{N}}_\nu$.

Proposition 2.2.

(a) $\widehat{\mathbb{N}}_{+1}^u$ *consists of two connected components, each of which is diffeomorphic to the direct product* $\mathbb{B}^{n-1} \times \mathbb{S}^1$.

(b) $\widehat{\mathbb{N}}_{-1}^u$ *is diffeomorphic to* $\mathbb{B}^{n-1} \times \mathbb{S}^1$.

(c) $\widehat{\mathbb{N}}_{+1}^s$ *is diffeomorphic to* $\mathbb{K}_{+1}^{n-1} \times [-1, 1]$.

(d) $\widehat{\mathbb{N}}_{-1}^s$ *is a tubular neighbourhood of a zero section of non-orientable one-dimensional vector bundle over* $\mathbb{K}_-^{n-1}$, *the boundary* $\partial \widehat{\mathbb{N}}_-$ *of* $\widehat{\mathbb{N}}_-$ *is diffeomorphic to* $\mathbb{S}^{n-2} \times \mathbb{S}^1$, *and if* $i_* : \pi_1(\partial \widehat{\mathbb{N}}_-) \to \pi_1(\widehat{\mathbb{N}}_-)$ *is a homomorphism induced by an inclusion, then* $\eta_{\widehat{\mathbb{N}}_-}(i_*(\pi_1(\partial \widehat{\mathbb{N}}_-))) = 2\mathbb{Z}$.

On figure 3, the neighbourhoods $\mathbb{N}^s$ and $\mathbb{N}^u$ and the fundamental domains

$$\widetilde{N}^s = \{(x_1, \ldots, x_n) \in \mathbb{N}^s | \frac{1}{4} \leqslant x_2^2 + \cdots + x_n^2 \leqslant 1\},$$

$$\widetilde{N}^u = \{(x_1, \ldots, x_n) \in \mathbb{N}^u | |x_1| \in [1, 2]\}$$

of action of the diffeomorphism b_{+1} on them are shown. Put

$$\mathcal{C} = \left\{ \{(x_1, \ldots, x_n) \in \mathbb{R}^n | \frac{1}{4} \leqslant x_2^2 + \cdots + x_n^2 \leqslant 1\}. \right.$$

The set $\mathbb{N}^s$ is the union of hyperplanes

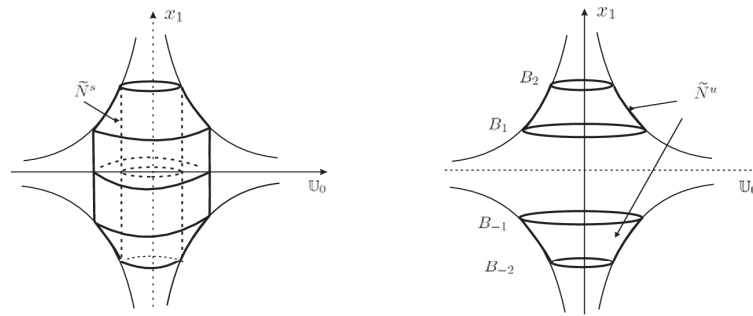$$\mathcal{L}_t = \{(x_1, \ldots, x_n) \in N^s | x_1^2(x_2^2 + \cdots + x_n^2) = t^2\}, \quad t \in [-1, 1].$$

Figure 3. The fundamental domains $\widetilde{N}^s$ and $\widetilde{N}^u$ of action of the diffeomorphism b_{+1} on the sets $\mathbb{N}^s$ and $\mathbb{N}^u$

Then, the fundamental domain $\widetilde{N}^s_{+1}$ is the union of pairs of annuli $\mathcal{K}_t = \mathcal{L}_t \cap \mathcal{C}, t \in [-1, 1]$ and the space $\widehat{\mathbb{N}}^s_{+1}$ can be obtained from $\widetilde{N}^s$ by gluing connected components of the boundary of each annulus by means of the diffeomorphism b_{+1}. The set $\widetilde{N}^u$ consists of two connected components, each of which is homeomorphic to $\mathbb{B}^{n-1} \times [0, 1]$. The space $\widehat{\mathbb{N}}^u_{+1}$ can be obtained from $\widetilde{N}^u$ by gluing the disk $B_1 = \{(x_1, \ldots, x_n) \in \mathbb{N}^u | x_1 = 1\}$ with the disk $B_2 = \{(x_1, \ldots, x_n) \in \mathbb{N}^u | x_1 = 2\}$ and the disk $B_{-1} = \{(x_1, \ldots, x_n) \in \mathbb{N}^u | x_1 = -1\}$ with the disk $B_{-2} = \{(x_1, \ldots, x_n) \in \mathbb{N}^u | x_1 = -2\}$ by means of b_{+1}.

The fundamental domain of action of the diffeomorphism b_{-1} on the set $\mathbb{N}^u$ is the set $\widetilde{N}^u_{-1} = \{(x_1, \ldots, x_n) \in \mathbb{N}^u | |x_1| \in [1, 4]\}$. The space $\widehat{\mathbb{N}}^u_{-1}$ can be obtained from $\widetilde{N}^u_{-1}$ by gluing the disk $B_1 = \{(x_1, \ldots, x_n) \in \mathbb{N}^u | x_1 = 1\}$ with the disk $B_4 = \{(x_1, \ldots, x_n) \in \mathbb{N}^u | x_1 = 4\}$ by means of b_{-1}. The structure of vector bundle on the space $\widehat{\mathbb{N}}^s_{-1}$ is defined by the natural projection of the one-dimensional foliation of the set $\mathbb{N}^s$ by straight lines parallel to the Ox_1-axis. This bundle is non-orientable as a wind along a loop $p^s_{b_\nu}(l_\nu)$, where l_ν is the segment of $0x_1$ with the endpoints $(0, \ldots, 0, 1)$ and $(0, \ldots, 1/2)$, induces a reversing orientation map.

Now came back to diffeomorphisms from the class $G(S^n)$. For a saddle point $\sigma \in \Omega^1_f \cup \Omega^{n-1}_f$, denote by m_σ its period.

We say that a saddle point σ of period m_σ has *an orientation type* $\nu_\sigma = +1$ ($\nu_\sigma = -1$) if the restriction $f^{m_\sigma}|_{W^u_\sigma}$ preserves (reverses) an orientation of W^u_σ.

Due to [18, theorem 2.1.2] (see also [16, proposition 4.3]), the following proposition holds.

Proposition 2.3.   *For any diffeomorphism $f \in G(S^n)$, there exists a set of pairwise disjoint f-invariant neighbourhoods $\{N_\sigma, \sigma \in \Omega^1_f \cup \Omega^{n-1}_f\}$, such that for any N_σ there exists a homeomorphism $\chi_\sigma : N_\sigma \to \mathbb{U}$, such that*

(a) *if $\sigma \in \Omega^1_f$, then $\chi_\sigma f^{m_\sigma}|_{N_\sigma} = b_{\nu_\sigma} \chi_\sigma|_{N_\sigma}$,*
(b) *if $\sigma \in \Omega^{n-1}_f$, then $\chi_\sigma f^{m_\sigma}|_{N_\sigma} = b^{-1}_{\nu_\sigma} \chi_\sigma|_{N_\sigma}$.*

We call the neighbourhood N_σ as *a linearizing neighbourhood*.

2.3. The scheme S_f of a diffeomorphism $f \in G(S^n)$

In this subsection, we introduce the notion of *a scheme* of a diffeomorphism $f \in G(S^n)$, which is an effective tool for studying the dynamics. Moreover, due to [15, theorem 1], the scheme is a complete invariant in the class $G(S^n)$, so the problem of topological classification is reduced to

a proof of the fact that diffeomorphisms, having isomorphic coloured graphs, have equivalent schemes too. This fact will be given in lemma 2.3. In section 3, we adduce an adaptation of the proof of [15, theorem 1] for the class $G(S^n)$ to provide the possibility of an independent reading of this paper.

Represent the sphere S^n as the union of pairwise disjoint sets

$$A_f = (\bigcup_{\sigma \in \Omega_f^1} W_\sigma^u) \cup \Omega_f^0, R_f = (\bigcup_{\sigma \in \Omega_f^{n-1}} W_\sigma^s) \cup \Omega_f^n, V_f = S^n \backslash (A_f \cup R_f).$$

Similar to [20], one can prove that the sets A_f, R_f, V_f are connected, the set A_f is an attractor, the set R_f is a repeller, and V_f consists of the wandering orbits of f moving from R_f to A_f.

Denote by $\widehat{V}_f = V_f / f$ the orbit space of the action of f on V_f, by $p_f : V_f \to \widehat{V}_f$ the natural projection. In virtue of [30] (theorem 3.5.7 and proposition 3.6.7), p_f is a covering map and the space $\widehat{V}_f$ is a manifold.

The following lemma is proved in section 4.2.

Lemma 2.1. *Let $f \in G(S^n)$. Then, $\widehat{V}_f$ is homeomorphic to the direct product $\mathbb{S}^{n-1} \times \mathbb{S}^1$.*

Notice that for $n = 3$ lemma 2.1 is not true in general (see, for example, [19, section 5]).
Denote by

$$\eta_f : \pi_1(\widehat{V}_f) \to \mathbb{Z}$$

a homomorphism defined in the following way. Let $\hat{c} \subset \widehat{V}_f$ be a loop non-homotopic to zero in $\widehat{V}_f$ and $[\hat{c}] \in \pi_1(\widehat{V}_f)$ be a homotopy class of $\hat{c}$. Choose an arbitrary point $\hat{x} \in \hat{c}$, denote by $p_f^{-1}(\hat{x})$ the complete inverse image of $\hat{x}$, and fix a point $\tilde{x} \in p_f^{-1}(\hat{x})$. As p_f is the covering map, then there is an unique path $\tilde{c}(t)$ beginning at the point $\tilde{x}$ ($\tilde{c}(0) = \tilde{x}$) and covering the loop c, such that $p_f(\tilde{c}(t)) = \hat{c}$. Then, there exists the element $n \in \mathbb{Z}$, such that $\tilde{c}(1) = f^n(\tilde{x})$. Set

$$\eta_f([\hat{c}]) = n.$$

Since V_f is simply connected, it is the universal cover for $\widehat{V}_f$ and the homomorphism η_f is an isomorphism.

Set

$$\hat{L}_f^s = \{p_f(W_p^s \backslash p), p \in \Omega_f^1\}), \quad \hat{L}_f^u = \{p_f(W_q^u \backslash q), q \in \Omega_f^{n-1}\}.$$

For a periodic saddle point $\sigma \in \Omega_f^1$ ($\sigma \in \Omega_f^{n-1}$), set

$$l_\sigma = W_\sigma^s \backslash \sigma \, (l_\sigma = W_\sigma^u \backslash \sigma), \hat{l}_\sigma = p_f(l_\sigma), \hat{N}_\sigma = p_f(N_\sigma \cap V_f),$$

so $\hat{l}_\sigma$ is an element of the set $\hat{L}_f^s \cup \hat{L}_f^u$.

Due to [24, theorem 5.5] (see also [2, proposition 1.2.3]) the composition $\hat{\chi}_\sigma = p_f \chi_\sigma^{-1}(p_{b_\nu}^s)^{-1}$ define a homeomorphism from $\mathbb{N}_{\nu_\sigma}^s$ to $\hat{N}_\sigma$, such that $\hat{\chi}_\sigma(\mathbb{K}_{\nu_\sigma}^{n-1}) = \hat{l}_\sigma$ and the following corollary of the proposition 2.3 holds. Denote by $\hat{\chi}_{\sigma*} : \pi_1(\mathbb{K}_{\nu_\sigma}^{n-1}) \to \pi_1(\widehat{V}_f)$ the homomorphism induced by $\hat{\chi}_\sigma$.

Corollary 2.1. $\eta_f \hat{\chi}_{\sigma*}(\pi_1(\mathbb{K}_{\nu_\sigma}^{n-1})) = m_\sigma \mathbb{Z}.$

Definition 2.1. The collection

$$S_f = (\widehat{V}_f, \hat{L}_f^s, \hat{L}_f^u, \eta_f)$$

is called *the scheme* of a diffeomorphism $f \in G(S^n)$.

Definition 2.2. The schemes S_f and $S_{f'}$ of diffeomorphisms f and f' are called *equivalent* if there exists a homeomorphism $\hat{\varphi} : \widehat{V}_f \to \widehat{V}_{f'}$, such that $\hat{\varphi}(\hat{L}_f^s) = \hat{L}_{f'}^s$, $\hat{\varphi}(\hat{L}_f^u) = \hat{L}_{f'}^u$, and $\eta_f = \eta_{f'}\hat{\varphi}_*$.

2.4. Interrelation between the scheme and the coloured graph

In this subsection, we recall some notions of graph theory, establish properties of the coloured graphs of diffeomorphisms from the class $G(S^n)$ and their interrelation with schemes that play a key role in the proof of theorem 1.

Recall that a loop-less graph without multiple and oriented edges is said to be *simple*. A *tree* is a simple, connected, cycle-free graph. A vertex, incident to exactly one edge, is said to be *a pendant vertex or a leaf*.

Proposition 2.4. *The coloured graph Γ_f of a diffeomorphism $f \in G(S^n)$ is a tree.*

Proof. By definition, any edge e of the graph Γ_f corresponds to an $(n-1)$-dimensional sphere, which cuts the ambient sphere S^n into two connected components. Then the edge e cuts the graph Γ_f into two connected components, so the graph Γ_f does not contain cycles. Now we are left to prove that the graph is connected. It is well-known that a connected graph with $k+1$ vertices is a tree if and only if it has k edges. If the graph Γ_f is not connected, then it consists of connected subgraphs $\Gamma_1, \ldots, \Gamma_m, m \geqslant 2$. Then adding $(m-1)$ edges transforms the disjoint union of Γ_i to a connected graph without cycles (i.e., a tree) with $k_f + 1$ vertices and $k_f + m$ edges, contradicting to the mentioned property of a tree. So, the graph Γ_f is connected and it does not contain cycles. Hence, it is a tree. $\square$

Recall that we denoted by $V(\Gamma_f)$ and $E(\Gamma_f)$ the sets of vertices and edges of the graph Γ_f, correspondingly. Denote by $uv \subset E(\Gamma_f)$ an edge, connecting vertices $v, u \in V(\Gamma_f)$.

We associate with the graph Γ_f a sequence

$$\Gamma_{f,0}, \Gamma_{f,1}, \ldots, \Gamma_{f,s}$$

of trees, such that $\Gamma_{f,0} = \Gamma_f$, $\Gamma_{f,s}$ contains one or two vertices and, for any $i \in \{1, \ldots, s\}$, a tree $\Gamma_{f,i}$ is obtained from $\Gamma_{f,i-1}$ by deletion of all its leaves. All the vertices of $\Gamma_{f,s}$ are called the *central vertices* of the tree Γ_f, and if $\Gamma_{f,s}$ has an edge, then it is called *the central edge* of the tree Γ_f. The tree Γ_f is said to be *central*, if it has exactly one central vertex, and *bicentral*, otherwise. *The rank of a vertex $x \in V(\Gamma_f)$, denoted by* rank(x), *is the number* $\max\{i | x \in V(\Gamma_{f,i})\}$. It follows from the definition that if vw is a non-central edge, then $|\text{rank}(v) - \text{rank}(w)| = 1$, and the central vertices of a bicentral tree have the same rank. On the figure 4 two types of graphs: central and bicentral with marked ranks of vertices, and corresponding sequences of trees are shown.

An *automorphism P of a tree Γ* is a bijective self-map of $V(\Gamma)$ that keeps the adjacency, i.e.

$$\forall\, u, v \in V(\Gamma)\ [uv \in E(\Gamma) \Leftrightarrow P(u)P(v) \in E(\Gamma)].$$

An automorphism P_f can be represented as the product of cyclic sub-permutations, and the set $V(\Gamma_f)$ can be decomposed into subsets, invariant under the sub-permutations, which are said to be *orbits*. Clearly, every orbit of P_f consists of the same rank vertices of Γ_f and if the tree is central (bicentral), then its central vertex (central edge) stay fixed under any automorphism.
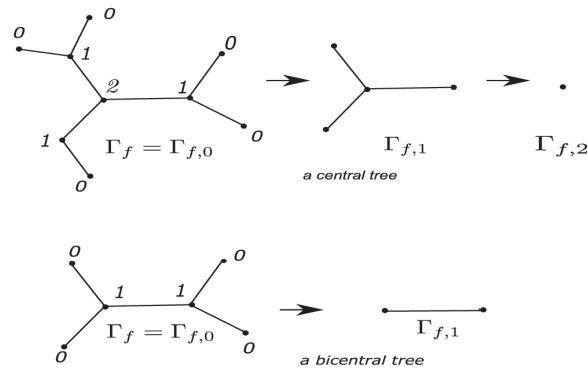
Figure 4. Central and bicentral trees.

Orbits O_1 and O_2 of P_f will be called *neighbour*, if there are adjacent vertices $x_1 \in O_1$ and $y_1 \in O_2$, such that $|\text{rank}(x_1) - \text{rank}(y_1)| = 1$.

Proposition 2.5. *Let v_1 and w_1 be vertices of the graph Γ_f, where $\text{rank}(v_1) - \text{rank}(w_1) = 1$, and $O_1 = (v_1, v_2, \ldots, v_p)$ and $O_2 = (w_1, w_2, \ldots, w_q)$ be orbits of v_1, w_1, correspondingly, such that $P_f(v_i) = v_{i+1}$, $P_f(w_j) = w_{j+1}$, taking the indices modulo p and q, correspondingly. Then the following properties are true:*

(a) p divides q;

(b) for any $i \in \{1, 2, \ldots, p\}$, the set of neighbours of v_i, belonging to O_2, coincides with $\{w_i, w_{i+p}, w_{i+2p}, \ldots, w_{i+(\frac{q}{p}-1)\cdot p}\}$;

(c) all the edges, simultaneously incident to a vertex O_1 and to a vertex of O_2, have the same colour.

Proof. Assume that p does not divide q. Since $v_1 w_1 \in E(\Gamma_f)$ and P_f is an automorphism of Γ_f, we have $v_i w_i \in E(\Gamma_f)$ for any $i \in \{1, \ldots, q\}$. Therefore $P_f(v_q)P_f(w_q) = v_{q+1}w_1 \in E(\Gamma_f)$. As p does not divide q, then $q + 1$ modulo p is not equal to 1. Hence, the edges $v_1 w_1$ and $v_{q+1}w_1$ are distinct. Hence, the graph Γ_f must contain a cycle, as there are two distinct paths from w_1 to the central vertices of Γ_f. We have a contradiction. Hence, $q = l \cdot p$, where l is a natural number.

Since $v_1 w_1 \in E(\Gamma_f)$ and P_f is an automorphism, we have $v_i w_i \in E(\Gamma_f)$ for any $i \in \{1, \ldots, p\}$. From this fact and as $q = l \cdot p$ it follows that for any $j \in \{1, \ldots, q\} v_i w_j$ is an edge if and only if $j \equiv i(\text{mod } p)$. Therefore the set of all neighbours of v_i, belonging to O_2, coincides with $\{w_i, w_{i+p}, w_{i+2p}, \ldots, w_{i+q-p}\}$.

As $P_f(v_1)P_f(w_1) = v_2 w_2$, then the edges $v_1 w_1$ and $v_2 w_2$ have the same colour and it is also true for all the edges, simultaneously incident to a vertex O_1 and to a vertex of O_2. $\square$

The automorphism P_f naturally induces a map of the set $E(\Gamma_f)$, which we also will denote by P_f. Proposition 2.5 immediately leads to corollary 2.2.

Corollary 2.2. *Let $vw \in E(\Gamma_f)$ be a non-central edge and $\text{rank}(v) < \text{rank}(w)$. Then the period of the edge vw equals to the period of the vertex v.*

Two trees Γ and Γ' are said to be *isomorphic*, if there is a bijection $\xi : V(\Gamma) \to V(\Gamma')$, called an *isomorphism*, such that $\forall u, v \in V(\Gamma)[uv \in E(\Gamma) \Leftrightarrow \xi(u)\xi(v) \in E(\Gamma')]$.
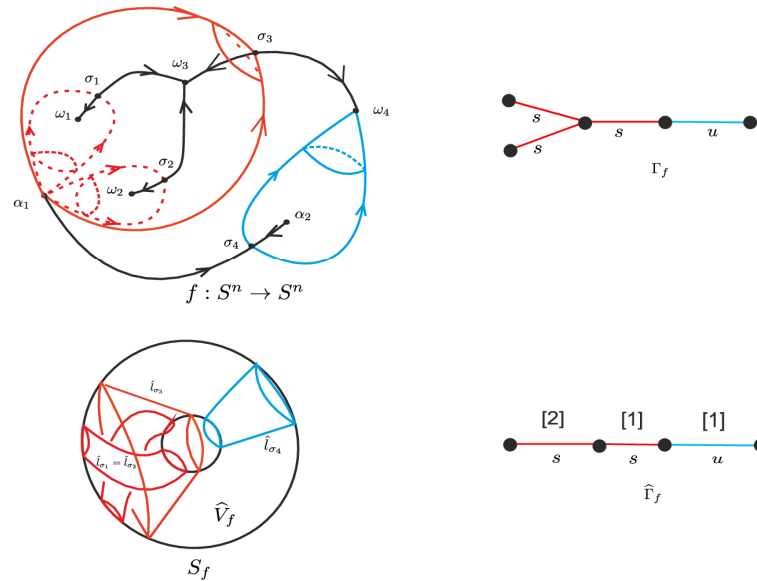
Figure 5. The phase portrait of a diffeomorphism $f \in G(S^n)$, its coloured graph Γ_f, weighted graph $\widehat{\Gamma}_f$, and elements of the scheme S_f.

Obviously, under any isomorphism of Γ, any its vertex must be mapped into a same rank vertex.

For any $f \in G(S^n)$, we define *a weighted graph* $\widehat{\Gamma}_f$ in the following way: (1) glue vertices in the graph Γ_f, belonging to the same orbit of the automorphism P_f, and the corresponding edges; (2) enrich each edge of the graph $\widehat{\Gamma}_f$ with a weight equal to the period of the corresponding separatrix of the diffeomorphism f. It follows from proposition 2.5 that this gluing operation keeps colours of edges. If the tree Γ_f is bicentral and the central vertices generate a period 2 orbit, then $\widehat{\Gamma}_f$ has the unique loop, corresponding to the central edge of Γ_f. Otherwise, the graph $\widehat{\Gamma}_f$ is a tree with the same central vertices as Γ_f. In both cases we will say that a vertex $\hat{v} \in \widehat{\Gamma}_f$ has the rank k if the rank of the corresponding vertex $v \in \Gamma_f$ equals k (figure 5).

The observation above immediately leads to the following proposition.

Proposition 2.6. *Let f and f' be two diffeomorphisms from $G(S^n)$ with isomorphic associated coloured graphs $\Gamma_f, \Gamma_{f'}$. Then, the weighted graphs $\widehat{\Gamma}_f$ and $\widehat{\Gamma}_{f'}$ are also isomorphic by means of an isomorphism, preserving weights of edges.*

Denote by $\widehat{\mathcal{D}}_f$ the set of connected components of the set $\widehat{V}_f \backslash (\hat{L}_f^u \cup \hat{L}_f^s)$. Since deleting 0- and one-dimensional set from a manifold with a dimension at most three does not change the number of connected components, there is a natural one-to-one correspondence ξ_* between the sets $E(\Gamma_f)$, $V(\Gamma_f)$ and $\mathcal{L}_f \cap V_f$, $\mathcal{D}_f \cap V_f$, such that $\xi_* P_f = f \xi_*$. This fact immediately provides the following proposition.

Proposition 2.7. *There is one-to-one correspondence $\hat{\xi} : E(\widehat{\Gamma}_f) \cup V(\widehat{\Gamma}_f) \to (\hat{L}_f^u \cup \hat{L}_f^s) \cup \widehat{\mathcal{D}}_f$, such that $\hat{\xi}(E(\widehat{\Gamma}_f)) = \hat{L}_f^u \cup \hat{L}_f^s$, $\hat{\xi}(V(\widehat{\Gamma}_f)) = \widehat{\mathcal{D}}_f$.*

Proposition 2.8. *The set $\Omega_f^1 \cup \Omega_f^{n-1}$ contains at most one saddle point of the negative orientation type. If such a point σ exists, then its period equals 1, the manifold $\hat{l}_\sigma$ is homeomorphic to the generalized Klein Bottle, and the $(n-1)$-dimensional invariant manifold of σ corresponds to the loop of the graph $\widehat{\Gamma}_f$.*

Proof. Suppose that the set Ω_f^1 contains a point σ of the negative orientation type and the period m_σ. To consider the case, when a point of the negative orientation type belongs to Ω_f^{n-1} one can replace f with f^{-1}. Due to proposition 2.3, there exists a neighbourhood N_σ and the homeomorphism $\chi_\sigma : N_\sigma \to \mathbb{U}$, such that $f^{m_\sigma}|_{N_\sigma} = \chi_\sigma^{-1} b_- \chi_\sigma|_{N_\sigma}$. Then, one-dimensional separatrices of σ both have the period $2m_\sigma$. The closure of the stable manifold of the point σ is a locally flat $(n-1)$-sphere cutting the ambient sphere S^n on two open n-balls B_1 and B_2, each of which contains an one-dimensional separatrix of σ. Hence, $f^{m_\sigma}(B_1) = B_2$ and $f^{m_\sigma}(B_2) = B_1$. Therefore, if we delete from the graph Γ_f the edge e_σ corresponding to the manifold W_σ^s, we get two graphs isomorphic by means of $P_f^{m_\sigma}$, so the edge e_σ is central. Since any tree has at most one central edge, there are at most one saddle point of the negative orientation type. $\square$

For $\widehat{\Gamma}_f$, we construct a sequence $\widehat{\Gamma}_{f,0} = \widehat{\Gamma}_f, \widehat{\Gamma}_{f,1}, \dots, \widehat{\Gamma}_{f,s}$ by consequence deleting verities of the smallest rank and the edges incident to them, similar to the sequence $\Gamma_{f,0}, \Gamma_{f,1}, \dots, \Gamma_{f,s}$. These two sequences consist of the same number of elements, since $\widehat{\Gamma}_f$ is obtained from Γ_f by gluing vertices and edges of the same orbits of P_f and this operation preserves ranks as we have defined.

For $i \in \{0, \dots, s-1\}$, set $\Delta_{f,i} = \hat{\xi}(\widehat{\Gamma}_f \backslash \widehat{\Gamma}_{f,i+1})$. If the graph Γ_f is bicentral, we define the set $\Delta_{f,s}$ depending on two possible cases: (1) the graph $\widehat{\Gamma}_f$ has the central edge e_*; (2) $\widehat{\Gamma}_f$ has a loop. In the first case, choose an arbitrary connected component $\widehat{\Gamma}_{f,*}$ of the graph $\widehat{\Gamma}_f \backslash e_*$ and set $\Delta_{f,s} = \hat{\xi}(\widehat{\Gamma}_{f,*} \cup e_*)$ (see figure 6). In the second case, there exists a saddle point $\sigma_* \in \Omega_f^1 \cup \Omega_f^{n-1}$ with the negative orientation type. In this case set $\Delta_{f,s} = \mathrm{cl}(\widehat{V}_f \backslash \mathrm{int}\, \widehat{N}_{\sigma_*})$, where N_{σ_*} is the linearizing neighbourhood of the point σ_*.

Proposition 2.9. *The chain of inclusions $\Delta_{f,0} \subset \mathrm{int}\, \Delta_{f,1} \subset \cdots \subset \mathrm{int}\, \Delta_{f,s-1}$ is true. If the graph $\widehat{\Gamma}_f$ is either central or has a loop, then $\Delta_{f,s-1} \subset \mathrm{int}\, \Delta_{f,s}$. Otherwise $\mathrm{int}\, \Delta_{f,s} \cap \Delta_{f,i} \neq \emptyset$, $i \in \{0, \dots, s-1\}$.*

Proof. Let $\sigma_* \in \Omega_f^1 \cup \Omega_f^{n-1}$ be a point of negative orientation type and an edge $e_{\sigma_*} \in E(\widehat{\Gamma}_f)$ corresponds to the manifold $\hat{l}_{\sigma_*} \subset \widehat{V}_f$. Then e_{σ_*} is a loop and the graph $\widehat{\Gamma}_f \backslash e_{\sigma_*}$ is connected. Hence, due to an existence of an isomorphism $\hat{\xi} : E(\widehat{\Gamma}_f) \cup V(\widehat{\Gamma}_f) \to (\hat{L}_f^s \cup \hat{L}_f^u) \cup \widehat{\mathcal{D}}_f$, the set $\widehat{V}_f \backslash \hat{l}_{\sigma_*}$, as well as the set $\widehat{V}_f \backslash \widehat{N}_{\sigma_*}$, is connected. More over, for any vertex $v \in V(\widehat{\Gamma}_f \backslash e_{\sigma_*})$ and any edge $e \in V(\widehat{\Gamma}_f \backslash e_{\sigma_*})$ the sets $\hat{\xi}(v)$ and $\hat{\xi}(e)$ belongs to $\widehat{V}_f \backslash \widehat{N}_{\sigma_*}$. Then, for the case when $\widehat{\Gamma}_f$ has a loop, the inclusion $\Delta_{f,i} \subset \mathrm{int}\, \Delta_{f,s}$ holds for any $i \in \{0, \dots, s-1\}$.

Now, let $\sigma \in \Omega_f^1 \cup \Omega_f^{n-1}$ be a point of positive orientation type and an edge $e_\sigma \in E(\widehat{\Gamma}_f)$ corresponds to the manifold $\hat{l}_\sigma \subset \widehat{V}_f$. Then the edge e_σ is incident exactly to two different vertices $v_{\sigma,+}, v_{\sigma,-}$ and cuts $\widehat{\Gamma}_f$ into two connected components $\widehat{\Gamma}_+, \widehat{\Gamma}_-, v_{\sigma,+} \subset \widehat{\Gamma}_+, v_{\sigma,-} \subset \widehat{\Gamma}_-$. Therefore the set $\widehat{V}_f \backslash \hat{l}_\sigma$ consists of two connected components $\hat{D}_{\sigma,+} = \xi(\widehat{\Gamma}_+), \hat{D}_{\sigma,-} = \xi(\widehat{\Gamma}_-)$.

Suppose $0 < \mathrm{rank}(v_{\sigma,-}) < \mathrm{rank}(v_{\sigma,+})$. Then a connected component of the set $\widehat{\Gamma}_f \backslash \widehat{\Gamma}_{f,\mathrm{rank}(v_{\sigma,+})}$ containing the edge e_σ is exactly $\widehat{\Gamma}_-$ and $\hat{D}_{\sigma,-}$ is a connected component of the set $\Delta_{f,\mathrm{rank}(v_{\sigma,-})}$.

Let $\sigma' \subset \Omega_f^1 \cup \Omega_f^{n-1}$ be a point such that the corresponding edge $e_{\sigma'}$ belongs to $\widehat{\Gamma}_-$. We determine for the point σ' all objects similar to ones for the point σ. Then the connected component
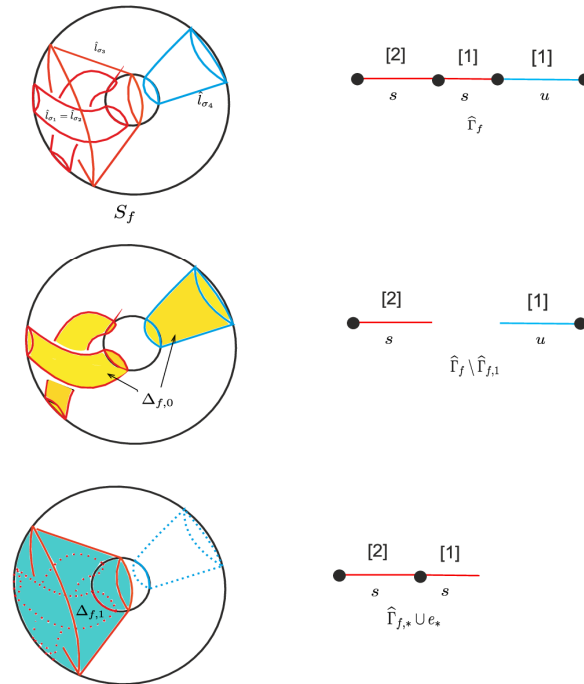
Figure 6. The scheme S_f, the graph $\widehat{\Gamma}_f$ and the elements $\Delta_{f,0}, \Delta_{f,1}$.

of the set $\Delta_{f,\mathrm{rank}(v_{\sigma',-})}$ containing $e_{\sigma'}$ belongs to $\hat{D}_{\sigma,-}$. It proves the inclusion $\Delta_{f,i} \subset \mathrm{int}\,\Delta_{f,i+1}$ for any $i \in \{0,\ldots,s-1\}$. In case $\mathrm{rank}(v_{\sigma,-}) = \mathrm{rank}(v_{\sigma,+}) = s$ we fix a connected component of the set $\widehat{\Gamma}_f \backslash e_\sigma$ and then apply similar arguments to prove that there are connected components of the set $\Delta_{f,s-1}$ that belongs to the interior of $\Delta_{f,s}$. □

The following lemma is proved in section 4.2.

Lemma 2.2. *Let $f \in G(S^n)$. Then every connected component of the set $\Delta_{f,i}$ is homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$.*

Now, to prove that the existence of an isomorphism between the coloured graphs Γ_f and $\Gamma_{f'}$ leads to the equivalence between the schemes S_f and $S_{f'}$, we need one more auxiliary proposition.

Let M^n be a topological manifold, possibly, with a non-empty boundary ∂M^n, $\beta \subset M^n$ be a knot, and $g: \mathbb{B}^{n-1} \times \mathbb{S}^1 \to M^n$ be a topological embedding, such that $g(\{O\} \times \mathbb{S}^1) = \beta$. The image $N_\beta = g(\mathbb{B}^{n-1} \times \mathbb{S}^1)$ is called *a tubular neighbourhood* of the knot β.

The following proposition is a slight modification of [16, proposition 5.1, 5.3]. We prove it in section 4.1.

Proposition 2.10. *Suppose that $n \geqslant 4$, $\mathbb{P}^{n-1}$ is either the ball $\mathbb{B}^{n-1}$ or the sphere $\mathbb{S}^{n-1}$. Let $\{\beta_i\}, \{\beta'_i\} \subset \mathrm{int}\,\mathbb{P}^{n-1} \times \mathbb{S}^1$ be two families of pairwise disjoint knots, such that β_i, β'_i are*

homotopic, $i \in \{1, \ldots, k\}$ and $\{N_{\beta_i}\}, \{N_{\beta'_i}\} \subset \mathbb{P}^{n-1} \times \mathbb{S}^1$ are pairwise disjoint tubular neighbourhoods of $\{\beta_i\}, \{\beta'_j\}$. Then, there exists a homeomorphism $h : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$, such that $h(\beta_i) = \beta'_i, h(N_{\beta_i}) = N_{\beta'_i}, i \in \{1, \ldots, k\}$, and if $\mathbb{P}^n = \mathbb{B}^n$, then $h|_{\partial \mathbb{P}^{n-1} \times \mathbb{S}^1} = id.$

Corollary 2.3. *Under the assumptions in proposition* 2.10, *let* $\tilde{h} : \partial \mathbb{B}^{n-1} \times \mathbb{S}^1 \to \partial \mathbb{B}^{n-1} \times \mathbb{S}^1$ *be a homeomorphism. Then, there exists a homeomorphism* $h : \mathbb{B}^{n-1} \times \mathbb{S}^1 \to \mathbb{B}^{n-1} \times \mathbb{S}^1$, *such that* $h(\beta_i) = \beta'_i, h(N_{\beta_i}) = N_{\beta'_i}, i \in \{1, \ldots, k\}$ *and* $h|_{\partial \mathbb{B}^{n-1} \times \mathbb{S}^1} = \tilde{h}.$

Lemma 2.3. *If the coloured graphs* Γ_f *and* $\Gamma_{f'}$ *of diffeomorphisms* $f, f' \in G(S^n)$ *are isomorphic, then the schemes* S_f *and* $S_{f'}$ *are equivalent.*

Proof. Any isomorphism ζ between the graphs Γ_f and $\Gamma_{f'}$ induces an isomorphism $\hat{\zeta} : \widehat{\Gamma}_f \to \widehat{\Gamma}_{f'}$ that preserves the weights and ranks of edges. Then, for any $i \in \{0, \ldots, s\}$, the isomorphism $\hat{\zeta}$ induces an one-to one correspondence ζ_i between the connected components of the sets $\Delta_{f,i}$ and $\Delta_{f',i}$. If the graph $\widehat{\Gamma}_f$ is bicentral, then there are two ways to choose the set $\Delta_{f,s}$, but, since the graphs $\widehat{\Gamma}_f$ and $\widehat{\Gamma}_{f'}$ are isomorphic, it is possible to choose the sets $\Delta_{f,s}$ and $\Delta_{f',s}$ in such a way that for any pair of connected components $P_{f,i} \subset \Delta_{f,i}, P_{f,j} \subset \Delta_{f,j}$, such that $P_{f,i} \subset \text{int} P_{f,j}$, the inclusion $\zeta_i(P_{f,i}) \subset \text{int} \zeta_j(P_{f,j})$ holds, $i \in \{0, \ldots, s-1\}, j \in \{1, \ldots, s\}$, $i < j$.

The following two cases are possible: (1) there are no saddle points of the negative orientation type; (2) there is a saddle point of the negative orientation type.

Let us consider the case (1). Without loss of generality, assume that the graph $\widehat{\Gamma}_f$ is bicentral. For the case, where the graph is central, the arguments are similar.

It follows from lemma 2.2 and proposition 2.10 that there exists a homeomorphism $\psi_s : \widehat{V}_f \to \widehat{V}_{f'}$, such that $\psi_s(\Delta_{f,s}) = \Delta_{f',s}$. If $s = 0$, then the proof is complete and $\hat{\varphi} = \psi_s$.

Let $s > 0$. Denote the images of the sets $\Delta_{f,0}, \ldots, \Delta_{f,s-1}$ under the homeomorphism ψ_s by the same symbols as their originals.

Due to corollary 2.3 from proposition 2.10, there exists a homeomorphism $\psi_{s-1} : \widehat{V}_{f'} \to \widehat{V}_{f'}$, such that $\psi_{s-1}|_{\widehat{V}_{f'} \setminus \text{int} \Delta_{f',s}} = id, \psi_{s-1}(P_{f,s-1}) = \zeta_i(P_{f,s-1})$, for any connected component $P_{f,s-1} \in \Delta_{f,s-1}$. If $s = 1$, then the proof is complete and $\hat{\varphi} = \psi_{s-1}\psi_s$. Otherwise, continue the process and after a finite number of steps we get the desired homeomorphism $\hat{\varphi}$.

In the case (2) denote by $\sigma_* \in \Omega_f, \sigma'_* \in \Omega_{f'}$ the points of the negative orientation type. Due to corollary 2.1, there exists a homeomorphism $\psi_* : \hat{N}_{\sigma_*} \to \hat{N}_{\sigma'_*}$. Due to lemma 2.2, the manifold $\text{cl}(\widehat{V}_f \setminus \hat{N}_{\sigma_*})$ is homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$. Then, using corollary 2.3 similar to the case (1), it is possible to continue the homeomorphism ψ_* up to the desired homeomorphism of $\widehat{V}_f$. $\qquad \square$

3. Topological classification of diffeomorphisms from $G(S^n)$

It is clear that if diffeomorphisms $f, f' \in G(S^n)$ are topologically conjugated, then their coloured graphs are isomorphic. This section is devoted to prove the inverse fact.

Let f and f' be diffeomorphisms from $G(S^n)$ with isomorphic coloured graphs Γ_f and Γ'_f. According to lemma 2.3, the schemes S_f and $S_{f'}$ of diffeomorphisms are equivalent, that is there exists a homeomorphism $\hat{\varphi} : \widehat{V}_f \to \widehat{V}_{f'}$, such that $\hat{\varphi}(\hat{L}^s_f) = \hat{L}^s_{f'}, \hat{\varphi}(\hat{L}^u_f) = \hat{L}^u_{f'}$, and $\eta_f = \eta_{f'}\hat{\varphi}_*$.

Now we start to prove that the diffeomorphisms f and f' are topologically conjugated. The homeomorphism $\hat{\varphi} : \widehat{V}_f \to \widehat{V}_{f'}$ induces a homeomorphism $\varphi : V_f \to V_{f'}$, such that $f|_{V_f} = \varphi^{-1} f' \varphi|_{V_{f'}}$ and, for any saddle point $\sigma \in \Omega^1_f$ ($\sigma \in \Omega^{n-1}_f$), there is a point

$\sigma' \in \Omega^1_{f'}$ ($\sigma' \in \Omega^{n-1}_{f'}$), such that $\varphi(W^s_\sigma \setminus \sigma) = W^s_{\sigma'} \setminus \sigma'$ ($\varphi(W^u_\sigma \setminus \sigma) = W^u_{\sigma'} \setminus \sigma'$). The homeomorphism φ extends univocally to the set Ω_f.

Let us extend the homeomorphism φ to the sets A_f and R_f. First, construct a conjugating homeomorphisms H_1 and H_{n-1} on the linearizing neighbourhoods $\{N_\sigma\}$ of saddle points from sets Ω^1_f and Ω^{n-1}_f that coincide with φ on the boundaries of the linearizing neighbourhoods. Then, the desired homeomorphism $H : S^n \to S^n$ conjugating f and f' will be defined by

$$H(x) = \begin{cases} \varphi(x), & x \in S^n \setminus \displaystyle\bigcup_{\sigma \in \Omega^1_f \cup \Omega^{n-1}_f} N_\sigma, \\ H_\delta(x), & x \in N_\sigma, \sigma \in \Omega^\delta_f, \delta \in \{1, n-1\}. \end{cases}$$

To built the homeomorphism H_1, we choose one point in each index-1 saddle orbit and denote the obtained set by $\widetilde{\Omega}^1_f$. Due to proposition 2.3, there exists a family of pairwise disjoint neighbourhoods $\{N_\sigma\}$ ($\{N'_\sigma\}$) of points from $\widetilde{\Omega}^1_f$ ($\widetilde{\Omega}^1_{f'}$) and the homeomorphisms $\chi_\sigma : N_\sigma \to \mathbb{U}$ ($\chi_{\sigma'} : N_{\sigma'} \to \mathbb{U}$) conjugating the restriction of the diffeomorphism f^{m_σ} ($f'^{m_{\sigma'}}$) on the set N_σ ($N_{\sigma'}$) with the diffeomorphism $b_\nu|_{\mathbb{U}}$.

Set

$$\mathbb{U}_\tau = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2(x_2^2 + \cdots + x_n^2) \leqslant \tau^2\}, \quad \tau \in (0, 1].$$

Recall that we set $\mathbb{U}_0 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}$. Let us define the homeomorphisms $\varphi^u_\sigma : \mathbb{U}_0 \to \mathbb{U}_0$, $\psi : \mathbb{U} \to \mathbb{U}$ by

$$\varphi^u_\sigma = \chi_{\sigma'} \varphi \chi^{-1}_\sigma|_{\mathbb{U}_0}, \Psi(x_1, x_2, \ldots, x_n) = (x_1, \varphi^u_\sigma(x_2, \ldots, x_n)).$$

Set $N^\tau_\sigma = \chi^{-1}_\sigma(\mathbb{U}_\tau)$. Choose $\tau \in (0, 1]$, such that the map $\psi : N^\tau_\sigma \to N_{\sigma'}$ defined by $\psi(x) = \chi^{-1}_{\sigma'} \Psi \chi_\sigma|_{N^\tau_\sigma}$ would be a well-defined topological embedding and $\psi(N^\tau_\sigma \setminus W^u_\sigma) \subset \varphi(N^\tau_\sigma \setminus W^u_\sigma)$.

Define a topological embedding $\theta_\sigma : N^\tau_\sigma \to N_\sigma$ by $\theta = \varphi^{-1}\psi$. Since $\theta|_{W^u_\sigma} = id$, it follows from [18, corollary 4.3.2] that there exist $0 < \tau_1 < \tau$ and a homeomorphism $\Theta : N_\sigma \to N_\sigma$ coinciding with θ on the set $N^{\tau_1}_\sigma$ and identical on ∂N_σ.

Define homeomorphisms

$$h_{\sigma,\sigma'} : N_\sigma \to N'_\sigma, h_{O(\sigma),O(\sigma')} : \bigcup_{i=0}^{m_\sigma-1} N_{f^i(\sigma)} \to \bigcup_{i=0}^{m_\sigma-1} N_{f'^i(\sigma')}$$

by $h_{\sigma,\sigma'} = \varphi\Theta$, $h_{O(\sigma),O(\sigma')} = f'^i h_{\sigma,\sigma'} f^{-i}(x)$, for $x \in N_{f^i(\sigma)}$.

Denote by $H_1 : \bigcup_{\sigma \in \Omega^1_f} N_\sigma \to \bigcup_{\sigma' \in \Omega^1_{f'}} N_{\sigma'}$ the homeomorphism coinciding for any point $\sigma \in \Omega^1_f$ with the homeomorphism $h_{O(\sigma),O(\sigma')}$.

To define the homeomorphism $H_{n-1} : \bigcup_{\sigma \in \Omega^{n-1}_f} N_\sigma \to \bigcup_{\sigma' \in \Omega^{n-1}_{f'}} N_{\sigma'}$, use the similar construction for points from the set Ω^{n-1}_f using formal replace s with u and b_ν with b^{-1}_ν.

4. Proofs of auxiliary results

4.1. On embedding of families of closed curves and their tubular neighbourhoods

In this section, we prove proposition 2.10 after all necessary definitions and auxiliary facts.

A topological space X is called *m-connected*, for $m > 0$, if it is non-empty, path-connected and its first m homotopy groups $\pi_i(X)$, $i \in \{1, \ldots, m\}$ are trivial. Any non-empty space X

can be interpreted as (-1)-connected, and any path-connected space X can be interpreted as 0-connected.

Let Q^q and M^n be two compact topological manifolds with dimensions q and n, where $q \leqslant n$.

Topological embeddings $e, e' : Q^q \to M^n$ are *homotopic* if there is a continuous map $H : Q^q \times [0, 1] \to M^n$, called *a homotopy*, such that $H|_{Q^q \times \{0\}} = e, H|_{Q^q \times \{1\}} = e'$. They are *concordant* if there is a topological embedding $H : Q^q \times [0, 1] \to M^n \times [0, 1]$ (*a concordance*), such that $H(Q^q, 0) = (e(Q^q), 0)$ and $H(Q^q, 1) = (e'(Q^q), 1)$. Topological embeddings $e, e' : Q^q \to M^n$ are *ambient isotopic with a subset $N \subset M^n$ fixed*, if there is a family h_t of homeomorphisms $h_t : M^n \to M^n$, $t \in [0, 1]$ (*an ambient isotopy*), such that $h_0 = id$, $h_1 e = e'$, $h_t|_N = id$, for any $t \in [0, 1]$. For $\varepsilon > 0$, an ambient isotopy is called *an ε-isotopy* if $d(h_t(x), x) \leqslant \varepsilon$, for any $x \in M^n, t \in [0, 1]$.

Let K be a *finite simplicial complex* in $\mathbb{R}^n$, that is a finite set of simplices that satisfies the following conditions:

 (a) Every face of a simplex from K is also in K.
 (b) The non-empty intersection of any two simplices $\Delta_1, \Delta_2 \in K$ is a face of both Δ_1 and Δ_2.

Considering K as a topological subspace of $\mathbb{R}$ (with the topology induced by an inclusion), we get a topological space P, which is called *a polyhedron generated by K*. The complex K is called *the partition* or *the triangulation* of the polyhedron P.

A complex K' is called *a subdivision* of a complex K if every simplex of K' belongs to a simplex of K.

Let K and L be the two complexes and P, Q be the two polyhedra generated by K and L. A map $h : P \to Q$ is called *piecewise linear* if there exists subdivision K', L' of K, L, correspondingly, such that h moves each simplex of the complex K' into a simplex of the complex L' (see, for example, [28]).

A polyhedron P is called *a piecewise linear manifold* of dimension n with boundary if it is a topological manifold with boundary and for any point $x \in \text{int} P$ ($y \in \partial P$) there is a neighbourhood U_x (U_y) and a piecewise linear homeomorphism $h_x : U_x \to \mathbb{R}^n$ ($h_y : U_y \to \mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \subset \mathbb{R}^n | x_1 \geqslant 0\}$).

The following important statement follows from theorem 4 of [22].

Statement 4.1. Suppose that Q^q and M^n are two compact piecewise linear manifolds of dimension q and n, respectively, Q^q is the manifold without boundary, M^n possibly has a non-empty boundary, $e, e' : Q^q \to \text{int} M^n$ are piecewise linear homotopic piecewise linear embeddings, and the following conditions hold:

 (a) $q \leqslant n - 3$;
 (b) Q^q is $(2q - n + 1)$-connected;
 (c) M^n is $(2q - n + 2)$-connected.

Then, e, e' are piecewise linear ambient isotopic with ∂M^n fixed.

To reduce the problem of embedding of families of curves to statement 4.1, we will use the following results.

Theorems 1 and 4 of R Miller's paper [26], see also A. Chernavskii paper [9], ensure us next statement.

Statement 4.2. Let Q^q and M^n be two compact piecewise linear manifolds of dimensions q and n, correspondingly, $q \leqslant n - 3$, $\varepsilon > 0$, and $e : N^k \to M^n$ be a locally flat embedding. Then, there is an ambient ε-isotopy of M^n connecting e with a piecewise embedding.

　　　G Weller's result [31, theorem 3] gives the following.

Statement 4.3.　　Let Q^q and M^n be two topological manifolds of dimensions q and n, correspondingly. Further, we assume that Q^q is compact, $e, e' : Q^q \to \text{int } M^n$ are two homotopic embeddings, and the following conditions hold:

(a) $q \leqslant n - 3$;
(b) Q^q is $(2q - n + 1)$-connected;
(c) M^n is $(2q - n + 2)$-connected.

　　　Then e and e' are locally flat concordant, that is there is a locally flat embedding $H : Q^q \times [0, 1] \to M^n \times [0, 1]$, such that $H(Q^q, 0) = (e(Q^q), 0)$ and $H(Q^q, 1) = (e'(Q^q), 1)$.

Proposition 4.1.　　*Suppose that* $\mathbb{P}^{n-1}$ *is either* $\mathbb{S}^{n-1}$ *or* $\mathbb{B}^{n-1}$, $\{\beta_i\}, \{\beta_i'\} \subset \text{int } \mathbb{P}^{n-1} \times \mathbb{S}^1$ *are two families of pairwise disjoint knots, such that the knots* β_i, β_i' *are homotopic,* $i \in \{1, \dots, k\}$, $n \geqslant 4$. *Then, there exists a homeomorphism* $H : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$, *such that* $H(\beta_i) = \beta_i'$, *for any* $i \in \{1, \dots, k\}$. *Moreover, if* $\mathbb{P}^{n-1} = \mathbb{B}^{n-1}$, *then* $H|_{\partial \mathbb{P}^{n-1} \times \mathbb{S}^1} = id$.

Proof.　　Denote by Δ^n the simplex of dimension n and by $\partial \Delta^n$ its boundary. Then, $\mathbb{P}^{n-1} \times \mathbb{S}^1$ is homeomorphic either to $\partial \Delta^n \times \partial \Delta^2$ or to $\Delta^{n-1} \times \partial \Delta^2$ and, without loss of generality, we will identify $\mathbb{P}^{n-1} \times \mathbb{S}^1$ with one of these piecewise linear objects.

　　　It follows from statement 4.2 that there exist homeomorphisms $g, g' : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$, such that for any $i \in \{1, \dots, k\}$ the sets $g(\beta_i)$ and $g'(\beta_i')$ are subpolyhedra. It is sufficient to construct a homeomorphism $H : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$, such that $H(g(\beta_i)) = g'(\beta_i')$, for any $i \in \{1, \dots, k\}$, and $H|_{\partial \mathbb{P}^{n-1} \times \mathbb{S}^1} = id$. Then, the homeomorphism $g'^{-1}Hg$ will be the desired map. So, without loss of generality, assume that the knots β_i and β_i' are sub-polyhedra, for $i \in \{1, \dots, k\}$.

　　　By assumption, the piecewise linear embeddings $e_i : \partial \Delta^2 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$, $e_i' : \partial \Delta^2 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$, such that $e_i(\partial \Delta^2) = \beta_1$ and $e_i'(\partial \Delta^2) = \beta_i'$ are homotopic. By statement 4.3, there is a locally flat embedding $\Psi : \partial \Delta^2 \times [0, 1] \to \mathbb{P}^{n-1} \times \mathbb{S}^1 \times [0, 1]$, such that $\Psi(\partial \Delta^2, 0) = (e(\partial \Delta^2), 0), \Psi(\partial \Delta^2, 1) = (e'(\partial \Delta^2), 1)$. By statement 4.2, there is a homeomorphism $\Phi : \mathbb{P}^{n-1} \times \mathbb{S}^1 \times [0, 1] \to \mathbb{P}^{n-1} \times \mathbb{S}^1 \times [0, 1]$, such that the composition $\Phi \Psi$ is a piecewise linear embedding. Denote by $\text{pr} : \mathbb{P}^{n-1} \times \mathbb{S}^1 \times [0, 1] \to \mathbb{P}^{n-1} \times \mathbb{S}^1 \times \{0\}$ the projection given by $\text{pr}(x, t) = (x, 0), x \in \mathbb{P}^{n-1} \times \mathbb{S}^1, t \in [0, 1]$. Then, the composition $\text{pr}\Phi\Psi$ is the piecewise linear homotopy.

　　　Now, we will construct the homeomorphism H by induction on i. By statement 4.1, there exists a piecewise linear homeomorphism $H_1 : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$ that maps β_1 to β_1'.

　　　Suppose that for some $j < k$ there exists a homeomorphism $H_j : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$, such that $H_j(\beta_i) = \beta_i'$, for $i \in \{1, \dots, j\}$, $H_j|_{\partial \mathbb{P}^{n-1} \times \mathbb{S}^1} = id$. We can construct a homeomorphism $H_{j+1} : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$, such that $H_{j+1}(\beta_i) = \beta_i'$, for $i \in \{1, \dots, j+1\}$, $H_{j+1}|_{\partial \mathbb{P}^{n-1} \times \mathbb{S}^1} = id$.

　　　Preserve the notation β_i for the images of the knots β_i, $i \in \{1, \dots, k\}$ by means of the homeomorphism H_j. Then, the knots β_i and β_i' coincide, for $i \in \{1, \dots, j\}$. Denote by $e_{j+1}, e_{j+1}' : \partial \Delta^2 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$ piecewise linear embeddings, such that $e_{j+1}(\partial \Delta^2) = \beta_{j+1}$ and $e_{j+1}'(\partial \Delta^2) = \beta_{j+1}'$.

　　　Now, we show that e_{j+1} and e_{j+1}' are piecewise linear homotopic in $\mathbb{P}^{n-1} \times \mathbb{S}^1 \setminus \bigcup_{i=1}^{j} \beta_i$. It follows from statement 4.1 that there exists a family of piecewise linear homeomorphisms $h_t^{j+1} : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$, $t \in [0, 1]$, such that $h_0^{j+1} = id$, $h_1^{j+1}e_{j+1} = e_{j+1}'$, $h_t^{j+1}|_{\partial \mathbb{P}^{n-1} \times \mathbb{S}^1} = id$, for any $t \in [0, 1]$. This family defines a piecewise linear homeomorphism $\overline{H}^{j+1} : \mathbb{S}^1 \times [0, 1] \to \mathbb{P}^{n-1} \times \mathbb{S}^1 \times [0, 1]$ by $\overline{H}^{j+1}(x, t) = (h_t^{j+1}(x), t)$. It follows from the general position theorem (see, for example, theorem 5.3 of chapter 5 of [28]) that there exists a

piecewise linear homeomorphism $\Phi : \mathbb{P}^{n-1} \times \mathbb{S}^1 \times [0,1] \to \mathbb{P}^{n-1} \times \mathbb{S}^1 \times [0,1]$ leaving the set $\partial \mathbb{P}^{n-1} \times \mathbb{S}^1 \times [0,1] \cup \mathbb{P}^{n-1} \times \mathbb{S}^1 \times \{0,1\}$ fixed and mapping the cylinder $\overline{H}^{j+1}(\mathbb{S}^1 \times [0,1])$ to one that has no intersection with all the cylinders $\beta_i \times [0,1]$, $i \in \{1, \ldots, j\}$. Then, it is possible to choose regular tubular neighbourhoods $N_{\beta'_1}, \ldots, N_{\beta'_j}$ of the knots $\beta'_1, \ldots, \beta'_j$ in $\mathbb{P}^{n-1} \times \mathbb{S}^1$, such that $\Phi(\overline{H}^{j+1}(\mathbb{S}^1 \times [0,1])) \cap \bigcup_{i=1}^{j} N_{\beta'_j} \times [0,1] = \emptyset$. So, the map $\Phi\overline{H}^{j+1} : \mathbb{S}^1 \times [0,1] \to (\mathbb{P}^{n-1} \times \mathbb{S}^1 \backslash \bigcup_{i=1}^{j} \text{int } N_{\beta'_j}) \times [0,1]$ is a piecewise concordance and the composition of it with the projection $\text{pr} : \mathbb{P}^{n-1} \times \mathbb{S}^1 \times [0,1] \to \mathbb{P}^{n-1} \times \mathbb{S}^1$ gives the desired homotopy. Applying statement 4.1 once more, one can obtain the piecewise linear homeomorphisms $\tilde{H}^{j+1} : \mathbb{P}^{n-1} \times \mathbb{S}^1 \backslash \bigcup_{i=1}^{j} \text{int } N_{\beta'_j} \to \mathbb{P}^{n-1} \times \mathbb{S}^1 \backslash \bigcup_{i=1}^{j} \text{int } N_{\beta'_j}$, $t \in [0,1]$, such that $\tilde{H}^{j+1} e_{j+1} = e'_{j+1}$, $\tilde{H}^{j+1}|_{\partial(\mathbb{P}^{n-1} \times \mathbb{S}^1 \backslash \bigcup_{i=1}^{j} \text{int } N_{\beta'_j})} = id$, for any $t \in [0,1]$. Now, the desired homeomorphism $H^{j+1} : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$ is defined by $H^{j+1}(x) = \tilde{H}^{j+1}(x)$, for $x \in \mathbb{P}^{n-1} \times \mathbb{S}^1 \backslash \bigcup_{i=1}^{j} \text{int } N_{\beta'_j}$, and $H^{j+1}(x) = x$, for $x \in \bigcup_{i=1}^{j} \text{int } N_{\beta'_j}$.

$\square$

The following statement 4.4 is proved in [13, lemma 2.1].

Statement 4.4. Let $h : \mathbb{B}^{n-1} \times \mathbb{S}^1 \to \text{int } \mathbb{B}^{n-1} \times \mathbb{S}^1$ be a topological embedding, such that $h(\{O\} \times \mathbb{S}^1) = \{O\} \times \mathbb{S}^1$. Then, the manifold $\mathbb{B}^{n-1} \times \mathbb{S}^1 \backslash \text{int } h(\mathbb{B}^{n-1} \times \mathbb{S}^1)$ is homeomorphic to the direct product $\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0,1]$.

The proof of the following statement one can find in [17, lemma 2].

Statement 4.5. Suppose that Y is a topological manifold with boundary, X is a connected component of its boundary, Y_1 is a manifold homeomorphic to $X \times [0,1]$, and $Y \cap Y_1 = X$. Then the manifold $Y \cup Y_1$ is homeomorphic to Y. Moreover, if the manifold Y is homeomorphic to the direct product $X \times [0,1]$, then there exists a homeomorphism $h : X \times [0,1] \to Y \cup Y_1$, such that $h(X \times \{\frac{1}{2}\}) = X$.

Proof of proposition 2.10. Suppose that $\mathbb{P}^{n-1}$ is either the ball $\mathbb{B}^{n-1}$ or the sphere $\mathbb{S}^{n-1}$, $\{\beta_i\}, \{\beta'_i\} \subset \text{int } \mathbb{P}^{n-1} \times \mathbb{S}^1$ are two families of knots, such that the knots β_i, β'_i are homotopic and $\{N_{\beta_i}\}, \{N_{\beta'_i}\} \subset \mathbb{P}^{n-1} \times \mathbb{S}^1$ are pairwise disjoint neighbourhoods of the knots $\{\beta_i\}, \{\beta'_i\}$. Let us prove that there exists a homeomorphism $h : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$, such that $h(\beta_i) = \beta'_i, h(N_{\beta_i}) = N_{\beta'_i}, i \in \{1, \ldots, k\}$, and $h|_{\partial \mathbb{P}^{n-1} \times \mathbb{S}^1} = id$.

By proposition 4.1, there exists a homeomorphism $h_0 : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$, such that $h_0(\beta_i) = \beta'_i$, $h_0|_{\partial \mathbb{P}^{n-1} \times \mathbb{S}^1} = id$. Set $\tilde{N}_i = h_0(N_{\beta_i})$. For $r \in (0,1)$, set

$$\mathbb{B}^{n-1}_r = \{(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} | \ x_1^2 + \cdots + x_{n-1}^2 \leqslant r^2\}.$$

It follows from [5] that there exist topological embeddings $\tilde{e}_i : \mathbb{B}^{n-1} \times \mathbb{S}^1 \to \text{int } \mathbb{P}^{n-1} \times \mathbb{S}^1$, such that $\tilde{e}_i(\{O\} \times \mathbb{S}^1) = \beta'_i$, $\tilde{e}_i(\mathbb{B}^{n-1}_r \times \mathbb{S}^1) = \tilde{N}_i$, for some $r \in (0,1)$, $\tilde{e}_i(\mathbb{B}^{n-1} \times \mathbb{S}^1) \cap \tilde{e}_j(\mathbb{B}^{n-1} \times \mathbb{S}^1) = \emptyset$, for $i \neq j, i, j \in \{1, \ldots, k\}$. Set $U_i = \tilde{e}_i(\mathbb{B}^{n-1} \times \mathbb{S}^1)$.

Denote by $e'_i : \mathbb{B}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$ a topological embedding, such that $e'_i(\mathbb{B}^{n-1} \times \mathbb{S}^1) = N_{\beta'_i}$, and choose r_0, r_1, such that $0 < r_0 < r_1 < 1$ and $e'_i(\mathbb{B}^{n-1}_{r_1} \times \mathbb{S}^1) \subset \tilde{N}_i$. Set $N'_{0,i} = e'_i(\mathbb{B}^{n-1}_{r_0} \times \mathbb{S}^1)$ and $N'_{1,i} = e'_i(\mathbb{B}^{n-1}_{r_1} \times \mathbb{S}^1)$.

By statement 4.4, the set $\tilde{N}_i \backslash N'_{1,i}$ is homeomorphic to the direct product $\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0,1]$, $i \in \{1, \ldots, k\}$. By statement 4.5, there exists a homeomorphism $g_i : \mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0,1] \to U_i \backslash \text{int } N'_{0,i}$ and $t_1, t_2 \subset (0,1)$, such that

$$g_i(\mathbb{S}^{n-2} \times \mathbb{S}^1 \times \{t_1\}) = \partial \tilde{N}_i, g_i(\mathbb{S}^{n-2} \times \mathbb{S}^1 \times \{t_2\}) = \partial N'_{1,i}.$$

Let $\xi : [0, 1] \to [0, 1]$ be a homeomorphism that is the identity on the ends of the interval $[0, 1]$ and such that $\xi(t_1) = t_2$. Define a homeomorphism $\tilde{g}_i : \mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0, 1] \to \mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0, 1]$ by $\tilde{g}_i(x, t) = (x, \xi(t))$ and a homeomorphism $h_i : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$ by

$$h_i(x) = \begin{cases} g_i(\tilde{g}_i(g_i^{-1}(x))), \ x \in U_i \setminus \text{int } N'_{0,i}; \\ x, \ x \in (\mathbb{P}^{n-1} \times \mathbb{S}^1 \setminus U_i). \end{cases}$$

The superposition $\eta = h_k \cdots h_1 h_0$ maps every knot β_i into the knot β'_i, the neighbourhood N_{β_i} into the set $N'_{1,i} \subset N_{\beta'_i}$, and keeps the set $\partial \mathbb{P}^{n-1} \times \mathbb{S}^1$ fixed. Since the set $N_{\beta'_i} \setminus \text{int } N'_{1,i}$ is homeomorphic to the direct product $\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0, 1]$, it is possible to apply the described construction once more and get the desired homeomorphism. The proof is complete.

Let M^n be an n-dimensional manifold homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$. A knot $\beta \subset \text{int } M^n$ is called *essential* if the homomorphism $i_* : \pi_1(\beta) \to \pi_1(M^n)$ induced by the inclusion $i : \beta \to M^n$ is an isomorphism.

The next statement follows immediately from proposition 2.10.

Corollary 4.1. *If $\beta \subset M^n$ is an essential knot and N_β is its tubular neighbourhood, then the manifold $M^n \setminus \text{int } N_\beta$ is homeomorphic to the direct product $\mathbb{B}^{n-1} \times \mathbb{S}^1$.*

One more corollary was stated in subsection 2.4, let us prove it. We will use the following statement proved in [25, theorem 2].

Statement 4.6. Let $\psi : \mathbb{S}^{n-2} \times \mathbb{S}^1 \to \mathbb{S}^{n-2} \times \mathbb{S}^1$ be an arbitrary homeomorphism. Then, there exists a homeomorphism $\Psi : \mathbb{B}^{n-1} \times \mathbb{S}^1 \to \mathbb{B}^{n-1} \times \mathbb{S}^1$, such that $\Psi|_{\mathbb{S}^{n-2} \times \mathbb{S}^1} = \psi|_{\mathbb{S}^{n-2} \times \mathbb{S}^1}$.

Proof of corollary 2.3. Let $\tilde{h} : \partial \mathbb{B}^{n-1} \times \mathbb{S}^1 \to \partial \mathbb{B}^{n-1} \times \mathbb{S}^1$ be a homeomorphism. Let us prove that there exists a homeomorphism $h : \mathbb{B}^{n-1} \times \mathbb{S}^1 \to \mathbb{B}^{n-1} \times \mathbb{S}^1$, such that $h(\beta_i) = \{x_i\} \times \mathbb{S}^1, h(N_{\beta_i}) = N_{\beta'_i}, i \in \{1, \dots, k\}$, and $h|_{\partial \mathbb{B}^{n-1} \times \mathbb{S}^1} = \tilde{h}$. Due to statement 4.6, the homeomorphism $\tilde{h}$ can be extended to a homeomorphism $h_0 : \mathbb{B}^{n-1} \times \mathbb{S}^1 \to \mathbb{B}^{n-1} \times \mathbb{S}^1$. Due to proposition 2.10, there exists a homeomorphism $h_1 : \mathbb{B}^{n-1} \times \mathbb{S}^1 \to \mathbb{B}^{n-1} \times \mathbb{S}^1$, such that $h_1(h_0(\beta_i)) = \beta'_i, h_1(h_0(N_{\beta_i})) = N_{\beta'_i}, i \in \{1, \dots, k\}$, and $h_1|_{\partial \mathbb{B}^{n-1} \times \mathbb{S}^1} = id$. Then, the composition $h_1 h_0$ is the desired homeomorphism h. □

4.2. Characteristic space and embedding of separatrices of dimension $(n - 1)$

Here we prove lemmas 2.1 and 2.2. If all saddle points of a diffeomorphism f are fixed and have the positive orientation type, then the two lemmas follow from [16, lemm 3.1]. Now, we prove them for the general case. The main tool of the proof is a surgery along knots that, in contrast with the case $n = 3$, does not change the topology of the manifold.

Let M^n be a topological manifold with dimension $n \geqslant 4$, possibly with non-empty boundary, $\beta \subset \text{int } M^n$ be a knot and $N_\beta \subset \text{int } M^n$ be its tubular neighbourhood. Glue manifolds $M^n \setminus \text{int } N_\beta$ and $\mathbb{B}^{n-1} \times \mathbb{S}^1$ by means of an arbitrary reversing the natural orientation homeomorphism $\varphi : \partial N_\beta \to \mathbb{S}^{n-2} \times \mathbb{S}^1$ and denote the obtained manifold by Q^n. We say that Q^n is obtained from M^n by *a surgery along the knot β*.

Proposition 4.2. Q^n *is homeomorphic to M^n.*

Proof. Set $N' = M^n \setminus \text{int } N_\beta$, then $Q^n = N' \cup_\varphi \mathbb{B}^{n-1} \times \mathbb{S}^1$, and, for any subset $X \subset N' \cup \mathbb{B}^{n-1} \times \mathbb{S}^1$, the projection $\pi : X \to Q^n$ is defined.

Set $\psi = \varphi^{-1} \pi^{-1}|_{\pi(\mathbb{S}^{n-2} \times \mathbb{S}^1)}$. Due to statement 4.6, the homeomorphism ψ can be extended

to a homeomorphism $\Psi : \pi(\mathbb{B}^{n-1} \times \mathbb{S}^1) \to N_\beta$. Let us define a map $H : Q^n \to M^n$ by $H(x) = \pi^{-1}(x) = x$, for $x \in \pi(\mathrm{int}\, N')$, and by $H(x) = \Psi(x)$, for $x \in \pi(\mathbb{B}^{n-1} \times \mathbb{S}^1)$. One can easily check that H is the desired homeomorphism. $\qquad\square$

Recall that we represent the sphere S^n as the union of three pairwise disjoint sets

$$A_f = (\bigcup_{\sigma \in \Omega_f^1} W_\sigma^u) \cup \Omega_f^0,\, R_f = (\bigcup_{\sigma \in \Omega_f^{n-1}} W_\sigma^s) \cup \Omega_f^n,\, V_f = S^n \backslash (A_f \cup R_f).$$

We denoted by $\widehat{V}_f = V_f/f$ the space under the action f on V_f, by $p_f : V_f \to \widehat{V}_f$ the natural projection, and we introduced the homomorphism $\eta_f : \pi_1(\widehat{V}_f) \to \mathbb{Z}$.

Suppose that the set Ω_f^1 is non-empty, in the opposite case we consider f^{-1}. Then, A_f can be represented as a graph embedded in S^n, whose vertices are sink periodic points and edges are one-dimensional unstable manifolds of saddle periodic points. Since the closure of all stable separatrix of dimension $(n-1)$ cuts the ambient sphere S^n into the union of some pairwise disjoint domains, so that each of them contains exactly one sink periodic point, we have $|\Omega_f^0| = |\Omega_f^1| + 1$. It follows from [20] that the set A_f is connected. Then, A_f is a tree and it is possible to define *ranks of sink periodic points* as ranks of the vertices of A_f.

Construct a series of attractors $A_0, A_1, \ldots, A_r$, where $A_0 = \bigcup_{\omega \in \Omega_f^0} \omega$, $A_r = A_f$, A_i consists of all vertices of A_f of ranks at most i and joining them edges of A_f. Denote by V_i the union of stable manifolds of all periodic points belonging to A_i, set $\widehat{V}_i = V_i/f$, and denote by $p_i : V_i \to \widehat{V}_i$ the natural projection. It follows from the definition that the number of connected components equals the number of $f-$invariant components of A_i.

Lemma 2.1 immediately follows from the next proposition.

Proposition 4.3. $\widehat{V}_i$ *is the union of manifolds homeomorphic to* $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

Proof. We will prove the proposition by induction on i. For a point $\omega \in \Omega_f^0$ of period m_ω, set $V_{\mathcal{O}_\omega}^s = \bigcup_{i=0}^{m_\omega - 1} f(W_\omega^s \backslash \omega)$ and $\widehat{V}_{\mathcal{O}_\omega}^s = V_\omega^s/f$. It follows from the hyperbolicity of the point ω and [24, theorem 5.5] (see also [4, propositions 1.2.3 and 1.2.4]) that $\widehat{V}_{\mathcal{O}_\omega}^s$ is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$. So, $\widehat{V}_0$ is the union of manifolds homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

Suppose the statement is proved for $k = i$ and prove it for $i + 1$.

For a point $\sigma \in \Omega_f^1$ of period m_σ, denote by ω_- and ω_+ the sink points belonging to cl W_σ^u and by $l_{\sigma,-}^u, l_{\sigma,+}^u$ the unstable separatrices of σ, such that $l_{\sigma,-}^u \subset W_{\omega_+}^u$ and $l_{\sigma,+}^u \subset W_{\omega_+}^u$. Set

$$\mathcal{O}_\sigma = \bigcup_{i=0}^{m_\sigma - 1} f^i(\sigma), l_{\mathcal{O}_\sigma,-}^u = \bigcup_{i=0}^{m_\sigma - 1} f^i(l_{\sigma,-}^u), l_{\mathcal{O}_\sigma,+}^u = \bigcup_{i=0}^{m_\sigma - 1} f^i(l_{\sigma,+}^u).$$

Let N_σ be the linearizing neighbourhood of the point σ and $N_\sigma^s = N_\sigma \backslash W_\sigma^s$, $N_\sigma^u = N_\sigma \backslash W_\sigma^u$. Denote by $N_{\sigma,-}^u$ and $N_{\sigma,+}^u$ connected components of the set N_σ^u containing separatrices $l_{\sigma,-}, l_{\sigma,+}$ correspondingly, set $\widehat{N}_\sigma^u = N_\sigma^u/f$, $\hat{N}_\sigma^s = N_\sigma^s/f$, denote by $p_{\hat{N}_\sigma^u} : N_\sigma^u \to \hat{N}_\sigma^u$, $p_{\hat{N}_\sigma^s} : N_\sigma^s \to \hat{N}_\sigma^s$ the natural projections and define a homeomorphism $\varphi : \partial \hat{N}_\sigma^u \to \hat{N}_\sigma^s$ by $\varphi = p_{\hat{N}_\sigma^s} p_{\hat{N}_\sigma^u}^{-1}$.

Suppose that $W_\sigma^u \subset A_{i+1} \backslash A_i$ and $\omega_- \subset A_i$. There are two possible cases: (1) rank$(\omega_-) <$ rank(ω_+), then $\omega_+ \subset A_{i+1} \backslash A_i$; (2) rank$(\omega_-) = $ rank(ω_+), then $\omega_+ \subset A_i$.

We first consider the case (1). It follows from proposition 2.2 that $m_\sigma = m_{\omega_-}$, moreover, the period of the connected component V_i^σ of V_i having non-empty intersection with the set $W_{\omega_-}^s$ also equals m_σ. Then, the set $\hat{l}_{\mathcal{O}_\sigma,-}^u = p_i(l_{\mathcal{O}_\sigma,-}^u)$ is an essential knot in $\widehat{V}_i^\sigma = V_i^\sigma/f$ and, due to corollary 4.1, the set $\widehat{V}_i^\sigma \backslash$ int $\hat{N}_{\sigma,-}^u$ is homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$.

Denote by $p_{\widehat{V}_{\mathcal{O}_{\omega_+}}} : V_{\mathcal{O}_{\omega_+}} \to \widehat{V}_{\mathcal{O}_{\omega_+}}$ the natural projection. Then, the set $\hat{l}^u_{\mathcal{O}_{\sigma,+}} = p_{\widehat{V}_{\mathcal{O}_{\omega_+}}}(l^u_{\mathcal{O}_{\sigma,+}})$ is a knot in $\widehat{V}_{\omega_{\sigma,+}}$ and $\hat{N}^u_{\sigma,+} = p_{\widehat{V}_{\mathcal{O}_{\omega_+}}}(N^u_{\sigma,+})$ is a tubular neighbourhood.

Set

$$V^\sigma_{i+1} = V^\sigma_i \cup V_{\mathcal{O}_{\omega_+}} \cup W^s_{\mathcal{O}_\sigma} \backslash W^u_{\mathcal{O}_\sigma} = V^\sigma_i \cup V_{\mathcal{O}_{\omega_+}} \cup N^s_{\mathcal{O}_\sigma} \backslash \text{int } N^u_{\mathcal{O}_\sigma}$$

$$= (V^\sigma_i \backslash \text{int } N^u_{\mathcal{O}_{\sigma,-}}) \cup N^s_{\mathcal{O}_\sigma} \cup (V_{\mathcal{O}_{\omega_+}} \backslash \text{int } N^u_{\mathcal{O}_{\sigma,+}}).$$

Then

$$\widehat{V}^\sigma_{i+1} = V^\sigma_{i+1}/f = ((\widehat{V}_{\mathcal{O}_-} \backslash \text{int } \hat{N}^u_{\sigma,-}) \cup_{\varphi_-} \hat{N}^s_\sigma) \cup_{\varphi_+} (\widehat{V}^\sigma \backslash \text{int } \hat{N}^u_{\sigma,-}),$$

where $\varphi_- = \varphi|_{\partial \hat{N}^u_{\sigma,-}}$ and $\varphi_+ = (\varphi|_{\partial \hat{N}^u_{\sigma,+}})^{-1}$.

Due to proposition 2.2, the manifold $\hat{N}^s_\sigma$ is homeomorphic to $\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [-1, 1]$. Due to statement 4.5, $(\widehat{V}_{\mathcal{O}_{\omega_+}} \backslash \text{int } \hat{N}^u_{\sigma,-}) \cup_{\varphi_-} \hat{N}^s_\sigma$ is homeomorphic to $\widehat{V}_{\mathcal{O}_{\omega_-}} \backslash \text{int } \hat{N}^u_{\sigma,-}$, so, it is homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$. Then, $\widehat{V}^\sigma_{i+1}$ is obtained from $\widehat{V}_{\mathcal{O}_{\omega_+}} \backslash \text{int } \hat{N}^u_{\sigma,+}$ by surgery along knot $\hat{l}^u_{\sigma,+}$ and, due to proposition 4.2, is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$. To get $\widehat{V}_{i+1}$, join $\widehat{V}^\sigma_{i+1}$ to the union $\bigcup_{\text{rank}(\omega)=i+1} \widehat{V}_{\mathcal{O}_\omega}$ and then repeat a similar procedure for all saddles $\tilde{\sigma}$, such that $W^u_{\tilde{\sigma}} \subset A_{i+1} \backslash A_i$ and $A_i \cap \text{cl } W^u_{\tilde{\sigma}} \neq \emptyset$. At every step, one gets the union of manifolds homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$ and after a finite number of steps, one gets either to one sink of the maximal rank (so, the last manifold is connected) or to case (2).

In the case (2) there are two possibilities: the point σ has either the positive or the negative orientation type. If σ has the positive orientation type, then $m_\sigma = m_{\omega,+} = m_{\omega,-} = 1$ and the manifold $\widehat{V}_i = \widehat{V}_{r-1}$ is the union of two connected components homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$, each of which contains one of the knots $\hat{l}^u_{\sigma,+}, \hat{l}^u_{\sigma,-}$. Then surgery and arguments similar to ones above prove that $\widehat{V}_r$ is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

If σ has the negative orientation type, then $m_\sigma = 1$, the points ω_+ and ω_- together form a 2-periodic orbit, and V_i consists of two 2-periodic connected components D and $f(D)$. Consider the map $g = f^2$, set $\widehat{V}_g = V_f/g$ and denote by $p_g : V_f \to \widehat{V}_g$ the natural projection. It follows from the arguments above that the factor-space V_f/f^2 is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$. Set $\tau = p_g f p_g^{-1}$. Then, $\tau^2 = id$, so τ is involution and $\widehat{V}_f = \widehat{V}_g/\tau$. It follows from [23] that $\widehat{V}_f$ is homeomorphic to one of the following manifolds: the direct product $\mathbb{S}^{n-1} \times \mathbb{S}^1$, the non-oriented fibre bundle $\mathbb{S}^{n-1} \tilde{\times} \mathbb{S}^1$ over the circle with the fibre $\mathbb{S}^{n-1}$, the direct product $\mathbb{S}^1 \times \mathbb{R}P^{n-1}$ or to the connected sum $\mathbb{R}P^n \times \mathbb{R}P^n$.

The set $\widehat{V}_f$ is covered by $\widehat{V}_g$ and, consequently, by $\mathbb{S}^{n-1} \times \mathbb{R}$, which is the universal cover of $\widehat{V}_f$. Then, due to [24, corollary 19.4], the fundamental group $\pi_1(\widehat{V}_f)$ is isomorphic to the group $\{f^n\}$ and, consequently, to the group $\mathbb{Z}$. So, $\widehat{V}_f$ cannot be homeomorphic to the direct product $\mathbb{S}^1 \times \mathbb{R}P^{n-1}$ or to $\mathbb{R}P^n \times \mathbb{R}P^n$. Since f is orientation preserving, the orbit space $\widehat{V}_f$ is orientable, so it is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$. $\qquad \square$

Proof of lemma 2.2. Let us prove that $\Delta_{f,i} = \hat{\xi}(\widehat{\Gamma}_f \backslash \widehat{\Gamma}_{f,i+1})$ is the union of $\mathbb{S}^{n-1} \times \mathbb{S}^1$, for any $i \in \{0, \ldots, s\}$.

If the graph $\widehat{\Gamma}_f$ has no loops, then, for any $i \in \{0, \ldots, s\}$, a connected component of the set $\widehat{\Gamma}_f \backslash \widehat{\Gamma}_{f,i+1}$ is the union of the following objects: the edge e_σ, the vertex v incident to e_σ, and all paths of the graph $\widehat{\Gamma}_f$ connecting the vertex v_σ with leaves and crossing the vertices in order of decreasing of ranks. Denote this component by T^σ and set $\Delta^\sigma_{f,i} = \hat{\xi}(T^\sigma)$. Let us prove that $\Delta^\sigma_{f,i}$ is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

Without loss of generality, suppose that the edge e_σ corresponds to the union of stable manifolds of points belonging to the orbit $\mathcal{O}_\sigma$ of the point $\sigma \in \Omega^1_f$. Denote by m_σ the period of the point σ.

The set $\mathrm{cl}\, W^s_{\mathcal{O}_\sigma}$ cuts the sphere S^{n-1} in $m_\sigma + 1$ connected components that belong to two f-invariant sets $D_{\sigma,-}$ and $D_{\sigma,+} = S^n \backslash \mathrm{cl}\, D_-$. Suppose that the domain, which is corresponding to the vertex v_σ, belongs to D_-. Then, $\Delta^\sigma_{f,i} = \mathrm{cl}\,(p_f(D_{\sigma,-}\backslash(A_f \cup R_f)))$ and the set $D_{\sigma,-}$ is the union of the open balls $D_0, D_1 = f(D_0), \ldots, D_{m_\sigma - 1} = f^{m_\sigma - 1}(D_0)$ bounded by $\mathrm{cl}\, W^s_{\mathcal{O}_\sigma}$.

On the other hand, the set $D_{\sigma,-}$ is the union of stable manifolds of periodic points belonging to the set $A_f \cap D_{\sigma,-}$. Hence, due to proposition 4.3, the set $\hat{D}_{\sigma,-} = p_f(D_{\sigma,-}\backslash(A_f \cup R_f))$ is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

Denote by $l_{\sigma,-}$ and $l_{\sigma,+}$ the one-dimensional separatrices of the point σ, such that $l_{\sigma,-} \subset D_{\sigma,-}$ and $l_{\sigma,+} \subset D_{\sigma,+}$. Let N_σ be a linearizing neighbourhood of the point σ, $N^s_\sigma = N_\sigma \backslash W^s_\sigma$, $N^u_\sigma = N_\sigma \backslash W^u_\sigma$, $\hat{N}^u_\sigma = N^u_\sigma /_f$, $\hat{N}^s_\sigma = N^s_\sigma /_f$.

It follows from propositions 2.2 and 2.3 that the set $\mathrm{cl}\, \hat{D}_{\sigma,-} \cap \hat{N}^s_\sigma$ is homeomorphic to the direct product $\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0, 1]$. Then, the set $\mathrm{cl}\, \widehat{D}_{\sigma,-}$ is homeomorphic to the set $\widehat{D}_{\sigma,-}\backslash \mathrm{int}\ \hat{N}^s_\sigma$.

The set $\hat{l}_{\sigma,-} = l_{\sigma,-}/_f$ is an essential knot in $\hat{D}_{\sigma,-}$ and the set $\hat{N}^u_{\sigma,-} = (N^u_\sigma \cap D_{\sigma,+})/_f$ is a tubular neighbourhood of it. Due to corollary 4.1, the set $\hat{D}_+ \backslash \mathrm{int}\ \hat{N}^u_{\sigma,+}$ is homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$. Since $\hat{D}_+ \backslash \mathrm{int}\ \hat{N}^u_{\sigma,+} = \widehat{D}_{\sigma,-}\backslash \mathrm{int}\ \hat{N}^s_\sigma$, $\widehat{D}_{\sigma,-}\backslash \mathrm{int}\, \hat{N}^s_\sigma$ and $\mathrm{cl}\, \widehat{D}_{\sigma,-}$ are also homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$.

Now, suppose that the graph $\widehat{\Gamma}_f$ has a loop. Then, the arguments above prove the lemma for $i \in \{0, \ldots, s-1\}$ and the set $\Delta_{f,s} = \mathrm{cl}\,(\widehat{V}_f \backslash \mathrm{int}\ \hat{N}_{\sigma_*})$, where N_{σ_*} is the linearizing neighbourhood of the point σ_* of the negative orientation type and $\hat{N}_{\sigma_*} = N_{\sigma_*}/f$. Let us prove that $\Delta_{f,s}$ is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

Without loss of generality, suppose that $\sigma_* \in \Omega^1_f$. The set $\mathrm{cl}\, W^s_{\sigma_*}$ cuts the sphere S^n into two connected components D and $f(D)$ of period 2. Hence, the point σ_* cuts the set A_f into two symmetrical parts of period 2. Denote by A_+ and A_- the connected components of the set $A_f \backslash W^u_{\sigma_*}$ lying in the sets D and $f(D)$, correspondingly, and by $l^u_+ \subset D$ and $l^u_- \subset f(D)$ unstable separatrices of the point σ_*. Set $D_+ = D\backslash(A_+ \cup R_f)$ and $D_- = f(D)\backslash(A_- \cup R_f)$.

It follows from proposition 4.3 that the orbit spaces $\hat{D}_+ = D_+/f^2$, $\hat{D}_- = D_-/f^2$ are homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$. Since the one-dimensional separatrices of the point σ_* are fixed with respect to f^2, their projections in $\hat{D}_+, \hat{D}_-$ are essential knots. Due to propositions 2.2 and 2.10, the sets $\hat{D}_+\backslash\ \mathrm{int}\ \hat{N}^u_{\sigma_*}, \hat{D}_-\backslash\ \mathrm{int}\ \hat{N}^u_{\sigma_*}$ are homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$. Then, the set $D_+\backslash N^u_{\sigma_*}$ is homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{R}$ and it is possible to find a fundamental domain $B_+ \subset D_+\backslash N^u_{\sigma_*}$ of the action of f^2 on $D_+\backslash N^u_{\sigma_*}$ homeomorphic to $\mathbb{B}^{n-1} \times [0, 1]$. Since the sets $D_+ \cup D_-$ and $N^u_{\sigma_*}$ are f-invariant and 2-periodic, the domain B_+ is also the fundamental domain for the action of f on $(D_+ \cup D_-)\backslash N^u_{\sigma_*}$ and the quotient-space $((D_+ \cup D_-)\backslash N^u_{\sigma_*})/_f$ is homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$.

Since $V_f \backslash \mathrm{int}\ N^s_{\sigma_*} = (D_+ \cup D_-)\backslash \mathrm{int}\ N^u_{\sigma_*}$ and $\mathrm{cl}\,(\widehat{V}_f \backslash \hat{N}^s_{\sigma_*}) = \widehat{V}_f \backslash \mathrm{int}\ \hat{N}^s_{\sigma_*}$, we have $\mathrm{cl}\,(\widehat{V}_f \backslash \hat{N}^s_{\sigma_*}) = ((D_+ \cup D_-)\backslash N^u_{\sigma_*})/_f$, so, $cl\,(\widehat{V}_f \backslash \hat{N}^s_{\sigma_*})$ is homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$.

5. A linear-time algorithm for distinguishing edge-coloured trees, equipped with automorphisms

In section 5, we prove the existence of a linear-time algorithm for distinguishing coloured graphs of cascades from the class $G(S^n)$ that proves theorem 2. At first, we will recall some basic definitions and define some new notation.

Let k be a fixed natural number. Let T be a tree and $c : E(T) \to [k]$, where $[k] = \{1, 2, \dots, k\}$, be some mapping, called an *edge k-colouring of T*. Elements of the set $[k]$ are called *colours*. The pair (T, c) is said to be an *edge-k-coloured tree*. Let P be some *automorphism* of (T, c), i.e. an automorphism of T, such that, for any edge $uv \in E(T)$, we have $c(uv) = c(P(u)P(v))$. We will refer to the triple (T, c, P) as an *equipped tree*. Two equipped trees (T_1, c_1, P_1) and (T_2, c_2, P_2) will said to be *isomorphic*, if there is an isomorphism ξ between T_1 and T_2, keeping edge colours, i.e. $\forall uv \in E(T_1)[c(uv) = c(\xi(u)\xi(v))]$, and conjugating P_1 and P_2, i.e. $\xi P_1 = P_2 \xi$.

Meeting the equality $|V(T_1)| = |V(T_2)|$ is a necessary condition for the existence of an isomorphism of (T_1, c_1, P_1) and (T_2, c_2, P_2). Further, we will consider that both trees T_1 and T_2 have k vertices. We will also assume that P_1 and P_2 are given by tables, in which, for any vertex $v \in V(T_i)$, a vertex $P_i(v), i \in \{1, 2\}$ is also given. In other words, for any $i \in \{1, 2\}$, the table for P_i has 2 rows and k columns, where, for any j, the content of the jth column are the jth vertex of T_i and its image with respect to P_i. We consider that c_1 and c_2 are also given by similar tables, whose the first rows are filled by vertices of the corresponding trees and the second rows are filled by the corresponding vertices' colours. Finally, we suppose that T_1 and T_2 are stored by the adjacency lists, i.e. all neighbours are listed, for any vertex of the trees. It will be shown that the isomorphism problem for two k-vertex equipped trees can be solved in $O(k)$ time.

Let (T, c, P) be some equipped tree. Recall that in section 2.4 the notion of the rank of a vertex was given in the following way. We associate with T a sequence $T_0, T_1, \dots, T_s$ of trees, such that $T_0 = T$, T_s contains one or two vertices and, for any $i \in [s]$, a tree T_i is obtained from T_{i-1} by deletion of all its *leaves*, i.e. degree one vertices. All the vertices of T_s are called the *central vertices* of the tree T and if T_s has an edge, then it is called *the central edge* of the tree T. The tree T is called *central*, if it has exactly one central vertex, and *bicentral*, otherwise. *The rank of a vertex* $x \in V(T)$, denoted by $\mathrm{rank}(x)$, is the number $\max\{i | x \in V(T_i)\}$.

From the equipped tree (T, c, P), we will construct a weighted edge-k-coloured tree $(\hat{T}, c, w)$. Vertices of $\hat{T}$ are all the orbits of P, two vertices of $\hat{T}$ are connected by an edge if and only if they are neighbours in T. As the weight of a vertex of the tree $\hat{T}$, we take the number of elements in the corresponding orbit of P. As the colour of an edge $O'O''$ of $\hat{T}$, we take the colour of any edge of T, simultaneously incident to a vertex from O' and to a vertex from O''. The tree $(\hat{T}, c, w)$ can be uniquely constructed from (T, c, P), see the third part of proposition 2.5.

Knowing $\hat{T}$, it is possible to uniquely restore the set of central vertices of T. Hence, by the second and the third parts of proposition 2.5, the equipped tree (T, c, P) can be uniquely restored from $(\hat{T}, c, w)$.

Let us construct a simple graph G from $(\hat{T}, c, w)$ in the following way. The operation of *s-subdivision* of some edge xy of a graph consists in deleting xy, then adding vertices $z_1, z_2, \dots, z_s$ and the edges $xz_1, z_1 z_2, z_2 z_3, \dots, z_{s-1} z_s, z_s y$. The operation of *joining an s-cycle to a vertex* v of some graph consists in adding vertices $u_1, \dots, u_{s-1}$ and the edges $vu_1, u_1 u_2, \dots, u_{s-2} u_{s-1}, u_{s-1} v$ to the graph. For any vertex v, we join a cycle of length $w(v) + 2$, where $w(v)$ is the weight of v. For any edge $e \in E(\hat{T})$, we apply its $c(e)$-subdivision, where $c(e)$ is the colour of e. The resultant graph is G. In figure 7 weighted coloured tree and the corresponding graph are depicted, where we apply 1-subdivision to green edges, 2-subdivision to blue ones, 3-subdivision to red edges (figure 7).

The graph G can be uniquely obtained from $(\hat{T}, c, w)$. Conversely, from G the triple $(\hat{T}, c, w)$ can also be restored in unique way. Indeed, vertices of G of degree more than two correspond to vertices of $\hat{T}$, lengths of cycles are equal to weights of the corresponding vertices minus two.
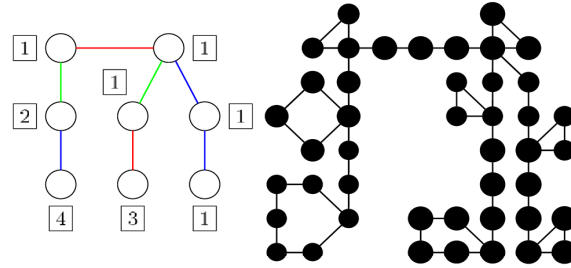
Figure 7. A coloured weighted tree and its corresponding graph.

Lengths of paths of G, whose end vertices are of degree at least three and all internal vertices have degree two, define colours of edges of $\hat{T}$.

The graph G is *planar*, i.e. a simple graph, which can be drawn on the plane, such that its vertices are points of the plane and edges are Jordan curves, not intersecting in internal points. There is known an algorithm with the complexity $O(n)$ for distinguishing n-vertex planar graph [1]. Our linear-time algorithm to recognize isomorphism of the equipped trees (T, c, P) and (T', c', P') is based on this fact.

Lemma 5.1. *One can consider that each of the trees T and T' is bicentral, P maps every central vertex of T into itself, and P' maps every central vertex of T' into itself.*

Proof. Since T and T' are given by the adjacency lists, the sets of their central vertices can be computed in $O(n)$ time. Hence, one can consider that T and T' are either simultaneously central or simultaneously bicentral, otherwise they would not be isomorphic. Hence, the algorithm outputs the negative answer for the isomorphism problem of (T, c, P) and (T', c', P').

Assume that T and T' are simultaneously bicentral. Denote by v_1 and v_2 the central vertices of T, and denote by u_1 and u_2 the central vertices of T'. One can consider that $c(v_1 v_2) = c'(u_1 u_2)$, otherwise the algorithm outputs the negative answer for the isomorphism problem of (T, c, P) and (T', c', P'). Similarly, we may assume that

$$P(v_1) = v_1, P(v_2) = v_2, P'(u_1) = u_1, P'(u_2) = u_2 \text{ or}$$

$$P(v_1) = v_2, P(v_2) = v_1, P'(u_1) = u_2, P'(u_2) = u_1.$$

In the first situation, we have a case from the statement. In the second one, we define mappings $\tilde{P}$ and $\tilde{P}'$ in the following way:

$$\tilde{P}(v_1) = v_1, \tilde{P}(v_2) = v_2, \quad \forall\, v \in V(T) \backslash \{v_1, v_2\}\ [\tilde{P}(v) = P(v)],$$

$$\tilde{P}'(u_1) = u_1, \tilde{P}'(u_2) = u_2, \quad \forall\, u \in V(T') \backslash \{u_1, u_2\}\ [\tilde{P}'(u) = P(u)].$$

The mappings $\tilde{P}$ and $\tilde{P}'$ are isomorphisms of (T, c) and (T', c'), correspondingly. The trees (T, c, P) and (T', c', P') are isomorphic if and only if $(T, c, \tilde{P})$ and $(T', c', \tilde{P}')$ are isomorphic.

Let us consider the case, when both T and T' have exactly one central vertex. Each of the isomorphisms P and P' maps the central vertex into itself. We construct coloured trees $(\tilde{T}, \tilde{c})$ and $(\tilde{T}', \tilde{c}')$ and some their automorphisms $\tilde{P}$ and $\tilde{P}'$. To construct $\tilde{P}$, we take two copies of (T, c, P), whose central vertices are denoted by v_1 and v_2. Connect them by an edge, and colour

it in the first colour. We obtain the tree $(\tilde{T}, \tilde{c}, \tilde{P})$. The tree $(\tilde{T}', \tilde{c}', \tilde{P}')$ is defined by analogy. Both these trees can be obtained in linear time on n. The equipped trees $(\tilde{T}, \tilde{c}, \tilde{P})$ and $(\tilde{T}', \tilde{c}', \tilde{P}')$ are isomorphic if and only if $(\tilde{T}, \tilde{c}, \tilde{P})$ and $(\tilde{T}', \tilde{c}', \tilde{P}')$ are isomorphic. This finishes the proof of this lemma. □

Proof of Theorem 2.   We suppose that trees Γ_f and $\Gamma_{f'}$ are bicentral and the automorphisms P_f and $P_{f'}$ map each of the central vertices into itself. Using the table representation of P_f, the set of all its orbits can be computed in linear on n time. To this end, we take an arbitrary vertex $v \in V(\Gamma_f)$, compute $P_f(v), P_f(P_f(v)), \ldots$, until $P_f^r(v) \neq v$ and, thereby, we find the orbit of P_f, containing v. From all the orbits of P_f, the adjacency list of Γ_f, and the mapping c_f, the triple $(\tilde{\Gamma}_f, \tilde{c}_f, \tilde{P}_f)$ can be computed in linear time. From this triple, the planar graph G_f can be computed in $O(|V(G_f)|)$ time. In any tree, the edges number is less by one then the number of vertices. The following inequalities are true:

$$|V(G_f)| \leqslant |V(\tilde{\Gamma}_f)| + k \cdot |E(\tilde{\Gamma}_f)| + \sum_{v \in V(\tilde{\Gamma}_f)} (\tilde{w}(v) + 1)$$

$$\leqslant |V(\tilde{\Gamma}_f)| + k \cdot |V(\tilde{\Gamma}_f)| + n + |V(\tilde{\Gamma}_f)| \leqslant (k + 3) \cdot n.$$

The equipped trees (Γ_f, c_f, P_f) and $(\Gamma_{f'}, c_{f'}, P_{f'})$ are isomorphic if and only if the graphs G_f and $G_{f'}$ are isomorphic. Since k is fixed, for recognizing isomorphism of G_f and $G_{f'}$, it is possible to use an algorithm from [21], having the complexity $O(n)$. Therefore, theorem 2 is true.

Acknowledgments

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