ISSN 1990-4789, Journal of Applied and Industrial Mathematics, 2020, Vol. 14, No. 3, pp. 480–489. © Pleiades Publishing, Ltd., 2020.
Russian Text © The Authors, 2020, published in Diskretnyi Analiz i Issledovanie Operatsii, 2020, Vol. 27, No. 3, pp. 71–87.

Efficient Solvability of the Weighted Vertex Coloring Problem for Some Hereditary Class of Graphs with 5-Vertex Prohibitions

D. V. Gribanov[1,2*], D. S. Malyshev[1,2], and D. B. Mokeev[2,1***]**

[1]*National Research University "Higher School of Economics," ul. Bolshaya Pecherskaya 25/12, Nizhny Novgorod, 603155 Russia*

[2]*Lobachevsky State University of Nizhny Novgorod, pr. Gagarina 23, Nizhny Novgorod, 603950 Russia*

Received July 25, 2019; in final form, January 6, 2020; accepted February 19, 2020

Abstract—We consider the problem of minimizing the number of colors in the colorings of the vertices of a given graph so that, to each vertex there is assigned some set of colors whose number is equal to the given weight of the vertex; and adjacent vertices receive disjoint sets. For all hereditary classes defined by a pair of forbidden induced connected subgraphs on 5 vertices but four cases, the computational complexity of the weighted vertex coloring problem with unit weights is known. We prove the polynomial solvability on the sum of the vertex weights for this problem and the intersection of two of the four open cases. We hope that our result will be helpful in resolving the computational complexity of the weighted vertex coloring problem in the above-mentioned forbidden subgraphs.

DOI: 10.1134/S1990478920030072

Keywords: *weighted vertex coloring problem, hereditary class, computational complexity*

INTRODUCTION

We consider *simple graphs*, i.e. undirected unlabeled graphs without loops and multiple edges.

The *Weighted Vertex Coloring Problem* (further, briefly, WVC) for a graph $G = (V, E)$ and function $w\colon V \to \mathbb{N}$ consists in finding the minimal number k (denoted by $\chi_w(G)$) such that there exists a mapping $c\colon V \to 2^{\{1,2,\dots,k\}}$ for which $|c(v)| = w(v)$ for every $v \in V$ and $c(u) \cap c(v) = \varnothing$ for every $uv \in E$. The unweighted version (i.e., with unit vertex weights) of WVC is called the *Vertex Coloring Problem* (henceforth, VC). In other words, VC consists in finding the minimum number of sets of pairwise nonadjacent vertices (called *independent*) into which the vertex set of the given graph can be partitioned. A *clique* in a graph is a subset of pairwise adjacent vertices. VC and WVC on graphs are classical NP-complete problems [1].

A graph *class* is a set of graphs closed under isomorphism. A graph class is called *hereditary* if it is closed under vertex removal. It is well known that every hereditary graph class $\mathcal{X}$ can be defined by the set of its forbidden induced subgraphs $\mathcal{Y}$, and this is written as follows: $\mathcal{X} = \text{Free}(\mathcal{Y})$. The graphs of the class $\mathcal{X}$ are also called $\mathcal{Y}$-*free*.

The VC problem is polynomially solvable for the class $\text{Free}(\{H\})$ if H is an induced subgraph of the graph P_4 or $P_3 + K_1$; otherwise, it is NP-complete in the given class (see [2]). However, if two induced subgraphs are forbidden then it is already impossible to obtain a complete complexity classification. For example, for all but three hereditary classes defined by prohibitions with at most 4 vertices each, the complexity status of VC is known (see [3]). For the remaining three cases, this status is unknown but for them it is possible to construct a polynomial approximation algorithm (see [4]). Some recent

*E-mail: dimitry.gribanov@gmail.com
**E-mail: dsmalyshev@rambler.ru
***E-mail: mokeevdb@gmail.com

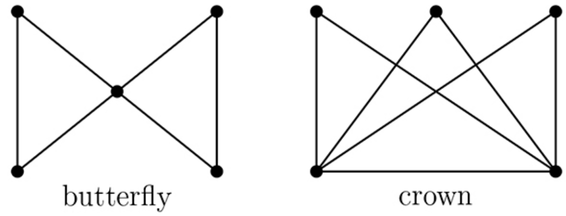

Fig. 1.

results on the complexity of VC in the hereditary classes defined by small size prohibitions are presented in [5–14].

In [9–14], the algorithmic complexity of VC was considered for a pair of connected forbidden induced fragments, each on 5 vertices. At present, the complexity of VC is an open question for the following four pairs of such a kind:

- $\{K_{1,3},$ butterfly$\}$,
- $\{P_5, H\}$, where $H \in \{K_{2,3},$ crown$, W_4\}$.

Unfortunately, it has been hard to clarify the status of VC for each of these four cases. Therefore, it occurred to consider the intersections of the corresponding hereditary classes and construct polynomial algorithms for them. Maybe this will help to develop polynomial algorithms for solving VC in the initial classes. In the present article, we prove that the WVC problem is solvable in polynomial time on the sum of the weights for the class of $\{P_5, K_{2,3}, W_4\}$-free graphs. Consequently, VC is solvable in polynomial time for the graphs of class $\text{Free}(\{P_5, K_{2,3}, W_4\})$.

1. NOTATIONS

Denote the neighborhood of the vertex x by $N(x)$. Denote a simple path, a simple cycle, and an empty graph on n vertices by P_n, C_n, and O_n respectively. Designate as $K_{p,q}$ the complete bipartite graph with p vertices in one part and q vertices in the other part. Denote by W_4 the graph obtained from a 4-cycle by adding one vertex and four edges each of which is incident to the added vertex and to its own vertex of the 4-cycle.

The graphs *butterfly* and *crown* are depicted in Fig. 1.

Let G be a graph and let $V' \subseteq V(G)$. Then $G[V']$ is the subgraph in G induced by the set of vertices V' and $G \setminus V'$ is the result of removing all elements of V' from G (together with all edges incident to them). Denote the complement of G by $\overline{G}$.

Let A and B be nonempty disjoint subsets of the vertex set of some graph $G = (V, E)$. The notation $A * B$ means that there is no edge $vu \in E$ for which $v \in A$ and $u \in B$, while $A \bullet B$ means that $vu \in E$ for all $v \in A$ and $u \in B$.

2. ATOMIC GRAPHS, PERFECT GRAPHS, AND THEIR MEANING

Let $G = (V, E)$ be a graph. A subset $M \subseteq V$ is called a *module* in G if each vertex in $V \setminus M$ is either adjacent to all vertices in M or is adjacent to none of them. Obviously, the empty set, each vertex of a graph, and the vertex set are modules. A module M is called *nontrivial* in G if $1 < |M| < |V(G)|$.

Let (G, w) be the input data in WVC and let M be a nontrivial module of G. Form a pair (G', w') from the triple (G, M, w). Given G, remove all vertices of M, then add a vertex x and the edges joining x to all vertices of the subgraph $G \setminus M$ to which the vertices of M were previously adjacent, and obtain the graph G'. Assign $w'(v) = w(v)$ for each $v \in V(G') \setminus \{x\}$ and $w'(x) = \chi_w(G[M])$ for x. Obviously,

$$|V(G')| < |V(G)|, \qquad \chi_{w'}(G') = \chi_w(G)$$

(see, for example, [14]). It turns out that, for an arbitrary graph with n vertices and m edges, all inclusion maximal nontrivial modules are pairwise disjoint and they all can be computed in time $O(n+m)$ (see [15]).

A clique Q in G is called *separating* if the number of connected components of $G \setminus Q$ is greater than the number of connected components in G. Let $A \sqcup B$ be an arbitrary partition of the vertex set of $G \setminus Q$ into some parts $A \neq \varnothing$ and $B \neq \varnothing$; moreover, the vertex set of each connected component of the graph is entirely embedded either in A or in B. Then, for every input data (G, w) of Problem WVC, we have

$$\chi_w(G) = \max(\chi_w(G[A \cup Q]), \chi_w(G[B \cup Q]))$$

(see, for example, [16]). The procedure of separating $V(G \setminus Q)$ into two parts is representable in the form of a binary tree, which is defined nonuniquely. Some tree of this kind for a graph with n vertices and m edges can be found in time $O(n \cdot m)$ [16].

A connected graph without separating trees and nontrivial modules is called *atomic*. Obviously, we have

Lemma 1. *The* WVC *problem in each hereditary class is polynomially reducible to the same problem for the atomic graphs of the class.*

A graph is called the *Berge graph* if it belongs to the class

$$\mathrm{Free}(\{C_{2i+1} \mid i > 1\} \cup \{\overline{C}_{2i+1} \mid i > 1\}).$$

A graph is called *perfect* if its chromatic and clique numbers are equal, and this holds for each of its induced subgraphs. It was proved in [17] that a graph is perfect if and only if it is a Berge graph. The following is known (see [18]):

Lemma 2. *The* WVC *problem is polynomially solvable for perfect graphs.*

It is not hard to see that $C_5 = \overline{C}_5$ and every $\{P_5, K_{2,3}, W_4, C_5, \overline{C}_7\}$-free graph is perfect. Indeed, the prohibition of an induced subgraph P_5 also forbids each cycle C_k, where $k \geq 6$. The graph $\overline{W}_4$ has three connected components two of which are isomorphic to P_2 and one is isomorphic to P_1; therefore, the prohibition of an induced subgraph $\overline{W}_4$ also forbids each subgraph $\overline{C}_k$, where $k \geq 8$.

In the following two sections, we will prove some results on the structure of atomic $\{P_5, K_{2,3}, W_4\}$-free graphs containing either an induced subgraph $\overline{C}_7$ or an induced subgraph C_5.

3. ATOMIC $\{P_5, K_{2,3}, W_4\}$-FREE GRAPHS CONTAINING THE COMPLEMENT OF AN INDUCED 7-CYCLE

Let H be an atomic $\{P_5, K_{2,3}, W_4\}$-free graph containing an induced subgraph $\overline{C}_7$. Since it is more convenient to work with the complement of H, consider the graph $G \triangleq \overline{H}$ containing an induced subgraph C_7.

Lemma 3. *The graph $G = (V, E)$ is isomorphic to a* 7*-cycle or contains an induced* 5*-cycle.*

Proof. Suppose that G is $\{C_5\}$-free. Since G is also $\{\overline{W}_4\}$-free, each vertex in G is adjacent to two vertices in a cycle $C_7 \triangleq (v_1, v_2, v_3, v_4, v_5, v_6, v_7)$. Henceforth, the indices of the vertices of the given cycle are understood modulo 7. Let $u \notin V(C_7)$ and let $\{v_i, v_{i+1} \ldots, v_{i+k}\}$ be the largest set of neighbors of u on the 7-cycle consisting of consecutive vertices of the cycle. Suppose that $1 < k < 7$. Thus, $uv_{i-1} \notin E$ and $uv_{i+k+1} \notin E$; otherwise, we get a contradiction to the choice of the set. We have $uv_{i-2} \notin E$ and $uv_{i+k+2} \notin E$; otherwise, v_{i-2}, v_{i-1}, v_i, v_{i+1}, and u or v_{i+k+2}, v_{i+k+1}, v_{i+k}, v_{i+k-1}, and u induce $\overline{P}_5$. Therefore, $k \neq 6$. We have $uv_{i-3} \notin E$ and $uv_{i+k+3} \notin E$; otherwise, v_{i-3}, v_{i-2}, v_{i-1}, v_i and u or v_{i+k+3}, v_{i+k+2}, v_{i+k+1}, v_{i+k}, and u induce C_5; thus, $k \neq 5$. Hence,

$$k \in \{2, 3, 4\}, \qquad N(u) \cap V(C_7) = \{v_i, v_{i+1} \ldots, v_{i+k}\};$$

therefore, G is not $\{\overline{K}_{2,3}\}$-free. Hence, either $k = 7$ or u cannot be adjacent to two consecutive vertices of a 7-cycle. Since G is $\{C_5\}$-free, in the last case, $N(u) \cap V(C_7) = \{v_i, v_{i+2}\}$ for some i.

Recall that the graph H contains no nontrivial modules. Consequently, neither does G. Since G is $\{\overline{P}_5\}$-free, every vertex adjacent to all vertices of the 7-cycle must be adjacent to each vertex having exactly two neighbors on the 7-cycle. Suppose that

$$u \notin V(C_7), \qquad N(u) \cap V(C_7) = \{v_i, v_{i+2}\},$$

and the set $\{u, v_{i+1}\}$ is not a module. Then there exists a vertex

$$u' \notin V(C_7), \quad N(u') \cap V(C_7) = \{v_j, v_{j+2}\}$$

for which $uu' \in E$ and $v_{i+1}u' \notin E$ or $uu' \notin E$ and $v_{i+1}u' \in E$. The second case is possible only for $j = i \pm 1$. Then u, v_i, u', v_{i+3}, and v_{i+5} or u, v_{i-1}, u', v_{i+2}, and v_{i+4} induce a subgraph $\overline{W}_4$. Consider the first case. Then $j \notin \{i-1, i+1\}$ and $j \notin \{i-2, i+2\}$ since otherwise u, u', v_{i+2}, v_{i+3}, and v_{i+4} or u, u', v_{i-2}, v_{i-1}, and v_i induce a subgraph $\overline{P}_5$. Nevertheless, $j \notin \{i-3, i+3\}$; otherwise, v_{i-3}, u', u, v_{i+2}, and v_{i+3} or v_{i-1}, v_i, u, u', and v_{i+5} induce a subgraph C_5. We have $j \neq i$ since otherwise u, u', v_{i+1}, v_{i+3}, and v_{i+4} induce a subgraph $\overline{W}_4$; a contradiction. Hence, there is no vertex belonging to the 7-cycle and having exactly two neighbors on it. Thus, $V(C_7)$ is a module itself. Hence, G is isomorphic to a 7-cycle.

Lemma 3 is proved. □

4. ATOMIC $\{P_5, K_{2,3}, W_4\}$-FREE GPAPHS CONTAINING AN INDUCED 5-CYCLE

Let $G = (V, E)$ be atomic $\{P_5, K_{2,3}, W_4\}$-free and contain an induced cycle $C_5 = (v_1, v_2, v_3, v_4, v_5)$. Everywhere below, the indices of the vertices of the cycle are understood modulo 5. Introduce the following notations for G:

- $X_i \triangleq \{x \notin V(C_5) \mid N(x) \cap V(C_5) = \{v_i, v_{i+2}\}\}$,
- $Y_i \triangleq \{y \notin V(C_5) \mid N(y) \cap V(C_5) = \{v_i, v_{i+1}, v_{i+2}\}\}$,
- $Z_i \triangleq \{z \notin V(C_5) \mid N(z) \cap V(C_5) = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}\}$,
- $T_i \triangleq \{t \notin V(C_5) \mid N(t) \cap V(C_5) = \{v_i, v_{i+2}, v_{i+3}\}\}$,
- W is the set of vertices adjacent to all vertices in the 5-cycle $(v_1, v_2, v_3, v_4, v_5)$,
- $S \triangleq V \setminus \Big\langle V(C_5) \cup \bigcup\limits_{i=1}^{5} (X_i \cup Y_i \cup Z_i \cup T_i) \cup W \Big\rangle$.

Below we formulate and prove several assertions about the structure of the sets X_i, Y_i, Z_i, T_i, and W, as well as the edges between them.

Lemma 4. *Every vertex not belonging to a* 5*-cycle and adjacent to one or several of its vertices belongs to the set*

$$\bigcup_{i=1}^{5} \big(X_i \cup Y_i \cup Z_i \cup T_i\big) \cup W.$$

Proof. If a vertex does not belong to $V(C_5) \cup S$ then it cannot be adjacent to exactly one or to exactly two adjacent vertices of the 5-cycle since G is $\{P_5\}$-free. Therefore, this vertex belongs to the desired set. The proof of Lemma 4 is over. □

Lemma 5. X_i*,* Y_i*,* Z_i*,* T_i*, and* W *are cliques for all* i*.*

Proof. Every two vertices in X_i must be adjacent; otherwise, together with the vertices v_i, v_{i+1}, and v_{i+2} they induce a subgraph $K_{2,3}$. Every two vertices of Y_i must be adjacent; otherwise, together with the vertices v_i, v_{i+1}, and v_{i+2} they induce a subgraph W_4. The same holds for the sets Z_i and W. Every two vertices in T_i must be adjacent; otherwise, together with the vertices v_{i+4}, v_i, and v_{i+3} they induce a subgraph $K_{2,3}$. Lemma 5 is proved. □

In the following lemma, we consider the sets X_1–X_5:

Lemma 6. *If X_i is not empty then*

$$X_i * (Y_i \cup Y_{i+2} \cup Y_{i+3} \cup Z_i \cup Z_{i+4} \cup T_{i+3} \cup T_{i+4} \cup W),$$

$$X_i \bullet (X_{i+1} \cup X_{i+4} \cup Y_{i+1} \cup Y_{i+4} \cup Z_{i+2} \cup T_i \cup T_{i+2}),$$

$$Z_{i+1} = Z_{i+3} = T_{i+1} = \varnothing.$$

Proof. Let x be an arbitrary element in X_i.

If $xy \in E$ and $y \in Y_i$ then y, x, v_i, v_{i+1}, and v_{i+2} induce a subgraph W_4. Therefore, $X_i * Y_i$. If $xy \in E$ and $y \in Y_{i+2}$ then v_{i+3}, y, x, v_i, and v_{i+1} induce a subgraph P_5. Hence, $X_i * Y_{i+2}$. By analogy, $X_i * Y_{i+3}$. Indeed, if $xy \in E$ for some vertex $y \in Y_{i+3}$ then v_{i+4}, y, x, v_{i+2}, and v_{i+1} induce a subgraph P_5. If $xz \in E$ and $z \in Z_i$ then z, x, v_i, v_{i+1}, v_{i+2} induce a subgraph W_4; and so $X_i * Z_i$. By analogy, $X_i * Z_{i+4}$. Indeed, if $xz \in E$ for some vertex $z \in Z_{i+4}$ then z, v_i, v_{i+1}, v_{i+2}, and x induce W_4. If $xt \in E$ and $t \in T_{i+3}$ then x, v_{i+1}, v_{i+3}, t, and v_{i+2} induce a subgraph $K_{2,3}$. Therefore, $X_i * T_{i+3}$. By analogy, $X_i * T_{i+4}$. Indeed, if $xt \in E$ for some vertex $t \in T_{i+4}$ then v_i, v_{i+1}, v_{i+4}, x, and t induce a subgraph $K_{2,3}$. If $xw \in E$ and $w \in W$ then w, x, v_i, v_{i+1}, and v_{i+2} induce a subgraph W_4. Hence, $X_i * W$.

If $xx' \notin E$ and $x' \in X_{i+1}$ then x, v_i, v_{i+4}, v_{i+3}, and x' induce a subgraph P_5. Therefore, $X_i \bullet X_{i+1}$. By analogy, $X_i \bullet X_{i+4}$. Indeed, if $xx' \notin E$ and $x' \in X_{i+4}$ then x', v_{i+4}, v_i, x, and v_{i+2} induce a subgraph P_5. If $xy \notin E$ and $y \in Y_{i+1}$ then x, v_i, v_{i+4}, v_{i+3}, and y induce a subgraph P_5. Hence, $X_i \bullet Y_{i+1}$. By analogy, $X_i \bullet Y_{i+4}$. Indeed, if $xy \notin E$ and $y \in Y_{i+4}$ then x, v_{i+2}, v_{i+1}, y, and v_{i+4} induce a subgraph P_5. If $xz \notin E$ and $z \in Z_{i+2}$ then x, v_{i+1}, z, v_i, and v_{i+2} induce a subgraph $K_{2,3}$. Therefore, $X_i \bullet Z_{i+2}$. If $xt \notin E$ and $t \in T_i$ then x, v_{i+1}, t, v_i, and v_{i+2} induce a subgraph $K_{2,3}$. Hence, $X_i \bullet T_i$. By analogy, $X_i \bullet T_{i+2}$. Indeed, if $xt \notin E$ and $t \in T_{i+2}$ then x, v_{i+1}, t, v_i, and v_{i+2} induce a subgraph $K_{2,3}$.

If $z \in Z_{i+1}$ and $xz \notin E$ then x, v_i, v_{i+1}, z, and v_{i+3} induce P_5. If $z \in Z_{i+1}$ and $xz \in E$ then x, v_{i+1}, v_{i+4}, v_i, and z induce $K_{2,3}$. Therefore, $Z_{i+1} = \varnothing$. By analogy, $Z_{i+3} = \varnothing$. Indeed, if $z \in Z_{i+3}$ and $xz \notin E$ then x, v_{i+2}, v_{i+1}, z, and v_{i+4} induce P_5; and if $z \in Z_{i+3}$ and $xz \in E$ then x, v_{i+1}, v_{i+3}, z, and v_{i+2} induce P_5. If $t \in T_{i+1}$ and $xt \in E$ then t, v_i, v_{i+2}, x, and v_{i+1} induce $K_{2,3}$. If $t \in T_{i+1}$ and $xt \notin E$ then x, v_i, v_{i+1}, t, and v_{i+3} induce P_5. Thus, $T_{i+1} = \varnothing$.

The proof of Lemma 6 is complete. □

We now consider the sets Y_1–Y_5:

Lemma 7. *If Y_i is nonempty then*

$$Y_i * (Z_{i+2} \cup T_i \cup T_{i+2}),$$

$$Y_i \bullet \left(Y_{i+1} \cup Y_{i+4} \cup Z_i \cup Z_{i+4} \cup T_{i+1} \cup T_{i+3} \cup T_{i+4} \cup W\right).$$

Proof. Let $y \in Y_i$. If $yz \in E$ and $z \in Z_{i+2}$ then y, v_i, v_{i+1}, v_{i+2}, and z induce a subgraph W_4. Hence, $Y_i * Z_{i+2}$. If $yt \in E$ and $t \in T_i$, then y, v_i, v_{i+1}, v_{i+2}, and t induce a subgraph W_4. Therefore, $Y_i * T_i$. By analogy, $Y_i * T_{i+2}$.

If $yy' \notin E$ and $y' \in Y_{i+1}$ then y, v_i, v_{i+4}, v_{i+3}, and y' induce a subgraph P_5. Hence, $Y_i \bullet Y_{i+1}$. By analogy, $Y_i \bullet Y_{i+4}$. If $yz \notin E$ and $z \in Z_i$ then v_{i+1}, v_i, y, v_{i+2}, and z induce a subgraph W_4. Therefore, $Y_i \bullet Z_i$. By analogy, $Y_i \bullet Z_{i+4}$. If $yt \notin E$ and $t \in T_{i+1}$ then v_{i+2}, y, v_i, v_{i+4}, and t induce a subgraph P_5. Hence, $Y_i \bullet T_{i+1}$. If $yt \notin E$ and $t \in T_{i+3}$ then y, v_{i+1}, t, v_{i+3}, and v_{i+4} induce a subgraph P_5; and so, $Y_i \bullet T_{i+3}$. By analogy, $Y_i \bullet T_{i+4}$. If $yw \notin E$ and $w \in W$ then v_{i+1}, v_i, y, v_{i+2}, and w induce a subgraph W_4. Therefore, $Y_i \bullet W$.

Lemma 7 is proved. □

We consider the sets Z_1–Z_5:

Lemma 8. *If Z_i is nonempty then*

$$Z_i * (Z_{i+2} \cup Z_{i+3} \cup T_{i+4}), \qquad Z_i \bullet (Z_{i+1} \cup Z_{i+4} \cup W),$$
$$T_i = T_{i+1} = T_{i+2} = T_{i+3} = \varnothing.$$

Proof. Consider $z \in Z_i$. If $zz' \in E$ and $z' \in Z_{i+2}$ then z, v_i, v_{i+1}, v_{i+2}, and z' induce a subgraph W_4. Hence, $Z_i * Z_{i+2}$. By analogy, $Z_i * Z_{i+3}$. If $zt \in E$ and $t \in T_{i+4}$ then v_i, t, v_{i+3}, v_{i+4}, and z induce a subgraph $K_{2,3}$. Therefore, $Z_i * T_{i+4}$.

If $zz' \notin E$ and $z' \in Z_{i+1}$ then v_{i+2}, v_{i+1}, z, v_{i+3}, and z' induce a subgraph W_4. Hence, $Z_i \bullet Z_{i+1}$. By analogy, $Z_i \bullet Z_{i+4}$.

If $zw \notin E$ and $w \in W$ then v_{i+1}, v_i, z, v_{i+2}, and w induce a subgraph W_4; and so, $Z_i \bullet W$.

If $zt \in E$ and $t \in T_i$ then z, v_i, v_{i+1}, v_{i+2}, and t induce a subgraph W_4. If $zt \notin E$ and $t \in T_i$ then z, t, v_{i+4}, v_i, and v_{i+3} induce a subgraph $K_{2,3}$. Hence, $T_i = \varnothing$. By analogy, $T_{i+3} = \varnothing$.

If $zt \in E$ and $t \in T_{i+1}$ then z, v_{i+1}, v_{i+2}, v_{i+3}, and t induce a subgraph W_4. If $zt \notin E$ and $t \in T_{i+1}$ then v_{i+2}, z, v_i, v_{i+4}, and t induce a subgraph P_5. Therefore, $T_{i+1} = \varnothing$. By analogy, $T_{i+2} = \varnothing$.

Lemma 8 is proved. □

In the following lemma, we consider the sets T_1–T_5:

Lemma 9. *If T_i is nonempty then*

$$T_i * W, \qquad T_i \bullet (T_{i+2} \cup T_{i+3}), \qquad T_{i+1} = T_{i+4} = \varnothing.$$

Proof. Let $t \in T_i$. If $tw \in E$ and $w \in W$ then w, v_i, v_{i+1}, v_{i+2}, and t induce a subgraph W_4. Therefore, $T_i * W$. If $tt' \notin E$ and $t' \in T_{i+2}$ then t, t', v_{i+1}, v_i, and v_{i+2} induce a subgraph $K_{2,3}$; and so, $T_i \bullet T_{i+2}$. By analogy, $T_i \bullet T_{i+3}$.

If $tt' \notin E$ and $t' \in T_{i+1}$ then v_{i+4}, t', v_{i+1}, v_{i+2}, and t induce a subgraph P_5. If $tt' \in E$ and $t' \in T_{i+1}$ then v_i, t', v_{i+2}, t, and v_{i+1} induce a subgraph $K_{2,3}$. Hence, $T_{i+1} = \varnothing$. By analogy, $T_{i+4} = \varnothing$.

Lemma 9 is proved. □

Lemma 10. *If $|X_i| \geq 2$ then $X_i * X_j$ or $X_i \bullet X_j$ for all $j \in \{i+2,\ i+3\}$.*

Proof. Suppose on the contrary that there exist vertices $x_1, x_2 \in X_i$ and $x' \in X_j$ for which $x_1x' \in E$ and $x_2x' \notin E$. By Lemma 5, we have $x_1x_2 \in E$. Symmetry considerations enable us to consider only the case $j = i+2$. Then x_2, x_1, x', v_{i+4}, and v_{i+3} induce a subgraph P_5. Hence, $X_i * X_{i+2}$ or $X_i \bullet X_{i+2}$. Lemma 10 is proved. □

In the next two lemmas, we formulate and prove some sufficient conditions for the emptiness of all X_i and T_i for $i = 1, \dots, 5$:

Lemma 11. *If some element of $Y_i \neq \varnothing$ is not adjacent to some element of $Z_{i+1} \cup Z_{i+3} \neq \varnothing$ then*

$$\bigcup_{i=1}^{5} X_i = \bigcup_{i=1}^{5} T_i = \varnothing.$$

Proof. Let $y \in Y_i$ and $z \in Z_{i+1}$; assume in addition that $yz \notin E$. The case when $z \in Z_{i+3}$ is considered by analogy from symmetry considerations. Since $Z_{i+1} \neq \varnothing$, we conclude that $X_i = X_{i+3} = \varnothing$ by Lemma 6. Since $Z_{i+1} \neq \varnothing$, we have $T_{i+1} = T_{i+2} = T_{i+3} = T_{i+4} = \varnothing$ by Lemma 8.

Suppose that $x \in X_{i+1}$. Then $xy \in E$ and $xz \notin E$ by Lemma 6. Thus, x, y, v_i, v_{i+4}, and z induce a subgraph P_5; and hence, $X_{i+1} = \varnothing$.

Let $x \in X_{i+2}$. Then $xy \notin E$ and $xz \notin E$ by Lemma 6. Thus, y, v_{i+1}, z, v_{i+4}, and x induce a subgraph P_5; and so, $X_{i+2} = \varnothing$.

Suppose that $x \in X_{i+4}$. Then $xy \in E$ and $xz \in E$ by Lemma 6. Thus, v_{i+3}, z, x, y, and v_i induce a subgraph P_5 and, therefore, $X_{i+4} = \varnothing$.

Let $t \in T_i$. Then $tz \notin E$ and $ty \notin E$ by Lemmas 7 and 8. Thus, y, v_i, t, v_{i+3}, and z induce a subgraph P_5; and so, $T_i = \varnothing$.

Lemma 11 is proved. □

Lemma 12. *If an element of Y_i is adjacent to an element of $Y_{i+2} \cup Y_{i+3}$ then*

$$\bigcup_{i=1}^{5} X_i = \bigcup_{i=1}^{5} T_i = \varnothing.$$

Proof. Let $y \in Y_i$ and $y' \in Y_{i+2}$ (the case of $y' \in Y_{i+3}$ can be considered similarly). Assume in addition that $yy' \in E$.

If $x \in X_{i+1}$ then $xy \in E$ and $xy' \in E$ by Lemma 6. Then y, x, y', v_{i+2}, and v_{i+1} induce a subgraph W_4; and so, $X_{i+1} = \varnothing$.

If $x \in X_i$ then $xy \notin E$ and $xy' \notin E$ by Lemma 6. Then x, v_i, y, y', and v_{i+3} induce a subgraph P_5; and hence, $X_i = \varnothing$. By analogy, $X_{i+2} = \varnothing$.

If $x \in X_{i+3}$ then $xy \notin E$ and $xy' \in E$ by Lemma 6. Then x, y, v_{i+4}, y', and v_i induce a subgraph $K_{2,3}$ and, therefore, $X_{i+3} = \varnothing$. By analogy, $X_{i+4} = \varnothing$.

If $t \in T_{i+2}$ then $ty \notin E$ and $ty' \notin E$ by Lemma 7. Then t, v_i, y, y', and v_{i+3} induce a subgraph P_5; and so, $T_{i+2} = \varnothing$.

If $t \in T_{i+1}$ then $ty \in E$ and $ty' \in E$ by Lemma 7. Then y', t, v_{i+3}, v_{i+2}, and y induce a subgraph W_4; and hence, $T_{i+1} = \varnothing$. By analogy, $T_{i+3} = \varnothing$.

If $t \in T_i$ then $ty \notin E$ and $ty' \in E$ by Lemma 7. Then y, v_{i+4}, t, y', and v_i induce a subgraph $K_{2,3}$; therefore, $T_i = \varnothing$. By analogy, $T_{i+4} = \varnothing$.

Lemma 12 is proved. □

Using the above, we will prove that if $S \neq \varnothing$ then G contains few vertices:

Lemma 13. *If $S \neq \varnothing$ then $|V| \leq 13$.*

Proof. Suppose the contrary.

Denote by V' the set of vertices that do not belong to S and are adjacent to the vertices of S. Since $G \in \mathrm{Free}(\{P_5\})$, we infer

$$V' \subseteq \bigcup_{i=1}^{5} (Z_i \cup T_i) \cup W.$$

The graph G is atomic; therefore, if $V' \neq \varnothing$ then V' must not be a clique since otherwise it would be separating.

Suppose that there exist two different nonadjacent vertices $u \in V'$ and $v \in V'$ that are adjacent to vertices $a \in S$ and $b \in S$ respectively. If u and v have no common neighbor in S and $ab \notin E$ then the vertices a, b, u, v and some vertices of a 5-cycle induce P_k, where $k \geq 5$. If u and v have no common neighbor in S and $ab \in E$ then $u, v \in W$; otherwise, G contains an induced subgraph P_5. Then u and v are adjacent by Lemma 5; therefore, we may assume that $a = b$.

Consider the possible cases:

If $u \in W$ then $v \in T_i$ by Lemmas 5 and 8. Then a, v_i, v_{i+2}, u, and v induce a subgraph $K_{2,3}$. If $u \in T_i$ then, by Lemmas 5, 8, and 9, we have $v \in W \cup Z_{i+1}$. The case $v \in W$ was examined earlier, and the case $v \in Z_{i+1}$ will be considered below. If $u \in Z_i$ then $v \in Z_{i+2} \cup Z_{i+3} \cup T_{i+4}$ by Lemmas 5 and 8.

The cases $v \in Z_{i+2}$ and $v \in Z_{i+3}$ are symmetric; therefore, we will consider only the first of them. Then v_i, v_{i+3}, a, v, and u induce a subgraph $K_{2,3}$.

The cases $u \in Z_i$, $v \in T_{i+4}$ and $u \in T_i$, $v \in Z_{i+1}$ are equivalent. Suppose that $u \in T_i$ and $v \in Z_{i+1}$. Since $T_i \neq \varnothing$; therefore, by Lemma 9, we have $T_{i+1} = T_{i+4} = \varnothing$. Since $Z_{i+1} \neq \varnothing$, we have $T_{i+2} = T_{i+3} = \varnothing$ by Lemma 8. Furthermore, Lemma 8 implies that

$$Z_i = Z_{i+2} = Z_{i+3} = Z_{i+4} = \varnothing$$

since otherwise $T_i = \varnothing$.

The set X_i must be empty since otherwise, by Lemma 6, we would have $Z_{i+1} = \varnothing$. By analogy, X_{i+3} must be empty since, otherwise, $Z_{i+1} = \varnothing$ by Lemma 6. 7

Check that

$$X_{i+1} = X_{i+2} = X_{i+4} = W = \varnothing.$$

If $x \in X_{i+1}$ then $xv \notin E$ and $xu \notin E$ by Lemma 6. Then a, u, v_i, v_{i+1}, and x induce a subgraph P_5; and so, $X_{i+1} = \varnothing$. By analogy, $X_{i+2} = \varnothing$. If $x \in X_{i+4}$ then $xu \in E$ and $xv \in E$ by Lemma 6. Then v_{i+2}, a, x, v, and u induce a subgraph $K_{2,3}$. Hence, $X_{i+4} = \varnothing$. If $w \in W$ then $vw \in E$ and $uw \notin E$ by Lemmas 8 and 9. But $aw \notin E$; otherwise, v_i, v_{i+2}, a, w, and u induce a subgraph $K_{2,3}$. Then v_{i+4}, w, v_{i+2}, u, and a induce a subgraph P_5; therefore, $W = \varnothing$.

Recall that $Z_{i+1} * T_i$ by Lemma 8. If $u' \in T_i$ and $u' \neq u$ then $uu' \in E$ by Lemma 5. Then $u'a \in E$; otherwise, u', u, a, v, and v_{i+1} induce a subgraph P_5. If $v' \in Z_{i+1}$ and $v' \neq v$ then $vv' \in E$ by Lemma 5. Then $v'a \in E$; otherwise; v', v, a, u, and v_i induce a subgraph P_5. Thus, if some element S is adjacent to a vertex in T_i or to a vertex in Z_{i+1} then it is adjacent to all elements in $Z_{i+1} \cup T_i$ simultaneously. Consequently, S is a module in G.

By Lemma 11, no vertex in Z_{i+1} is adjacent to a vertex in the set $Y_i \cup Y_{i+3}$. By Lemma 12, no vertex in Y_j is adjacent to a vertex in $Y_{j-2} \cup Y_{j+2}$ for any j. Thus, by Lemmas 5 and 7, each of the sets Y_1–Y_5 is a module in G. The same is true for Z_{i+1} and T_i. Hence, each of the sets S, Y_1–Y_5, Z_{i+1}, and T_i contains at most one element; therefore, $|V| \leq 13$.

The proof of Lemma 13 is complete. □

In the following lemma, we prove an important consequence of the emptiness of S and all X_i and T_i for $i = 1, \ldots, 5$:

Lemma 14. *If*

$$\bigcup_{i=1}^{5} X_i = \bigcup_{i=1}^{5} T_i = S = \varnothing$$

then G is the $\{O_3\}$-free graph.

Proof. Suppose on the contrary that G contains pairwise nonadjacent vertices x, y, and z. Lemmas 4, 5, 7, and 8 together with the hypotheses of Lemma 14 imply that

$$W \cap \{x, y, z\} = V(C_5) \cap \{x, y, z\} = \varnothing.$$

Indeed,

$$V = \bigcup_{i=1}^{5} (Y_i \cup Z_i) \cup W \cup V(C_5)$$

by Lemma 4; and for every i, by Lemma 5, the sets Y_i and Z_i as well as W are cliques. By Lemmas 7 and 8, $W \bullet Y_i$ and $W \bullet Z_i$ for each i. Thus, $W \cap \{x, y, z\} = \varnothing$. Let $x = v_i$. Then

$$\{y, z\} \subseteq \{v_{i+2}, v_{i+3}\} \cup Z_{i+1} \cup Y_{i+1} \cup Y_{i+2},$$

whence $\{y, z\} \cap \{v_{i+2}, v_{i+3}\} = \varnothing$. By Lemma 7, $Z_{i+1} \bullet (Y_{i+1} \cup Y_{i+2})$ and $Y_{i+1} \bullet Y_{i+2}$. Therefore, y and z must be adjacent.

If each of the vertices x, y, and z has exactly three neighbors on a 5-cycle then, by Lemma 5, one of them belongs to Y_i and the other, to Y_{i+1}. Therefore, by Lemma 7, they must be adjacent.

If x and y have exactly three neighbors on a 5-cycle and z has exactly four neighbors on it then, by Lemmas 5 and 7, we may assume that $x \in Y_i$ and $y \in Y_{i+2}$. This and Lemma 7 imply that $z \in Z_{i+3}$. Then the vertices x, v_{i+1}, z, v_{i+3}, and y induce a subgraph P_5.

If x has exactly three neighbors on a 5-cycle, while y and z have exactly four neighbors on it; then, by Lemmas 5 and 8, we may assume that $y \in Z_i$ and $z \in Z_{i+2}$. This and Lemma 7 implies that $x \in Y_{i+4}$. Then the vertices y, v_{i+1}, x, v_{i+4}, and z induce a subgraph P_5.

If each of the three vertices x, y, and z has exactly four neighbors on a 5-cycle then, by Lemma 5, we may assume that one of them belongs to Z_i and the other, to Z_{i+1}; therefore, by Lemma 8, they must be adjacent.

The proof of Lemma 14 is complete. □

5. SOME RESULTS ON THE COMPLEXITY OF WVC

The following two lemmas were proved in [14]:

Lemma 15. *The* WVC *problem for an* $\{O_3\}$*-free graph* $G = (V, E)$ *and a function* $w\colon V \to \mathbb{N}$ *is solvable in time*

$$O\left(\Big(\sum_{v\in V} w(v)\Big)^3\right).$$

Lemma 16. *For every fixed* C*, the* WVC *problem is solvable in polynomial time on the sum of the vertex weights in the class of graphs with at most* C *vertices.*

6. THE MAIN RESULT

The main result of the present article is as follows:

Theorem. *The* WVC *problem is solvable in polynomial time on the sum of the vertex weights in the class of* $\{P_5, K_{2,3}, W_4\}$*-free graphs.*

Proof. Let $G \in \mathrm{Free}(\{P_5, K_{2,3}, W_4\})$.

By Lemmas 1 and 2 and the remark after Lemma 2, we may assume that G is atomic and contains either an induced subgraph C_5 or an induced subgraph $\overline{C}_7$. By Lemmas 3 and 16, we may assume that G contains an induced subgraph C_5. By Lemmas 13 and 16, we may assume that $S = \varnothing$. Lemma 6 implies that, for each i, every vertex in $V \setminus (X_{i+2} \cup X_{i+3})$ is either adjacent to all vertices in X_i simultaneously or adjacent to none of them. By Lemma 10, the same holds for X_{i+2} and X_{i+3}. Therefore, X_i is a module in G.

By Lemmas 12, 14, and 15, we may assume that if $Y_i \neq \varnothing$ then $Y_i * (Y_{i+2} \cup Y_{i+3})$. By Lemmas 11, 14, and 15, we may assume that if $Y_i \neq \varnothing$ and $Z_{i+1} \cup Z_{i+3} \neq \varnothing$ then $Y_i \bullet (Z_{i+1} \cup Z_{i+3})$. This and Lemma 7 imply that Y_i is a module for all i (recall that X_i is a module for each i).

Since X_i and Y_i are modules for each i; therefore, by Lemma 8, Z_i is also a module for each i.

Since the sets X_i, Y_i, and Z_i are modules for each i; therefore, by Lemma 9, T_i is a module for each i. Hence, W is also a module in G.

Therefore, for each i, the sets Y_i, Z_i, T_i, and W and also each of the sets X_1–X_5 contain at most one vertex. Hence, $|V| \leq 26$. This and Lemma 16 complete the proof of the theorem. ☐

FUNDING

The authors were supported by the Russian Foundation for Basic Research (project no. 18–31–2001–mol-a-ved).

REFERENCES

1. M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness* (Freeman, San Francisco, 1979; Mir, Moscow, 1982).
2. D. Král, J. Kratochvíl, Z. Tuza, and G. Woeginger, "Complexity of Coloring Graphs without Forbidden Induced Subgraphs," in *Proceedings of 27th International Workshop on Graph-Theoretic Concepts in Computer Science (Boltenhagen, Germany, June 14–16, 2001)* (Springer, Heidelberg, 2001), pp. 254–262 [*Lecture Notes in Computer Science*, Vol. 2204].
3. V. V. Lozin and D. S. Malyshev, "Vertex Coloring of Graphs with Few Obstructions," Discrete Appl. Math. **216**, 273–280 (2017).
4. D. S. Malyshev, "Polynomial-Time Approximation Algorithms for the Coloring Problem in Some Cases," J. Combin. Optim. **33** (3), 809–813 (2017).
5. K. Dabrowski, P. A. Golovach, and D. Paulusma, "Colouring of Graphs with Ramsey-Type Forbidden Subgraphs," Theor. Comput. Sci. **522**, 34–43 (2014).
6. K. Dabrowski, V. V. Lozin, R. Raman, and B. Ries, "Colouring Vertices of Triangle-Free Graphs without Forests," Discrete Math. **312** (7), 1372–1385 (2012).

7. P. A. Golovach, M. Johnson, D. Paulusma, and J. Song, "A Survey on the Computational Complexity of Coloring Graphs with Forbidden Subgraphs," J. Graph Theory **84**, 331–363 (2017).
8. P. A. Golovach, D. Paulusma, and B. Ries, "Coloring Graphs Characterized by a Forbidden Subgraph," Discrete Appl. Math. **180**, 101–110 (2015).
9. C. Hoàng and D. Lazzarato, "Polynomial-Time Algorithms for Minimum Weighted Colorings of $(P_5, \overline{P}_5)$-Free Graphs and Similar Graph Classes," Discrete Appl. Math. **186**, 105–111 (2015).
10. T. Karthick, F. Maffray, and L. Pastor, "Polynomial Cases for the Vertex Coloring Problem," Algorithmica **81** (3), 1053–1074 (2017).
11. D. S. Malyshev, "The Coloring Problem for Classes with Two Small Obstructions," Optim. Lett. **8** (8), 2261–2702 (2014).
12. D. S. Malyshev, "Two Cases of Polynomial-Time Solvability for the Coloring Problem," J. Combin. Optim. **31** (2), 833–845 (2015).
13. D. S. Malyshev, "The Weighted Coloring Problem for Two Graph Classes Characterized by Small Forbidden Induced Structures," Discrete Appl. Math. **47**, 423–432 (2018).
14. D. S. Malyshev and O. O. Lobanova, "Two Complexity Results for the Vertex Coloring Problem," Discrete Appl. Math. **219**, 158–166 (2017).
15. A. Cournier and M. Habib, "A New Linear Algorithm for Modular Decomposition," in *Trees in Algebra and Programming—CAAP'94 (Proceedings of 19th International Colloquium, Edinburgh, UK, April 11–13, 1994)* (Elsevier, Heidelberg,1994), pp. 68–84. [*Lecture Notes in Computer Science*, Vol. 787].
16. R. E. Tarjan, "Decomposition by Clique Separators," Discrete Math. **55** (2), 221–232 (1985).
17. M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, "The Strong Perfect Graph Theorem," Ann. Math. **164**, 51–229 (2006).
18. M. Grötschel, L. Lovász, and A. Schrijver, "Polynomial Algorithms for Perfect Graphs," Ann. Discrete Math. **21**, 325–356 (1984).