Existence of a singular projective variety with an arbitrary set of characteristic numbers

A. Y. Buryak *

It is known that Chern characteristic numbers of compact complex manifolds cannot have arbitrary values. They satisfy certain divisability conditions. For example (see, e.g., [5])

$$\begin{aligned} 2 &\mid \langle c_1(X), [X]\rangle, \text{ for } dimX = 1,\\ 12 &\mid \left\langle c_1^2(X) + c_2(X), [X]\right\rangle, \text{ for } dimX = 2,\\ 24 &\mid \langle c_1(X)c_2(X), [X]\rangle, \text{ for } dimX = 3. \end{aligned}$$

W. Ebeling and S. M. Gusein-Zade ([1]) offer a definition of characteristic numbers of singular compact complex analytic varieties. For an n-dimensional singular analytic variety X, let $\nu\colon \widehat{X} \to X$ be its Nash transform and let $\widehat{T}X$ be the tautological bundle over $\widehat{X}$ (see, e.g, [1]). If X is embedded into a smooth complex analytic manifold M, then over the nonsingular part X_{reg} of X there is a section of $Gr_n(TM)$ given by the tangent space to X. The Nash transform $\widehat{X}$ is the closure in $Gr_n(TM)$ of the image of this section. The bundle $\widehat{T}X$ is the restriction to $\widehat{X}$ of the tautological bundle over $Gr_n(TM)$. Let the variety X be compact. For a partition $I = i_1, \ldots, i_r, i_1 + \ldots + i_r = n$ of n the corresponding characteristic number $c_I[X]$ of the variety X is defined by

$$c_I[X] := \left\langle c_{i_1}(\widehat{T}X)c_{i_2}(\widehat{T}X)\cdots c_{i_r}(\widehat{T}X), \left[\widehat{X}\right]\right\rangle,$$

where $\left[\widehat{X}\right]$ is the fundamental class of the variety $\widehat{X}$. Let $\overline{c}[X]$ be the vector $(c_I[X]) \in \mathbb{Z}^{p(n)}$, where $p(n)$ is the number of partitions of n.

Theorem. *For any vector $\overline{v} \in \mathbb{Z}^{p(n)}$ there exists a projective variety X of dimension n such that $\overline{c}[X] = \overline{v}$.*

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Let V be an algebraic variety. R. MacPherson ([6]) defined the local Euler obstruction $Eu_p(V)$ of the variety V at a point p. He proved that it is a constructible function on the variety V. Denote this function by $Eu(X)$. The notion of the integral with respect to the Euler characteristic was defined in [9]. The proof of the Theorem will use the following fact.

Lemma 1. *Let X be a compact algebraic variety of dimension n; then $c_n[X]$ is equal to the following integral with respect to the Euler characteristic*

$$c_n[X] = \int\limits_X Eu(X)d\chi.$$

Proof. For any constructible function α on the variety X R. MacPherson ([6]) defined an element $c_*(\alpha) \in H_*(X)$. From his construction it follows that

$$c_n[X] = \int\limits_X c_*(Eu(X)),$$

where the integral means the degree of the class $c_*(Eu(X))$. L. Ernström ([4]) proved that for any constructible function α on a variety X

$$\int\limits_X \alpha d\chi = \int\limits_X c_*(\alpha).$$

Lemma 1 follows from these two formulas. $\square$

Proof of Theorem 1. We need some combinations of characteristic numbers (see, e.g., [7]). Define two monomials in $t_1, \ldots, t_k$ to be equivalent if some permutation of $t_1, \ldots, t_k$ transforms one into the other. Define $\sum t_1^{i_1} \cdots t_r^{i_r}$ to be the sum of all monomials in $t_1, \cdots, t_k$ equivalent to $t_1^{i_1} \cdots t_r^{i_r}$. For any partition $I = i_1, \ldots, i_r$ of n, define a polynomial s_I in n variables as follows. For $k \geq n$ elementary symmetric functions $\sigma_1, \ldots, \sigma_n$ of $t_1, \ldots, t_k$ are algebraically independent. Let s_I be the unique polynomial satisfying

$$s_I(\sigma_1, \ldots, \sigma_n) = \sum t_1^{i_1} \cdots t_r^{i_r}.$$

This polynomial does not depend on k. Let F be a complex vector bundle over a topological space Y. For a partition I of n the cohomology class $s_I(c_1(F), \ldots, c_k(F)) \in H^{2n}(Y)$ will be denoted by $s_I(F)$. For a compact analytic variety X of dimension n and a partition I of n let the number $s_I[X]$ be defined by

$$s_I[X] := \left\langle s_I(\widehat{T}X), \left[\widehat{X}\right] \right\rangle.$$

Let $\overline{s}[X]$ be the vector $(s_I[X]) \in \mathbb{Z}^{p(n)}$. We have the following relationship between the vectors $\overline{c}[X]$ and $\overline{s}[X]$ (see, e.g., [7]). There exists a $p(n) \times p(n)$ matrix A with integer coefficients and $det(A) = \pm 1$ such that, for any compact analytic variety X of dimension n, one has $\overline{c}[X] = A\overline{s}[X]$. Hence it is sufficient to prove that for any vector $\overline{v} \in \mathbb{Z}^{p(n)}$ there exists a projective variety X such that $\overline{s}[X] = \overline{v}$.

For two complex bundles F, F' the characteristic class $s_I(F \oplus F')$ is equal to

$$s_I(F \oplus F') = \sum_{JK=I} s_J(F) s_K(F'), \tag{1}$$

where the sum is over all partitions J and K with union JK equal to I ([7]).

Let X_1, X_2 be two compact analytic varieties and $\nu_1\colon \widehat{X_1} \to X_1, \nu_2\colon \widehat{X_2} \to X_2$ be their Nash transforms. It is clear that the map $(\nu_1, \nu_2)\colon \widehat{X_1} \times \widehat{X_2} \to X_1 \times X_2$ is the Nash transform of $X_1 \times X_2$. Let $p_{1,2}\colon \widehat{X_1} \times \widehat{X_2} \to \widehat{X_{1,2}}$ be projections; then $\widehat{T}(X_1 \times X_2) = p_1^* \widehat{T} X_1 \oplus p_2^* \widehat{T} X_2$. Let n_1 and n_2 be the dimensions of X_1 and X_2. Let I be a partition of $n_1 + n_2$. From (1) it follows that

$$s_I[X_1 \times X_2] = \sum_{\substack{JK=I \\ |J|=n_1 \\ |K|=n_2}} s_J[X_1] s_K[X_2]. \tag{2}$$

Lemma 2. *For any $i \geq 1$ there exist projective varieties K_+^i and K_-^i of dimension i such that $s_i[K_\pm^i] = \pm 1$.*

We shall prove Lemma 2 later. Before that we shall deduce the statement of the Theorem from Lemma 2. Let $J = j_1, \ldots, j_q$ be a partition of n. Let

$$\begin{aligned} K_+^J &= K_+^{j_1} \times K_+^{j_2} \times \cdots \times K_+^{j_q}, \\ K_-^J &= K_-^{j_1} \times K_+^{j_2} \times \cdots \times K_+^{j_q}. \end{aligned}$$

From (2) it follows that

$$s_I[K_\pm^J] = \sum_{\substack{I_1 \cdots I_q = I \\ |I_l| = j_l}} s_{I_1}[K_\pm^{j_1}] s_{I_2}[K_+^{j_2}] \cdots s_{I_q}[K_+^{j_q}].$$

A refinement of a partition J means any partition which can be written as a union $J_1 \cdots J_q$ where each J_l is a partition of j_l. Consider the lexicographical order on partitions of n. It is obvious that if I is a refinement of J then $I \leq J$. We see that the characteristic number $s_I[K_\pm^J]$ is zero unless the partition I

is a refinement of J, hence $s_I[K^J_\pm] = 0$, if $I > J$. We have $s_I[K^I_\pm] = \pm 1$. Now it is clear that the vectors $\overline{s}[K^J_\pm]$ generate the whole lattice $\mathbb{Z}^{p(n)}$ as a semigroup. This finishes the proof of the theorem. □

Proof of Lemma 2. It is known that, for any smooth compact algebraic variety W of dimension n, there exists a smooth compact algebraic variety V of dimension n such that for any partition I of the number n we have $c_I[V] = -c_I[W]$ (see e.g. [8]). Denote the variety V by $-W$. We have (see e.g. [7])

$$s_n[\mathbb{CP}^n] = n + 1. \tag{3}$$

We see that existence of a variety K^n_- immediately follows from existence of a variety K^n_+ because $s_n[(-\mathbb{CP}^n) + nK^n_+] = -1$. We also see that it is sufficient to construct a projective variety $\widetilde{K}^n_+$ such that $s_n[\widetilde{K}^n_+] \equiv 1 \mod n+1$.

Let $n = 1$. Let $\widetilde{K}^1_+$ be the closure in $\mathbb{CP}^2$ of the semicubical parabola $\{x^2 - y^3 = 0\} \subset \mathbb{C}^2$. From Lemma 1 and properties of the local Euler obstruction ([6]) it follows that $s_1[\widetilde{K}^1_+] = c_1[\widetilde{K}^1_+] = 3 \equiv 1 \mod 2$.

Let us construct varieties $\widetilde{K}^n_+$ for any $n \geq 2$. For a smooth subvariety $X \subset \mathbb{CP}^{N-1}$ of dimension $n-1$, let $CX \subset \mathbb{CP}^N$ be the cone over X. Let $h \in H^2(\mathbb{CP}^{N-1})$ be the hyperplane class.

Lemma 3. *Suppose the element $h|_X \in H^2(X)$ is divisible by d; then*

$$s_n[CX] \equiv n s_{n-1}[X] \mod d.$$

Proof. Let $\mathbb{F}_{i_1,\dots,i_s}$ be the variety consisting of flags $(V^{i_1}, \dots, V^{i_{s-1}})$ with $V^{i_1} \subset \dots \subset V^{i_{s-1}} \subset \mathbb{C}^{i_s}$ and $dim V^{i_k} = i_k$. Denote by D_{i_k} the tautological bundle over $\mathbb{F}_{i_1,\dots,i_s}$. Let p be a point of $\mathbb{CP}^N$ and let $V \subset T_p\mathbb{CP}^N$ be a d-dimensional subspace. Denote by $g(V)$ the unique d-dimensional projective subspace of $\mathbb{CP}^N$ such that $p \in g(V)$ and $T_p(g(V)) = V$. Let $G \subset \mathbb{CP}^N$ be a d-dimensional projective subspace. By $k(G)$ denote the associated $(d+1)$-dimensional vector subspace of $\mathbb{C}^{N+1}$. Let $Y \subset \mathbb{CP}^N$ be an n-dimensional subvariety. Consider the map

$$\sigma\colon Y_{reg} \to \mathbb{F}_{1,n+1,N+1}, Y_{reg} \ni p \mapsto (k(p), k(g(T_pY_{reg}))) \in \mathbb{F}_{1,n+1,N+1}.$$

By definition the closure $\overline{\sigma(Y_{reg})}$ is the Nash transform of Y. The bundle $\widehat{T}Y$ is isomorphic to $Hom(D_1, (D_{n+1}/D_1))|_{\widehat{Y}}$.

Let $\widehat{X} \subset \mathbb{F}_{1,n,N}$ and $\widehat{CX} \subset \mathbb{F}_{1,n+1,N+1}$ be the Nash transforms of X and CX respectively. Consider the diagram

$$\begin{array}{ccccc} & & \mathbb{F}_{1,2,n+1,N+1} & \xrightarrow{\pi_2} & \mathbb{F}_{1,n+1,N+1} \\ & & \big\downarrow{\scriptstyle \pi_1} & & \\ \mathbb{F}_{1,n,N} & \xrightarrow{i} & \mathbb{F}_{2,n+1,N+1} & & \end{array}$$

where π_1, π_2 are the natural projections and the map i is defined by

$$i\colon (V^1, V^n) \mapsto (V^1 \oplus k(O), V^n \oplus k(O)),$$

where $O \in \mathbb{CP}^N$ is the vertex of the cone CX. Obviously the map i is injective. Let $Y = \pi_1^{-1}(i(\widehat{X}))$.

Lemma 4. *The image of Y under the map π_2 is $\widehat{CX}$. The map $\pi_2\colon Y \to \widehat{CX}$ is birational.*

Proof. Denote by $\overline{pq}$ the line, which goes through two different points $p, q \in \mathbb{CP}^N$. From the definition of the variety Y it follows that

$$\begin{aligned} Y = \{\, (L, k(q) \oplus k(O), k(g(T_qX)) \oplus k(O)) \in \mathbb{F}_{1,2,n+1,N+1} |\\ q \in X, L \subset k(q) \oplus k(O)\} =\\ = \{\, (k(p), k(q) \oplus k(O), k(g(T_qX)) \oplus k(O)) \in \mathbb{F}_{1,2,n+1,N+1} |\\ q \in X, p \in CX, p \in \overline{qO}\}. \end{aligned} \tag{4}$$

Note that, if $p \neq O$, then q is uniquely determined by p. Denote by Y' the subset of triples from (4) such that $p \neq O$.

It is clear that for any point $p \in CX \backslash \{O\}$ we have

$$k(g(T_pCX)) = k(g(T_{\overline{pO} \cap X}X)) \oplus k(O).$$

We see that for any element $(V^1, V^{n+1}) \in \widehat{CX} \subset \mathbb{F}_{1,n+1,N+1}$ there exist points $p \in CX$ and $q \in X$ such that $p \in \overline{qO}$ and

$$V^1 = k(p), V^{n+1} = k(g(T_qX)) \oplus k(O). \tag{5}$$

Note that a point q is not uniquely determined by the element (V^1, V^{n+1}).

The map π_2 just forgets the second element of the triple from (4) and it is clear that we obtain the pair (V^1, V^{n+1}) from (5). This completes the proof of the first part of the lemma.

Let $\widehat{CX}' = \{(V^1, V^{n+1}) \in \widehat{CX} \Big| V^1 \neq k(O)\}$. Note that if $(V^1, V^{n+1}) \in \widehat{CX}'$ then a point q from (5) is uniquely determined and $q = \overline{pO} \cap X$. Now it is clear that the map π_2 sends Y' isomorphically onto $\widehat{CX}'$. This concludes the proof of the second part of the lemma. $\square$

By $\widetilde{D}_i$ we denote the tautological bundles over $\mathbb{F}_{2,n+1,N+1}, \mathbb{F}_{1,2,n+1,N+1}, \mathbb{F}_{1,n+1,N+1}$. By D_i we denote the tautological bundles over $\mathbb{F}_{1,n,N}$. We have

$$s_n[CX] = \left\langle s_n(\widehat{T}(CX)), \left[\widehat{CX}\right] \right\rangle = \left\langle s_n(\widetilde{D}_1^* \otimes (\widetilde{D}_{n+1}/\widetilde{D}_1)), \left[\widehat{CX}\right] \right\rangle \overset{\text{lemma 4}}{=}$$
$$= \left\langle s_n(\widetilde{D}_1^* \otimes (\widetilde{D}_{n+1}/\widetilde{D}_1)), [Y] \right\rangle.$$

The map $\pi_1\colon \mathbb{F}_{1,2,n+1,N+1} \to \mathbb{F}_{2,n+1,N+1}$ is the projectivization of the bundle $\widetilde{D}_2$ over $\mathbb{F}_{2,n+1,N+1}$. We have that $i^*\widetilde{D}_2 \cong D_1 \oplus \mathbb{C}$ and $i^*\widetilde{D}_{n+1} \cong D_n \oplus \mathbb{C}$. We see that the variety Y is the total space $\mathbb{P}(D_1 \oplus \mathbb{C})$ of the projectivization of the bundle $D_1 \oplus \mathbb{C}$ over $\widehat{X}$. By τ we denote the tautological bundle over $\mathbb{P}(D_1 \oplus \mathbb{C})$. It is clear that $\tau = \widetilde{D}_1\Big|_Y$. Therefore we have

$$\left\langle s_n(\widetilde{D}_1^* \otimes (\widetilde{D}_{n+1}/\widetilde{D}_1)), [Y] \right\rangle = \langle s_n(\tau^* \otimes ((D_n \oplus \mathbb{C})/\tau)), [\mathbb{P}(D_1 \oplus \mathbb{C})] \rangle .$$

Moreover

$$\begin{aligned} s_n(\tau^* \otimes ((D_n \oplus \mathbb{C})/\tau)) &= s_n(\tau^* \otimes ((D_n/D_1) \oplus D_1 \oplus \mathbb{C})) = \\ &= s_n(\tau^* \otimes (D_n/D_1)) + s_n(\tau^* \otimes D_1) + s_n(\tau^*) = \\ &= s_n(\tau^* \otimes D_1 \otimes (D_1^* \otimes (D_n/D_1))) + s_n(\tau^* \otimes D_1) + s_n(\tau^*) = \\ &= s_n(\tau^* \otimes D_1 \otimes \widehat{T}X) + s_n(\tau^* \otimes D_1) + s_n(\tau^*). \end{aligned}$$

Let $c_1(\tau^*) = u \in H^2(\mathbb{P}(D_1 \oplus \mathbb{C}))$. We have $u^2 = uh$. Therefore from the assumption of the lemma it follows that for any $k \geq 2$ the element $u^k \in H^{2k}(\mathbb{P}(D_1 \oplus \mathbb{C}))$ is divisible by d. Hence we have

$$\begin{aligned} \langle s_n(\tau^* \otimes D_1), [\mathbb{P}(D_1 \oplus \mathbb{C})] \rangle = \langle (u-h)^n, [\mathbb{P}(D_1 \oplus \mathbb{C})] \rangle \equiv 0 \mod d, \\ \langle s_n(\tau^*), [\mathbb{P}(D_1 \oplus \mathbb{C})] \rangle = \langle u^n, [\mathbb{P}(D_1 \oplus \mathbb{C})] \rangle \equiv 0 \mod d. \end{aligned}$$

Let $x_1, \dots, x_{n-1}$ be Chern roots of the bundle $\widehat{T}X$. Then $x_1+u-h, \dots, x_{n-1}+u-h$ are Chern roots of the bundle $\tau^* \otimes D_1 \otimes \widehat{T}X$. Hence

$$\begin{aligned} &\left\langle s_n(\tau^* \otimes D_1 \otimes \widehat{T}X), [\mathbb{P}(D_1 \oplus \mathbb{C})] \right\rangle = \\ &= \left\langle \sum_{i=1}^{n-1} (x_i+u-h)^n, [\mathbb{P}(D_1 \oplus \mathbb{C})] \right\rangle \equiv \left\langle \sum_{i=1}^{n-1} (x_i+u)^n, [\mathbb{P}(D_1 \oplus \mathbb{C})] \right\rangle \equiv \\ &\equiv \left\langle \sum_{i=1}^{n-1} x_i^n + nu \sum_{i=1}^{n-1} x_i^{n-1}, [\mathbb{P}(D_1 \oplus \mathbb{C})] \right\rangle \mod d. \end{aligned}$$

The class $\sum_{i=1}^{n-1} x_i^n \in H^{2n}(\widehat{X})$ is equal to zero because $dim_{\mathbb{R}}\widehat{X} = 2n-2$. Therefore

$$\begin{aligned} &\left\langle \sum_{i=1}^{n-1} x_i^n + nu \sum_{i=1}^{n-1} x_i^{n-1}, [\mathbb{P}(D_1 \oplus \mathbb{C})] \right\rangle = \left\langle nus_{n-1}(\widehat{T}X), [\mathbb{P}(D_1 \oplus \mathbb{C})] \right\rangle = \\ &= n\left\langle (\pi_{1*}u)s_{n-1}(\widehat{T}X), \left[\widehat{X}\right] \right\rangle = n\left\langle s_{n-1}(\widehat{T}X), \left[\widehat{X}\right] \right\rangle = ns_{n-1}[X]. \end{aligned}$$

This completes the proof of Lemma 3. $\square$

Let $X = \mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^{\binom{2n}{n-1}-1}$ be the image of the Veronese embedding of degree $n+1$. Let $\widetilde{K}_+^n = CX$. From (3) and lemma 3 it follows that $s_n[\widetilde{K}_+^n] \equiv n^2 \equiv 1 \mod n+1$. This concludes the proof of Lemma 2. $\square$

Remark. *It seems to be interesting to construct a cobordism theory of singular varieties associated to this notion of characteristic numbers.*

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