BOTT'S RESIDUE FORMULA FOR SINGULAR VARIETIES

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Abstract. In this paper we develop a differential-geometric approach to the characteristic numbers of singular varieties. In particular we generalize Bott's residue formula for singular varieties.

1. Introduction

Characteristic numbers of compact smooth manifolds are important topological invariants. In [5] W. Ebeling and S. M. Gusein-Zade offered a definition of characteristic numbers of singular compact complex analytic varieties. In [4] the author proved that there exists a singular projective variety with an arbitrary given set of characteristic numbers. It is well known that this fact is not true for smooth varieties (see e.g. [7]).

There is the well known construction of characteristic classes using the curvature tensor. Hence we can compute characteristic numbers by integration of certain differential forms. In this paper we generalize this approach to singular varieties. We prove that characteristic numbers of a singular variety are equal to integrals of certain differential forms over the smooth part of the variety.

In [3] R. Bott gave a method for a computation of characteristic numbers using holomorphic vector fields. We give a generalization of this result for singular varieties. As a byproduct of his construction R. Bott defined new invariants of a holomorphic vector field near its singular point and proved that the sum of these invariants over all singular points of a holomorphic vector field on a smooth compact analytic manifold is equal to zero. In this paper we also give a partial generalization of this result.

Acknowledgements. The author is grateful to professor S. M. Gusein-Zade for suggesting the area of the research and for constant attention to this work.

The author is partially supported by the grants RFBR-07-01-00593, NSh-709.2008.1, Vidi grant of NWO and by the Moebius Contest Foundation for Young Scientists.

2. Main results

2.1. **Characteristic numbers of singular varieties.** For a singular analytic variety X of dimension n, let $\nu_X\colon \widehat{X} \to X$ be its Nash transform and let $\widehat{T}X$ be the tautological bundle over $\widehat{X}$ (see, e.g, [6]). If X is embedded into a smooth complex analytic manifold M, then over the nonsingular part X_{reg} of X there is a section of $Gr_n(TM)$ given by the tangent space to X. The Nash transform $\widehat{X}$ is the closure in $Gr_n(TM)$ of the image of this section and the map ν_X is the restriction of the projection $Gr_n(TM) \to M$ to $\widehat{X}$. The bundle $\widehat{T}X$ is the restriction to $\widehat{X}$ of the tautological bundle over $Gr_n(TM)$. Let the variety X be compact. For a partition $I = i_1, \ldots, i_r, i_1 + \ldots + i_r = n$ of n the corresponding characteristic number $c_I[X]$ of the variety X is defined by

$$c_I[X] := \left\langle \prod_{j=1}^{r} c_{i_j}(\widehat{T}X), \left[\widehat{X}\right] \right\rangle,$$

where $\left[\widehat{X}\right]$ is the fundamental class of the variety $\widehat{X}$.

2.2. **Differential-geometric construction.** Let $X \subset M$ be an n-dimensional compact complex analytic subvariety of a smooth complex manifold M. Let us choose a hermitian structure on M and restrict it to X_{reg}. Let ∇ be the canonical connection in TX_{reg} and K its curvature. Consider differential forms $\widetilde{c}_r \in \Omega^{2r}(X_{reg})$ defined as follows

$$\sum_i \lambda^i \widetilde{c}_i = det\left(1 + \frac{i}{2\pi}\lambda K\right).$$

Let $I = i_1, \ldots, i_k$ be a partition of n. The first result of this paper is the following theorem.

Theorem 2.1. $c_I[X] = \int_{X_{reg}} \prod_{j=1}^{k} \widetilde{c}_{i_j}$.

Proof. Let $\widehat{X} \subset Gr_n(TM)$ be the Nash transform of X. The hermitian structure on M defines the hermitian structure in the tautological bundle τ_n over $Gr_n(TM)$. Let $\overline{\nabla}$ be the canonical connection in τ_n and $\overline{K}$ its curvature. Consider differential forms $\overline{\widetilde{c}}_r \in \Omega^{2r}(Gr_n(TM))$ defined as follows

$$\sum_i \lambda^i \overline{\widetilde{c}}_i = det\left(1 + \frac{i}{2\pi}\lambda \overline{K}\right).$$

It is clear that $\overline{\widetilde{c}}_r|_{\nu_X^{-1}(X_{reg})} = \widetilde{c}_r$. The form $\overline{\widetilde{c}}_r$ represents the class $c_r(\tau_n) \in H^{2r}(Gr_n(TM))$. The variety $\widehat{X}$ is analytic, so

$$\left\langle c_{i_1}(\tau_n) \ldots c_{i_k}(\tau_n), \left[\widehat{X}\right] \right\rangle = \int_{\widehat{X}} \overline{\widetilde{c}}_{i_1} \ldots \overline{\widetilde{c}}_{i_k}.$$

This concludes the proof of the theorem. $\square$

2.3. **Bott's residue formula for singular varieties.** Let $X \subset M$ be an n-dimensional complex analytic subvariety of a smooth complex manifold M. Let $Z \subset X$ be a compact analytic subset such that $X_{sing} \subset Z$. Let V be a holomorphic vector field on $X \backslash Z$ such that for any $p \in X \backslash Z$ we have $V(p) \neq 0$. Let $\phi(c_1, \ldots, c_n)$ be a homogeneous polynomial of degree n, where the degree of c_i is equal to i. We construct a residue $Res_\phi(Z)$ of the field V near the set Z. In fact we show that Bott's construction from [3] works in our situation. Suppose X is compact. Let $\phi[X] = \left\langle \phi(c(\widehat{T}X)), \left[\widehat{X}\right] \right\rangle$. We prove the following theorem.

Theorem 2.2. *$\phi[X] = Res_\phi(Z)$.*

We show a few examples of a computation of these residues.

2.4. **The residue Res_1 for singular varieties.** Let $X \subset M$ be an n-dimensional complex analytic subvariety of a smooth complex manifold M. Let V be a holomorphic vector field on X_{reg}. We say that V is holomorphic on X if for any point $p \in X_{sing}$ there exists an open set $U \subset M, p \in U$ and a holomorphic vector field W on U such that W is tangent to $X_{reg} \cap U$ and $W|_{X_{reg} \cap U} = V$. We say that the field V is not equal to zero at a point $p \in X_{sing}$ if $W(p) \neq 0$.

Let $Z \subset X$ be a compact complex analytic subset and V be a holomorphic vector field on $X \backslash Z$ such that for any point $p \in X \backslash Z$ we have $V(p) \neq 0$. We construct a residue $Res_1(Z)$ of the field V near the set Z. Suppose X is compact. We prove the following theorem.

Theorem 2.3. *$Res_1(Z) = 0$.*

We obtain a simple formula for this residue in the following situation. Consider the vector field $V = \sum_{i=1}^N \lambda_i z^i \frac{\partial}{\partial z^i}$ in $\mathbb{C}^N$, where $\lambda_i \neq 0$. Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a germ of an n-dimensional variety such that the field V is tangent to it. Consider a subset $A \subset \{1, \ldots, N\}, |A| = n$. Let $\Pi_A \subset \mathbb{C}^N$ be the coordinate vector subspace. Suppose the projection $p_A \colon (X, 0) \to (\Pi_A, 0)$ is a branched covering of degree d_A. We obtain the following theorem.

Theorem 2.4. *$Res_1(0) = \frac{d_A}{\prod_{i \in A} \lambda_i}$.*

3. Constructions and proofs

3.1. **The residue Res_ϕ.** We follow the notations of Section 2.3. We have the decomposition $T_{\mathbb{C}}M = T'_{\mathbb{C}}M \oplus T''_{\mathbb{C}}M$, where $T_{\mathbb{C}}M$ is the complexified tangent bundle of M, $T'_{\mathbb{C}}M$ is in duality with the forms of type $(1, 0)$ and $T''_{\mathbb{C}}M$ is in duality with the forms of type $(0, 1)$. Let $U = X \backslash Z$. Consider an arbitrary hermitian metric in the bundle $T'_{\mathbb{C}}M$

and its restriction to $T'_{\mathbb{C}}U$. Let ∇ be the canonical connection in the bundle $T'_{\mathbb{C}}U$. Consider a section $L \in \Gamma(End(T'_{\mathbb{C}}U))$ defined as follows

$$Ls = [V, s] - \nabla_V s,$$

where $s \in \Gamma(T'_{\mathbb{C}}U)$. Let π be an arbitrary differential $(1,0)$-form on U such that $\pi(V) = 1$. Let K be the curvature of ∇. For an arbitrary $n \times n$ matrix A let $\phi(A) = \phi(\sigma_1(A), \ldots, \sigma_n(A))$, where $\sigma_i(A)$ is the i-th symmetric function of the eigenvalues of the matrix A. Let $\eta^{(k)}$ be the coefficient of t^k in the series $\eta = \frac{\phi(L+tK)\pi}{1-t\bar{\partial}\pi}$. Let $Z_\varepsilon \subset X$ be an ε-neighbourhood of Z. Define the residue $Res_\phi(Z)$ by the following formula

$$Res_\phi(Z) = \left(\frac{i}{2\pi}\right)^n \lim_{\varepsilon \to 0} \int\limits_{\partial Z_\varepsilon} \eta^{(n-1)}. \tag{1}$$

Let us prove the existence of the limit on the right hand side of (1). From [2] we know that

$$\phi(K) + d\eta^{(n-1)} = 0. \tag{2}$$

From the proof of Theorem 2.1 it follows that the integral $\int_{Z_\delta} \phi(K)$ is well defined. Using (2) and Stokes' theorem, we get $\int_{Z_\delta} \phi(K) + \int_{\partial Z_\delta} \eta^{(n-1)} = \lim_{\varepsilon \to 0} \int_{\partial Z_\varepsilon} \eta^{(n-1)}$. This equation proves the existence of the limit. It is easy to show that the residue $Res_\phi(Z)$ doesn't depend on the metric on M, on the form π and on the embedding of X into M. Let the variety X be compact. Theorem 2.2 immediately follows from Theorem 2.1 and the equation (2).

Example 3.1. For a smooth subvariety $X \subset \mathbb{CP}^N$ of dimension n let $CX \subset \mathbb{C}^{N+1}$ be the cone over X. Consider the vector field $V = \sum_{i=1}^{N+1} z^i \frac{\partial}{\partial z^i}$ in $\mathbb{C}^{N+1}$. The field V is tangent to CX. Let O be the vertex of the cone CX. We can compute all residues $Res_\phi(O)$ by the following procedure. Let $\overline{CX}$ be the closure of CX in $\mathbb{CP}^{N+1}$. From [4] we know how to compute all characteristic numbers of $\overline{CX}$. On the other hand the field V can be extended to the holomorphic field on $\overline{CX}\backslash O$ with a zero of order 1 along the divisor $D = \overline{CX}\backslash CX$. In [1] there are formulas for all residues $Res_\phi(D)$. Hence using Theorem 2.2 we can compute all residues $Res_\phi(O)$. Let us show a few examples. Let d be the degree of X. Let $H \subset \mathbb{CP}^N$ be a general hyperplane.

$$\begin{aligned}
dimX = 1, \qquad & Res_{c_2}(O) = c_1[X] - d, \\
& Res_{c_1^2}(O) = 4c_1[X] - 4d, \\
dimX = 2, \qquad & Res_{c_3}(O) = c_2[X] - c_1[X \cap H], \\
& Res_{c_1 c_2}(O) = 3c_2[X] + 2c_1^2[X] - 9c_1[X \cap H], \\
& Res_{c_1^3}(O) = 9c_1^2[X] - 27c_1[X \cap H].
\end{aligned}$$

If X is a hypersurface then we have the following

$$\begin{aligned} dimX = 1, \qquad & Res_{c_2}(O) = 2d - d^2, \\ & Res_{c_1^2}(O) = 8d - 4d^2, \\ dimX = 2, \qquad & Res_{c_3}(O) = d^3 - 3d^2 + 3d, \\ & Res_{c_1 c_2}(O) = 5d^3 - 19d^2 + 23d, \\ & Res_{c_1^3}(O) = 9d^3 - 45d^2 + 63d. \end{aligned}$$

3.2. **The residue Res_1.** Again we follow the notations of Section 2.4 Let $h(\cdot,\cdot)$ be an arbitrary hermitian form in the bundle $T'_{\mathbb{C}}M$ such that for any point $p \in X\backslash Z$ we have $h(V_p, V_p) \neq 0$. Consider the differential $(1,0)$-form $\pi_{h,V}$ on $(X\backslash Z)_{reg}$ defined as follows $\pi_{h,V}(A) = \frac{h(A,V)}{h(V,V)}$, where $A \in \Gamma(T'_{\mathbb{C}}(X\backslash Z)_{reg})$. For any point $p \in X\backslash Z$ there exist a neighbourhood $U \subset M, p \in U$ and a $(1,0)$-form π_U on U such that $\pi_{h,V}|_{U\cap(X\backslash Z)_{reg}} = \pi_U|_{U\cap(X\backslash Z)_{reg}}$, so we can integrate the form $\pi_{h,V}(\overline{\partial}\pi_{h,V})^{n-1}$ over an arbitrary cycle in $X\backslash Z$. The inner product of a form θ by a vector field W is denoted by $i_W\theta$. It is easy to see that $i_V\overline{\partial}\pi_{h,V} = 0$, hence $i_V(\overline{\partial}\pi_{h,V})^n = 0$ and $(\overline{\partial}\pi_{h,V})^n = 0$. We see that an integral of the form $\pi_{h,V}(\overline{\partial}\pi_{h,V})^{n-1}$ over a cycle doesn't depend on its homology class. Let $Z_\varepsilon \subset X$ be an ε-neighbourhood of Z. Now we shall give the following definition.

$$Res_1(Z) = \left(\frac{i}{2\pi}\right)^n \int\limits_{\partial Z_\varepsilon} \pi_{h,V}(\overline{\partial}\pi_{h,V})^{n-1}.$$

Theorem 2.3 immediately follows from this definition. This residue doesn't depend on a choice of the form h and on an embedding of X into M.

Proof of Theorem 2.4. Consider the form $h = \sum\limits_{i\in A} dz^i d\overline{z}^i$ in space $\mathbb{C}^N$ and the form $\pi_{h,V}$. By definition

$$Res_1(0) = \left(\frac{i}{2\pi}\right)^n \int\limits_{S_\varepsilon^{2N-1}\cap X} \pi_{h,V}(\overline{\partial}\pi_{h,V})^{n-1}.$$

Consider the vector field $V_A = \sum_{i\in A} \lambda_i z^i \frac{\partial}{\partial z^i}$ and the form $h_A = \sum_{i\in A} dz^i d\overline{z}^i$ on the space Π_A. It is clear that $p_A^*\pi_{h_A,V_A} = \pi_{h,V}$, hence

$$\left(\frac{i}{2\pi}\right)^n \int\limits_{S_\varepsilon^{2N-1}\cap X} \pi_{h,V}(\overline{\partial}\pi_{h,V})^{n-1} = \left(\frac{i}{2\pi}\right)^n \int\limits_{p_{A*}(S_\varepsilon^{2N-1}\cap X)} \pi_{h_A,V_A}(\overline{\partial}\pi_{h_A,V_A})^{n-1} =$$

$$= d_A \left[\left(\frac{i}{2\pi}\right)^n \int\limits_{S_\varepsilon^{2n-1}} \pi_{h_A,V_A}(\overline{\partial}\pi_{h_A,V_A})^{n-1} \right].$$

The expression in the brackets was computed in [3] and is equal to $\frac{1}{\prod\limits_{i\in A}\lambda_i}$. This completes the proof of the theorem. $\square$

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