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The vertex cover game: Application to transport networks☆

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ABSTRACT

As communications are progressing, transport networks need to be monitored, more specifically, camera surveillance of violations is needed. Standards are being developed on how to install cameras, and the question of efficiently distributing surveillance devices across the road network ensue. This task is addressed in this paper by using the methods of the cooperative game theory. The vertex cover game is introduced, and its properties are studied. Since surveillance cameras are to cover all areas of the network, the characteristic function depends on the vertex covers of the graph. The Shapley–Shubik index is used as the measure of centrality. The Shapley–Shubik index is shown to be efficient in a vertex cover game for the allocation of cameras in a transport network. Proceeding from the Shapley–Shubik indices calculated in this study, recommendations were given for the allocation of surveillance cameras in a specific transport network in a district of the City of Petrozavodsk, Russia.

1. Introduction

There is a concept of the vertex cover of an undirected graph in the graph theory [34]. A vertex cover of a graph $G = \langle N, E\rangle, E \subseteq \{\{a, b\}|a, b \in N\}$ is any subset S of the set of graph vertices N such that any edge of this graph is incident to least one vertex in the set S. We are interested in the vertex cover with a minimal number of vertices, i.e. so-called minimum vertex cover of a graph.

Let us list some of the applications for the problem of finding the minimum vertex cover.

Vertex cover in transport networks. Let the undirected graph G be a transport network. Each vertex is a crossroad, an edge is a road. If surveillance cameras are deployed at vertices of the minimum vertex cover, then every road portion will be monitored by a camera and the costs of purchasing cameras will be minimized [1,61].

Vertex cover in a society. Let each vertex of the graph G be a person. If there is a conflict between two people, then there exists an edge between the vertices representing these people. Denote by S the minimum vertex cover. There are no conflicts between people in the set $N\backslash S$, and the dimensionality of this set is maximal [14,40].

Vertex cover in computer networks. Let the graph G be a network of servers. Each vertex of the graph G is a server. There is an edge between two vertices if their corresponding servers are interconnected. It is demonstrated in [22] that worm propagation on a network depends on the topology of the graph G. Timely handling of the servers in the minimum vertex cover can mitigate or terminate worm propagation [12,59,60].

Vertex cover in business processes [9]. Let each vertex of the graph G be a state of a certain process. If transition from one state to another is possible, then there exists an edge between the corresponding vertices. Some work is performed during the transition from one state of the process to another. Errors may arise while the work is being implemented. Checks are required to control errors. If such checks are made at vertices in the minimum vertex cover, the number of checks will be minimized. Also, if a process has been in two states, at least one check will be performed.

Potential applications of the vertex cover of a graph are related in more detail in [53]. The question arises of which vertex in the identified cover is the most significant.

Depending on the sphere of application, the questions to be answered are following. How many surveillance cameras are required for a crossroads depending on the road layout? Which individual in a community is the most conflictive and why? Which server requires higher defense than other servers? How should checks be distributed among states of a process? This paper will give the answer to the first question for the transport network of one Petrozavodsk city district. One must note here that for some practical rea-

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sons the minimum vertex cover may be inferior in its properties to non-minimum vertex cover. Minimum vertex cover can change if some edges are added to or removed from the graph. That is why the focus when considering the mathematical model in which the topology of the graph will be variable should rather be placed on a cover that is not minimal. Such a cover would very likely remain a cover if the graphs topology is transformed. In transport networks however, major roads emerge or disappear rarely, so decision-making based on vertex centrality in a minimum vertex cover is acceptable.

To determine the power of vertices in the vertex cover, we use the methods of the cooperative game theory. In the terms of the game theory, each vertex can be regarded as an individual player. We consider the terms vertex and player to be synonyms. The vertex cover is a coalition of players. If a group of vertices contains at least one vertex cover, the payoff of this group of vertices is unity; otherwise it is zero. The characteristic function of the cooperative game shall take only two values: 0 & 1 [11]. Shapley suggested using the cooperative game theory for measuring the influence of parties. The application of Shapley values and cooperative games to allocate resources can be found in [5,15,33,36]. An application of the Shapley value to transport and computer networks is described in [30,39,52]. The Shapley–Shubik index is also used in benefit allocation in games on electricity networks [66], and for the analysis of hierarchical structures [27], and human behavior [45,46,50,64]. In this study, the Shapley–Shubik index is applied to estimate the power of graph vertices with regard to vertex covers. Methods of decision-making on networks can be found in [17,65,67]. For more information about network games, see [7,44]. The problems of resource allocation on the network are studied in [8,54].

For voting games with quota q, where each player has some weight w_i, $i \in N$, methods based on generating functions have been worked out for calculating the Shapley–Shubik index [4,16,51]. In reference [16], the Public Good Index is calculated and in [51] an extension of the Shapley value. The computation of the Shapley–Shubik index in simple games with restrictions on the set of minimal winning coalitions is studied in [3,13]. However, not any game with a simple monotone characteristic function would have weights w_i and quota q. Some algorithms for calculating the Shapley–Shubik index can be found in [10,42].

Structure of the article. The section Preliminaries gives the key concepts and definitions. Section 3 introduces a definition of the vertex cover game and studies its properties. Section 4 deals with calculating the Shapley–Shubik index in the vertex cover game. In Section 5 specific transport network was considered. Recommendations are given on how to distribute resources in a specific transport network. The Appendix provides proof of the results of the paper.

2. Preliminaries

Let $N = \{1, 2, \dots, n\}$ be the set of players. Denote by 2^N the set of all kinds of subsets of the set N.

Consider a cooperative game $\langle N, v\rangle$, where v is a characteristic function, $v : 2^N \to \mathbb{R}, v(\emptyset) = 0$. Characteristic function v is *monotone* if $\forall S, T \subseteq N$: $S \subseteq T \Rightarrow v(S) \le v(T)$. A game $\langle N, v\rangle$ is a *simple game* when 1. $\forall S \subseteq N : v(S) = 0$ or $v(S) = 1$; 2. $v(N) = 1$; 3. Monotone is true [62].

A coalition S is *winning* if $v(S) = 1$ and *losing* otherwise. The set of winning coalitions is denoted by $W = W(v)$. K is called *a minimal winning coalition* if $v(K) = 1$ and $\forall i \in K : v(K \setminus \{i\}) = 0$. The set of minimal winning coalitions is denoted by $W^m = W^m(v)$.

Player $i \in N$ is *critical* in a coalition S if and only if $i \in S \in W$ and $S \setminus \{i\} \notin W$. Critical player and *pivotal* player are synonyms [62].

Let $S \subseteq N$ and $S \neq \emptyset$. A pair $\langle N, v_S\rangle$ is an *unanimity game* [57], where

$$v_S(K) = \begin{cases} 1, & \text{if } S \subseteq K; \\ 0, & \text{otherwise.} \end{cases}$$

Any simple monotone characteristic function can be defined through the set of minimal winning coalitions as follows:

$$v(K) = \begin{cases} 1, & \text{if } \exists A \in W^m : A \subseteq K; \\ 0, & \text{otherwise.} \end{cases}$$

The *union* (*intersection*) of the simple games $\langle N, v\rangle$ and $\langle N, w\rangle$ is the game $\langle N, (v \vee w)\rangle$ (resp. $\langle N, (v \wedge w)\rangle$) in which the set of winning coalitions is the union (intersection) of the sets of winning coalitions for $\langle N, v\rangle$ and $\langle N, w\rangle$ [26,63].

Let $\langle N, v\rangle$ is simple game. The Shapley–Shubik index for the player $i \in N$ is determined by the formula

$$\phi_i(v) = \sum_{\substack{K \in W(v): \\ K \setminus \{i\} \notin W(v)}} \frac{(|K| - 1)!(|N| - |K|)!}{|N|!}$$

[18,20,43].

A simple game $\langle N, v\rangle$ is a *weighted majority game* if it admits a representation by means of $n + 1$ nonnegative real numbers $[q; w_1, \dots, w_n]$ such that

$$v(K) = \begin{cases} 1, & \sum_{i \in K} w_i \ge q \\ 0, & \sum_{i \in K} w_i < q \end{cases}, \forall K \subseteq N$$

[24,26].

Weighted majority games is a subclass of simple monotonic games. Simple game can be represented by union [62] and intersection [25] of weighted majority games.

3. Cooperative vertex cover game

This section investigates vertex cover game properties. Section 3.1 gives a definition of the vertex cover game. Section 3.2 proves the graph decomposition theorem, finds the necessary and sufficient conditions for a simple game to be a vertex cover game. Section 3.3 studies the dimensionality of a vertex cover game.

3.1. Definition of the vertex cover game

Let us introduce several concepts to define the cooperative vertex cover game. A *vertex cover* S of an undirected graph $G = \langle N, E\rangle$ is a subset of N such that $\forall (u, v) \in E \Rightarrow u \in S$ or $v \in S$ [22,32]. The *minimum vertex cover* of the graph $G = \langle N, E\rangle$ is the vertex cover consisting of the smallest possible number of vertices. A vertex cover S of the graph G is called *least vertex cover* if for $\forall i \in S$ the set $S \setminus \{i\}$ is not a vertex cover. Denote by $M(G)$ the set of least vertex covers of the graph G.

The set of least vertex covers for the graph shown in Fig. 1 equals $M(G) = \{\{1, 3, 5\}, \{2, 3, 6\}, \{1, 2, 4, 5\}, \{2, 4, 5, 6\}\}$. Any other vertex cover for the graph in question contains at least one element of the set $M(G)$.

Definition 1. Let $G = \langle N, E\rangle, E \neq \emptyset$ an undirected graph, $M(G)$ is the set of least vertex covers of the graph G. A simple game $\langle N, v\rangle$ is a vertex cover game of G if $W^m(v) = M(G)$, that is

$$v(K) = \begin{cases} 1, & \text{if } \exists A \in M(G) : A \subseteq K; \\ 0, & \text{otherwise.} \end{cases} \quad \forall K \subseteq N.$$

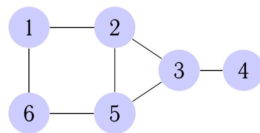

Fig. 1. Graph G with 6 vertices.

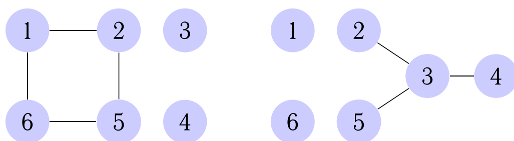

Fig. 2. Decomposition of the graph G from Fig. 1. Graph G_1 on the left, G_2 on the right.

3.2. Decomposition theorem

Theorem 1 shows how the characteristic function of the vertex cover game can be represented in the form of an intersection of simple monotone characteristic functions.

Theorem 1. *Let* $Y = \{1, 2, \ldots, r\}, G = \langle N, E\rangle, G_j = \langle N, E_j\rangle, j \in Y$ *be undirected graphs, where* $E = \bigcup_{j\in Y} E_j$. *Then,* $\forall K \subseteq N$ *we have*

$$v(K) = (v_1 \wedge v_2 \wedge \ldots \wedge v_r)(K), \qquad (1)$$

where $\forall j \in Y \langle N, v\rangle, \langle N, v_j\rangle$ *are vertex cover game with* $W^m(v) = M(G), W^m(v_j) = M(G_j)$.

The proof is in Appendix.

Theorem 1 is based on the properties of vertex covers of a graph. No papers have been found that analyze simple games in which the set of minimal winning coalitions is the set of least vertex covers of a graph. Applications for the union and intersection of simple functions can be found in the papers [2,23,47].

Example 1. Let us demonstrate the application of Theorem 1. Consider the graph $G = \langle N, E\rangle$ in Fig. 1. The graph G can be decomposed into the graphs $G_1 = \langle N, E_1\rangle, G_2 = \langle N, E_2\rangle$, where G_1, G_2 are shown in Fig. 2.

We get $W^m(v_1) = \{\{1, 5\}, \{2, 6\}\}, W^m(v_2) = \{\{3\}, \{2, 4, 5\}\}$. Since $E_1 \cup E_2 = E$, then, according to Theorem 1, $v(K) = (v_1 \wedge v_2)(K)$, $W^m(v) = M(G)$.

Consider two simple games $\langle N, v\rangle$, $\langle N, w\rangle$, such that $|W^m(v)| = |W^m(w)| = a$ and $\forall A \in W^m(v)\ \forall B \in W^m(w) : A \not\subseteq B$ and $B \not\subseteq A$. Then,

$$W^m(v \vee w) = \{A | A \in W^m(v) \text{ or } A \in W^m(w)\},$$

$$W^m(v \wedge w) = \{A \cup B | A \in W^m(v) \text{ and } B \in W^m(w)\}.$$

Hence, $|W^m(v \wedge w)| = a^2$, and $|W^m(v \vee w)| = 2a$. Since the equality $(v \wedge w)(K) = v(K) + w(K) - (v \vee w)(K)$, holds for simple games, then instead of considering the characteristic function with a^2 minimal winning coalitions we can simultaneously consider three games, the total number of minimal winning coalitions in the three games being $4a$. Knowing the representation of the characteristic function in the form of a conjunction of simple games, one can consider games with a smaller number of minimal winning coalitions.

Value ϕ has linearity property if $\phi(\alpha v + \beta w) = \alpha\phi(v) + \beta\phi(w), \alpha, \beta \in \mathbb{R}; v, w$ are characteristic functions. If the characteristic function of a cooperative vertex cover game can be represented in the form of a linear combination of characteristic functions, then the linearity property can be used to calculate the Shapley–Shubik index. The original graph can be decomposed into subgraphs so as to fulfill the conditions of Theorem 1. Then, using $(v \wedge w)(K) = v(K) + w(K) - (v \vee w)(K)$, the characteristic function of the vertex cover game can be represented in the form of a linear combination of other functions. It is convenient to use the decomposition procedure if a graph is large enough, e.g. a network- or a communication graph. With this approach, it is not necessary to know the minimal winning coalitions of the original characteristic function. The same is true for linear values in games, such as the Banzhaf value, Owen value and others.

Not any simple game is a vertex cover game. Taking $N = \{1, 2, 3\}$, consider the connected, undirected graphs $G_1 = \langle N, \{\{1, 2\}, \{2, 3\}\}\rangle$ and $G_2 = \langle N, \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}\rangle$. Then, $W^m(v_1) = \{\{2\}, \{1, 3\}\}, W^m(v_2) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, where $\langle N, v_1\rangle$ and $\langle N, v_2\rangle$ are vertex cover games on graphs G_1 and G_2, respectively. If we consider disconnected graphs with three vertices, there will be vertices not belonging to any vertex cover of the graph. Hence, simple games with a set of minimal winning coalitions $W^m(v') = \{\{1\}, \{2\}, \{3\}\}$ and $W^m(v'') = \{\{1, 2, 3\}\}$ are not vertex cover games.

Theorem 2. *A simple game* $\langle N, v\rangle$, *is a vertex cover game on graph* G *if and only if there exist simple games* $\langle N, v_l(K)\rangle, l \in \{1, 2, \ldots, r\}, W^m(v_l) = \{\{i_l\}, \{k_l\}\}$ *for which the equality*

$$v(K) = (v_1 \wedge v_2 \wedge \ldots \wedge v_r)(K) \qquad (2)$$

holds, and $G = \langle N, E\rangle, E = \{\{i_l, k_l\} | 1 \le l \le r\}$.

The proof is in Appendix.

For a given graph G it suffices to find all of its least vertex covers to compose a simple vertex cover game for the graph. Rewriting a graph from a set of minimal winning coalitions is a more challenging task. Hypothetically, such a problem is NP complete or NP hard, since finding the minimum vertex cover is NP complete problem. One of the ways to reconstruct a graph is demonstrated in Example 2.

Example 2. Let $\langle N, v\rangle, N = \{1, 2, \ldots, 7\}$ is the simple game,

$$W^m(v) = \{\{1, 3, 4, 6\}, \{1, 3, 4, 5, 7\}, \{2, 5, 6\}, \{2, 5, 7\}, \\ \{2, 3, 4, 6\}, \{2, 3, 4, 5, 7\}\}.$$

In order to determine whether the game $\langle N, v\rangle$ is a vertex cover game, special-form characteristic functions need to be selected to fulfill the equality from Theorem 2. In the papers [37,41], the vertex and the graph are compared to the Boolean variable and Boolean function, respectively. Consider the function $f(W^m(v), x) = \underset{A\in W^m(v)}{\vee}\left(\underset{i\in A}{\wedge} x_i\right)$. For the aforementioned set $W^m(v)$ the function $f(W^m(v), x)$ takes the form

$$f(W^m(v), x) = (x_1 \wedge x_3 \wedge x_4 \wedge x_6) \vee (x_1 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_7) \\ \vee (x_2 \wedge x_5 \wedge x_6) \vee (x_2 \wedge x_5 \wedge x_7) \vee (x_2 \wedge x_3 \wedge x_4 \wedge x_6) \\ \vee (x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_7).$$

Transforming $f(W^m(v), x)$, we get

$$f(W^m(v), x) = (x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (x_2 \vee x_4) \wedge (x_3 \vee x_5) \\ \wedge (x_4 \vee x_5) \wedge (x_5 \vee x_6) \wedge (x_6 \vee x_7).$$

Consider the simple games $\langle N, v_j\rangle, j \in \{1, 2, \ldots, 7\}, W^m(v_1) = \{\{1\}, \{2\}\}, W^m(v_2) = \{\{2\}, \{3\}\}, W^m(v_3) = \{\{2\}, \{4\}\}, W^m(v_4) = \{\{3\}, \{5\}\}, W^m(v_5) = \{\{4\}, \{5\}\}, W^m(v_6) = \{\{5\}, \{6\}\}, W^m(v_7) = \{\{6\}, \{7\}\}$.

Since $v(K) = (v_1 \wedge v_2 \wedge \ldots \wedge v_7)(K), \forall K \subseteq N$, then, according to Theorem 2, $\langle N, v\rangle$ is the vertex cover game on the graph $G = \langle N, E\rangle, E = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 5\}, \{4, 5\}, \{5, 6\}, \{6, 7\}\}$.

Statement 1. Let $\langle N, v\rangle$ is the vertex cover game on $G = \langle N, E\rangle$. Then

$$v(K) = \prod_{(i,j)\in E} \left(v_{\{i\}}(K) + v_{\{j\}}(K) - v_{\{i,j\}}(K)\right),$$

where $\forall i \in N$, $\forall \{i, j\} \in E$, $\langle N, v_{\{i\}}\rangle$ and $\langle N, v_{\{i,j\}}\rangle$ are unanimity games.

The proof is in Appendix.

3.3. Dimension of the vertex cover game

The *dimensionality* of $\langle N, v\rangle$ is the least r such that there exists weighted majority games $\langle N, v_1\rangle, \ldots, \langle N, v_r\rangle$ for which

$$W(v) = W(v_1) \cap \ldots \cap W(v_r)$$

[26,62].

The vertex cover game on a complete graph and a star graph can be represented in the form $[1; 1, 1, \ldots, 1]$ and $[1; 1, \frac{1}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}]$, respectively. Hence, the dimensionality of such games is 1.

It follows from Theorem 2 that the vertex cover game dimensionality does not exceed the number of edges in the graph, i.e. $|E|$. The upper bound can, however, be specified by $|L^m|$ and codimensionality by $|W^m|$.

The vertex cover game $\langle N, v\rangle$ of $G = \langle\{1, 2, 3, 4\}, \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}\rangle$, where $W^m(v) = \{\{1, 3\}, \{2, 4\}\}$ is not a weighted majority game.

Statement 2. Let $\langle N, v\rangle$ is the vertex cover game on G. Decompose the graph G in the graphs $G_j = \langle N, E_j\rangle, j \in Y, Y = \{1, 2, \ldots, r\}, \cup_{j\in Y} E_j = E$. Then

$$\dim(v) \le \sum_{j\in Y} \dim(v_j),$$

where $\langle N, v_j\rangle$ is the vertex cover game on the graph G_j, $j \in Y$.

The proof is in Appendix.

Statement 3. The dimensionality of vertex cover game does not exceed the number of edges in the minimal vertex cover of the graph G.

The proof is in Appendix.

Theorem 3. *Among all vertex cover games on trees only the vertex cover game on a star graph is a weighted majority game.*

The proof is in Appendix.

4. Calculation of the Shapley–Shubik index in the vertex cover game

Decompose the characteristic function v over the basis (Möbius transformation) [57], and set $u(x) = \sum_{S\subseteq N}\left(\lambda_S(v)\prod_{i\in S} x_i\right)$, $\forall x \in \{0, 1\}^n$, where $\lambda_S(v) = \sum_{R\subseteq S}(-1)^{|S|-|R|} v(R)$. In that case, if $f : [0, 1]^n \to \mathbb{R}$ is a multilinear extension of $u : \{0, 1\}^n \to \mathbb{R}$, then $\phi_i(N, v) = \int_0^1 \frac{\partial f}{\partial x}(t, t, \ldots, t)dt$ [31,48]. If the minimal winning coalitions do not intersect, the following statement is true.

Lemma 1. *Let $W^m(v) = \{A_1, A_2, \ldots, A_m\}$, $|A_j| = a_j, j = 1, 2, \ldots, m; \forall i, j, i \ne j : A_i \cap A_j = \emptyset$. Then, in the simple game $\langle N, v\rangle$, the Shapley–Shubik index for the player $k \in N$ is calculated by the formula*

$$\phi_k(v) = \int_0^1 x^{a_i-1} \prod_{j=1, j\ne i}^{m} (1 - x^{a_j})dx,$$

where $k \in A_i, a_i = |A_i|, i \in \{1, 2, \ldots, m\}$.

The proof is in Appendix.

Statement 4 deals with the case where the intersection of any two given minimal winning coalitions is one and the same player.

Statement 4. Let $N = \{1, 2, \ldots, n\}, i \in N, W^m(v) = \{A_1, \ldots, A_m\}, \cup_{j=1}^m A_j = N; \forall j, l, j \ne l : A_j \cap A_l = \{i\}$. Then, in the game $\langle N, v\rangle$, the Shapley–Shubik index for the player $k \in N$ is equal to

$$\phi_k(v) = \begin{cases} 1 - \int\limits_0^1 \prod\limits_{j=1}^{m} (1 - x^{a_j-1})dx, & k = i; \\ \int\limits_0^1 x^{a_k-1} \prod\limits_{\substack{j=1,2,\ldots,m \\ j\ne k}} (1 - x^{a_j-1})dx, & \text{otherwise}. \end{cases}$$

The proof is in Appendix.

Example 3. Let N be the set of players, and $1 \in N$. $W^m(v) = \{A_1, \ldots, A_m\}$ is the set of minimal winning coalitions, and elements of the set $W^m(v)$ fulfill the following restrictions: 1. $\forall i, j, i \ne j : A_i \cap A_j = \{1\}$; 2. $|A_i| = i + 1$. E.g., $A_1 = \{1, 2\}, A_2 = \{1, 3, 4\}, A_3 = \{1, 5, 6, 7\}$, etc. Find the limit payoff of player 1 in the game $\langle N, v\rangle$, where the number of minimal winning coalitions tends to infinity. We get

$$\lim_{m\to\infty} \phi_1(v) = \lim_{m\to\infty}\left(1 - \int_0^1 \prod_{k=1}^{m}\left(1 - x^{a_k-1}\right)dx\right)$$
$$= 1 - \int_0^1 \prod_{k=1}^{\infty}\left(1 - x^k\right)dx = 1 - \frac{4\pi\sqrt{3}}{\sqrt{23}} \cdot \frac{\sinh\frac{\pi\sqrt{23}}{3}}{\cosh\frac{\pi\sqrt{23}}{2}} \approx 0.6316.$$

We find that as the number of minimal winning coalitions in the set $W^m(v)$ increases, the payoff of player 1 tends to a finite limit. Some limit theorems for the Penrose–Banzhaf value can be found in [35].

The Shapley–Shubik indices for the linear graph consisting of n vertices, $n = 2, 3, \ldots, 10$ are given in Table 1. Numbers from 2 to 10 in the first line indicate the number of vertices in the linear graph. Numbers from 1 to 10 in the first column are the players numbers. The index will be the highest for the vertices connected to end vertices.

Let $G = \langle N, E\rangle$ be a star graph, for which $E = \{\{1, 2\}\{1, 3\}, \ldots, \{1, n\}\}$. Consider the vertex cover game $\langle N, v\rangle$ of G, where $W^m(v) = \{\{1\}, \{2, 3, \ldots, n\}\}$. Calculate Shapley–Shubik index of each player. Elements of the set of minimal winning coalitions do not intersect each other, wherefore Lemma 1 can be applied:

$$\phi_1(v) = \int_0^1 (1 - x^{n-1})dx = 1 - \frac{1}{n},$$

$$\phi_i(v) = \int_0^1 x^{n-2}(1 - x)dx = \frac{1}{n(n-1)}, i \ne 1.$$

Statement 5. Let $G = \langle N, E\rangle$ be a complete bipartite graph, $L \cup R = N, L \cap R = \emptyset, E = \{\{a, b\} | a \in L, b \in R\}$. Then, the Shapley–Shubik index for the player $i \in N$ in the vertex cover game $\langle N, v\rangle$ has the following form:

$$\phi_i(v) = \begin{cases} \frac{1}{|L|} - \frac{1}{|L|+|R|}, & i \in L; \\ \frac{1}{|R|} - \frac{1}{|L|+|R|}, & i \in R. \end{cases}$$

The proof is in Appendix.

Statement 6. Let $G = \langle N, E\rangle, E = \{\{1, 2\}, \{\{1, a_p\}_{p=1}^k\}, \{\{2, b_q\}\}_{q=1}^r\}$, $\forall p, q : a_p \ne 2, b_q \ne 1, a_p \ne b_q$. Then, the Shapley–Shubik index for the player $i \in N$ in the vertex cover game $\langle N, v\rangle$ has the following form:

$$\phi_i(v) = \begin{cases} \frac{1}{2} - \frac{1}{k+2} + \frac{1}{r+1} - \frac{1}{r+2}, & i = 1; \\ \frac{1}{2} - \frac{1}{r+2} + \frac{1}{k+1} - \frac{1}{k+2}, & i = 2; \\ \frac{1}{k+1} - \frac{1}{k+2}, & i = a_p, p \in \{1, \ldots, k\}; \\ \frac{1}{r+1} - \frac{1}{r+2}, & i = b_q, q \in \{1, \ldots, r\}. \end{cases}$$

The proof is in Appendix.

An example of the graph used in Statement 6 is shown in Fig. 3. We get $\phi_1(v) = \frac{34}{105} \approx 0.32, \phi_2(v) = \frac{57}{140} \approx 0.41, \phi_{a_p}(v) = \frac{1}{20} = 0.05, \phi_{b_q}(v) = \frac{1}{42} \approx 0.02, p \in \{1, 2, 3\}, q \in \{1, 2, 3, 4, 5\}$.

Table 1
Solution for the linear graph.

	2	3	4	5	6	7	8	9	10
1	1/2	1/6	1/6	7/60	1/10	1/12	61/840	23/360	2/35
2	1/2	2/3	1/3	17/60	13/60	11/60	131/840	43/315	61/504
3	-	1/6	1/3	1/5	11/60	3/20	37/280	293/2520	263/2520
4	-	-	1/6	17/60	11/60	1/6	39/280	311/2520	23/210
5	-	-	-	7/60	13/60	3/20	39/280	151/1260	34/315
6	-	-	-	-	1/10	11/60	37/280	311/2520	34/315
7	-	-	-	-	-	1/12	131/840	293/2520	23/210
8	-	-	-	-	-	-	61/840	43/315	263/2520
9	-	-	-	-	-	-	-	23/360	61/504
10	-	-	-	-	-	-	-	-	2/35

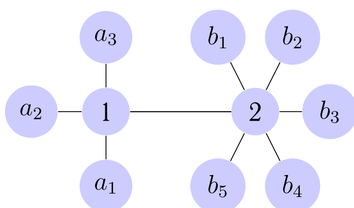

Fig. 3. Star graph with two centers, $k = 3, r = 5$.

5. Vertex cover game for transport networks

Section 5 deals with the application of a vertex cover game in transport networks. Section 5.1 provides argumentation that the allocation of cameras based on the values of the Shapley–Shubik index is efficient. In Section 5.2, the Shapley–Shubik index is calculated for a specific transport network.

5.1. Application of the Shapley–Shubik index for estimating the efficiency of vertices in the vertex cover of a graph

Let the graph G be a transport network. A vertex in this graph is a crossroads, an edge is a road. The task is to optimally distribute surveillance cameras. Knowing the power of graph vertices, one can deploy the cameras accordingly.

Surveillance cameras are to provide for full coverage of the transport network. If the existing cameras capture the transport network entirely, no more budget allocations are needed to purchase new cameras. Let us consider all possible rearrangements of graph vertices. Let σ denote a rearrangement of vertices. In the rearrangement σ, enumerate vertices as $1, 2, \ldots, |V|$. Denote by $\sigma(k)$ the set of vertices occupying in the rearrangement σ positions before and including the vertex with the number k. The coalition of vertices $\sigma(k)$ in the rearrangement σ is losing if it does not cover the transport network, and winning otherwise. If $\sigma(k-1)$ is the losing coalition and $\sigma(k)$ is the winning one, then the vertex numbered as k is called pivotal for the given rearrangement. Vertices occupying positions preceding the pivotal vertex in the rearrangement do not cover the network. Vertices after the pivotal vertex make no further contribution since the transport network is already covered. Hence, the essential question when arranging cameras is whether a vertex is the pivotal one. Knowing this, the efficiency of each vertex can be calculated by the formula

$$\phi_i = \frac{\text{the number of rearrangements in which the vertex i is pivotal with regard to vertex covers}}{n!},$$

where $n!$ is the number of all possible rearrangements among n vertices. The value of ϕ_i is the Shapley–Shubik index for the vertex cover game. The higher is the number of the rearrangements where the vertex is pivotal, the higher is the power of this vertex.

Denote SG_n the set of all simple games with n players.

Since the vertex cover game is a simple game and the efficiency axiom (for all $v \in SG_n$, $\sum_{i=1}^{n} \phi_i(v) = 1$ [18]) is fulfilled for the Shapley–Shubik index, the power of each vertex is not less than zero and the sum of all values is equal to unity.

Let us demonstrate what properties an array of surveillance cameras will have if the cameras are arranged proportionately to the values of the Shapley–Shubik index in a vertex cover game. The Shapley–Shubik index conforms to the null player, anonymity, symmetry, transfer axioms.

Null player axiom: for any $v \in SG_n$ and any $i \in N$, if i is a null player in game v, then $\phi_i(v) = 0$. The player $i \in N$ is called the null player if $v(S) = v(S \setminus \{i\})$ for all $i \in S \subseteq N$. If the vertex degree is 0, this means there are no roads running across the given crossroads. In the vertex cover game, such a vertex is the null player. Owing to the null player property, cameras will not be deployed in the vertices in which they are unnecessary.

Anonymity axiom: for all $v \in SG_n$, any permutation π of N, and any $i \in N$, $\phi_i(\pi v) = \phi_{\pi(i)}(v)$, where $(\pi v)(S) := v(\pi(S))$. The numbers assigned to vertices have no effect on the distribution of cameras. If the vertex numbering scheme is changed but the transport network topology remains the same, the distribution of cameras will not be affected. Owing to the anonymity axiom, all vertices are in an equal position.

The Shapley–Shubik index has the symmetry axiom, i.e. if $i, j \in N$, $i \neq j$ $v(S \cup \{i\}) = v(S \cup \{j\})$ $\forall S \subseteq N \setminus \{i, j\}$ then $\phi_i(v) = \phi_j(v)$. If there are two vertices symmetrical with respect to the graphâs vertex cover, then these vertices will have equal Shapley–Shubik index in the vertex cover game. This axiom ensures that symmetric vertices are allocated equal numbers of surveillance cameras.

Transfer axiom: for any $v, w \in SG_n$ such that $v \vee w \in SG_n$, $\phi(v) + \phi(w) = \phi(v \wedge w) + \phi(v \vee w)$. This axiom implies that when winning coalitions are added, changes in the solution of the game depend only on the added coalitions. This interpretation of the axiom can be found in [19]. If, for instance, a new road appeared in the transport network or, vice versa, a road has been closed, changes in the distribution of cameras will depend solely on the respective changes in the graph topology, but not on any other factors.

5.2. Shapley–Shubik index for a specific transport network

For the graph shown in Fig. 1, the vector of the Shapley–Shubik indices in the cooperative vertex cover game has the form

$$\phi = \left\{ \frac{3}{20}, \frac{1}{5}, \frac{7}{30}, \frac{1}{15}, \frac{1}{5}, \frac{3}{20} \right\}.$$

One of its minimum vertex covers is the set $S = \{1, 3, 5\}$. Vertices 3 and 5 have the same degree, which is equal to 3. From the Shapley–Shubik index perspective, however, vertex 3 has the highest weight. Normalize the numbers ϕ_1, ϕ_3, ϕ_5 and convert the resultant values to percentages. We get 25.71%, 40.00%, 34.29%, respectively. If the graph G is a transport network, the budget

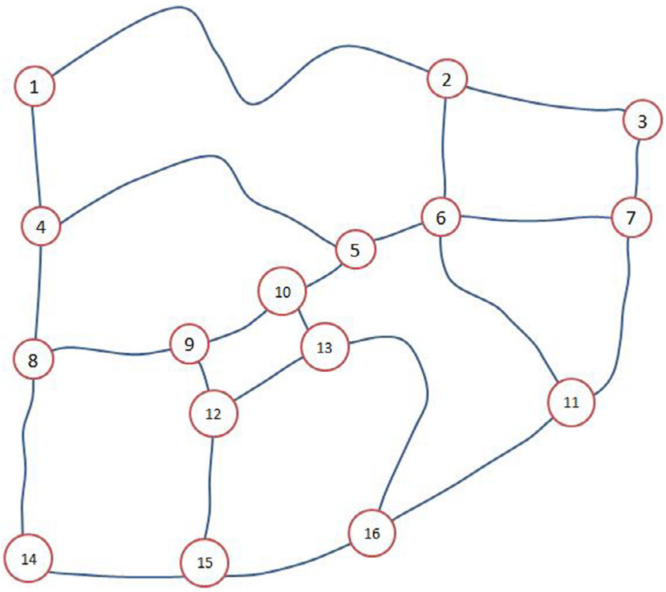

Fig. 4. Layout of the main roads of Kukkovka district, Petrozavodsk, Russia.

allocated to purchasing cameras can be distributed in accordance with the resultant per cent values.

Fig. 4 shows the layout of major roads in Kukkovka district in Petrozavodsk, Russia. Construct the transport network graph $G = \langle N, E\rangle, N = \{1, 2, \dots, 16\}$, which is shown in Fig. 5.

$\{8, 9\}, \{9, 10\}, \{10, 5\}, \{5, 6\}, \{6, 7\}$	form Rovio Street
$\{1, 4\}, \{4, 8\}, \{8, 14\}$	form Komsomolskij Avenue
$\{14, 15\}, \{15, 16\}, \{16, 11\}$	form Karelskiy Avenue
$\{10, 13\}, \{12, 13\}, \{13, 16\}$	form Pitkyarantskaya Street
$\{3, 7\}, \{7, 11\}$	form Lyzhnaya Street
$\{9, 12\}, \{12, 15\}$	form Sortovalskaya Street
$\{1, 2\}, \{2, 3\}$	form Baltiyskaya Street
$\{4, 5\}$	form General Frolova Street
$\{2, 6\}$	form Parfenova Street
$\{6, 11\}$	form Torneva Street

The names and locations of the streets can be looked up online.[1]

If a simple game is a weighted game, then the Shapley–Shubik index can be calculated using the theory of generating functions. A vertex cover game is, however, not always a weighted game. Finding the minimum vertex cover is recognized as an NP-complete problem [29], and finding all least vertex covers is a challenging computational problem. It is therefore more convenient to handle the original vertex cover game using the decomposition theorem and to calculate the Shapley–Shubik index for new simple games. We decompose the graph into subgraphs and find the least vertex covers for the new graphs. It is advisable to decompose the original graph so that the number of vertices and edges in the new graphs is roughly equal. This approach will reduce the number of vertex covers and the number of vertices in the least vertex covers compared to the original graph. Calculate the Shapley–Shubik index in the vertex cover game $\langle N, v\rangle$.

To this end, decompose the graph G into the graphs G_1, G_2, G_3 as shown in Figs. 6–8, respectively. The sets of minimal winning coalitions $W^m(v_1)$, $W^m(v_2)$, $W^m(v_3)$ are

$$W^m(v_1) = \{\{4, 5, 6\}, \{1, 5, 6, 8\}, \{4, 6, 10\}, \{2, 4, 5, 7, 11\},$$

[1] https://www.google.ru/maps/@61.7653959,34.3725932,15z?hl=en.

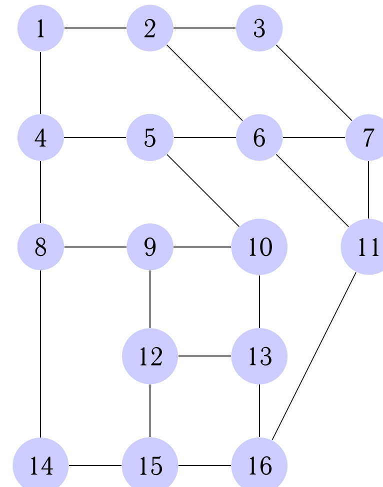

Fig. 5. Graph G of the transport network.

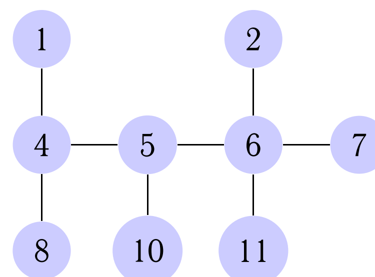

Fig. 6. Graph G_1, subgraph of the graph G shown in Fig. 5.

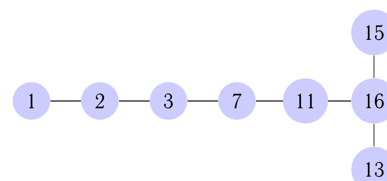

Fig. 7. Graph G_2, subgraph of the graph G shown in Fig. 5.

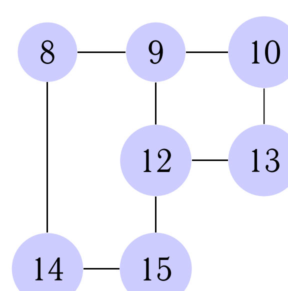

Fig. 8. Graph G_3, subgraph of the graph G shown in Fig. 5.

$$\{1, 2, 5, 7, 8, 11\}\};$$

$$W^m(v_2) = \{\{1, 3, 7, 16\}, \{1, 3, 11, 16\}, \{1, 3, 11, 13, 15\}, \{2, 7, 16\}, \{2, 7, 11, 13, 15\}, \{2, 3, 11, 16\}, \{2, 3, 11, 13, 15\}\};$$

$$W^m(v_3) = \{\{9, 12, 13, 14\}, \{9, 13, 14, 15\}, \{8, 9, 13, 15\}, \{8, 10, 12, 14\}, \{9, 10, 12, 14\}, \{8, 10, 12, 15\}\}.$$

Write $v(K)$, $K \subseteq N$ in the form

$$v(K) = v_1(K) + v_2(K) + v_3(K) - (v_1 \vee v_2)(K) - (v_1 \vee v_3)(K) - (v_2 \vee v_3)(K) + (v_1 \vee v_2 \vee v_3)(K).$$

Table 2
Shapley–Shubik index values.

i	$w_1(K)$	$w_2(K)$	$w_3(K)$	$w_4(K)$	$w_5(K)$	$w_6(K)$	$w_7(K)$	$v(K)$
1	0.0560	0.0762	-	0.0678	0.0231	0.0378	0.0419	0.0453
2	0.0393	0.1690	-	0.1075	0.0158	0.0891	0.0714	0.0674
3	-	0.1357	-	0.0706	-	0.0651	0.0448	0.0447
4	0.2321	-	-	0.1227	0.1296	-	0.0877	0.0675
5	0.1726	-	-	0.0738	0.0940	-	0.0569	0.0617
6	0.2821	-	-	0.1627	0.1435	-	0.1063	0.0822
7	0.0393	0.1595	-	0.0976	0.0158	0.0855	0.0672	0.0672
8	0.0560	-	0.1262	0.0222	0.0889	0.0575	0.0535	0.0671
9	-	-	0.1762	-	0.0894	0.0770	0.0542	0.0640
10	0.0833	-	0.1262	0.0461	0.1115	0.0639	0.0767	0.0648
11	0.0393	0.1190	-	0.0679	0.0158	0.0501	0.0399	0.0645
12	-	-	0.1762	-	0.0770	0.0897	0.0538	0.0634
13	-	0.0524	0.1262	0.0206	0.0652	0.0841	0.0551	0.0637
14	-	-	0.1429	-	0.0726	0.0728	0.0480	0.0455
15	-	0.0524	0.1262	0.0206	0.0579	0.0841	0.0513	0.0672
16	-	0.2357	-	0.1199	-	0.1433	0.0911	0.0637

Table 3
Percentage distribution of the budget.

i	6	15	8	11	9	13	5	1	3
%	14.66	12.00	11.98	11.51	11.43	11.37	11.00	8.08	7.97

For brevity, denote

$$w_1(K) = v_1(K), w_2(K) = v_2(K), w_3(K) = v_3(K), w_4(K) = (v_1 \vee v_2)(K),$$
$$w_5(K) = (v_1 \vee v_3)(K), w_6(K) = (v_2 \vee v_3)(K), w_7(K) = (v_1 \vee v_2 \vee v_3)(K).$$

Perform the Shapley–Shubik index calculations in Table 2.

Numbers in Table 2 are Shapley–Shubik indices. The players number and the characteristic function are written in the first column and first line, respectively. We are now interested only in the last column of Table 2.

Consider two situations.

Situation 1. The number of cameras is equal to the number of vertices in the graphs minimum vertex cover. Cameras are ranked by quality. The question is which vertices should higher quality cameras be deployed to.

The vertex cover of the graph is the set of vertices {1, 3, 5, 6, 8, 9, 11, 13, 15}.

Arranging the vertices in the order of decreasing Shapley–Shubik index we get the vector

(6, 15, 8, 11, 9, 13, 5, 1, 3).

Hence, the best camera among the nine available should be deployed in vertex 6, the next best in quality in vertex 15, and so forth. Vertices 15, 8, 11, 9, 13, 5 have the degree 3, and vertices 1 and 3 have the degree 2. Yet, vertex 15 is the most powerful among other vertices with the degree 3 in the minimum vertex cover.

Situation 2. The task is to distribute the budget for purchasing cameras which will be deployed in the vertices constituting a minimum vertex cover of the graph. Normalize the values of Shapley–Shubik indices and convert the resultant values to percentages. We get the Table 3.

The share of the budget to be allocated for the purchase of cameras and related equipment (outdoor power supply, connecting cable, fasteners, etc.) is determined according to Table 3. Financial applications of power indices are presented in detail in the papers [28,55].

6. Conclusions and future work

Games on graphs have become a popular field in game theory because the solution of applied problems requires the analysis of transport, communication, or computer networks. As the society and social interactions develop, this field is moving even further into the foreground. The questions of centrality and significance of objects in a network structure come up.

Relying on the graph theory and methods of the cooperative game theory, a graph decomposition technique for Shapley–Shubik index estimation was suggested. This procedure permits representing the characteristic function of the original game in the form of a linear combination of simpler characteristic functions. Using this approach, the Shapley–Shubik index was calculated in the cooperative vertex cover game for a transport network and some classes of graphs.

The cooperative game in this paper depends on vertex covers of the graph. Since there exist also other covers of a graph, such as the edge cover, it is interesting to consider the corresponding simple games, too. If the covers are interrelated, is there also a relationship between the respective simple games?

The usual procedure in mathematical models of resource allocation is that a functional is composed to be then optimized. One of the properties of the solution is that the composed functional reaches the required value. The cooperative game theory has a different approach to resource allocation. The optimal distribution fulfils several axioms, these axioms having an applied interpretation.

Where a graph has several minimum vertex covers, a measure of centrality can be composed based only on minimum vertex covers.

Cooperative vertex cover games are useful in problems where the vertex cover is essential. For example, each vertex is either a source or a receiver of information. The source vertices form a vertex cover of the information network. A vertex cover game can be applied to find the power of sources and receivers, rank vertices in the information network, give recommendations on how to allocate resources [21,38,56].

Theorem 1 proved in this paper can serve as the basis for the analysis of networks by means of Owen [49], Aumann-Dreze values

[6], C-core [58]. The statements related in the paper can be used to analyze network objects by cooperative game theory methods.

Acknowledgements

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Proof of Theorem 1. The equality $2^N = A \cup (2^N \setminus A)$, where

$A = \{K | K \in 2^N \wedge (\forall i \in Y, \exists S_i \in M(G_i) : S_i \subseteq K)\}$.

is valid. Consider 2 possible cases.

1. Let $K \in A$. The coalition K contains the sets S_i, $i \in Y$, which are vertex covers of the graphs G_i, $i \in Y$, respectively. The condition is that $E = \bigcup_{i \in Y} E_i$. Hence, $\bigcup_{i \in Y} S_i$ is a vertex cover of graph G. Since $\bigcup_{i \in Y} S_i \subseteq K$, we have $v(K) = 1$; $(v_1 \wedge v_2 \wedge \ldots \wedge v_r)(K) = 1$.
2. Let $K \in 2^N \setminus A$. The coalition K contains no vertex covers of the graph G, wherefore $v(K) = 0$. Since $E = \bigcup_{i \in Y} E_i$, there exists such graph G_i that K does not contain a vertex cover of the graph G_i. Hence, $(v_1 \wedge v_2 \wedge \ldots \wedge v_r)(K) = 0$.

In each of the cases, $v(K) = (v_1 \wedge v_2 \wedge \ldots \wedge v_r)(K)$ holds, which proves the theorem. □

Proof of Theorem 2.

1) Let $\langle N, v\rangle$ be a simple vertex cover game on the graph G. Let us demonstrate that (2) holds. Decompose $G = \langle N, E\rangle$ into the subgraphs $G_j = \langle N, e_j\rangle, e_j \in E, j = 1, \ldots, |E|$. Applying Theorem 1, we get the equality (2).
2) Let the equality (2) be true. Let us demonstrate that $\langle N, v\rangle$ is a vertex cover game on the graph G. Do this by proving the equality $W^m(v) = M(G)$. Since the functions $v_j(K), j \in Y$ exist, then $E \neq \emptyset$. Note that the graph G may be disconnected.

Let $A \in W^m(v)$. Then, $v(A) = 1, (v_1 \wedge v_2 \wedge \ldots \wedge v_r)(A) = 1$. So, $\forall j \in Y : v_j(A) = 1 \Rightarrow i_j \in A$ or $k_j \in A$. Hence, A is the vertex cover of the graph G.

Let $K \subset A$, $K \neq \emptyset$. Then, $v(A \setminus K) = 0, (v_1 \wedge v_2 \wedge \ldots \wedge v_r)(A \setminus K) = 0$. So, $\exists j \in Y : v_j(A \setminus K) = 0 \Rightarrow i_j \notin A \setminus K$ and $k_j \notin A \setminus K$. Hence, A is the minimal vertex cover of the graph. It follows that $W^m(v) \subseteq M(G)$.

Let $A \in M(G)$. Then, $\forall (i_j, k_j) \in E$: $i_j \in A$ or $k_j \in A$. So, $\forall j \in Y$: $v_j(A) = 1 \Rightarrow (v_1 \wedge v_2 \wedge \ldots \wedge v_r)(A) = 1, v(A) = 1$. Hence, A is the winning coalition.

Let $K \subset A$, $K \neq \emptyset$. Then, $A \setminus K \notin M(G)$, $\exists j \in Y$: $i_j \notin A \setminus K$ and $k_j \notin A \setminus K$. So, $v_j(A \setminus K) = 0 \Rightarrow (v_1 \wedge v_2 \wedge \ldots \wedge v_r)(A \setminus K) = 0, v(A \setminus K) = 0$. Hence, A is the minimal winning coalition. It follows that $M(G) \subseteq W^m(v)$.

Since $M(G) \subseteq W^m(v)$ and $W^m(v) \subseteq M(G)$, then $W^m(v) = M(G)$. By definition, $\langle N, v\rangle$ is a vertex cover game. □

Proof of Lemma 1. Fix the player k. Since the minimal winning coalitions do not intersect, there exists only one minimal winning coalition containing the player k. Denote this coalition by A_i. Let $L \subseteq \{1, 2, \ldots, m\}, L \neq \emptyset$. Then

$$\left|\underset{j \in L}{\cup} A_j\right| = |A_i| + \left|\underset{j \in L \setminus \{i\}}{\cup} A_j\right| = a_i + \sum_{j \in L \setminus \{i\}} a_j.$$

The following sequence of equations is valid:

$$\begin{aligned}
\phi_k(v) &= \sum_{\substack{L \subseteq \{1,2,\ldots,m\} \\ k \in \bigcup_{j \in L} A_j}} \frac{(-1)^{|L|-1}}{\left|\bigcup_{j \in L} A_j\right|} = \sum_{\substack{L \subseteq \{1,2,\ldots,m\} \\ k \in \bigcup_{j \in L} A_j}} \frac{(-1)^{|L|-1}}{a_i + \sum_{j \in L \setminus \{i\}} a_j} \\
&= \sum_{L \subseteq \{1,2,\ldots,m\} \setminus \{i\}} \frac{(-1)^{|L|}}{a_i + \sum_{j \in L} a_j} \\
&= \sum_{L \subseteq \{1,2,\ldots,m\} \setminus \{i\}} (-1)^{|L|} \int_0^1 x^{a_i - 1 + \sum_{j \in L} a_j} dx \\
&= \int_0^1 \left(\sum_{L \subseteq \{1,2,\ldots,m\} \setminus \{i\}} (-1)^{|L|} \cdot x^{a_i - 1 + \sum_{j \in L} a_j} \right) dx \\
&= \int_0^1 x^{a_i - 1} \left(\sum_{L \subseteq \{1,2,\ldots,m\} \setminus \{i\}} (-1)^{|L|} \cdot x^{\sum_{j \in L} a_j} \right) dx \\
&= \int_0^1 x^{a_i - 1} \prod_{j=1, j \neq i}^{m} (1 - x^{a_j}) dx.
\end{aligned}$$

The lemma is thus proven. □

Proof of Statement 1. Since $\langle N, v\rangle$ is a vertex cover game, then $v(K) = (v_1 \wedge v_2 \wedge \ldots \wedge v_{|E|})(K) = v_1(K) \cdot v_2(K) \cdot \ldots \cdot v_{|E|}(K)$, following from Theorem 2. Since $W^m(v_j) = \{\{i_j\}, \{k_j\}\}$, then $v_j(K) = v_{\{i_j\}}(K) + v_{\{k_j\}}(K) - v_{\{i_j, k_j\}}(K)$. Substituting the representation of the functions $v_j(K)$ over the basis into the product, we get that the statement is true. □

Proof of Statement 2. Since any simple game has a dimensionality, then $\forall j \in Y : W(v_j) = W(v_{j,1}) \cap W(v_{j,2}) \cap \ldots \cap W(v_{j,\dim(v_j)})$, where $\langle N, v_{j,k}\rangle, k = 1, 2, \ldots, \dim(v_j)$ is a weighted majority game. Since the conditions of Theorem 1 are fulfilled, then $v(K) = (v_1 \wedge v_2 \wedge \ldots \wedge v_r)(K)$. So, $W(v) = \bigcap_{j \in Y} \bigcap_{k=1}^{\dim(v_j)} W(v_{j,k})$. Hence, $\dim(v) \leq \sum_{j \in Y} \dim(v_j)$. □

Proof of Statement 3. Let $S = \{j_1, j_2, \ldots, j_r\}$ be the minimal vertex cover of the graph G. Decompose the graph G into the graphs $G_i = \langle N, E_i\rangle, E_i = \{\{j_i, k\} | j_i \in S, \{j_i, k\} \in E\}, i = 1, 2, \ldots, r$. The graph G_i consists of a star graph and, possibly, several disconnected vertices. Each star graph is centered around a vertex from the minimal vertex cover S. Since $\cup_{i \in \{1,2,\ldots,r\}} E_i = E$, then conformant to Statement 2, we get that $\dim(v) \leq \sum_{i \in \{1,\ldots,r\}} \dim(v_i)$, where $\langle N, v_i\rangle$ is the vertex cover game on the graph G_i. Since $\dim(v_i) = 1 \; \forall i \in \{1, 2, \ldots, r\}$, then $\dim(v) \leq |S|$, which proves the statement. □

Proof of Statement 4. Let $W^m(v_1) = \{\{i\}\}, W^m(v_2) = \{A_1 \setminus \{i\}, \ldots, A_m \setminus \{i\}\}$. Then $v(K) = (v_1 \wedge v_2)(K), W^m(v_1 \vee v_2) = \{\{i\}, A_1 \setminus \{i\}, \ldots,$
$A_m \setminus \{i\}\}$. Observe that elements in the sets $W^m(v_2)$, $W^m(v_1 \vee v_2)$ do not intersect pairwise, respectively. Make use of Shapley–Shubik index linearity property, and Lemma 1. For the player i we get the following sequence of equalities:

$$\begin{aligned}
\phi_i(v) &= \phi_i(v_1 \wedge v_2) = \phi_i(v_1) + \phi_i(v_2) - \phi_i(v_1 \vee v_2) \\
&= 1 + 0 - \int_0^1 \prod_{j=1}^{m} (1 - x^{a_j - 1}) dx = 1 - \int_0^1 \prod_{j=1}^{m} (1 - x^{a_j - 1}) dx.
\end{aligned}$$

For the player $k \in N$, $k \neq i$ we get

$$\begin{aligned}
\phi_k(v) &= \phi_k(v_1) + \phi_k(v_2) - \phi_k(v_1 \vee v_2) \\
&= 0 + \int_0^1 x^{a_k - 2} \prod_{\substack{j=1,2,\ldots,m \\ j \neq k}} (1 - x^{a_j - 1}) dx \\
&\quad - \int_0^1 x^{a_k - 2} (1 - x) \prod_{\substack{j=1,2,\ldots,m \\ j \neq k}} (1 - x^{a_j - 1}) dx \\
&= \int_0^1 (x^{a_k - 2} - x^{a_k - 2}(1 - x)) \prod_{\substack{j=1,2,\ldots,m \\ j \neq k}} (1 - x^{a_j - 1}) dx \\
&= \int_0^1 x^{a_k - 1} \prod_{\substack{j=1,2,\ldots,m \\ j \neq k}} (1 - x^{a_j - 1}) dx.
\end{aligned}$$

□

Proof of Theorem 3. The vertex cover game on a star graph is a weighted majority game, $[1; 1, \frac{1}{n-1}, \frac{1}{n-1}, \dots, \frac{1}{n-1}]$. Consider the tree $G = \langle N, E\rangle$, that is not a star. The set of vertices $\{1, 2, 3, 4\} \subseteq N$, $\{\{1, 2\}, \{2, 3\}, \{3, 4\}\} \subseteq E$, $\{1, 3\} \notin E$, $\{2, 4\} \notin E$, $\{1, 4\} \notin E$ will then exist.

Decompose the graph G into the graphs $G_1 = \langle N, E_1\rangle, E_1 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}, G_2 = \langle N, E \setminus E_1\rangle$. If $E \setminus E_1 \neq \emptyset$, then we compose the vertex cover S of the graph G_2 by the following algorithm: 1. $S := \emptyset$; 2. Run the direct search through all edges of the graph G_2. For all edge $\{i, x\}$, $i \in \{1, 2, 3, 4\}$, $x \in N \setminus \{1, 2, 3, 4\}$, $S := S \cup \{x\}$. For all edge $\{x, y\}$, $x, y \in N \setminus \{1, 2, 3, 4\}$, $S := S \cup \{x\}$ or $S := S \cup \{y\}$.

According to Theorem 1, $v(K) = (v_1 \wedge v_2)(K)$, where $\langle N, v_1\rangle$, $\langle N, v_2\rangle$ are vertex cover games on the graphs G_1, G_2, respectively. Hence, $v(\{1, 3\} \cup S) = 1, v(\{2, 4\} \cup S) = 1, v(\{1, 2\} \cup S) = 0$ (since the edge $\{3, 4\}$ is not covered), $v(\{3, 4\} \cup S) = 0$ (since the edge $\{1, 2\}$ is not covered). Suppose that the game $\langle N, v\rangle$ can be represented in the form $\langle q; w_1, w_2, \dots, w_{|N|}\rangle$. Then, $w_1 + w_3 + \sum_{i\in S} w_i \geq q, w_2 + w_4 + \sum_{i\in S} w_i \geq q, w_1 + w_2 + \sum_{i\in S} w_i < q, w_3 + w_4 + \sum_{i\in S} w_i < q$. Adding the first and the second inequalities together and the third and the fourth inequalities together, we get a contradiction. □

Proof of Statement 5. For a complete bipartite graph, $W^m(G) = \{L, R\}$ is valid. Elements of the set of minimal winning coalitions do not intersect with one another, wherefore Lemma 1 can be applied.

$$\phi_i(v) = \int_0^1 x^{|L|-1}(1 - x^{|R|})dx = \frac{1}{|L|} - \frac{1}{|L| + |R|}, i \in L$$

$$\phi_j(v) = \int_0^1 x^{|R|-1}(1 - x^{|L|})dx = \frac{1}{|R|} - \frac{1}{|L| + |R|}, j \in R.$$

□

Proof of Statement 6. Decompose the graph $G = \langle N, E\rangle$ into subgraph

$$G_1 = \langle N, E_1\rangle, G_2 = \langle N, E_2\rangle$$

$$E_1 = \{\{1, 2\}, \{1, a_p\}\}, p \in \{1, \dots, k\}, E_2 = \{\{2, b_q\}\}, q \in \{1, \dots, r\},$$

$$W^m(v_1) = \{\{1\}, \{2, a_1, \dots, a_k\}\}, W^m(v_2) = \{\{2\}, \{b_1, \dots, b_r\}\},$$

$$W^m(v_1 \vee v_2) = \{\{1\}, \{2\}, \{b_1, \dots, b_r\}\}.$$

Apply Theorem 1, Shapley–Shubik index linearity property, and Lemma 1. We get

$$\phi(v) = \phi(v_1 \wedge v_2) = \phi(v_1) + \phi(v_2) - \phi(v_1 \vee v_2).$$

$$\begin{aligned}\phi_1(v) &= \int_0^1 (1 - x^{k+1})dx - \int_0^1 (1 - x)(1 - x^r)dx \\ &= \frac{1}{2} - \frac{1}{k+2} + \frac{1}{r+1} - \frac{1}{r+2};\end{aligned}$$

$$\begin{aligned}\phi_2(v) &= \int_0^1 x^k(1 - x)dx + \int_0^1 (1 - x^r)dx - \int_0^1 (1 - x)(1 - x^r)dx \\ &= \frac{1}{2} - \frac{1}{r+2} + \frac{1}{k+1} - \frac{1}{k+2};\end{aligned}$$

$$\phi_{a_p}(v) = \int_0^1 x^k(1 - x)dx = \frac{1}{k+1} - \frac{1}{k+2}.$$

Calculate $\phi_{b_q}(v)$. Since $W^m(v_1 \vee v_2) = \{\{1\}, \{2\}, \{b_1, \dots, b_r\}\}$, $|\{1\}| = 1, |\{2\}| = 1, |\{b_1, \dots, b_r\}| = r$, then

$$\begin{aligned}\phi_{b_q}(v) &= \int_0^1 x^{r-1}(1 - x)dx - \int_0^1 (1 - x)^{r-1}(1 - x)^1(1 - x)^1 dx \\ &= \int_0^1 x^{r-1}(1 - x)dx - \int_0^1 (1 - x)^{r-1}(1 - x)^2 dx \\ &= \frac{1}{r+1} - \frac{1}{r+2}.\end{aligned}$$

□

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