

# A SIMPLE PROOF OF THE FORMULA FOR THE BETTI NUMBERS OF THE QUASIHOMOGENEOUS HILBERT SCHEMES.

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ABSTRACT. In a recent paper the first two authors proved that the generating series of the Poincare polynomials of the quasihomogeneous Hilbert schemes of points in the plane has a simple decomposition in an infinite product. In this paper we give a very short geometrical proof of that formula.

## 1. INTRODUCTION

The Hilbert scheme  $(\mathbb{C}^2)^{[n]}$  of  $n$  points in the plane  $\mathbb{C}^2$  parametrizes ideals  $I \subset \mathbb{C}[x, y]$  of colength  $n$ :  $\dim_{\mathbb{C}} \mathbb{C}[x, y]/I = n$ . It is a nonsingular, irreducible, quasiprojective algebraic variety of dimension  $2n$  with a rich and much studied geometry, see [7, 13] for an introduction.

The cohomology groups of  $(\mathbb{C}^2)^{[n]}$  were computed in [5], and the ring structure in the cohomology was determined independently in the papers [10] and [14].

There is a  $(\mathbb{C}^*)^2$ -action on  $(\mathbb{C}^2)^{[n]}$  that plays a central role in this subject. The algebraic torus  $(\mathbb{C}^*)^2$  acts on  $\mathbb{C}^2$  by scaling the coordinates,  $(t_1, t_2) \cdot (x, y) = (t_1x, t_2y)$ . This action lifts to the  $(\mathbb{C}^*)^2$ -action on the Hilbert scheme  $(\mathbb{C}^2)^{[n]}$ .

For arbitrary non-negative integers  $\alpha$  and  $\beta$ , such that  $\alpha + \beta \geq 1$ , let  $T_{\alpha, \beta} = \{(t^\alpha, t^\beta) \in (\mathbb{C}^*)^2 \mid t \in \mathbb{C}^*\}$  be a one-dimensional subtorus of  $(\mathbb{C}^*)^2$ . If  $\alpha$  and  $\beta$  are non-zero, then the fixed point set  $((\mathbb{C}^2)^{[n]})^{T_{\alpha, \beta}}$  is called the quasihomogeneous Hilbert scheme of points on the plane  $\mathbb{C}^2$ .

The quasihomogeneous Hilbert scheme  $((\mathbb{C}^2)^{[n]})^{T_{\alpha, \beta}}$  is compact and in general has many irreducible components. They were described in [6]. In the case  $\alpha = 1$  the Poincare polynomials of the irreducible components were computed in [3].

The Poincare polynomial of a manifold  $X$  is defined by  $P_q(X) = \sum_{i \geq 0} \dim H_i(X; \mathbb{Q}) q^{\frac{i}{2}}$ . In [4] the first two authors proved the following theorem.

**Theorem 1.1.** *Suppose  $\alpha$  and  $\beta$  are positive coprime integers, then*

$$\sum_{n \geq 0} P_q \left( ((\mathbb{C}^2)^{[n]})^{T_{\alpha, \beta}} \right) t^n = \prod_{\substack{i \geq 1 \\ (\alpha + \beta) \nmid i}} \frac{1}{1 - t^i} \prod_{i \geq 1} \frac{1}{1 - qt^{(\alpha + \beta)i}}.$$

In this paper we give another proof of this theorem. In [4] the large part of the proof consists of non-trivial combinatorial computations with Young diagrams. Our new proof is more geometrical and is much shorter. In fact, we prove a slightly more general statement.

Let  $\Gamma_m$  be the finite subgroup of  $(\mathbb{C}^*)^2$  defined by

$$\Gamma_m = \left\{ (\zeta^j, \zeta^{-j}) \in (\mathbb{C}^*)^2 \mid \zeta = \exp \left( \frac{2\pi i}{m} \right), j = 0, 1, \dots, m - 1 \right\}.$$

For a manifold  $X$  let  $H_*^{BM}(X; \mathbb{Q})$  denote the Borel-Moore homology group of  $X$  with rational coefficients. Let  $P_q^{BM}(X) = \sum_{i \geq 0} \dim H_i^{BM}(X; \mathbb{Q}) q^{\frac{i}{2}}$ .

We prove the following theorem.

**Theorem 1.2.** *Let  $\alpha$  and  $\beta$  be any two non-negative integers, such that  $\alpha + \beta \geq 1$ . Then we have*

$$(1) \quad \sum_{n \geq 0} P_q^{BM} \left( ((\mathbb{C}^2)^{[n]})^{T_{\alpha, \beta} \times \Gamma_{\alpha + \beta}} \right) t^n = \prod_{\substack{i \geq 1 \\ (\alpha + \beta) \nmid i}} \frac{1}{1 - t^i} \prod_{i \geq 1} \frac{1}{1 - qt^{(\alpha + \beta)i}}.$$

Here we use Borel-Moore homology, because the variety  $((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta} \times \Gamma_{\alpha+\beta}}$  is in general not compact, if  $\alpha = 0$ .

If  $\alpha$  and  $\beta$  are coprime, then  $\Gamma_{\alpha+\beta} \subset T_{\alpha,\beta}$ . Hence, Theorem 1.1 follows from Theorem 1.2.

Our proof of Theorem 1.2 consists of two steps. First, we prove that the left-hand side of (1) depends only on the sum  $\alpha + \beta$ . We use an argument with an equivariant symplectic form that is very similar to the one that was applied by the third author in [12] (proof of Proposition 5.7). After that the case  $\alpha = 0$  can be done using a notion of a power structure over the Grothendieck ring of quasiprojective varieties.

In [4], as a corollary of Theorem 1.1, there was derived a combinatorial identity. In the same way Theorem 1.2 leads to a more general combinatorial identity. Denote by  $\mathcal{Y}$  the set of all Young diagrams. The number of boxes in a Young diagram  $Y$  is denoted by  $|Y|$ . For a box  $s \in Y$  we define the numbers  $l_Y(s)$  and  $a_Y(s)$ , as it is shown on Fig. 1.

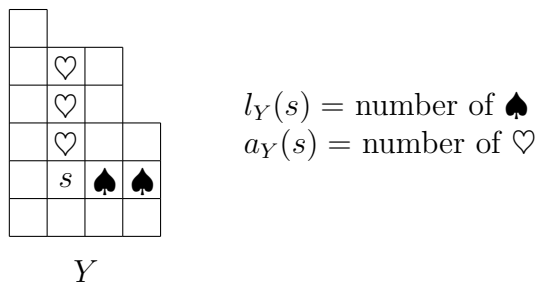


FIGURE 1. Definition of the numbers  $a_Y(s)$  and  $l_Y(s)$

For a Young diagram  $Y$  define the number  $h_{\alpha,\beta}(Y)$  by

$$h_{\alpha,\beta}(Y) = \left\{ s \in Y \mid \begin{array}{l} \alpha l_Y(s) = \beta(a_Y(s)+1) \\ (\alpha+\beta) | l_Y(s) + a_Y(s) + 1 \end{array} \right\}.$$

The following corollary is a generalization of Theorem 1.2 from [4].

**Corollary 1.3.** *Let  $\alpha$  and  $\beta$  be arbitrary non-negative integers, such that  $\alpha + \beta \geq 1$ . Then we have*

$$(2) \quad \sum_{Y \in \mathcal{Y}} q^{h_{\alpha,\beta}(Y)} t^{|Y|} = \prod_{\substack{i \geq 1 \\ (\alpha+\beta) \nmid i}} \frac{1}{1-t^i} \prod_{i \geq 1} \frac{1}{1-qt^{(\alpha+\beta)i}}.$$

*Proof.* The proof is similar to the proof of Theorem 1.2 in [4]. We apply the results from [1, 2], in order to construct a cell decomposition of the variety  $((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta} \times \Gamma_{\alpha+\beta}}$ , and show that the left-hand side of (1) is equal to the left-hand side of (2).  $\square$

We thank Ole Warnaar for suggesting this more general combinatorial identity after the paper [4] was published in arXiv.

1.1. *Organization of the paper.* In Section 2 we recall the definition of the Grothendieck ring of complex quasiprojective varieties and the properties of the natural power structure over it. Section 3 contains the proof of Theorem 1.2.

## 2. POWER STRUCTURE OVER THE GROTHENDIECK RING $K_0(\nu_{\mathbb{C}})$

In this section we review the definition of the Grothendieck ring of complex quasiprojective varieties and the power structure over it.

2.1. *Grothendieck ring.* The Grothendieck ring  $K_0(\nu_{\mathbb{C}})$  of complex quasiprojective varieties is the abelian group generated by the classes  $[X]$  of all complex quasiprojective varieties  $X$  modulo the relations:

- (1) if varieties  $X$  and  $Y$  are isomorphic, then  $[X] = [Y]$ ;
- (2) if  $Y$  is a Zariski closed subvariety of  $X$ , then  $[X] = [Y] + [X \setminus Y]$ .

The multiplication in  $K_0(\nu_{\mathbb{C}})$  is defined by the Cartesian product of varieties:  $[X_1] \cdot [X_2] = [X_1 \times X_2]$ . The class  $[\mathbb{A}_{\mathbb{C}}^1] \in K_0(\nu_{\mathbb{C}})$  of the complex affine line is denoted by  $\mathbb{L}$ .

We will need the following property of the Grothendieck ring  $K_0(\nu_{\mathbb{C}})$ . There is a natural homomorphism  $\theta: \mathbb{Z}[z] \rightarrow K_0(\nu_{\mathbb{C}})$ , defined by  $z \mapsto \mathbb{L}$ . This homomorphism is injective (see e.g. [11]).

2.2. *Power structure.* In [8] there was defined a notion of a power structure over a ring and there was described a natural power structure over the Grothendieck ring  $K_0(\nu_{\mathbb{C}})$ . This means that for a series  $A(t) = 1 + a_1 t + a_2 t^2 + \dots \in 1 + t \cdot K_0(\nu_{\mathbb{C}})[[t]]$  and for an element  $m \in K_0(\nu_{\mathbb{C}})$  one defines a series  $(A(t))^m \in 1 + t \cdot K_0(\nu_{\mathbb{C}})[[t]]$ , so that all the usual properties of the exponential function hold.

For a series  $A(t) = (1 - t)^{-1}$  and a quasiprojective variety  $M$  the series  $A(t)^{[M]}$  coincides with the motivic zeta function  $\zeta_{[M]}(t)$  introduced by M. Kapranov ([9]):

$$(1 - t)^{-[M]} = 1 + \sum_{i \geq 1} [S^i M] t^i,$$

where  $S^i M$  is the  $i$ -th symmetric product of the variety  $M$ . There is the following generalization of this formula. Suppose that  $M_1, M_2, \dots$  and  $N$  are quasiprojective varieties. Then we have

$$(3) \quad \left( 1 + \sum_{i \geq 1} [M_i] t^i \right)^{[N]} = 1 + \sum_{n \geq 1} X_n t^n, \quad \text{where}$$

$$X_n = \sum_{\sum_{i \geq 1} i d_i = n} \left[ \left( (N^{\sum d_i} \setminus \Delta) \times \left( \prod M_i^{d_i} \right) \right) / \prod S_{d_i} \right].$$

Here  $\Delta$  is the "large diagonal" in  $N^{\sum d_i}$ , which consists of  $(\sum d_i)$  points of  $N$  with at least two coinciding ones. The permutation group  $S_{d_i}$  acts by permuting corresponding  $d_i$  factors in  $\prod N^{d_i}$  and  $\prod M_i^{d_i}$  simultaneously.

We also need the following property of the power structure over  $K_0(\nu_{\mathbb{C}})$ . For any  $i \geq 1$  and  $j \geq 0$  we have

$$(4) \quad (1 - \mathbb{L}^j t^i)^{-\mathbb{L}} = (1 - \mathbb{L}^{j+1} t^i)^{-1}.$$

It can be derived from several statements from [8] as follows. Let  $a_i, i \geq 1$ , and  $m$  be from the Grothendieck ring  $K_0(\nu_{\mathbb{C}})$  and  $A(t) = 1 + \sum_{i \geq 1} a_i t^i$ . Then for any  $s \geq 0$  we have

$$(5) \quad A(\mathbb{L}^s t)^m = (A(t)^m)|_{t \rightarrow \mathbb{L}^s t},$$

$$(6) \quad (1 - t)^{-\mathbb{L}^s m} = (1 - t)^{-m}|_{t \rightarrow \mathbb{L}^s t}.$$

Formula (5) follows from Statement 2 in [8] and equation (6) follows from Statement 3 in [8]. Also for any  $s \geq 1$  we have (see [8])

$$(7) \quad A(t^s)^m = (A(t)^m)|_{t \rightarrow t^s}.$$

Obviously, formula (4) follows from (5), (6) and (7).

## 3. PROOF OF THEOREM 1.2

Using the  $(\mathbb{C}^*)^2$ -action on  $(\mathbb{C}^2)^{[n]}$  and the results from [1, 2] one can easily construct a cell decomposition of  $((\mathbb{C}^2)^{[n]})^{\Gamma_{\alpha+\beta} \times T_{\alpha,\beta}}$ . Thus, Theorem 1.2 is equivalent to the following formula

$$(8) \quad \sum_{n \geq 0} \left[ ((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta} \times \Gamma_{\alpha+\beta}} \right] t^n = \prod_{\substack{i \geq 1 \\ (\alpha+\beta) \nmid i}} \frac{1}{1-t^i} \prod_{i \geq 1} \frac{1}{1-\mathbb{L}t^{(\alpha+\beta)i}}.$$

It is clear that equation (8) is a corollary of the following two lemmas.

**Lemma 3.1.** *For any  $\alpha, \beta \geq 0$ , such that  $\alpha + \beta \geq 1$ , we have*

$$\left[ ((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta} \times \Gamma_{\alpha+\beta}} \right] = \left[ ((\mathbb{C}^2)^{[n]})^{T_{0,\alpha+\beta} \times \Gamma_{\alpha+\beta}} \right].$$

**Lemma 3.2.** *For any  $m \geq 1$  we have*

$$\sum_{n \geq 0} \left[ ((\mathbb{C}^2)^{[n]})^{T_{0,m} \times \Gamma_m} \right] t^n = \prod_{\substack{i \geq 1 \\ m \nmid i}} \frac{1}{1-t^i} \prod_{i \geq 1} \frac{1}{1-\mathbb{L}t^{mi}}.$$

*Proof of Lemma 3.1.* Let  $((\mathbb{C}^2)^{[n]})^{\Gamma_{\alpha+\beta}} = \coprod_i ((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}}$  be the decomposition in the irreducible components. It is sufficient to prove that

$$\left[ ((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}} \right] = \mathbb{L}^{\frac{d_i}{2}} \left[ \left( ((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}} \right)^{T_{\alpha,\beta}} \right],$$

where  $d_i = \dim ((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}}$ . The subvarieties  $((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}}$  are quiver varieties of affine type  $\tilde{A}_{\alpha+\beta-1}$ . We prove the above equality by using the idea in [12, Proposition 5.7].

Let  $\left( ((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}} \right)^{T_{\alpha,\beta}} = \coprod_j ((\mathbb{C}^2)^{[n]})_{i,j}^{\Gamma_{\alpha+\beta} \times T_{\alpha,\beta}}$  be the decomposition in the irreducible components. Consider the  $\mathbb{C}^*$ -action on  $(\mathbb{C}^2)^{[n]}$  induced by the homomorphism  $\mathbb{C}^* \rightarrow (\mathbb{C}^*)^2, t \mapsto (t^\alpha, t^\beta)$ . Define the sets  $C_{i,j}$  by

$$C_{i,j} = \left\{ z \in ((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}} \mid \lim_{t \rightarrow 0, t \in \mathbb{C}^*} t \cdot z \in ((\mathbb{C}^2)^{[n]})_{i,j}^{\Gamma_{\alpha+\beta} \times T_{\alpha,\beta}} \right\}.$$

From [1, 2] it follows that the set  $C_{i,j}$  is a locally trivial fiber bundle over  $((\mathbb{C}^2)^{[n]})_{i,j}^{\Gamma_{\alpha+\beta} \times T_{\alpha,\beta}}$  with an affine space as a fiber. Let us denote by  $d_{i,j}$  the dimension of a fiber. For  $p \in ((\mathbb{C}^2)^{[n]})_{i,j}^{\Gamma_{\alpha+\beta} \times T_{\alpha,\beta}}$  the tangent space  $T_p ((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}}$  is a  $\mathbb{C}^*$ -module. Let

$$T_p ((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}} = \sum_{m \in \mathbb{Z}} H(m)$$

be the weight decomposition. It is clear that  $d_{i,j} = \dim \left( \bigoplus_{m \geq 1} H(m) \right)$ .

The Hilbert scheme  $(\mathbb{C}^2)^{[n]}$  has the canonical symplectic form  $\omega$  that is induced from the symplectic form  $dx \wedge dy$  on  $\mathbb{C}^2$  (see e.g. [13]). The form  $\omega$  has weight  $-\alpha - \beta$  with respect to the  $\mathbb{C}^*$ -action on  $(\mathbb{C}^2)^{[n]}$ . The restriction  $\omega|_{((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}}}$  is the canonical symplectic form on the quiver variety (see [12]). Therefore, the spaces  $\bigoplus_{m \leq 0} H(m)$  and  $\bigoplus_{m \geq \alpha+\beta} H(m)$  are dual with respect to this form. Obviously, the  $(\alpha + \beta)$ -th root of unity  $\alpha+\beta\sqrt[1]{1}$  acts trivially on  $((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}}$ , thus,  $H(m) = 0$ , if  $(\alpha + \beta) \nmid m$ . We get  $\bigoplus_{m \geq \alpha+\beta} H(m) = \bigoplus_{m \geq 1} H(m)$  and  $d_{i,j} = \dim \left( \bigoplus_{m \geq 1} H(m) \right) = \frac{d_i}{2}$ . This completes the proof of the lemma.  $\square$

*Proof of Lemma 3.2.* Obviously, we have  $((\mathbb{C}^2)^{[n]})^{T_{0,m}} = ((\mathbb{C}^2)^{[n]})^{T_{0,1}}$ . For a partition  $\lambda = (\lambda_1, \dots, \lambda_l)$ ,  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l \geq 1$ , and a point  $x_0 \in \mathbb{C}$  define the ideal  $I_{\lambda, x_0} \subset \mathbb{C}[x, y]$  by

$$I_{\lambda, x_0} = (y^{\lambda_1}, (x - x_0)y^{\lambda_2}, \dots, (x - x_0)^{l-1}y^{\lambda_l}, (x - x_0)^l).$$

In [13] it is proved that each element  $I \in ((\mathbb{C}^2)^{[n]})^{T_{0,1}}$  can be uniquely expressed as

$$I = I_{\lambda^1, x_1} \cap \dots \cap I_{\lambda^k, x_k}$$

for some distinct points  $x_1, \dots, x_k \in \mathbb{C}$  and for some partitions  $\lambda^1, \dots, \lambda^k$  satisfying  $\sum_{i=1}^k |\lambda^i| = n$ .

Denote by  $\mathbb{C}_x$  the  $x$ -axis in the plane  $\mathbb{C}^2$ . Consider the map  $\pi_n: ((\mathbb{C}^2)^{[n]})^{T_{0,1}} \rightarrow S^n \mathbb{C}_x$  defined by

$$\pi_n (I_{\lambda^1, x_1} \cap \dots \cap I_{\lambda^k, x_k}) = \sum_{i=1}^k |\lambda^i| [x_i].$$

Suppose  $Z$  is an open subset of  $\mathbb{C}_x$ . From (3) it follows that

$$\sum_{n \geq 0} [\pi_n^{-1}(S^n Z)] t^n = \left( \prod_{i \geq 1} \frac{1}{1-t^i} \right)^{[Z]}.$$

The  $\Gamma_m$ -action on  $\mathbb{C}_x \setminus \{0\}$  is free and  $(\mathbb{C}_x \setminus \{0\})/\Gamma_m \cong \mathbb{C}_x \setminus \{0\}$ , therefore,

$$(\pi_n^{-1}(S^n(\mathbb{C}_x \setminus \{0\})))^{\Gamma_m} \cong \begin{cases} \emptyset, & \text{if } m \nmid n, \\ \pi_l^{-1}(S^l(\mathbb{C}_x \setminus \{0\})), & \text{if } n = ml. \end{cases}$$

We obtain

$$\sum_{n \geq 0} [(\pi_n^{-1}(S^n(\mathbb{C}_x \setminus \{0\})))^{\Gamma_m}] t^n = \left( \prod_{i \geq 1} \frac{1}{1-t^{mi}} \right)^{\mathbb{L}-1}.$$

Therefore, we get

$$\begin{aligned} \sum_{n \geq 0} [((\mathbb{C}^2)^{[n]})^{T_{0,1} \times \Gamma_m}] t^n &= \left( \sum_{n \geq 0} [\pi_n^{-1}(n[0])] t^n \right) \left( \sum_{n \geq 0} [(\pi_n^{-1}(S^n(\mathbb{C}_x \setminus \{0\})))^{\Gamma_m}] t^n \right) = \\ &= \left( \prod_{i \geq 1} \frac{1}{1-t^i} \right) \left( \prod_{i \geq 1} \frac{1}{1-t^{mi}} \right)^{\mathbb{L}-1} = \prod_{i \geq 1} \frac{1}{1-t^i} \prod_{i \geq 1} \frac{1}{1-\mathbb{L}t^{mi}}. \end{aligned}$$

The lemma is proved.  $\square$

The theorem is proved.

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