DUBROVIN-ZHANG HIERARCHY FOR THE HODGE INTEGRALS

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Abstract. In this paper we prove that the generating series of the Hodge integrals over the moduli space of stable curves is a solution of a certain deformation of the KdV hierarchy. This hierarchy is constructed in the framework of the Dubrovin-Zhang theory of the hierarchies of the topological type. It occurs that our deformation of the KdV hierarchy is closely related to the hierarchy of the Intermediate Long Wave equation.

1. Introduction

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable complex algebraic curves with n labelled marked points. The intersection theory of $\overline{\mathcal{M}}_{g,n}$ is closely related to the theory of integrable systems of partial differential equations. The basic result in this subject is the famous Witten conjecture ([Wit91]) proved by M. Kontsevich (see [Kon92]). It tells the following. The class $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n};\mathbb{C})$ is defined as the first Chern class of the line bundle over $\overline{\mathcal{M}}_{g,n}$ formed by the cotangent lines at the i-th marked point. Intersection numbers $\langle\tau_{k_1}\tau_{k_2}\dots\tau_{k_n}\rangle_g$ are defined as follows:

$$\langle\tau_{k_1}\tau_{k_2}\dots\tau_{k_n}\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\psi_2^{k_2}\dots\psi_n^{k_n}.$$

Let us introduce variables $\hbar, t_0, t_1, t_2, \dots$ and consider the generating series

$$F := \sum_{\substack{g\ge 0, n\ge 1\\ 2g-2+n>0}} \frac{\hbar^g}{n!} \sum_{k_1,\dots,k_n\ge 0} \langle\tau_{k_1}\dots\tau_{k_n}\rangle_g t_{k_1}\dots t_{k_n}.$$

Witten's conjecture, proved by M. Kontsevich, says that the second derivative $\frac{\partial^2 F}{\partial t_0^2}$ is a solution of the KdV hierarchy. The first two equations of this hierarchy are

$$\begin{aligned} u_{t_1} &= uu_x + \frac{\hbar}{12}u_{xxx},\\ u_{t_2} &= \frac{1}{2}u^2u_x + \frac{\hbar}{12}(2u_xu_{xx} + uu_{xxx}) + \frac{\hbar^2}{240}u_{xxxxx}. \end{aligned}$$

Here we identify x with t_0.

In this paper we study the Hodge integrals over the moduli space $\overline{\mathcal{M}}_{g,n}$:

$$\langle\lambda_j\tau_{k_1}\dots\tau_{k_n}\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \lambda_j\psi_1^{k_1}\psi_2^{k_2}\dots\psi_n^{k_n},$$

where $\lambda_j \in H^{2j}(\overline{\mathcal{M}}_{g,n};\mathbb{C})$ is the j-th Chern class of the rank g Hodge vector bundle over $\overline{\mathcal{M}}_{g,n}$ whose fibers over smooth curves are the spaces of holomorphic one-forms. Consider the generating series

$$F^{Hodge} := \sum_{\substack{g,n\ge 0\\ 2g-2+n>0}} \sum_{0\le j\le g} \frac{\hbar^g\varepsilon^j}{n!} \sum_{k_1,\dots,k_n\ge 0} \langle\lambda_j\tau_{k_1}\dots\tau_{k_n}\rangle_g t_{k_1}\dots t_{k_n}.$$

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The main result of the paper is the following. In Section 2.2 we construct a certain hamiltonian deformation of the KdV hierarchy. The first two equations of this hierarchy are

$$
\begin{aligned}
(1.1)\qquad u_{t_1} &= uu_x + \sum_{g\ge 1}\hbar^g\varepsilon^{g-1}\frac{|B_{2g}|}{(2g)!}u_{2g+1},\\
u_{t_2} &= \frac{1}{2}u^2u_x + \sum_{g\ge 1}\frac{|B_{2g}|}{(2g)!}\hbar^g\frac{\varepsilon^{g-1}}{4}(2(uu_{2g})_x+\partial_x^{2g+1}(u^2)) + \sum_{g\ge 2}\frac{|B_{2g}|}{(2g)!}\hbar^g\varepsilon^{g-2}(g+1)u_{2g+1}.
\end{aligned}
$$

Here B_{2g} are Bernoulli numbers: $B_2=\frac{1}{6}, B_4=-\frac{1}{30},\ldots$; and we denote by u_i the derivative $\partial_x^i u$. We call this hierarchy the deformed KdV hierarchy. Let

$$
\widetilde{F}^{Hodge} := F^{Hodge} + \sum_{g\ge 1}\frac{(-1)^g}{2^{2g}(2g+1)!}\hbar^g\varepsilon^g\frac{\partial^{2g}F^{Hodge}}{\partial t_0^{2g}}. \tag{1.2}
$$

Theorem 1.1. *The series $\frac{\partial^2\widetilde{F}^{Hodge}}{\partial t_0^2}$ is a solution of the deformed KdV hierarchy.*

We remind the reader that we identify x with t_0.

Let us explain how to compute the series F^{Hodge} using this theorem. Since $\overline{\mathcal{M}}_{0,3}$ is a point and $\int_{\overline{\mathcal{M}}_{1,1}}\lambda_1=\frac{1}{24}$, we have

$$
F^{Hodge}\big|_{t_{\ge 1}=0} = \frac{t_0^3}{6}+\frac{\hbar\varepsilon}{24}t_0.
$$

Therefore,

$$
\left.\frac{\partial^2\widetilde{F}^{Hodge}}{\partial t_0^2}\right|_{t_{\ge 1}=0} = t_0.
$$

Using this equation as an initial condition for the deformed KdV hierarchy, Theorem 1.1 allows to determine the series $\frac{\partial^2\widetilde{F}^{Hodge}}{\partial t_0^2}$. Note that the transformation (1.2) is invertible, one can check that

$$
F^{Hodge} = \widetilde{F}^{Hodge} + \sum_{g\ge 1}\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}\hbar^g\varepsilon^g\frac{\partial^{2g}\widetilde{F}^{Hodge}}{\partial t_0^{2g}}.
$$

Therefore, using $\frac{\partial^2\widetilde{F}^{Hodge}}{\partial t_0^2}$ we can reconstruct $\frac{\partial^2 F^{Hodge}}{\partial t_0^2}$. After that the string equation allows to determine F^{Hodge}. This is the same argument as E. Witten used in [Wit91] in order to reconstruct the series F from the second derivative $\frac{\partial^2 F}{\partial t_0^2}$.

Remark 1.2. In [Kaz09] M. Kazarian proved that after a certain change of variables the series F^{Hodge} becomes a solution of the KP hierarchy. It seems to be interesting to relate his result to ours.

Equation (1.1) coincides (after several rescalings) with the Intermediate Long Wave (ILW) equation (see e.g. [SAK79]). We are very grateful to S. Ferapontov and D. Novikov for noticing this fact after the author's talk on the conference in Trieste (Hamiltonian PDEs, Frobenius manifolds and Deligne-Mumford moduli spaces, September 2013). An infinite sequence of conserved quantities of the ILW equation was constructed in [SAK79]. We compare these conserved quantities with the Hamiltonians of our deformed KdV hierarchy in Section 8.

Our approach is based on the B. Dubrovin and Y. Zhang theory of the integrable hierarchies of the topological type. In [DZ05] B. Dubrovin and Y. Zhang gave a construction of a bihamiltonian hierarchy associated to any conformal semisimple Frobenius manifold. They conjectured that the equations and the hamiltonian structures of this hierarchy are polynomial. In [BPS12a] the authors suggested a more general construction of a hamiltonian hierarchy associated to an arbitrary semisimple cohomological field theory and proved the polynomiality of the equations and of the hamiltonian structure (see also [BPS12b]). One of the simplest examples of a cohomological field theory is the one formed by the Hodge classes

$$1+\varepsilon\lambda_1+\varepsilon\lambda_2+\ldots+\varepsilon^g\lambda_g\in H^*(\overline{\mathcal{M}}_{g,n};\mathbb{C}). \tag{1.3}$$

The main step in the proof of Theorem 1.1 is the application of the polynomiality theorem from [BPS12a] to the Dubrovin-Zhang hierarchy associated to the cohomological field theory (1.3). We also prove the following theorem.

Theorem 1.3. *Consider the Dubrovin-Zhang hierarchy associated to the cohomological field theory* (1.3). *Then the Miura transformation*

$$u\mapsto\widetilde{u}=u+\sum_{g\geq 1}\frac{(-1)^g}{2^{2g}(2g+1)!}\hbar^g\varepsilon^g u_{2g} \tag{1.4}$$

transforms this hierarchy to the deformed KdV hierarchy.

One can see that the variable $\widetilde{u}$ is related to the variable u (eq. (1.4)) in the same way as the series $\widetilde{F}^{Hodge}$ is related to the series F^{Hodge} (eq. (1.2)). This is so, because, as it will be explained in Section 4, Theorem 1.1 is a consequence of Theorem 1.3.

1.1. **Organization of the paper.** In Section 2 we give a construction of the deformed KdV hierarchy. The main statement here is Proposition 2.3.

In Section 3 we recall the Dubrovin-Zhang theory of the hierarchies of the topological type.

In Section 4 we formulate three propositions and show that Theorems 1.1, 1.3 and Proposition 2.3 follow from them. These propositions are proved in Sections 5, 6 and 7 correspondingly.

In Section 8 we compare the deformed KdV hierarchy with the hierarchy of the Intermediate Long Wave equation.

Appendix is devoted to the proof of several technical statements.

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2. Deformed KdV hierarchy

In this section we construct the deformed KdV hierarchy. First, in Section 2.1 we recall basic facts about hamiltonian systems of partial differential equations. Then in Section 2.2 we present a construction of the deformed KdV hierarchy. The main statement here is Proposition 2.3. It says that there exists a unique sequence of local functionals with certain properties. The uniqueness part is simple. It is based on Lemma 2.4 that is proved in Section 2.3. The proof of the existence part is presented in Section 4.

2.1. **Hamiltonian systems of PDEs.** Here we recall the hamiltonian formalism in the theory of partial differential equations. The material of this section is mostly borrowed from [DZ05].

2.1.1. *Differential polynomials and local functionals.* Consider variables $u, u_1, u_2, \ldots$. We will often denote u by u_0 and use an alternative notation for the variables $u_1, u_2, \ldots$:

$$u_x := u_1, \quad u_{xx} := u_2, \ldots.$$

Let $\mathcal{A}$ be the space of polynomials in the variables u_s, $s = 1, 2, \ldots$,

$$f(u; u_x, u_{xx}, \ldots) = \sum_{m\ge 0}\sum_{s_1,\ldots,s_m\ge 1} f^{s_1,s_2,\ldots,s_m}(u) u_{s_1} u_{s_2} \ldots u_{s_m}$$

with the coefficients $f^{s_1,\ldots,s_m}(u)$ being power series in u. Such an expression will be called differential polynomial.

The operator $\partial_x\colon \mathcal{A} \to \mathcal{A}$ is defined as follows:

$$\partial_x := \sum_{s\ge 0} u_{s+1}\frac{\partial}{\partial u_s}.$$

Let $\Lambda = \mathcal{A}/\operatorname{im}(\partial_x)$. We have the projection $\pi\colon \mathcal{A} \to \mathcal{A}/\operatorname{im}(\partial_x)$. We will use the following notation:

$$\int h dx := \pi(h),$$

for any $h \in \mathcal{A}$. The elements of the space Λ will be called local functionals.

For a local functional $\overline{h} = \int h dx \in \Lambda$, the variational derivative $\frac{\delta \overline{h}}{\delta u} \in \mathcal{A}$ is defined as follows:

$$\frac{\delta \overline{h}}{\delta u} := \sum_{i\ge 0} (-\partial_x)^i \frac{\partial h}{\partial u_i}.$$

Let us introduce a gradation $\deg_{dif}$ on the ring $\mathcal{A}$ of differential polynomials putting

$$\deg_{dif} u_k = k,\ k \ge 1; \quad \deg_{dif} f(u) = 0.$$

This gradation will be called differential degree. The gradation on $\mathcal{A}$ induces the gradation on the space Λ. There is an important lemma (see e.g. [DZ05]).

Lemma 2.1. *Let f be an arbitrary differential polynomial such that $f|_{u_i=0} = 0$. Then the local functional $\overline{f} = \int f dx$ is equal to zero, if and only if $\frac{\delta \overline{f}}{\delta u} = 0$.*

Let $\mathcal{A}' \subset \mathcal{A}$ be the subring of polynomials in $u, u_1, u_2, \ldots$. Sometimes we will use another gradation on the ring $\mathcal{A}'$ assigning to $u_i, i \ge 0$, degree 1. This second gradation will be just called degree.

2.1.2. *Extended spaces.* Introduce a formal indeterminate $\hbar$ of the differential degree

$$\deg_{dif} \hbar = -2.$$

Let $\widehat{\mathcal{A}} := \mathcal{A} \otimes \mathbb{C}[[\hbar]]$ and $\widehat{\mathcal{A}}^{[k]} \subset \widehat{\mathcal{A}}$ be the subspace of elements of the total differential degree k, $k \ge 0$. The space $\widehat{\mathcal{A}}^{[k]}$ consists of elements of the form

$$f(u; u_1, u_2, \ldots; \hbar) = \sum_{i\ge 0} \hbar^i f_i(u; u_1, \ldots), \quad f_i \in \mathcal{A}, \quad \deg_{dif} f_i = 2i + k.$$

The elements of the space $\widehat{\mathcal{A}}^{[k]}$ will be also called differential polynomials.

Let $\widehat{\Lambda} := \Lambda \otimes \mathbb{C}[[\hbar]]$ and $\widehat{\Lambda}^{[k]} \subset \Lambda \otimes \mathbb{C}[[\hbar]]$ be the subspace of elements of the total differential degree k. The space $\widehat{\Lambda}^{[k]}$ consists of integrals of the form

$$\overline{f} = \int f(u; u_1, u_2, \ldots; \hbar) dx, \quad f \in \widehat{\mathcal{A}}^{[k]}.$$

They will also be called local functionals.

2.1.3. *Hamiltonian systems of PDEs.* Let K be a differential operator

$$K = \sum_{i,j\ge 0} f_{i,j}\hbar^i\partial_x^j, \tag{2.1}$$

where $f_{i,j}\in\mathcal{A}$ and $\deg_{dif} f_{i,j}+j=2i+1$. Let us define the bracket $\{\cdot,\cdot\}_K\colon \widehat{\Lambda}^{[k]}\times\widehat{\Lambda}^{[l]}\to\widehat{\Lambda}^{[k+l+1]}$ by

$$\{\overline{g},\overline{h}\}_K := \int \frac{\delta\overline{g}}{\delta u}K\frac{\delta\overline{h}}{\delta u}dx.$$

The operator K is called Poisson, if the bracket $\{\cdot,\cdot\}_K$ is antisymmetric and satisfies the Jacobi identity. It is well-known that the operator ∂_x is Poisson (see e.g. [DZ05]).

A system of partial differential equations

$$\frac{\partial u}{\partial t_i} = f_i(u;u_1,\ldots;\hbar), \quad i\ge 1, \tag{2.2}$$

where $f_i\in\widehat{\mathcal{A}}^{[1]}$, is called hamiltonian, if there exists a Poisson operator K and a sequence of local functionals $\overline{h}_i\in\widehat{\Lambda}^{[0]}$, $i\ge 1$, such that

$$\begin{aligned} &f_i = K\frac{\delta\overline{h}_i}{\delta u},\\ &\{\overline{h}_i,\overline{h}_j\}_K = 0, \quad \text{for } i,j\ge 1. \end{aligned}$$

The local functionals $\overline{h}_i$ are called the Hamiltonians of the system (2.2).

2.1.4. *Miura transformations.* Let us recall the Miura group action on hamiltonian hierarchies.

Consider transformations of the form

$$u\mapsto\widetilde{u} = u+\sum_{k\ge 1}\hbar^k f_k(u;u_1,\ldots,u_{2k}), \quad f_k\in\mathcal{A}, \quad \deg_{dif} f_k = 2k. \tag{2.3}$$

It is easy to see that transformations (2.3) form a group which is called the Miura group.

Let us define the Miura group action on hamiltonian hierarchies. Given a transformation (2.3), any differential polynomial from $\widehat{\mathcal{A}}^{[0]}$ can be rewritten in the variables $\widetilde{u}_i$. This defines the Miura group action on $\widehat{\mathcal{A}}^{[0]}$ and on $\widehat{\Lambda}^{[0]}$. The action on Poisson operators is defined as follows:

$$K\mapsto\widetilde{K} = \left(\sum_{p\ge 0}\frac{\partial\widetilde{u}}{\partial u_p}\partial_x^p\right)\circ K\circ\left(\sum_{q\ge 0}(-\partial_x)^q\circ\frac{\partial\widetilde{u}}{\partial u_q}\right).$$

The Miura group action transforms solutions of hamiltonian hierarchies in the following way (see e.g. [DZ05]).

Lemma 2.2. *Suppose we have a Poisson operator K and a sequence of commuting local functionals $\overline{h}_n\in\widehat{\Lambda}^{[0]}$: $\{\overline{h}_n,\overline{h}_m\}_K=0$. Let $u(x,t_1,\ldots;\hbar)$ be a solution of the corresponding hierarchy of PDEs: $\frac{\partial u}{\partial t_n} = K\frac{\delta\overline{h}_n}{\delta u}$. Consider a Miura transformation* (2.3)*. Then the series $\widetilde{u}(x,t_1,\ldots;\hbar)$ is a solution of the transformed hierarchy: $\frac{\partial\widetilde{u}}{\partial t_n}=\widetilde{K}\frac{\delta\overline{h}_n}{\delta\widetilde{u}}$.*

2.2. **Deformed KdV hierarchy.** In this section we give a construction of a deformation of the KdV hierarchy.

Proposition 2.3. *Let ε be any complex number. There exists a unique sequence of local functionals $\overline{h}_n \in \widehat{\Lambda}^{[0]}, n \geq 1$, such that*

$$\overline{h}_1 = \int \left(\frac{u^3}{6} + \sum_{g\geq 1} \hbar^g \varepsilon^{g-1} \frac{|B_{2g}|}{2(2g)!} u u_{2g} \right) dx, \tag{2.4}$$
$$\overline{h}_n = \int \left(\frac{u^{n+2}}{(n+2)!} + O(\hbar) \right) dx, \quad \text{for } n \geq 2,$$
$$\{\overline{h}_i, \overline{h}_j\}_{\partial_x} = 0, \quad \text{for } i,j \geq 1.$$

The hamiltonian system of partial differential equations corresponding to the sequence of local functionals $\overline{h}_n$ and the Poisson operator ∂_x will be called the deformed KdV hierarchy.

The uniqueness statement in Proposition 2.3 is a consequence of the following simple lemma that will be proved in the next section.

Lemma 2.4. *Let us fix a local functional $\overline{h} \in \widehat{\Lambda}^{[0]}$ of the form $\overline{h} = \int \left(\frac{u^3}{6} + O(\hbar) \right) dx$. Consider also an arbitrary power series $q_0(u)$. Suppose there exists a local functional $\overline{q} \in \widehat{\Lambda}^{[0]}$ of the form $\overline{q} = \int (q_0(u) + O(\hbar))\, dx$, such that $\{\overline{h}, \overline{q}\}_{\partial_x} = 0$. Then the local functional $\overline{q}$ is uniquely determined by $\overline{h}$ and $q_0(u)$.*

We thank B. Dubrovin for telling us about Lemma 2.4.

The proof of the existence part of Proposition 2.3 is presented in Section 4.

2.3. **Proof of Lemma 2.4.** The proof is based on the following lemma.

Lemma 2.5. *Let $p(u; u_1, u_2, \ldots)$ be an arbitrary homogeneous differential polynomial of positive differential degree. Suppose $\left\{ \int p dx, \int \frac{u^3}{6} dx \right\}_{\partial_x} = 0$, then $\int p dx = 0$.*

Proof. If $\deg_{dif} p = 1$, then automatically $\int p dx = 0$. Suppose $\deg_{dif} p \geq 2$. Define the bracket $[\cdot, \cdot]$ on differential polynomials as follows:

$$[q, r] := \sum_{s\geq 0} \left((\partial_x^s q) \frac{\partial r}{\partial u_s} - (\partial_x^s r) \frac{\partial q}{\partial u_s} \right).$$

We have

$$\int [u u_x, p] dx = \int \left(\sum_{s\geq 0} \partial_x^s (u u_x) \frac{\partial p}{\partial u_s} \right) dx - \int (p u_x + u \partial_x p) dx =$$
$$= \int \frac{\delta p}{\delta u} \partial_x \left(\frac{u^2}{2} \right) dx - \int \partial_x (pu) dx = \left\{ \int p dx, \int \frac{u^3}{6} dx \right\}_{\partial_x} = 0.$$

Thus, $[u u_x, p]$ is a ∂_x-derivative.

Let us consider the lexicographical order on monomials $\prod_{k=1}^m u_k^{\alpha_k}$. It is easy to compute that, for a monomial $f(u) \prod_{k=1}^m u_k^{\alpha_k}$, we have (see [LZ05])

$$[u u_x, f(u) \prod_{k=1}^m u_k^{\alpha_k}] = \left(\sum_{k=1}^m (k+1)\alpha_k - \alpha_1 - 1 \right) f(u) u_x \prod_{k=1}^m u_k^{\alpha_k} + \text{monomials with the lower lexicographical order}. \tag{2.5}$$

Let $f(u) \prod_{k=1}^m u_k^{\alpha_k}$ be the monomial in p with the highest lexicographical order. From (2.5) and the fact that $[u u_x, p]$ is a ∂_x-derivative it follows that $m \geq 2$ and $\alpha_m = 1$. The lexicographical order of the highest monomial in the polynomial

$$p - \partial_x \left(\frac{1}{\alpha_{m-1}+1} f(u) \left(\prod_{k=1}^{m-2} u_k^{\alpha_k} \right) u_{m-1}^{\alpha_{m-1}+1} \right)$$

is lower than the lexicographical order of the highest monomial in p. We can do the same process further and prove that p is a ∂_x-derivative and, therefore, $\int p dx = 0$. ☐

Now let us prove Lemma 2.4. Suppose that there exist two different local functionals $\overline{q}^1, \overline{q}^2 \in \widehat{\Lambda}^{[0]}$, such that $\{\overline{h}, \overline{q}^j\}_{\partial_x} = 0$ and $\overline{q}^j = \int \left(q_0(u) + \sum_{i\ge1} q_i^j(u; u_1, \ldots)\hbar^i\right) dx$. We have

$$\{\overline{h}, \overline{q}^1 - \overline{q}^2\}_{\partial_x} = 0. \tag{2.6}$$

Let i_0 be the smallest i, such that $\int (q_i^1 - q_i^2)dx \neq 0$. From (2.6) it obviously follows that $\left\{\int \frac{u^3}{6}dx, \int (q_{i_0}^1 - q_{i_0}^2)dx, \right\}_{\partial_x} = 0$. Hence, by Lemma 2.5, $\int (q_{i_0}^1 - q_{i_0}^2)dx = 0$. This contradiction proves the lemma.

3. Cohomological field theories and the Dubrovin-Zhang hierarchies

In this section we briefly recall the Dubrovin-Zhang theory of the hierarchies of the topological type. In Section 3.1 we review the definition of cohomological field theory. In Section 3.2 we describe the construction of the Dubrovin-Zhang hierarchy associated to a semisimple cohomological field theory.

3.1. Cohomological field theory.

Here we recall the definition of cohomological field theory. For simplicity, we consider only one-dimensional cohomological field theories[1]. We refer the reader to [Sha09] for a more detailed introduction to this subject.

A one-dimensional cohomological field theory is a collection of classes $\alpha_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{C})$ defined for all g and n and satisfying the following properties (axioms):

- $\alpha_{g,n}$ belongs to the S_n-invariant part in the cohomology $H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{C})$, where the S_n-action on $H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{C})$ is induced by the mappings $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ defined by permutations of marked points.
- We have $\alpha_{0,3} = 1 \in H^*(\overline{\mathcal{M}}_{0,3}; \mathbb{C}) = \mathbb{C}$.
- If $\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map, then $\pi^*\alpha_{g,n} = \alpha_{g,n+1}$.
- a) If $gl\colon \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$ is the gluing map, then $gl^*\alpha_{g_1+g_2,n_1+n_2} = \alpha_{g_1,n_1+1} \cdot \alpha_{g_2,n_2+1}$.
  b) If $gl\colon \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ is the gluing map, then $gl^*\alpha_{g,n} = \alpha_{g-1,n+2}$.

The potential F of the cohomological field theory is defined as follows. Introduce variables t_d, where $d \geq 0$. Then

$$F := \sum_{g\geq0} F_g \hbar^g, \quad \text{where}$$

$$F_g := \sum_{\substack{n\geq0 \\ 2g-2+n>0}} \frac{1}{n!} \sum_{d_1,\ldots,d_n\geq0} \left(\int_{\overline{\mathcal{M}}_{g,n}} \alpha_{g,n} \prod_{i=1}^n \psi_i^{d_i}\right) \prod_{i=1}^n t_{d_i}.$$

Example 3.1. Let ε be an arbitrary complex number. Then the classes

$$\alpha_{g,n} = 1 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots + \varepsilon^g\lambda_g \in H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{C})$$

form a one-dimensional cohomological field theory.

Example 3.2. Let $\varepsilon_1, \varepsilon_2, \ldots$ be an arbitrary sequence of complex numbers. Then the classes

$$\alpha_{g,n} = \exp\left(\sum_{i\geq1} \varepsilon_i \operatorname{ch}_{2i-1}(\Lambda)\right),$$

where $\operatorname{ch}_{2i-1}(\Lambda)$ are the Chern characters of the Hodge bundle, form a one-dimensional cohomological field theory. In fact, any one-dimensional cohomological field theory has this form (see [MZ00]).

[1]To be completely precise, we consider one-dimensional cohomological field theories, where the scalar product of the unit with itself is equal to 1.

3.2. **Dubrovin-Zhang hierarchy.** In [BPS12a] the authors gave a construction of a hamiltonian system of partial differential equations associated to an arbitrary semisimple cohomological field theory. In this section we recall that construction. For simplicity, we do it in the case of a one-dimensional cohomological field theory. Any one-dimensional cohomological field theory is semisimple.

We fix a one-dimensional cohomological field theory, $\alpha_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n};\mathbb{C})$, with a potential $F = \sum_{g\geq 0} \hbar^g F_g$. In Sections 3.2.1 and 3.2.2 we construct a sequence of local functionals and a Poisson operator. In Section 3.2.3 we present a solution of the constructed hierarchy.

3.2.1. *Local functionals.* Let

$$u := \frac{\partial^2 F}{\partial t_0^2}.$$

We identify x with t_0. Let $u_n := \partial_x^n u$. From the axioms of cohomological field theory it follows that

$$u_n = t_n + \delta_{n,1} + O(t^2) + O(\hbar), \quad n \geq 0.$$

Thus, any power series in $\hbar$ and $t_0, t_1, \ldots$ can be expressed as a power series in $\hbar$ and $u, u_1 - 1, u_2, u_3, \ldots$.

Let

$$\Omega_{p,q} := \frac{\partial^2 F}{\partial t_p \partial t_q}.$$

Let us express $\Omega_{p,q}$ as a power series in $\hbar$ and $u, u_1 - 1, u_2, \ldots$. In [BPS12a] it is proved that the coefficient of $\hbar^g$ in $\Omega_{p,q}$ is a differential polynomial of differential degree $2g$. So, we can consider $\Omega_{p,q}$ as an element of $\widehat{\mathcal{A}}^{[0]}$. Let $\overline{h}_n := \int \Omega_{0,n+1} dx \in \widehat{\Lambda}^{[0]}$, $n \geq 1$. The local functionals $\overline{h}_n$ will be the Hamiltonians of our hierarchy. It is easy to show that $\Omega_{0,n} = \frac{u^{n+1}}{(n+1)!} + O(\hbar)$.

3.2.2. *Poisson operator.* Let us construct a Poisson operator of our hierarchy. Let

$$v := \frac{\partial^2 F_0}{\partial t_0^2}$$

and $v_n := \partial_x^n v$. From the axioms of cohomological field theory it follows that

$$v_n = t_n + \delta_{n,1} + O(t^2).$$

Thus, any power series in $t_0, t_1, t_2, \ldots$ can be expressed as a power series in $v, v_1 - 1, v_2, \ldots$.

Consider u as a power series in $v, v_1 - 1, v_2, \ldots$. Consider the differential operator

$$K := \left(\sum_{p\geq 0} \frac{\partial u}{\partial v_p} \partial_x^p\right) \circ \partial_x \circ \left(\sum_{q\geq 0} (-\partial_x)^q \circ \frac{\partial u}{\partial v_q}\right).$$

We can express this operator in the following form

$$K = \sum_{i,j\geq 0} p_{i,j} \hbar^i \partial_x^j,$$

where $p_{i,j}$ is a power series in $u, u_1 - 1, u_2, \ldots$. In [BPS12a] it is proved that $p_{i,j}$ is a differential polynomial of differential degree $2i + 1 - j$. Thus, K is an operator of the form (2.1). In fact, the operator K is Poisson and the local functionals $\overline{h}_n$ commute with respect to the Poisson bracket defined by it: $\{\overline{h}_n, \overline{h}_m\}_K = 0$.

By definition (see [BPS12a]), the Dubrovin-Zhang hierarchy, associated to our cohomological field theory, is the hamiltonian hierarchy, formed by the local functionals $\overline{h}_n, n \geq 1$, and the Poisson operator K.

3.2.3. *Solution of the hierarchy.* We have the following lemma (see [BPS12a]).

Lemma 3.3. *The series $\frac{\partial^2 F}{\partial t_0^2}$ is a solution of the constructed hierarchy:*

$$\frac{\partial u}{\partial t_n} = K\frac{\delta \overline{h}_n}{\delta u}, \quad n \geq 1.$$

4. Reformulation of Theorems 1.1, 1.3 and of Proposition 2.3

In this section we formulate three propositions and show that Theorems 1.1, 1.3 and Proposition 2.3 follow from them. These propositions are proved in the next three sections of the paper.

Consider the cohomological field theory (1.3) and the corresponding Dubrovin-Zhang hierarchy.

Proposition 4.1. *The Miura transformation*

$$u \mapsto \widetilde{u} = u + \sum_{g\geq 1} \frac{(-1)^g}{2^{2g}(2g+1)!}\hbar^g \varepsilon^g u_{2g} \tag{4.1}$$

transforms the Poisson operator of the hierarchy to ∂_x and the Hamiltonian $\overline{h}_1$ to

$$\int \left(\frac{\widetilde{u}^3}{6} + \frac{\hbar}{24}\widetilde{u}\widetilde{u}_2 + \frac{\hbar^2\varepsilon}{1440}\widetilde{u}\widetilde{u}_4 + \sum_{g\geq 3} \hbar^g \varepsilon^{g-1} c_g \widetilde{u}\widetilde{u}_{2g} \right) dx, \tag{4.2}$$

where c_g, $g \geq 3$, are some complex constants.

Proposition 4.2. *The following two local functionals*

$$\overline{h}_1 = \int \left(\frac{u^3}{6} + \sum_{g\geq 1} \hbar^g \varepsilon^{g-1} \frac{|B_{2g}|}{2(2g)!} u u_{2g} \right) dx,$$

$$\overline{h}_2 = \int \left(\frac{u^4}{4!} + \frac{\hbar}{48} u^2 u_{xx} + \sum_{g\geq 2} \frac{|B_{2g}|}{(2g)!}\hbar^g \left(\varepsilon^{g-2}\frac{g+1}{2} u u_{2g} + \varepsilon^{g-1}\frac{1}{4}u^2 u_{2g} \right) \right) dx,$$

commute with respect to the bracket $\{\cdot,\cdot\}_{\partial_x}$.

Proposition 4.3. *Suppose there exists a sequence of complex numbers c_g, $g \geq 1$, $c_1 \neq 0$, that satisfies the following property: there exists a local functional $\overline{h}_2 \in \widehat{\Lambda}^{[0]}$ of the form*

$$\overline{h}_2 = \int \left(\frac{u^4}{24} + O(\hbar) \right) dx$$

that commutes with the local functional

$$\overline{h}_1 = \int \left(\frac{u^3}{6} + \sum_{g\geq 1} \hbar^g \varepsilon^{g-1} c_g u u_{2g} \right) dx$$

with respect to the bracket $\{\cdot,\cdot\}_{\partial_x}$. Then all numbers c_g, for $g \geq 3$, are uniquely determined by c_1 and c_2.

Let us show that Theorems 1.1, 1.3 and Proposition 2.3 follow from these propositions.

From the propositions it follows that the Miura transform of our Dubrovin-Zhang hierarchy is a hierarchy with ∂_x as a Poisson operator and the local functional (2.4) as the Hamiltonian $\overline{h}_1$. This proves the existence statement of Proposition 2.3. The uniqueness statement follows from Lemma 2.4. We also immediately get Theorem 1.3. Theorem 1.1 follows from Theorem 1.3, Lemma 3.3 and Lemma 2.2.

5. Proof of Proposition 4.1

We have $\overline{h}_n = \int \Omega_{0,n+1} dx$. The proof of the proposition is splitted in four steps. In Section 5.1 we derive a certain homogeneity property of the differential polynomials $\Omega_{p,q}$. In Section 5.2 we find the coefficient of $\hbar^g\varepsilon^g$ in the potential F^{Hodge}. In Section 5.3 we prove that substitution (4.1) kills the coefficients of $\hbar^g\varepsilon^g$ in the Hamiltonians $\overline{h}_n$ and show that

$$\overline{h}_1 = \int \left(\frac{\widetilde{u}^3}{6} + \sum_{g\ge 1} \hbar^g \varepsilon^{g-1} c_g \widetilde{u}\widetilde{u}_{2g} \right) dx.$$

We also show that $c_1 = \frac{1}{24}$. The computation of c_2 is quite technical, it is done in Appendix A. Section 5.4 is devoted to the computation of the Poisson operator of our Dubrovin-Zhang hierarchy.

Let us fix some notations. By F^{Hodge} we denote the potential of the cohomological field theory (1.3). We also use the notations from Section 3.2:

$$u_i := \partial_x^i \frac{\partial^2 F^{Hodge}}{\partial t_0^2}, \qquad v_i := \partial_x^i \frac{\partial^2 F_0^{Hodge}}{\partial t_0^2}.$$

Recall that we identify x with t_0.

5.1. **Homogeneity of $\Omega_{p,q}$.** The dimension of $\overline{\mathcal{M}}_{g,n}$ is equal to $3g-3+n$, thus, the coefficient of $\hbar^g\varepsilon^j \prod_{i\ge 0} t_i^{d_i}$ in F^{Hodge} is non-zero only if $\sum_{i\ge 0}(i-1)d_i + j = 3g-3$. Consider the linear differential operator O_1 defined by

$$O_1 := \sum_{i\ge 0}(i-1)t_i \frac{\partial}{\partial t_i} + \varepsilon\frac{\partial}{\partial\varepsilon} - 3\hbar\frac{\partial}{\partial\hbar}.$$

We get

$$O_1 F^{Hodge} = -3F^{Hodge}. \tag{5.1}$$

From (5.1) and the commutation relation (recall that $\partial_x = \frac{\partial}{\partial t_0}$)

$$[\partial_x, O_1] = -\partial_x \tag{5.2}$$

it is clear that

$$O_1 u_n = (n-1)u_n.$$

Thus,

$$O_1 = \sum_{i\ge 0}(i-1)u_i \frac{\partial}{\partial u_i} + \varepsilon\frac{\partial}{\partial\varepsilon} - 3\hbar\frac{\partial}{\partial\hbar}.$$

From (5.1) it is easy to see that

$$O_1\Omega_{p,q} = -(p+q+1)\Omega_{p,q}. \tag{5.3}$$

On the other hand, in [BPS12a] it is proved that $\Omega_{p,q}$ is a power series in $\hbar$, where the coefficient of $\hbar^g$ is a homogeneous differential polynomial of differential degree $2g$. This property can be written as

$$O_2\Omega_{p,q} = 0, \quad \text{where} \tag{5.4}$$

$$O_2 := \sum_{i\ge 0} i u_i \frac{\partial}{\partial u_i} - 2\hbar\frac{\partial}{\partial\hbar}.$$

If we subtract (5.3) from (5.4), we get

$$\left(\sum_{i\ge 0} u_i \frac{\partial}{\partial u_i} + \hbar\frac{\partial}{\partial\hbar} - \varepsilon\frac{\partial}{\partial\varepsilon} \right) \Omega_{p,q} = (p+q+1)\Omega_{p,q}. \tag{5.5}$$

We have that $\Omega_{p,q}$ is a power series in $\hbar$ and ε with the coefficients that are differential polynomials. It is easy to see that the coefficient of $\hbar^g\varepsilon^j$ is non-zero, only if $g\ge j$. From (5.5) it follows that the coefficient of $\hbar^g\varepsilon^j$ is a polynomial in $u,u_1,\dots$ of degree $p+q+1-g+j$.

5.2. **Coefficient of $\hbar^g\varepsilon^g$.** The so-called λ_g-conjecture, proved in [FP03], tells that

$$\int_{\overline{\mathcal{M}}_{g,n}}\lambda_g\psi_1^{d_1}\psi_2^{d_2}\dots\psi_n^{d_n}=\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}\frac{(2g-3+n)!}{d_1!d_2!\dots d_n!},\quad g\ge 1,\quad \sum_{i=1}^n d_i=2g-3+n. \tag{5.6}$$

We have

$$F_0^{Hodge}=\sum_{n\ge 3}\frac{1}{n!}\sum_{\substack{d_1,\dots,d_n\ge 0\\ d_1+\dots+d_n=n-3}}\frac{(n-3)!}{d_1!\dots d_n!}t_{d_1}\dots t_{d_n}.$$

Therefore, from (5.6) it follows that, for $g\ge 1$, the coefficient of $\hbar^g\varepsilon^g$ in F^{Hodge} is equal to $\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}v_{2g-2}$.

Consider now $\Omega_{0,n}=\frac{\partial^2F^{Hodge}}{\partial t_0\partial t_n}$ as a series in $\hbar,\varepsilon,v,v_1-1,v_2,\dots$. We get that the coefficient of $\hbar^g\varepsilon^g$ is equal to

$$\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}\partial_x^{2g-1}\left(\frac{\partial v}{\partial t_n}\right)=\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}\partial_x^{2g-1}\left(\frac{v^n}{n!}v_x\right)=\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}\partial_x^{2g}\left(\frac{v^{n+1}}{(n+1)!}\right).$$

Thus,

$$\Omega_{0,n}=\frac{v^{n+1}}{(n+1)!}+\sum_{g\ge 1}\hbar^g\varepsilon^g\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}\partial_x^{2g}\left(\frac{v^{n+1}}{(n+1)!}\right)+\sum_{g>j\ge 0}\hbar^g\varepsilon^j f_{g,j}^n(v,v_1-1,v_2,\dots),$$

where $f_{g,j}^n(v,v_1-1,v_2,\dots)$ are power series in $v,v_1-1,v_2,\dots$.

5.3. **Miura transformation.** We have

$$u=v+\sum_{g\ge 1}(\hbar\varepsilon)^g\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}v_{2g}+\sum_{g>j\ge 0}\hbar^g\varepsilon^j f_{g,j}^0(v,v_1-1,v_2,\dots). \tag{5.7}$$

It is easy to check that

$$\left(1+\sum_{g\ge 1}\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}z^{2g}\right)\left(1+\sum_{g\ge 1}\frac{(-1)^g}{2^{2g}(2g+1)!}z^{2g}\right)=1. \tag{5.8}$$

Therefore,

$$v=\widetilde{u}+\sum_{g>j\ge 0}\hbar^g\varepsilon^j q_{g,j}(\widetilde{u},\widetilde{u}_1-1,\widetilde{u}_2,\dots),$$

where $q_{g,j}(\widetilde{u},\widetilde{u}_1-1,\widetilde{u}_2,\dots)$ are power series in $\widetilde{u},\widetilde{u}_1-1,\widetilde{u}_2,\dots$.

We get

$$\Omega_{0,n}=\frac{\widetilde{u}^{n+1}}{(n+1)!}+\sum_{g\ge 1}\hbar^g\varepsilon^g\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}\partial_x^{2g}\left(\frac{\widetilde{u}^{n+1}}{(n+1)!}\right)+\sum_{g>j\ge 0}\hbar^g\varepsilon^j w_{g,j}^n(\widetilde{u},\widetilde{u}_1,\dots),$$

$$\overline{h}_n=\int\Omega_{0,n+1}dx=\int\left(\frac{\widetilde{u}^{n+2}}{(n+2)!}+\sum_{g>j\ge 0}\hbar^g\varepsilon^j w_{g,j}^{n+1}(\widetilde{u},\widetilde{u}_1,\dots)\right)dx.$$

Here $w_{g,j}^n$ are differential polynomials in $\widetilde{u}_i$. From (5.5) it follows that $w_{g,j}^n$ is a polynomial in $\widetilde{u},\widetilde{u}_1,\dots$ of degree $n+1-g+j$. If $g-j=n$, then $w_{g,j}^n=b_g^n\widetilde{u}_{2g}$, for some constant b_g^n and we have $\int w_{g,j}^n dx=0$. We obtain

$$\overline{h}_n=\int\left(\frac{\widetilde{u}^{n+2}}{(n+2)!}+\sum_{\substack{g,j\ge 0\\ n\ge g-j\ge 1}}\hbar^g\varepsilon^j w_{g,j}^{n+1}\right)dx.$$

In particular, we get

$$\overline{h}_1=\int\left(\frac{\widetilde{u}^3}{6}+\sum_{g\ge 1}\hbar^g\varepsilon^{g-1}q_g\right)dx,$$

where q_g are quadratic polynomials in $\widetilde{u},\widetilde{u}_1,\ldots$. It is cleat that $\int\widetilde{u}_i\widetilde{u}_j dx=(-1)^i\int\widetilde{u}\widetilde{u}_{i+j}dx$. Therefore, we have

$$\overline{h}_1=\int\left(\frac{\widetilde{u}^3}{6}+\sum_{g\ge 1}\hbar^g\varepsilon^{g-1}c_g\widetilde{u}\widetilde{u}_{2g}\right)dx, \tag{5.9}$$

for some constants c_g.

It remains to prove that $c_1=\frac{1}{24}$ and $c_2=\frac{1}{1440}$. If $\varepsilon=0$, then our cohomological field theory is trivial. The corresponding Dubrovin-Zhang hierarchy in this case is the KdV hierarchy (see [DZ05]). Thus, $c_1=\frac{1}{24}$. The computation of c_2 is done in Appendix A.

5.4. **Poisson operator.** Consider the operator O_1 from Section 5.1. Since $O_1v=-v$ and $O_1v_n=(n-1)v_n$, we get

$$O_1=\sum_{i\ge 0}(i-1)v_i\frac{\partial}{\partial v_i}+\varepsilon\frac{\partial}{\partial\varepsilon}-3\hbar\frac{\partial}{\partial\hbar}.$$

Thus,

$$O_1\frac{\partial u}{\partial v_n}=-n\frac{\partial u}{\partial v_n}. \tag{5.10}$$

The Poisson operator K of our hierarchy is equal to

$$K=\left(\sum_{m\ge 0}\frac{\partial u}{\partial v_m}\partial_x^m\right)\circ\partial_x\circ\left(\sum_{n\ge 0}(-\partial_x)^n\circ\frac{\partial u}{\partial v_n}\right). \tag{5.11}$$

Let us express it as $K=\sum_{n\ge 0}p_n\partial_x^n$. From (5.10) and (5.2) it follows that

$$O_1p_n=-(n-1)p_n. \tag{5.12}$$

On the other hand, in [BPS12a] it is proved that the coefficient of $\hbar^g\partial_x^n$ in K is a differential polynomial in $u,u_1,\ldots$ of differential degree $2g+1-n$. Therefore, we have

$$\left(-\sum_{i\ge 0}iu_i\frac{\partial}{\partial u_i}+2\hbar\frac{\partial}{\partial\hbar}\right)p_n=(n-1)p_n. \tag{5.13}$$

Let us sum (5.12) and (5.13), we get

$$\left(-\sum_{i\ge 0}u_i\frac{\partial}{\partial u_i}+\varepsilon\frac{\partial}{\partial\varepsilon}-\hbar\frac{\partial}{\partial\hbar}\right)p_n=0. \tag{5.14}$$

We know that p_n is a power series in $\hbar$ and ε with the coefficients that are differential polynomials in u_i. It is easy to see that the coefficient of $\hbar^g\varepsilon^j$ is zero, if $g<j$. Thus, from (5.14) it follows that $p_n=\sum_{g\ge 0}b_{g,n}\hbar^g\varepsilon^g$, where $b_{g,n}$ are complex numbers. From (5.13) it follows that $b_{g,n}=0$, if $2g\ne n-1$. Finally, we get

$$K=\sum_{g\ge 0}b_g\hbar^g\varepsilon^g\partial_x^{2g+1}, \tag{5.15}$$

where b_g are some complex numbers.

We have proved that in the operator K there are no terms with $\hbar^g\varepsilon^j$, for $g>j$. Thus, by (5.7) and (5.11),

$$K=\left[1+\sum_{g\ge 1}(\hbar\varepsilon)^g\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}\partial_x^{2g}\right]\circ\partial_x\circ\left[1+\sum_{g\ge 1}(\hbar\varepsilon)^g\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}\partial_x^{2g}\right]. \tag{5.16}$$

This equation together with (5.8) implies that the Miura transformation (4.1) transforms the operator K to ∂_x. This concludes the proof of the proposition.

Remark 5.1. Let us compute the product on the right-hand side of (5.16). By (5.8),

$$1+\sum_{g\ge 1}\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}z^{2g}=\frac{iz}{e^{\frac{iz}{2}}-e^{-\frac{iz}{2}}}.$$

Let $\psi(z):=\frac{iz}{2}\frac{e^{\frac{iz}{2}}+e^{-\frac{iz}{2}}}{e^{\frac{iz}{2}}-e^{-\frac{iz}{2}}}$. A direct computation shows that

$$\left(\frac{iz}{e^{\frac{iz}{2}}-e^{-\frac{iz}{2}}}\right)^2=\psi-z\psi'.$$

On the other hand, $\psi(z)=1-\sum_{g\ge 1}\frac{|B_{2g}|}{(2g)!}z^{2g}$. Therefore,

$$\psi-z\psi'=1+\sum_{g\ge 1}\frac{(2g-1)|B_{2g}|}{(2g)!}z^{2g}.$$

We conclude that

$$K=\partial_x+\sum_{g\ge 1}\hbar^g\varepsilon^g\frac{(2g-1)|B_{2g}|}{(2g)!}\partial_x^{2g+1}.$$

6. Proof of Proposition 4.2

Before the proof of the proposition let us state several useful formulas.

$$\left\{\int uu_{2g_1}dx,\int uu_{2g_2}dx\right\}_{\partial_x}=0, \tag{6.1}$$

$$\left\{\int \frac{u^3}{6}dx,\int u^2u_{2g}dx\right\}_{\partial_x}=-2\left\{\int uu_{2g}dx,\int \frac{u^4}{24}dx\right\}_{\partial_x}. \tag{6.2}$$

They can be easily checked by a direct computation.

From (6.1) and (6.2) it follows that

$$\{\overline{h}_1,\overline{h}_2\}_{\partial_x}=\sum_{g\ge 2}\hbar^g\varepsilon^{g-2}\times$$

(6.3)

$$\times\left[\frac{(g+1)|B_{2g}|}{(2g)!}\left\{\int\frac{u^3}{6}dx,\int\frac{uu_{2g}}{2}dx\right\}_{\partial_x}+\sum_{\substack{g_1+g_2=g\\ g_1,g_2\ge 1}}\frac{|B_{2g_1}||B_{2g_2}|}{8(2g_1)!(2g_2)!}\left\{\int uu_{2g_1}dx,\int u^2u_{2g_2}dx\right\}_{\partial_x}\right].$$

We have to prove that (6.3) is equal to 0. Expression (6.3) is equal to

$$\int\left[\frac{(g+1)|B_{2g}|}{2(2g)!}u^2u_{2g+1}-\sum_{\substack{g_1+g_2=g\\ g_1,g_2\ge 1}}\frac{|B_{2g_1}||B_{2g_2}|}{4(2g_1)!(2g_2)!}(2uu_{2g_2}+\partial_x^{2g_2}(u^2))u_{2g_1+1}\right]dx. \tag{6.4}$$

We have $\int\partial_x^{2g_2}(u^2)u_{2g_1+1}dx=\int u^2u_{2g+1}$. If $m\ge 2$, then (see e.g. [GKP94])

$$\sum_{\substack{m_1+m_2=m\\ m_1,m_2\ge 1}}\frac{|B_{2m_1}||B_{2m_2}|}{(2m_1)!(2m_2)!}=\frac{(2m+1)|B_{2m}|}{(2m)!}.$$

Therefore, (6.4) is equal to

$$\int\left[\frac{|B_{2g}|}{4(2g)!}u^2u_{2g+1}-\sum_{\substack{g_1+g_2=g\\ g_1,g_2\ge 1}}\frac{|B_{2g_1}||B_{2g_2}|}{2(2g_1)!(2g_2)!}uu_{2g_2}u_{2g_1+1}\right]dx.$$

The variational derivative of this integral is equal to

$$\frac{|B_{2g}|}{4(2g)!}(2uu_{2g+1}-\partial_x^{2g+1}(u^2))-\sum_{\substack{g_1+g_2=g\\ g_1,g_2\ge 1}}\frac{|B_{2g_1}||B_{2g_2}|}{2(2g_1)!(2g_2)!}(u_{2g_2}u_{2g_1+1}+\partial_x^{2g_2}(uu_{2g_1+1})-\partial_x^{2g_1+1}(uu_{2g_2}))$$

$$=\frac{|B_{2g}|}{4(2g)!}(2uu_{2g+1}-\partial_x^{2g+1}(u^2))-\sum_{\substack{g_1+g_2=g\\ g_1,g_2\ge 1}}\frac{|B_{2g_1}||B_{2g_2}|}{2(2g_1)!(2g_2)!}(u_{2g_2}u_{2g_1+1}-\partial_x^{2g_1}(u_1u_{2g_2}))$$

$$=\frac{(-1)^{g+1}}{2}\sum_{i=0}^{g}\frac{B_{2i}B_{2g-2i}}{(2i)!(2g-2i)!}(u_{2i}u_{2g-2i+1}-\partial_x^{2i}(u_1u_{2g-2i})).$$

Lemma 6.1. *We have the following identity:*

$$\sum_{i=0}^{g}\frac{B_{2i}B_{2g-2i}}{(2i)!(2g-2i)!}(u_{2i}u_{2g-2i+1}-\partial_x^{2i}(u_1u_{2g-2i}))=\begin{cases}-\frac{u_1u_2}{4}, & \text{if } g=1,\\ 0, & \text{if } g\ne 1.\end{cases}\qquad(6.5)$$

This lemma concludes the proof of Proposition 4.2. We prove it in Appendix B.

7. Proof of Proposition 4.3

We have

$$\overline{h}_2=\int\left(\frac{u^4}{24}+\sum_{g\ge 1}\hbar^g p_g\right)dx.\qquad(7.1)$$

It is easy to see that $\int p_1dx=\int\frac{c_1}{2}u^2u_2dx$. Denote $\frac{c_1}{2}u^2u_2$ by r_1.

Let us show that, for $g\ge 2$, we have

$$\int p_gdx=\int\left(\varepsilon^{g-2}q_g+\varepsilon^{g-1}r_g\right)dx,\qquad(7.2)$$

where q_g and r_g are polynomials in u_i of degrees 2 and 3 correspondingly. We prove it by induction on g. The coefficient of $\hbar^g$ in $\{\overline{h}_1,\overline{h}_2\}_{\partial_x}$ is equal to

$$\left\{\int\frac{u^3}{6}dx,\int p_gdx\right\}_{\partial_x}+\varepsilon^{g-2}\sum_{\substack{g_1+g_2=g\\ g_1,g_2\ge 1}}c_{g_1}\left\{\int uu_{2g_1}dx,\int r_{g_2}dx\right\}_{\partial_x}+$$
$$+\varepsilon^{g-1}c_g\left\{\int uu_{2g}dx,\int\frac{u^4}{24}dx\right\}_{\partial_x}=0.\qquad(7.3)$$

The second term in (7.3) has degree 3 and the third one has degree 4. Hence, we get (7.2).

From (7.3) and (6.2) it follows that $\int r_gdx=\frac{c_g}{2}\int u^2u_{2g}dx$. Clearly, we have $\int q_gdx=e_g\int uu_{2g}dx$, where e_g is a complex constant. Using (7.3), we get

$$e_g\left\{\int\frac{u^3}{6}dx,\int uu_{2g}dx\right\}_{\partial_x}+\sum_{\substack{g_1+g_2=g\\ g_1,g_2\ge 1}}\frac{c_{g_1}c_{g_2}}{2}\left\{\int uu_{2g_1}dx,\int u^2u_{2g_2}dx\right\}_{\partial_x}=0.\qquad(7.4)$$

Define the local functionals $\overline{f}_g, \overline{f}_{g_1,g_2} \in \Lambda^{[0]}$ as follows:

$$\overline{f}_g := \left\{ \int \frac{u^3}{6} dx, \int u u_{2g} dx \right\}_{\partial_x},$$
$$\overline{f}_{g_1,g_2} := \left\{ \int u u_{2g_1} dx, \int u^2 u_{2g_2} dx \right\}_{\partial_x} + \left\{ \int u u_{2g_2} dx, \int u^2 u_{2g_1} dx \right\}_{\partial_x}.$$

In these notations equation (7.4) looks as follows:

$$e_g \overline{f}_g + \frac{c_{g-1}c_1}{2} \overline{f}_{g-1,1} + \sum_{\substack{g_1+g_2=g\\ g_1\ge g_2\ge 2}} \frac{c_{g_1}c_{g_2}}{2} \overline{f}_{g_1,g_2} = 0.$$

Let us show that, for $g \geq 4$, this equation uniquely determines c_{g-1} from $c_{g-2}, c_{g-3}, \ldots, c_1$. For this we have to prove that the local functionals $\overline{f}_g$ and $\overline{f}_{g-1,1}$ are linearly independent. We have

$$\begin{aligned}
\overline{f}_g =& \int u^2 u_{2g+1} dx,\\
\overline{f}_{g-1,1} =& \int \left[-2(\partial_x^2(u^2) + 2uu_2) u_{2g-1} - 2(2uu_{2g-2} + \partial_x^{2g-2}(u^2)) u_3 \right] dx =\\
=& -4\overline{f}_g - 2 \int (2uu_2u_{2g-1} + 2uu_3u_{2g-2}) dx =\\
=& -4\overline{f}_g - 2 \int (\partial_x^2(u^2) u_{2g-1} - 2u_1^2 u_{2g-1} + \partial_x^3(u^2) u_{2g-2} - 6u_1u_2u_{2g-2}) dx =\\
=& -4\overline{f}_g - 2 \int u_1^2 u_{2g-1} dx.
\end{aligned}$$

We need to prove that $\frac{\delta}{\delta u} \int (u^2 u_{2g+1}) dx$ and $\frac{\delta}{\delta u} \int (u_x^2 u_{2g-1}) dx$ are linearly independent. We have

$$\frac{\delta}{\delta u} \int (u^2 u_{2g+1}) dx = -2 \sum_{i=1}^{g} \binom{2g+1}{i} u_i u_{2g+1-i}, \tag{7.5}$$

$$\frac{\delta}{\delta u} \int (u_x^2 u_{2g-1}) dx = -2u_1u_{2g} - 2u_2u_{2g-1} - 2 \sum_{i=1}^{g} \binom{2g-1}{i-1} u_i u_{2g+1-i}. \tag{7.6}$$

The matrix of coefficients of u_1u_{2g} and u_3u_{2g-2} in (7.5) and (7.6) is equal to

$$\begin{pmatrix} -2(2g+1) & -\frac{(2g+1)2g(2g-1)}{3} \\ -4 & -(2g-1)(2g-2) \end{pmatrix}$$

It is non-degenerate, if $g \geq 4$. This completes the proof of the proposition.

8. Deformed KdV hierarchy and the ILW equation

In this section we explain a relation of the deformed KdV hierachy to the hierarchy of the conserved quantities of the Intermediate Long Wave equation constructed in [SAK79].

In Section 8.1 we recall the definition of the ILW equation and show how to rescale the parameters in order to get the first equation (1.1) of the deformed KdV hierarchy. In Section 8.2 we introduce slight extensions of the spaces $\widehat{\mathcal{A}}^{[k]}$ and $\widehat{\Lambda}^{[k]}$. Section 8.3 contains a review of the construction of conserved quantities of the ILW equation from [SAK79]. In Section 8.4 we compare these conserved quantities with the Hamiltonians of the deformed KdV hierarchy.

8.1. **Intermediate Long Wave equation.** The Intermediate Long Wave equation looks as follows (see e.g. [SAK79]):

$$w_\tau + 2ww_x + T(w_{xx}) = 0, \tag{8.1}$$

where

$$T(f) := \sum_{n\ge 1} \delta^{2n-1} 2^{2n} \frac{|B_{2n}|}{(2n)!} \partial_x^{2n-1} f$$

and δ is a non-zero complex number.

Remark 8.1. In the physics literature the operator T is usually written in the following way:

$$T(f) = PV \int_{-\infty}^{\infty} \frac{1}{2\delta}\left(\operatorname{sgn}(x-\xi) - \coth\frac{\pi(x-\xi)}{2\delta}\right) f(\xi) d\xi.$$

Let μ be a formal variable and ε be a non-zero complex number. Let us make the following rescalings:

$$w = \frac{\sqrt{\varepsilon}}{\mu} u, \qquad \tau = -\frac{\mu}{2\sqrt{\varepsilon}} t, \qquad \delta = \frac{\mu\sqrt{\varepsilon}}{2}. \tag{8.2}$$

Then equation (8.1) is transformed to

$$u_t = uu_x + \sum_{g\ge 1} \mu^{2g} \varepsilon^{g-1} \frac{|B_{2g}|}{(2g)!} u_{2g+1}. \tag{8.3}$$

If we put $\hbar = \mu^2$, we get exactly the first equation (1.1) of the deformed KdV hierarchy.

8.2. **Extensions of $\widehat{\mathcal{A}}^{[k]}$ and of $\widehat{\Lambda}^{[k]}$.** We need to enlarge the spaces $\widehat{\mathcal{A}}^{[k]}$ and $\widehat{\Lambda}^{[k]}$.

Let $\widehat{\mathcal{A}}^{[k]}_\mu$ be the space of series of the form

$$f(u; u_1, u_2, \ldots; \mu) = \sum_{i\ge 0} \mu^i f_i(u; u_1, \ldots), \quad f_i \in \mathcal{A}, \quad \deg_{dif} f_i = i + k.$$

Denote by $\widehat{\Lambda}^{[k]}_\mu$ the space of integrals of the form

$$\overline{f} = \int f(u; u_1, u_2, \ldots; \mu) dx, \quad \text{where } f \in \widehat{\mathcal{A}}^{[k]}_\mu.$$

We have the following simple generalization of Lemma 2.4.

Lemma 8.2. *Let us fix a local functional $\overline{h} \in \widehat{\Lambda}^{[0]}_\mu$ of the form $\overline{h} = \int \left(\frac{u^3}{6} + O(\mu)\right) dx$. Consider also an arbitrary power series $q_0(u)$. Suppose there exists a local functional $\overline{q} \in \widehat{\Lambda}^{[0]}_\mu$ of the form $\overline{q} = \int (q_0(u) + O(\mu))\, dx$, such that $\{\overline{h}, \overline{q}\}_{\partial_x} = 0$. Then the local functional $\overline{q}$ is uniquely determined by $\overline{h}$ and $q_0(u)$.*

The proof is the same as the proof of Lemma 2.4.

8.3. **Conserved quantities.** Here we review the construction of an infinite sequence of conserved quantities of the ILW equation. We follow [SAK79] except for the fact that we make the rescalings (8.2).

Let us introduce the operator R by

$$R := \sum_{g\ge 1} \mu^{2g-1} \varepsilon^{g-1} \frac{|B_{2g}|}{(2g)!} \partial_x^{2g-1}.$$

Consider the following equation:

$$e^\sigma - 1 = \frac{1}{\lambda}\left(\frac{2}{\varepsilon}\sigma - \mu\left(\frac{i}{\sqrt{\varepsilon}} + 2R\right)\sigma_x + 2u\right).$$

It is easy to see that it has a unique solution of the form $\sigma=\sum_{n\ge 1}\frac{\sigma_n}{\lambda^n}$, where $\sigma_n\in\widehat{\mathcal{A}}^{[0]}_\mu$. For example,

$$\sigma_1=2u,$$
$$\sigma_2=-2u^2-2\mu\left(\frac{i}{\sqrt{\varepsilon}}+2R\right)u_x+\frac{4u}{\varepsilon}.$$

It is not hard to check that, if u is a solution of (8.3), then σ satisfies the following equation:

$$\sigma_t=\frac{\lambda}{2}\left(e^\sigma-1\right)\sigma_x-\varepsilon^{-1}\sigma\sigma_x+\mu\sigma_xR\sigma_x+\mu R\sigma_{xx}.$$

We can easily see that $\int\sigma_t dx=0$, therefore, all local functionals $\int\sigma_n dx$ are conserved quantities of the equation (8.3).

8.4. **Relation to the deformed KdV hierarchy.** In this section we express the conserved quantities $\int\sigma_n dx$ as linear combinations of the Hamiltonians $\overline{h}_n$.

Let $\hbar=\mu^2$ and consider the Hamiltonians $\overline{h}_n, n\ge 1$, of the deformed KdV hierarchy. Let $\overline{h}_0:=\int\frac{u^2}{2}dx$ and $\overline{h}_{-1}:=\int udx$. It is easy to see that

$$\sigma_n=(-1)^{n+1}\frac{2^n}{n}u^n+\sum_{i=1}^{n-1}\frac{a_{i,n}}{(n-i)!}\frac{u^{n-i}}{\varepsilon^i}+O(\mu),$$

where $a_{i,j}$, $1\le i<j$, are some complex coefficients. Thus, we have

$$\int\sigma_n dx=(-1)^{n+1}2^n(n-1)!\overline{h}_{n-2}+\sum_{i=1}^{n-1}\frac{a_{i,n}}{\varepsilon^i}\overline{h}_{n-i-2}+O(\mu).$$

Since $\int\sigma_n dx$ are conserved quantities, we have $\{\int\sigma_n dx,\overline{h}_1\}_{\partial_x}=0$. Therefore, from Lemma 8.2 it follows that

$$\int\sigma_n dx=(-1)^{n+1}2^n(n-1)!\overline{h}_{n-2}+\sum_{i=1}^{n-1}\frac{a_{i,n}}{\varepsilon^i}\overline{h}_{n-i-2}.$$

Appendix A. Coefficient of $\hbar^2$

Here we compute the coefficient c_2 in (5.9) and complete the proof of Proposition 4.1.

Consider the local functionals $\int\Omega_{0,2}dx$ before the Miura transformation (4.1). In order to compute the coefficient c_2 in (5.9), we only need to compute the coefficients of $\hbar$ and of $\hbar^2\varepsilon$ in $\int\Omega_{0,2}dx$. The coefficient of $\hbar$ is equal to $\frac{1}{24}\int uu_2dx$. Let us compute the coefficient of $\hbar^2\varepsilon$.

The series $\left.\frac{\partial\Omega_{0,2}}{\partial\varepsilon}\right|_{\varepsilon=0}$ can be computed using the Givental operators that act on potentials of cohomological field theories. We remind the general formulas for this in Section A.1. All technical computations are done in Section A.2.

A.1. **Deformations of cohomological field theories.** Consider a one-dimensional cohomological field theory, $\alpha_{g,n}\in H^*(\overline{\mathcal{M}}_{g,n};\mathbb{C})$. Let F be its potential. Consider the following deformation of the classes $\alpha_{g,n}$:

$$\alpha_{g,n}(\varepsilon)=\exp\left(\varepsilon\,\mathrm{ch}_{2l-1}(\Lambda)\right)\alpha_{g,n},$$

where $\mathrm{ch}_{2l-1}(\Lambda)$ is the Chern character of the Hodge bundle. It is well-known that the classes $\alpha_{g,n}(\varepsilon)$ form a cohomological field theory.

Let $F(\varepsilon)$ be the potential of the deformed cohomological field theory. There is the following formula (see e.g. [BPS12a]):

$$\left.\frac{\partial F(\varepsilon))}{\partial\varepsilon}\right|_{\varepsilon=0}=-\frac{B_{2l}}{(2l)!}\widehat{z^{2l-1}}(F),$$

where $\widehat{z^{2l-1}}$ is the operator that acts as follows:

$$\widehat{z^{2l-1}}(F) := -\frac{\partial F}{\partial t_{2l}} + \sum_{d\geq 0} t_d \frac{\partial F}{\partial t_{d+2l-1}} + \frac{\hbar}{2}\sum_{i+j=2l-2}(-1)^{i+1}\frac{\partial^2 F}{\partial t_i \partial t_j} + \frac{1}{2}\sum_{i+j=2l-2}(-1)^{i+1}\frac{\partial F}{\partial t_i}\frac{\partial F}{\partial t_j}.$$

Consider the second derivatives $\Omega_{p,q}(\varepsilon) := \frac{\partial^2 F(\varepsilon)}{\partial t_p \partial t_q}$. They are differential polynomials in $u_i(\varepsilon) := \partial_x^i \frac{\partial^2 F(\varepsilon)}{\partial t_0^2}$ (see [BPS12a]). Denote $u_i(\varepsilon)$ by u_i. Let $\frac{\partial \Omega_{p,q}(\varepsilon)}{\partial \varepsilon}[u]$ be the derivative of $\Omega_{p,q}(\varepsilon)$ as a differential polynomial in u_i. In other words,

$$\frac{\partial \Omega_{p,q}(\varepsilon)}{\partial \varepsilon}[u] := \frac{\partial \Omega_{p,q}(\varepsilon)}{\partial \varepsilon} - \sum_{i\geq 0}\frac{\partial \Omega_{p,q}(\varepsilon)}{\partial u_i}\frac{\partial u_i}{\partial \varepsilon}.$$

In [BPS12a] it is proved that

$$\left.\frac{\partial \Omega_{p,q}(\varepsilon)}{\partial \varepsilon}[u]\right|_{\varepsilon=0} = -\frac{B_{2l}}{(2l)!}\widehat{z^{2l-1}}[u](\Omega_{p,q}),$$

where

(A.1)

$$\begin{aligned}\widehat{z^{2l-1}}[u](\Omega_{p,q}) :=& \Omega_{p+2l-1,q} + \Omega_{p,q+2l-1} + \sum_{i=0}^{2l-2}(-1)^{i+1}\Omega_{p,i}\Omega_{2l-2-i,q}\\ &- \sum_{n\geq 0}\frac{\partial \Omega_{p,q}}{\partial u_n}\left((n+2)\partial_x^n\Omega_{0,2l-1} + \sum_{i=0}^{2l-2}\sum_{k=0}^{n-1}\binom{n}{k}(-1)^{i+1}\partial_x^{k+1}\Omega_{0,i}\partial_x^{n-k-1}\Omega_{2l-2-i,0}\right.\\ &\left.+ \sum_{i=0}^{2l-2}(-1)^{i+1}\partial_x^n(\Omega_{0,i}\Omega_{2l-2-i,0})\right)\\ &+ \frac{\hbar}{2}\sum_{n,m\geq 0}\frac{\partial^2\Omega_{0,2}}{\partial u_n \partial u_m}\sum_{i=0}^{2l-2}(-1)^{i+1}\partial_x^{n+1}\Omega_{0,i}\partial_x^{m+1}\Omega_{2l-2-i,0}.\end{aligned}$$

A.2. **Coefficient of $\hbar^2$.** Let us return to the case of the cohomological field theory (1.3): $\Omega_{p,q} = \frac{\partial^2 F^{Hodge}}{\partial t_p \partial t_q}$. Let F^{KdV} be the potential of the trivial cohomological field theory:

$$F^{KdV} := \sum_{\substack{g\geq 0, n\geq 1\\ 2g-2+n>0}}\frac{\hbar^g}{n!}\sum_{k_1,\ldots,k_n\geq 0}\langle\tau_{k_1}\ldots\tau_{k_n}\rangle_g t_{k_1}\ldots t_{k_n}$$

and $\Omega^{KdV}_{p,q} := \frac{\partial^2 F^{KdV}}{\partial t_p \partial t_q}$.

From Section A.1 it follows that

$$\left.\frac{\partial \Omega_{0,2}}{\partial \varepsilon}[u]\right|_{\varepsilon=0} = -\frac{1}{12}\widehat{z^1}[u]\left(\Omega^{KdV}_{0,2}\right),$$

where

$$\begin{aligned}\widehat{z^1}[u]\left(\Omega^{KdV}_{0,2}\right) =& \Omega^{KdV}_{1,2} + \Omega^{KdV}_{0,3} - \Omega_{0,0}\Omega^{KdV}_{0,2}\\ &- \sum_{n\geq 0}\frac{\partial \Omega^{KdV}_{0,2}}{\partial u_n}\left[(n+2)\partial_x^n\Omega^{KdV}_{0,1} - \sum_{k=0}^{n-1}\binom{n}{k}u_{k+1}u_{n-k-1} - \partial_x^n(u^2)\right]\\ &- \frac{\hbar}{2}\sum_{n,m\geq 0}\frac{\partial^2\Omega^{KdV}_{0,2}}{\partial u_m \partial u_n}u_{n+1}u_{m+1}.\end{aligned}$$

We have the following formulas (see e.g. [DZ05]):

$$
\begin{aligned}
\Omega_{0,1}^{KdV} &= \frac{u^2}{2} + \frac{\hbar}{12}u_2,\\
\Omega_{0,2}^{KdV} &= \frac{u^3}{6} + \frac{\hbar}{24}(u_1^2 + 2uu_2) + \frac{\hbar^2}{240}u_4,\\
\Omega_{0,3}^{KdV} &= \frac{u^4}{24} + \frac{\hbar}{24}(u^2u_2 + uu_1^2) + \frac{\hbar^2}{480}(2uu_4 + 4u_1u_3 + 3u_2^2) + \frac{\hbar^3}{6720}u_6,\\
\Omega_{1,2}^{KdV} &= \frac{u^4}{8} + \frac{\hbar}{24}(3u^2u_2 + 2uu_1^2) + \hbar^2\left(\frac{uu_4}{90} + \frac{23u_2^2}{1440} + \frac{u_1u_3}{60}\right) + \frac{\hbar^3}{2880}u_6.
\end{aligned}
$$

By direct computations, we get

$$\int \widehat{z^1}[u]\left(\Omega_{0,2}^{KdV}\right)dx = \int\left(\frac{\hbar}{4}u^2u_2 + \frac{\hbar^2}{30}uu_4\right)dx.$$

Thus, the coefficient of $\hbar^2\varepsilon$ in $\int \Omega_{0,2}dx$ is equal to $-\frac{1}{360}\int uu_4dx$. Now it is easy to compute that the coefficient c_2 in (5.9) is equal to $\frac{1}{1440}$. This completes the proof of Proposition 4.1.

Appendix B. Proof of Lemma 6.1

Introduce the function $\phi(z) := \sum_{i\ge 0}\frac{B_{2i}}{(2i)!}z^{2i}$. For a power series $f(z) = \sum_{i\ge 0} f_i z^i$, we denote by $[z^i]f$ the coefficient f_i. The coefficient of $u_{2k+1}u_{2g-2k}$ on the left-hand side of (6.5) is equal to

$$
\begin{aligned}
&\frac{B_{2k}B_{2g-2k}}{(2k)!(2g-2k)!} - \sum_{i=0}^{g}\binom{2i}{2k}\frac{B_{2i}B_{2g-2i}}{(2i)!(2g-2i)!} - \sum_{i=0}^{k}\binom{2g-2i}{2g-2k-1}\frac{B_{2i}B_{2g-2i}}{(2i)!(2g-2i)!} =\\
=&[z^{2g}]\left(\frac{B_{2k}\phi z^{2k}}{(2k)!} - \frac{\phi\phi^{(2k)}z^{2k}}{(2k)!} - \sum_{i=0}^{k}\frac{B_{2i}\phi^{(2k-2i+1)}z^{2k+1}}{(2i)!(2k-2i+1)!}\right).
\end{aligned}
$$

Therefore, the lemma is equivalent to the following identity.

$$\frac{B_{2k}\phi z^{2k}}{(2k)!} - \frac{\phi\phi^{(2k)}z^{2k}}{(2k)!} - \sum_{i=0}^{k}\frac{B_{2i}\phi^{(2k-2i+1)}z^{2k+1}}{(2i)!(2k-2i+1)!} = -\delta_{k,0}\frac{z^2}{4}.$$

Let us rewrite it in a bit different way:

$$\frac{\phi\phi^{(2k)}}{(2k)!} = \frac{B_{2k}\phi}{(2k)!} - \sum_{i=0}^{k}\frac{B_{2i}\phi^{(2k-2i+1)}z}{(2i)!(2k-2i+1)!} + \delta_{k,0}\frac{z^2}{4}. \tag{B.1}$$

Let us formulate another identity of this type.

$$\frac{\phi^{(2k+1)}\phi}{(2k+1)!} = -\sum_{i=0}^{k}\frac{B_{2i}\phi^{(2k+2-2i)}z}{(2i)!(2k+2-2i)!} + \delta_{k,0}\frac{z}{4}. \tag{B.2}$$

We prove (B.1) and (B.2) by induction on k. For $k = 0$, equation (B.1) looks as follows:

$$z\phi' = -\phi^2 + \phi + \frac{z^2}{4}. \tag{B.3}$$

It is equivalent to the following identity between the Bernoulli numbers (see e.g. [GKP94]).

$$\sum_{i=1}^{m}\frac{B_{2i}B_{2m-2i}}{(2i)!(2m-2i)!} = -\frac{2mB_{2m}}{(2m)!} + \frac{\delta_{m,1}}{4}.$$

Suppose that (B.1) is true and also (B.2) is true for $k' < k$. Let us prove (B.2). Let us differentiate (B.1), we get

$$\frac{\phi'\phi^{(2k)}}{(2k)!}+\frac{\phi\phi^{(2k+1)}}{(2k)!}=\frac{B_{2k}\phi'}{(2k)!}-\sum_{i=0}^{k}\frac{B_{2i}\phi^{(2k-2i+2)}z}{(2i)!(2k-2i+1)!}-\sum_{i=0}^{k}\frac{B_{2i}\phi^{(2k-2i+1)}}{(2i)!(2k-2i+1)!}+\delta_{k,0}\frac{z}{2}.$$

Using (B.3) and the induction assumption, we get

$$\left(-\frac{\phi^2}{z}+\frac{z}{4}\right)\frac{\phi^{(2k)}}{(2k)!}+\frac{\phi\phi^{(2k+1)}}{(2k)!}=\left(-\frac{\phi^2}{z}+\frac{z}{4}\right)\frac{B_{2k}}{(2k)!}-\sum_{i=0}^{k}\frac{B_{2i}\phi^{(2k-2i+2)}z}{(2i)!(2k-2i+1)!}+\delta_{k,0}\frac{z}{4}. \tag{B.4}$$

From the induction assumption it follows that

$$\frac{\phi^2\phi^{(2k)}}{(2k)!}=\frac{B_{2k}\phi^2}{(2k)!}-\frac{\phi\phi^{(2k+1)}z}{(2k+1)!}-\sum_{i=0}^{k}\frac{B_{2i}\phi^{(2k-2i+2)}z^2}{(2i-1)!(2k-2i+2)!}+\frac{\phi^{(2k)}z^2}{4(2k)!}-\frac{B_{2k}z^2}{4(2k)!}+\frac{\delta_{k,0}z^2}{4}. \tag{B.5}$$

After substituting (B.5) into (B.4) we get (B.2).

Suppose that (B.2) is true and also (B.1) is true, for any $k' \le k$. Then the proof of (B.1) for $k' = k+1$ can be done in a completely similar way. This concludes the proof of the lemma.

References

[BPS12a] A. Buryak, H. Posthuma, S. Shadrin. On deformations of quasi-Miura transformations and the Dubrovin-Zhang bracket. *Journal of Geometry and Physics* **62** (2012), no. 7, 1639-1651.

[BPS12b] A. Buryak, H. Posthuma, S. Shadrin. A polynomial bracket for the Dubrovin-Zhang hierarchies. *Journal of Differential Geometry* **92** (2012), no. 1, 153-185.

[DZ05] B. A. Dubrovin, Y. Zhang. Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants. A new 2005 version of *arXiv:math/0108160v1*, 295 pp.

[FP03] C. Faber, R. Pandharipande. Hodge integrals, partition matrices, and the λ_g-conjecture. *Annals of Mathematics* **157** (2003), no. 1, 97-124.

[GKP94] R. L. Graham, D. E. Knuth, O. Patashnik. Concrete Mathematics: A Foudation for Computer Science. Addison-Wesley Professional, 2 edition (March 10, 1994), 672 pp.

[Kaz09] M. Kazarian. KP hierarchy for Hodge integrals. *Advances in Mathematics* **221** (2009), 1-21.

[Kon92] M. Kontsevich. Intersection Theory on the Moduli Space of Curves and the Matrix Airy Function. *Communications in Mathematical Physics* **147** (1992), 1-23.

[LZ05] Si-Qi Liu, Y. Zhang. On quasi-triviality and integrability of a class of scalar evolutionary PDEs. *Journal of Geometry and Physics* **57** (2006), no. 1, 101-119.

[MZ00] Y. Manin, P. Zograf. Invertible cohomological field theories and Weil-Petersson volumes. *Annales de l'institut Fourier* **50** (2000), no. 2, 519-535.

[SAK79] J. Satsuma, M. J. Ablowitz, Y. Kodama. On an internal wave equation describing a stratified fluid with finite depth. *Physics Letters A* **73** (1979), no. 4, 283-286.

[Sha09] S. Shadrin. BCOV theory via Givental group action on cohomological field theories. *Moscow Mathematical Journal* **9** (2009), no. 2, 411-429.

[Wit91] E. Witten. Two-dimensional gravity and intersection theory on moduli space. *Surveys in differential geometry* (Cambridge, MA, 1990), 243-310, Lehigh Univ., Bethlehem, PA, 1991.

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