THE HYPERGEOMETRIC FUNCTIONS OF THE FABER-ZAGIER AND PIXTON RELATIONS

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Dedicated to E. Looijenga on the occasion of his 69^{th} birthday

Abstract. The relations in the tautological ring of the moduli space $\mathcal{M}_g$ of nonsingular curves conjectured by Faber-Zagier in 2000 and extended to the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves by Pixton in 2012 are based upon two hypergeometric series A and B. The question of the geometric origins of these series has been solved in at least two ways (via the Frobenius structures associated to 3-spin curves and to $\mathbb{P}^1$). The series A and B also appear in the study of descendent integration on the moduli spaces of open and closed curves. We survey here the various occurrences of A and B starting from their appearance in the asymptotic expansion of the Airy function (calculated by Stokes in the 19^{th} century). Several open questions are proposed.

Contents

0. Introduction

0.1. **Tautological classes.** For $g \geq 2$, let $\mathcal{M}_g$ be the moduli space of nonsingular, projective, genus g curves over $\mathbb{C}$, and let

$$\pi : \mathcal{C}_g \to \mathcal{M}_g$$

be the universal curve. The cotangent line class is defined via the line bundle ω_π of relative differentials of the morphism π,

$$\psi = c_1(\omega_\pi) \in A^1(\mathcal{C}_g, \mathbb{Q}) \ .$$

The κ classes are defined by push-forward,

$$\kappa_r = \pi_*(\psi^{r+1}) \in A^r(\mathcal{M}_g, \mathbb{Q}) \ .$$

The *tautological ring* in algebraic cycles,

$$R^*(\mathcal{M}_g) \subset A^*(\mathcal{M}_g, \mathbb{Q}) \, ,$$

is the $\mathbb{Q}$-subalgebra generated by all of the κ classes. Since

$$\kappa_0 = 2g - 2 \in \mathbb{Q}$$

is a multiple of the fundamental class, we need not take κ_0 as a generator. There is a canonical quotient

$$\mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots] \xrightarrow{q} R^*(\mathcal{M}_g) \longrightarrow 0 \ .$$

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The ideal of *tautological relations* among the κ classes is the kernel of q.

0.2. **Relations.** Faber and Zagier conjectured in 2000 a remarkable set of relations among the κ classes in $R^*(\mathcal{M}_g)$ which were first proven to hold in [27].

To write the Faber-Zagier relations, we will require the following notation. Let the variable set

$$\mathbf{p} = \{\ p_1, p_3, p_4, p_6, p_7, p_9, p_{10}, \dots\ \}$$

be indexed by positive integers *not* congruent to 2 modulo 3. Define the series

$$\begin{aligned}\Psi(t,\mathbf{p}) &= \big(1 + tp_3 + t^2p_6 + t^3p_9 + \dots\big)\sum_{i=0}^{\infty}\frac{(6i)!}{(3i)!(2i)!}t^i \\ &+ \big(p_1 + tp_4 + t^2p_7 + \dots\big)\sum_{i=0}^{\infty}\frac{(6i)!}{(3i)!(2i)!}\frac{6i+1}{6i-1}t^i\ .\end{aligned}$$

Since Ψ has constant term 1, we may take the logarithm. Define the constants $C_r^{\mathsf{FZ}}(\sigma)$ by the formula

$$\log(\Psi) = \sum_\sigma \sum_{r=0}^{\infty} C_r^{\mathsf{FZ}}(\sigma)\ t^r \mathbf{p}^\sigma\ .$$

The above sum is over all partitions σ of size $|\sigma|$ which avoid parts congruent to 2 modulo 3. The empty partition is included in the sum. To the partition $\sigma = 1^{n_1}3^{n_3}4^{n_4}\cdots$, we associate the monomial $\mathbf{p}^\sigma = p_1^{n_1}p_3^{n_3}p_4^{n_4}\cdots$. Let

$$\gamma^{\mathsf{FZ}} = \sum_\sigma \sum_{r=0}^{\infty} C_r^{\mathsf{FZ}}(\sigma)\ \kappa_r t^r \mathbf{p}^\sigma\ .$$

For a series $\Theta \in \mathbb{Q}[\kappa][[t,\mathbf{p}]]$ in the variables κ_i, t, and p_j, let $[\Theta]_{t^r\mathbf{p}^\sigma}$ denote the coefficient of $t^r\mathbf{p}^\sigma$ (which is a polynomial in the κ_i).

Theorem 1 (Pandharipande-Pixton)**.** *In $R^r(\mathcal{M}_g)$, the Faber-Zagier relation*

$$\big[\exp(-\gamma^{\mathsf{FZ}})\big]_{t^r\mathbf{p}^\sigma} = 0$$

holds when $g - 1 + |\sigma| < 3r$ and $g \equiv r + |\sigma| + 1 \mod 2$.

As a corollary of the proof [27] of Theorem 1, a stronger boundary result was obtained. If $g - 1 + |\sigma| < 3r$ and $g \equiv r + |\sigma| + 1 \mod 2$, then

$$\big[\exp(-\gamma^{\mathsf{FZ}})\big]_{t^r\mathbf{p}^\sigma} \in R^*(\partial\overline{\mathcal{M}}_g)\ . \tag{1}$$

Not only is the Faber-Zagier relation 0 in $R^*(\mathcal{M}_g)$, but the relation is equal to a tautological class supported on the boundary of the moduli space $\overline{\mathcal{M}}_g$.

A precise conjecture for the boundary terms (and much more) has been proposed by Pixton in [31]. We review the complete form of Pixton's relations in Appendix A, see also [28, 31]. Pixton has conjectured that his relations provide a *complete* set of tautological relations in the Chow rings of the moduli spaces $\overline{\mathcal{M}}_{g,n}$. Since Pixton's relations restrict to the Faber-Zagier relations, the hypergeometric series

$$\mathsf{A}(z) = \sum_{i=0}^{\infty}\frac{(6i)!}{(3i)!(2i)!}\left(\frac{z}{288}\right)^i, \quad \mathsf{B}(z) = \sum_{i=0}^{\infty}\frac{(6i)!}{(3i)!(2i)!}\frac{6i+1}{6i-1}\left(\frac{z}{288}\right)^i, \quad z = 288t$$

also occur in the formula of Pixton. In fact, just as above for Faber and Zagier, the series A and B are the *only* non-formal inputs for Pixton.

0.3. **Differential equations.** The series A and B are easily related via the following differential equation:

$$3z^2\,\frac{d\mathsf{A}}{dz}+\left(\frac{z}{2}-1\right)\mathsf{A}=\mathsf{B}\ .$$

Hence, we often view A as the more fundamental function. The main hypergeometric differential equation satisfied by A is:

$$3z^2\frac{d^2\mathsf{A}}{dz^2}+(6z-2)\frac{d\mathsf{A}}{dz}+\frac{5}{12}\mathsf{A}=0\ .$$

0.4. **Origins of A and B.** In order to prove the Faber-Zagier and Pixton relations, geometric sources for the series A and B were found. At present, two successful approaches are known: via the Frobenius geometries of 3-spin curves [28] and of $\mathbb{P}^1$ [16, 17, 27]. The two approaches lead to two different geometric origins for A and B.

More recently, occurances of A have been noticed [3] in the generating series of descendent integrals over the moduli spaces of open Riemann surfaces [30]. Remarkably, the series A can already be seen in the asymptotic expansion of the Airy function related to the Witten-Kontsevich theory of descendent integration over $\overline{\mathcal{M}}_{g,n}$. Our goal here is to survey these various appearances of the series A and B.

The occurances connected to descendent integration have not (yet) played a role in proofs of the Faber-Zagier and Pixton relations. Perhaps the reverse is more likely: the relations could be used to constrain descendent integration. For integration against the product of the top two Chern classes of the Hodge bundle,

$$\lambda_g\lambda_{g-1}\in A^{2g-1}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})\,,$$

corresponding to the geometry of the moduli space of nonsingular curves, a subset of Pixton's relations have been shown in [29, 32] to imply the $\lambda_g\lambda_{g-1}$ descendent formula [9, 11].

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1. Asymptotic expansion of the Airy function

Define the closely related functions $\mathcal{A}$ and $\mathcal{B}$ by

$$\mathcal{A}(x)=\mathsf{A}(-x^3)\ ,\quad \mathcal{B}(x)=\mathsf{B}(-x^3)\,.$$

We see

$$\mathcal{A}(x)=\sum_{j=0}^{\infty}a_jx^{3j}=1-\frac{5}{24}x^3+\dots\ ,\qquad -\mathcal{B}(x)=\sum_{j=0}^{\infty}b_jx^{3j}=1+\frac{7}{24}x^3+\dots\ .$$

where the coefficients a_j and b_j are

$$a_j=(-1)^j\frac{(6j)!}{288^j(2j)!(3j)!}\,,\qquad b_j=-\frac{6j+1}{6j-1}a_j\qquad j\geq 0\,.$$

The series $\mathcal{A}(x)$ is related to the classical Airy function in the following way. The Airy function $\mathrm{Ai}(x)$ is defined by

$$\mathrm{Ai}(x)=\int_0^\infty\cos\left(\frac{t^3}{3}+xt\right)dt\,,\quad x\in\mathbb{R}\,. \tag{2}$$

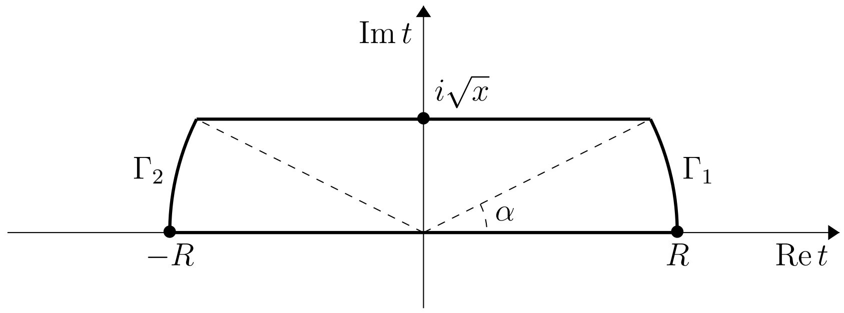

FIGURE 1. Shift of the integration contour

It is the unique (up to a scalar factor) bounded real solution of the Airy differential equation

$$y'' = xy\,. \tag{3}$$

The Airy function $\mathrm{Ai}(x)$ has the following asymptotic expansion for $x \to \infty$,

$$\mathrm{Ai}(x) \asymp_{x\to\infty} \frac{\sqrt{\pi}}{2} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}} \mathcal{A}\left(2^{-\frac{1}{3}} x^{-\frac{1}{2}}\right), \tag{4}$$

see [8, pages 22-23].

We review here the short derivation of asymptotic expansion (4). We may write the oscillatory integral defining the Airy function as

$$\mathrm{Ai}(x) = \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) \mathrm{d}t = \frac{1}{2}\int_{-\infty}^{\infty} e^{i\left(\frac{t^3}{3}+xt\right)} \mathrm{d}t\,. \tag{5}$$

Viewing (5) as a complex line integral, we move the integration contour from $\{\mathrm{Im}(t) = 0\}$ to $\{\mathrm{Im}(t) = i\sqrt{x}\}$ by shifting the integration variable t by $i\sqrt{x}$. There are no poles in the region

$$0 \le \mathrm{Im}\, t \le \sqrt{x}\,.$$

Let us check that the integrals of the function $\frac{1}{2}e^{i\left(\frac{t^3}{3}+xt\right)}$ over the arcs Γ_1 and Γ_2 (see Figure 1) go to zero, when R goes to infinity. If R is big enough, then $\alpha \le \frac{\pi}{6}$ and we have

$$\begin{aligned}
\left|\frac{1}{2}\int_{\Gamma_2} e^{i\left(\frac{t^3}{3}+xt\right)} dt\right| &= \left|\frac{1}{2}\int_{\Gamma_1} e^{i\left(\frac{t^3}{3}+xt\right)} dt\right| \\
&\le \frac{R}{2}\int_0^\alpha \left| e^{i\left(\frac{R^3}{3}e^{3i\phi}+xRe^{i\phi}\right)}\right| d\phi \\
&= \frac{R}{2}\int_0^\alpha e^{-\frac{R^3}{3}\sin 3\phi - xR\sin\phi} d\phi \\
&\overset{\substack{\text{by Jordan's} \\ \text{inequality}}}{\le} \frac{R}{2}\int_0^\alpha e^{-\frac{2}{\pi}(R^3+xR)\phi} d\phi \\
&= \frac{\pi}{4(R^2+x)}\left(1 - e^{-\frac{2}{\pi}(R^3+xR)\alpha}\right) \xrightarrow[R\to\infty]{} 0\,.
\end{aligned}$$

Therefore, we obtain

$$\mathrm{Ai}(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{i\frac{t^3}{3} - \sqrt{x}t^2 - \frac{2}{3}x^{3/2}} \mathrm{d}t\,.$$

Scaling t by $(2\sqrt{x})^{1/2}$ makes the integrand a deformed Gaussian integral:

$$\mathrm{Ai}(x) = e^{-\frac{2}{3}x^{3/2}} \frac{x^{-\frac{1}{4}}}{2\sqrt{2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} e^{ix^{-3/4}\frac{t^3}{6\sqrt{2}}} \mathrm{d}t\,.$$

By dominated convergence, we see

$$e^{-\frac{2}{3}x^{3/2}}\frac{x^{-\frac{1}{4}}}{2\sqrt{2}}\sum_{j=0}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}}\frac{1}{j!}\left(\frac{it^3}{6\sqrt{2}x^{3/4}}\right)^j\mathrm{d}t = e^{-\frac{2}{3}x^{3/2}}\frac{x^{-\frac{1}{4}}}{2\sqrt{2}}\sum_{j=0}^{\infty}\frac{x^{-3j/4}}{(-6\sqrt{2}i)^j j!}\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}}t^{3j}\mathrm{d}t$$

is an asymptotic expansion for $\mathrm{Ai}(x)$ for $x \to \infty$. Using

$$\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}}t^j\mathrm{d}t = \begin{cases}\sqrt{2\pi}\,(j-1)!!\,, & \text{if } j \text{ is even,}\\ 0\,, & \text{if } j \text{ is odd,}\end{cases}$$

we arrive at

$$\begin{aligned}\mathrm{Ai}(x) \quad &\asymp_{x\to\infty} \quad \frac{\sqrt{\pi}}{2}e^{-\frac{2}{3}x^{3/2}}x^{-\frac{1}{4}}\sum_{j=0}^{\infty}\frac{(6j-1)!!}{(6\sqrt{2}i)^{2j}(2j)!}x^{-3j/2}\\ &= \frac{\sqrt{\pi}}{2}e^{-\frac{2}{3}x^{3/2}}x^{-\frac{1}{4}}\sum_{j=0}^{\infty}\frac{(6j)!}{(2j)!(3j)!}\left(-\frac{x^{-3/2}}{576}\right)^j\,,\end{aligned}$$

which is exactly (4).

Similarly, for the derivative of $\mathrm{Ai}(x)$,

$$\mathrm{Ai}'(x) = \frac{1}{2}\int_{-\infty}^{\infty}e^{i\left(\frac{t^3}{3}+xt\right)}it\,\mathrm{d}t\,,$$

we obtain the asymptotic expansion

$$\mathrm{Ai}'(x) \asymp_{x\to\infty} \frac{\sqrt{\pi}}{2}e^{-\frac{2}{3}x^{3/2}}x^{\frac{1}{4}}\sum_{j=0}^{\infty}\frac{(6j)!}{(2j)!(3j)!}\frac{6j+1}{6j-1}\left(-\frac{x^{-3/2}}{576}\right)^j\,,$$

or equivalently,

$$\mathrm{Ai}'(x) \asymp_{x\to\infty} \frac{\sqrt{\pi}}{2}x^{\frac{1}{4}}e^{-\frac{2}{3}x^{\frac{3}{2}}}\mathcal{B}\left(2^{-\frac{1}{3}}x^{-\frac{1}{2}}\right)\,.$$

2. Moduli of stable curves and the infinite Grassmanian

2.1. **Overview.** The series $\mathcal{A}$ appears in the intersection theory of the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves. We start by reviewing Witten's conjecture governing descendent integration in Section 2.2. Kontsevich's formula for the generating series F^c of the descendents is expressed in terms of integrals over spaces of Hermitian matrices. Certain specializations of the descendent partition function $\exp(F^c)$ may be expressed as ratios of simple determinants. As an almost immediate consequence, the most basic of these specialization of $\exp(F^c)$ coincides with the hypergeometric series $\mathcal{A}$.

In Section 2.4, we review a well-known construction associating a tau-function of the KP hierarchy to any point of the infinite dimensional Grassmanian. We then present a result of Kac and Schwarz which explicitly describes the point in the Grassmanian corresponding to the partition function $\exp(F^c)$. The series $\mathcal{A}$ and $\mathcal{B}$ emerge here and play a prominent role.

2.2. **Witten's conjecture.** Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of genus g stable curves over $\mathbb{C}$ with n marked points. The first Chern class of the contangent line at the i^{th} marking is denoted by

$$\psi_i \in A^1(\overline{\mathcal{M}}_{g,n},\mathbb{Q})\,.$$

We define the *descendent integrals* by

$$\langle\tau_{k_1}\tau_{k_2}\dots\tau_{k_n}\rangle_g^c = \int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\psi_2^{k_2}\dots\psi_n^{k_n}\,. \tag{6}$$

The bracket (6) vanishes unless the dimension constraint

$$3g-3+n=\sum_{i=1}^{n} k_i$$

is satisfied. The superscript c here indicates integration over the moduli of compact Riemann surfaces. Integration over the moduli of open Riemann surfaces will be considered in Section 3.

We introduce variables $\{t_i\}_{i\geq 0}$ and define the generating series

$$F^c(t_0,t_1,\ldots)=\sum_{\substack{g\geq 0,\, n\geq 1\\ 2g-2+n>0}} \frac{1}{n!}\sum_{k_1,\ldots,k_n\geq 0}\langle\tau_{k_1}\tau_{k_2}\ldots\tau_{k_n}\rangle_g^c\, t_{k_1}t_{k_2}\ldots t_{k_n}\,.$$

Witten [36] conjectured that the partition function $\exp(F^c)$ is a tau-function of the KdV hierarchy. In particular,

$$u=\frac{\partial^2 F^c}{\partial t_0^2}$$

is a solution of the KdV hierarchy. The first equations of the hierarchy are

$$\begin{aligned}
u_{t_1}&=uu_x+\frac{1}{12}u_{xxx},\\
u_{t_2}&=\frac{1}{2}u^2u_x+\frac{1}{12}(2u_xu_{xx}+uu_{xxx})+\frac{1}{240}u_{xxxxx}\,,\\
&\vdots
\end{aligned}$$

where we have identified x here with t_0. Witten's conjecture was proven by Kontsevich [21]. See [20, 24, 26] for other proofs.

2.3. **Kontsevich's matrix integral.**

2.3.1. *Matrix integrals.* Kontsevich [21] proposed a representation of the partition function $\exp(F^c)$ in terms of integrals over spaces of Hermitian matrices. To connect $\exp(F^c)$ to the hypergeometric series A, we will use Kontsevich's matrix model.

Let $\mathcal{H}_N$ denote the N^2-dimensional real vector space of Hermitian $N\times N$ matrices,

$$\mathcal{H}_N=\left\{\ H=(h_{i,j})\in \mathrm{Mat}_{N,N}(\mathbb{C})\ \middle|\ h_{i,j}=\overline{h}_{j,i}\ \right\}.$$

We introduce coordinates $\{x_{i,i}\}_{1\leq i\leq N}$ and $\{x_{i,j},y_{i,j}\}_{1\leq i<j\leq N}$ on $\mathcal{H}_N$ by

$$\begin{aligned}
x_{i,i}&=h_{i,i},\\
x_{i,j}&=\mathrm{Re}(h_{i,j})\quad\text{and}\quad y_{i,j}=\mathrm{Im}(h_{i,j}),\quad i<j.
\end{aligned}$$

A volume form on $\mathcal{H}_N$ is defined by

$$dv(H)=\prod_i dx_{i,i}\prod_{i<j}dx_{i,j}dy_{i,j}\,.$$

Let Λ be a diagonal $N\times N$ matrix with positive real entries $\Lambda_1,\ldots,\Lambda_N$ along the diagonal. Define a Gaussian measure on the space $\mathcal{H}_N$ by

$$d\mu_\Lambda(H)=c_{\Lambda,N}\,e^{-\frac{1}{2}\mathrm{tr}(H^2\Lambda)}\,dv(H),$$

where the normalization

$$c_{\Lambda,N}=(2\pi)^{-N^2/2}\prod_i\Lambda_i^{1/2}\prod_{i<j}(\Lambda_i+\Lambda_j)$$

is determined by the constraint

$$\int_{\mathcal{H}_N}d\mu_{\Lambda,N}(H)=1\,.$$

Since $\left|e^{\frac{i}{6}\operatorname{tr}(H^3)}\right| \leq 1$, for $H \in \mathcal{H}_N$, the integral

$$\int_{\mathcal{H}_N} e^{\frac{i}{6}\operatorname{tr}(H^3)} d\mu_{\Lambda,N}(H) \tag{7}$$

is convergent and defines a function of $(\Lambda_1, \ldots, \Lambda_N) \in \mathbb{R}^N_{>0}$. When $\Lambda_j^{-1} \to 0$, the function (7) admits an asymptotic expansion given by

$$\int_{\mathcal{H}_N} e^{\frac{i}{6}\operatorname{tr}(H^3)} d\mu_{\Lambda,N}(H) \;\asymp_{\Lambda_j^{-1}\to 0}\; \exp(F^c)\big|_{t_i=-(2i-1)!!\operatorname{tr}(\Lambda^{-2i-1})}\,, \tag{8}$$

The above expansion (8) is called Kontsevich's formula [21].

2.3.2. *Determinantal formulas.* Via an averaging procedure over the unitary group applied to the left side of (8), the following formula can be obtained:

$$\exp(F^c)\big|_{t_i=-(2i-1)!!\operatorname{tr}(\Lambda^{-2i-1})} = \frac{\left|(D_j^{i-1}\mathcal{A}(\Lambda_j^{-1}))_{1\leq i,j\leq N}\right|}{\prod_{1\leq i<j\leq N}(\Lambda_j - \Lambda_i)}\,, \tag{9}$$

where D_j is the differential operator

$$D_j = -\frac{1}{\Lambda_j}\frac{d}{d\Lambda_j} + \Lambda_j + \frac{1}{2\Lambda_j^2}\,.$$

We recommend [15, Section 2.2] for a quick derivation of (9). In Section 2.3.3, we provide a direct proof for $N = 1$.

2.3.3. *Case $N = 1$.* For $N = 1$, Kontsevich's formula (8) yields

$$\frac{\sqrt{\lambda}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{\frac{i}{6}t^3 - \frac{1}{2}t^2\lambda} dt \;\asymp_{\lambda^{-1}\to 0}\; \exp(F^c)\big|_{t_i=-(2i-1)!!\lambda^{-2i-1}}\,.$$

Clearly, the left-hand side is equal to $\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{\frac{i}{6}\lambda^{-\frac{3}{2}}t^3} e^{-\frac{1}{2}t^2} dt$. We compute:

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{\frac{i}{6}\lambda^{-\frac{3}{2}}t^3} e^{-\frac{1}{2}t^2} dt \;\asymp_{\lambda^{-1}\to 0}\; & \sum_{j=0}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{i^j}{6^j j!} t^{3j} e^{-\frac{1}{2}t^2} dt\right)\lambda^{-\frac{3}{2}j} \\
&= \sum_{j=0}^{\infty}(-1)^j \frac{(6j-1)!!}{36^j (2j)!}\lambda^{-3j} \\
&= \mathcal{A}(\lambda^{-1}).
\end{aligned}$$

We have arrived at a direct connection between descendent integration and the series $\mathcal{A}(\lambda^{-1})$,

$$\exp(F^c)\big|_{t_i=-(2i-1)!!\lambda^{-2i-1}} = \mathcal{A}(\lambda^{-1}). \tag{10}$$

Pixton's relations constrain tautological classes on $\overline{\mathcal{M}}_{g,n}$ and hence also descendent integrals. In fact, Pixton's relations are expected to uniquely determine the descendent theory, but the implication is not yet proven. A simpler question, since both involve the hypergeometric series $\mathcal{A}$, is the following.

Question 1. *Can the specialization of the partition fuction* (10) *be derived from Pixton's relations?*

2.4. **The Infinite Grassmanian and tau-functions of the KP hierarchy.**

2.4.1. *Brief introduction to the KP hierarchy.* A *pseudo-differential* operator A is a Laurent series

$$A = \sum_{n=-\infty}^{m} a_n(T)\,\partial_x^n\,,$$

where $m \in \mathbb{Z}$ and the coefficients $a_n(T)$ are formal power series in the variables $\{T_i\}_{i\geq 1}$,

$$a_n(T) \in \mathbb{C}[[T_1, T_2, T_3, \ldots]]\,.$$

We identify the variable x with T_1. The non-negative and negative degree parts of the pseudo-differential operator A are defined by

$$A_+ = \sum_{n=0}^{m} a_n \partial_x^n \quad \text{and} \quad A_- = A - A_+\,.$$

The product of pseudo-differential operators is defined by the following commutation rule:

$$\partial_x^k \circ f = \sum_{l=0}^{\infty} \frac{k(k-1)\ldots(k-l+1)}{l!} \frac{\partial^l f}{\partial x^l} \partial_x^{k-l}\,,$$

where $k \in \mathbb{Z}$ and $f \in \mathbb{C}[[T_1, T_2, T_3, \ldots]]$.

Consider the pseudo-differential operator

$$L = \partial_x + \sum_{i\geq 1} w_i \partial_x^{-i}\,.$$

The *KP hierarchy* is the following system of partial differential equations for the power series w_i:

$$\frac{\partial L}{\partial T_n} = \left[(L^n)_+\,, L\right], \quad n = 1, 2, 3, \ldots. \tag{11}$$

For $n = 1$, the equation is equivalent to

$$\frac{\partial w_i}{\partial T_1} = \frac{\partial w_i}{\partial x}\,, \quad \forall i \geq 1\,,$$

compatible with our identification of x with T_1.

Suppose an operator L satisfies the system (11). Then there exists a pseudo-differential operator P of the form

$$P = 1 + \sum_{n\geq 1} p_n(T)\,\partial_x^{-n}, \tag{12}$$

satisfying $L = P \circ \partial_x \circ P^{-1}$ and

$$\frac{\partial P}{\partial T_n} = -\left(L^n\right)_- \circ P, \qquad n = 1, 2, 3, \ldots\,. \tag{13}$$

The operator P is the *dressing* operator and (13) are the Sato-Wilson equations. The Laurent series

$$\widehat{P}(T; z) = 1 + \sum_{n\geq 1} p_n(T)\, z^{-n}$$

is the *symbol* of the dressing operator P.

We can now introduce the notion of a tau-function. Denote by G_z the *shift operator* which acts on a power series $f \in \mathbb{C}[[T_1, T_2, T_3, \ldots]]$ as follows:

$$G_z(f)\Big(T_1, T_2, T_3, \ldots\Big) = f\Big(T_1 - \frac{1}{z}, T_2 - \frac{1}{2z^2}, T_3 - \frac{1}{3z^3}, \ldots\Big)\,.$$

Let $P = 1 + \sum_{n\geq 1} p_n(T)\,\partial_x^{-n}$ be the dressing operator of some operator L satisfying the KP hierarchy (11). Then there exists a series $\tau \in \mathbb{C}[[T_1, T_2, T_3, \ldots]]$ with constant term $\tau|_{\{T_i=0\}} = 1$ for which

$$\widehat{P} = \frac{G_z(\tau)}{\tau}\,.$$

The series τ is a *tau-function* of the KP hierarchy.

The KdV hierarchy is a certain reduction of the KP hierarchy. We do not discuss the details here, but only state the following property: a tau-function of the KdV hierarchy is a tau-function of the KP hierarchy which is independent of the variables $\{T_{2i}\}_{i\geq 1}$.

The precise form of Witten's conjecture may now be formulated:

$$\exp(F^c)|_{t_i=(2i+1)!!T_{2i+1}}$$

is a tau-function of the KdV hierarchy.

2.4.2. *The Infinite Grassmanian and the Fock space.* Consider the space of Laurent series $\mathbb{C}[z^{-1}, z]]$. There is a natural projection

$$p_- \colon \mathbb{C}[z^{-1}, z]] \to \mathbb{C}[z^{-1}]\,.$$

We denote by $\mathrm{Gr}^0\mathbb{C}[z^{-1}, z]]$ the set of all vector subspaces $H \subset \mathbb{C}[z^{-1}, z]]$ for which the projection

$$p_- \colon H \to \mathbb{C}[z^{-1}]$$

is an isomorphism.

The *Fock space* $\mathcal{F}$ is the vector space of (possibly infinite) linear combinations of wedge products of the form

$$z^{a_1} \wedge z^{a_2} \wedge z^{a_3} \wedge \ldots, \tag{14}$$

where $a_1, a_2, a_3, \ldots$ is a decreasing sequence of integers for which there exists an integer c (called the charge) satisfying

$$a_i = -i + c$$

for all i sufficiently large. Denote by $\mathcal{F}^{[c]} \subset \mathcal{F}$ the subspace consisting of vectors with charge c.

The *vacuum* vector

$$|0\rangle = z^{-1} \wedge z^{-2} \wedge z^{-3} \wedge z^{-4} \wedge z^{-5} \ldots \quad \in \mathcal{F}^{[0]}$$

plays a special role. For any vector $v \in \mathcal{F}$, we denote by $\langle 0\,|\,v\rangle$ the coefficient of $|0\rangle$ in the expression of v as a linear combination of the vectors (14).

We construct a map

$$\mathrm{pl} \colon \mathrm{Gr}^0\mathbb{C}[z^{-1}, z]] \to \mathcal{F}^{[0]}$$

by the following rule. Let $H \in \mathrm{Gr}^0\mathbb{C}[z^{-1}, z]]$. Let

$$f_1, f_2, f_3, \ldots \in \mathbb{C}[z^{-1}, z]]$$

be a basis in H of the form $f_i(z) = z^{-i}(1 + o(1))$. Let

$$\mathrm{pl}(H) = f_1 \wedge f_2 \wedge f_3 \wedge \ldots.$$

The infinite wedge product is defined by picking a single monomial summand in each f_i in such a way that the summand is z^{-i} for all but finitely many indices i (and summing over possible such choices). The resulting vector in $\mathcal{F}^{[0]}$ is easily seen to be independent of the basis choice $\{f_i\}$.

2.4.3. *Tau-functions from the infinite Grassmanian.* For any integer k, define the operator

$$\psi_k\colon \mathcal{F} \to \mathcal{F}\,, \qquad \psi_k\left(z^{a_1}\wedge z^{a_2}\wedge\ldots\right) = z^k \wedge z^{a_1}\wedge z^{a_2}\wedge\ldots\,.$$

The operator ψ_k increases the charge by 1. Denote by ψ_k^* the associated contraction operator,

$$\psi_k^*\colon z^{a_1}\wedge z^{a_2}\wedge\ldots \mapsto \begin{cases} (-1)^{j-1} z^{a_1}\wedge\ldots\wedge\widehat{z^{a_j}}\wedge\ldots, & \text{if there exists } j \text{ such that } a_j = k,\\ 0, & \text{otherwise.}\end{cases}$$

The hat above denotes an omitted element in the wedge product. The operator ψ_k^* decreases the charge by 1.

For $n \geq 1$, define $\alpha_n = \sum_{i\in\mathbb{Z}} \psi_i\psi_{i+n}^*$. These operators α_n do not change the charge and therefore leave invariant the space $\mathcal{F}^{[0]}$. The operator Γ is defined by

$$\Gamma = \exp\left(\sum_{n\geq 1} T_n\alpha_n\right).$$

For $H \in \mathrm{Gr}^0\mathbb{C}[z^{-1},z]]$, define a series $\tau_H(T_1,T_2,T_3,\ldots)$ by

$$\tau_H(T_1,T_2,T_3,\ldots) = \left\langle 0 \,\middle|\, \Gamma\big(\mathrm{pl}(H)\big)\right\rangle.$$

The series τ_H is a tau-function of the KP hierarchy, see [6].

2.4.4. *The partition function* $\exp(F^c)$ *as a point in the infinite Grassmanian.* For $i \geq 1$, define the Laurent series f_i:

$$f_i = \begin{cases} z^{-i}\mathcal{A}(-z), & \text{if } i \text{ is odd,}\\ -z^{-i}\mathcal{B}(-z), & \text{if } i \text{ is even.}\end{cases}$$

Let $H_{\mathcal{A},\mathcal{B}} \subset \mathbb{C}[z^{-1},z]]$ be the subspace spanned by the Laurent series f_i. The following result is proven by Kac and Schwarz [19]:

$$\exp(F^c)\big|_{t_i=(2i+1)!!T_{2i+1}} = \tau_{H_{\mathcal{A},\mathcal{B}}}\,. \tag{15}$$

In fact, (15) is essentially equivalent to (9), see [21, Lemma 4.2].

3. The moduli space of Riemann surfaces with boundary

3.1. **Overview.** The series A also appears in the intersection theory of the moduli space of Riemann surfaces with boundary (often viewed, with the boundary removed, as open Riemann surfaces). We recall the basics of the moduli of Riemann surfaces with boundary in Section 3.2 following [30]. In Section 3.3, we review two equivalent conjectural descriptions of the descendent theory: the open KdV and the open Virasoro equations. An explicit formula for the partition function of the open theory is discussed in Section 3.4. The hypergeometric series A plays a basic role in the formula.

3.2. **Moduli of Riemann surfaces with boundary.** Let $\Delta \in \mathbb{C}$ be the open unit disk, and let $\overline{\Delta}$ be its closure. An extendable embedding of the open disk Δ in a compact Riemann surface $f\colon \Delta \to C$ is a holomorphic map which can be extended to a holomorphic embedding of an open neighborhood of $\overline{\Delta}$. Two extendable embeddings are disjoint if the images of $\overline{\Delta}$ are disjoint.

A Riemann surface with boundary $(X,\partial X)$ is obtained by removing a finite positive number of disjoint extendable open disks from a connected compact Riemann surface. A compact Riemann surface is *not* viewed here as Riemann surface with boundary.

To a Riemann surface with boundary $(X,\partial X)$, we can canonically construct the double via the Schwartz reflection through the boundary. The double $D(X,\partial X)$ of $(X,\partial X)$ is a compact Riemann surface. The doubled genus of $(X,\partial X)$ is defined to be the usual genus of $D(X,\partial X)$.

On a Riemann surface with boundary $(X,\partial X)$, we consider two types of marked points. The markings of interior type are points of $X\backslash\partial X$. The markings of boundary type are points of ∂X. Let $\mathcal{M}_{g,k,l}$ denote the moduli space of Riemann surfaces with boundary of doubled genus

g with k distinct boundary markings and l distinct interior markings. The moduli space $\mathcal{M}_{g,k,l}$ is defined to be empty unless the stability condition

$$2g-2+k+2l>0$$

is satisfied. The moduli space $\mathcal{M}_{g,k,l}$ is a real orbifold of real dimension $3g-3+k+2l$.

The cotangent line classes $\psi_i \in H^2(\mathcal{M}_{g,k,l},\mathbb{Q})$ are defined (as before) as the first Chern classes of the cotangent line bundles associated to the interior markings. In [30], cotangent lines at the boundary points are not considered. Open intersection numbers are defined by

$$\left\langle \tau_{a_1}\tau_{a_2}\dots\tau_{a_l}\sigma^k\right\rangle_g^o = \int_{\overline{\mathcal{M}}_{g,k,l}} \psi_1^{a_1}\psi_2^{a_2}\dots\psi_l^{a_l}\,. \tag{16}$$

To rigorously define the right-hand side of (16), at least three significant steps must be taken:

- A natural compactification $\mathcal{M}_{g,k,l}\subset\overline{\mathcal{M}}_{g,k,l}$ must be constructed. Candidates for $\overline{\mathcal{M}}_{g,k,l}$ are themselves real orbifolds with boundary $\partial\overline{\mathcal{M}}_{g,k,l}$;
- For integration over $\overline{\mathcal{M}}_{g,k,l}$ to be well-defined, boundary conditions of the integrand along $\partial\overline{\mathcal{M}}_{g,k,l}$ must be specified;
- Orientation issues should be resolved, since the moduli space $\mathcal{M}_{g,k,l}$ is in general non-orientable.

All three steps are completed in genus 0 in [30]. The higher genus constructions will appear in upcoming work of Solomon and Tessler [34].

We introduce formal variables $t_0,t_1,t_2,\dots$ and s. The generating series F^o is defined by

$$F^o(t_0,t_1,\dots,s) = \sum_{\substack{g,k,l\ge 0\\ 2g-2+k+2l>0}} \frac{1}{k!l!}\sum_{a_1,\dots,a_l\ge 0}\left\langle \tau_{a_1}\dots\tau_{a_l}\sigma^k\right\rangle_g^o t_{a_1}\dots t_{a_l}s^k.$$

The series F^o is the *open potential.*

3.3. **Open KdV and open Virasoro equations.**

3.3.1. *Constraints.* KdV and Virasoro type constraints for the open intersection numbers (16) were conjectured in [30] for all genera (and proven in genus 0). The following initial condition follows easily from the definitions:

$$F^o\big|_{t_{i\ge 1}=0} = \frac{s^3}{6}+t_0 s\,. \tag{17}$$

3.3.2. *Open KdV equations.* The following system of partial differential equations for a series

$$F\in\mathbb{Q}[[t_0,t_1,t_2\dots,s]]$$

was introduced in [30]:

$$\frac{2n+1}{2}\frac{\partial F}{\partial t_n} = \frac{\partial F}{\partial s}\frac{\partial F}{\partial t_{n-1}} + \frac{\partial^2 F}{\partial s\partial t_{n-1}} + \frac{1}{2}\frac{\partial F}{\partial t_0}\frac{\partial^2 F^c}{\partial t_0\partial t_{n-1}} - \frac{1}{4}\frac{\partial^3 F^c}{\partial t_0^2\partial t_{n-1}}\,,\qquad n\ge 1. \tag{18}$$

The above system is called the *open KdV equations.* The open potential F^o was conjectured in [30] to be a solution of the open KdV equations. The open KdV equations (18), the initial condition (17), and the potential F^c together uniquely determine the series F^o. However, the existence of a such a solution, proven in [2] is non-trivial. We will denote the unique solution by $\widetilde{F}^o$.

3.3.3. *Open Virasoro equations.* The classical Virasoro operators $\{L_n\}_{n\geq -1}$ which appear in the descendent theory of closed Riemann surfaces are defined as follows:

$$\begin{aligned} L_n \;=\; & \sum_{i\geq 0} \frac{(2i+2n+1)!!}{2^{n+1}(2i-1)!!}(t_i-\delta_{i,1})\frac{\partial}{\partial t_{i+n}} \\ & + \frac{1}{2}\sum_{i=0}^{n-1}\frac{(2i+1)!!(2n-2i-1)!!}{2^{n+1}}\frac{\partial^2}{\partial t_i \partial t_{n-1-i}} + \delta_{n,-1}\frac{t_0^2}{2} + \delta_{n,0}\frac{1}{16}\,. \end{aligned}$$

The following modified operators,

$$\mathcal{L}_n = L_n + \left(s\frac{\partial^{n+1}}{\partial s^{n+1}} + \frac{3n+3}{4}\frac{\partial^n}{\partial s^n}\right), \quad n\geq -1,$$

were introduced in [30] and conjectured to constrain the partition function,

$$\mathcal{L}_n \exp(F^o+F^c) = 0, \quad n \geq -1\,. \tag{19}$$

The equations (19) are called the *open Virasoro equations.* Equations (19), the initial condition

$$F^o|_{t_{i\geq 0}} = \frac{s^3}{6}\,,$$

and the potential F^c together uniquely determine the series F^o.

The power series $\widetilde{F}^o$ is proven in [2] to satisfy the open Virasoro equations. Hence, the two conjectural descriptions of the open descendent theory, given by the open KdV equations and the open Virasoro equations, are equivalent.

3.4. **Formula for the open potential.** Let G_z be the shift operator which acts on a series $f(t_0,t_1,\ldots)\in\mathbb{C}[[t_0,t_1,\ldots]]$ by

$$G_z(f)\big(t_0,t_1,t_2,\ldots\big) = f\left(t_0-\frac{k_0}{z}, t_1-\frac{k_1}{z^3}, t_2-\frac{k_2}{z^5},\ldots\right),$$

where $k_n=(2n-1)!!$ and, by definition, $(-1)!!=1$.

Define the numbers $\{d_i\}_{i\geq 0}$ by

$$d_n = \sum_{i=0}^{n} 3^i |a_{n-i}| \prod_{k=1}^{i}\left(n+\frac{1}{2}-k\right),$$

where a_{n-i} is a coefficient of $\mathcal{A}$, see Section 1. The series $D(x)$, defined by

$$D(x) = 1 + \sum_{i\geq 1} d_i x^{3i}\,,$$

has the following equivalent description: $D(x)$ is the unique series solution of the differential equation

$$\left(-x^4\frac{\partial}{\partial x} - \frac{3}{2}x^3+1\right)D(x) = \mathcal{A}(-x).$$

The formula of [3] expressing the open descendent theory in terms of the closed also requires the series

$$\xi(t,s;z) = \frac{s}{2}z^2 + \sum_{i\geq 0}\frac{t_i}{(2i+1)!!}z^{2i+1}\,.$$

Theorem 2 (Buryak)**.** *We have*

$$\exp(\widetilde{F}^o) = \mathrm{Coef}_{z^0}\left[D\left(z^{-1}\right)\frac{G_z(\exp(F^c))}{\exp(F^c)}\exp(\xi)\right]. \tag{20}$$

The product $D\left(z^{-1}\right)\frac{G_z(\exp(F^c))}{\exp(F^c)}$ is a series in z^{-1}. On the other hand, $\exp(\xi)$ is a series in z. In general, the multiplication of two such series may not be well-defined. In our case, the issue is resolved as follows. We introduce a grading in the ring $\mathbb{C}[[t_0,t_1,t_2,\ldots,s]]$ assigning to t_i the degree $2i+1$ and to s the degree 2. Since the degree of the coefficient of z^i in $\exp(\xi)$ grows as i grows, the product in the square brackets is well defined.

In fact, a more general statement is proven in [3]. A natural way to include variables

$$s_1, s_2, s_3, \ldots$$

in the series $\widetilde{F}^o$ was proposed in [2]. The new variables s_i may be viewed as descendants of the boundary marked points. The extended power series is denoted by $\widetilde{F}^{o,ext}$. In [3], a formula similar to (20) is proven for the extended series $\widetilde{F}^{o,ext}$. The series

$$\exp(\widetilde{F}^{o,ext}+F^c)\Big|_{\substack{t_i=(2i+1)!!T_{2i+1}\\ s_i=2^{i+1}(i+1)!T_{2i+2}}}$$

is proven to be a tau-function of the KP hierarchy in [1] using the formula of [3].

Question 2. *Is there a system of tautological relations involving* A *and* B *in the cohomology of the moduli spaces* $\overline{\mathcal{M}}_{g,k,l}$ *parallel to Pixton's?*

4. Cohomological field theories and Witten's 3-spin class

4.1. **Overview.** We describe here how the hypergeometric series A and B appear in the study of Witten's 3-spin class via cohomological field theories.

In Section 4.2, we recall the basics of cohomological field theories and the Givental-Teleman classification [12, 35] in the semisimple case. In Section 4.3, we consider the example of Witten's r-spin class and see how the A and B series appear in the R-matrix of the 3-spin theory. Finally, in Sections 4.4 and 4.5, we see how the Airy differential equation is directly related to the flatness equation for the Dubrovin connection of the 3-spin theory.

4.2. **Cohomological field theories.** Cohomological field theories (CohFTs) were first defined by Kontsevich and Manin [22] in order to place the axioms of Gromov-Witten theory in an algebraic structure.

An n-dimensional CohFT, defined on an n-dimensional vector space V together with a non-degenerate bilinear form

$$\eta : V \times V \to V$$

and unit vector $\mathbf{1} \in V$, is a collection of S_n-symmetric, multilinear maps

$$\Omega_{g,n} : V^n \to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}),$$

for every g and n satisfying $2g-2+n>0$, for which the following two properties hold:

- *Splitting.* The pull-back of $\Omega_{g,n}(v_1,\ldots,v_n)$ via a gluing map

$$\prod_i \overline{\mathcal{M}}_{g_i,n_i} \to \overline{\mathcal{M}}_{g,n}$$

  is the product of the Ω_{g_i,n_i} corresponding to the components, with arguments $v_1,\ldots,v_n$ at the preimages of the marked points and the symmetric bivector η^{-1} at the points which are glued together.

- *Unit.* Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map. Then

$$\pi^*\Omega_{g,n}(v_1,\ldots,v_n) = \Omega_{g,n+1}(v_1,\ldots,v_n,\mathbf{1}).$$

  Additionally, $\Omega_{0,3}(v,w,\mathbf{1}) = \eta(v,w)$.

If all classes of Ω are of cohomological degree 0 on the moduli spaces of curves, Ω is a *topological field theory* (TQFT). Then, Ω is uniquely determined from $\Omega_{0,3}$ and η by calculating $\Omega_{g,n}$ – a multiple of the fundamental class – at a maximally degenerate curve with $2g-2+n$ rational irreducible components, each with three special points.

The study of CohFTs is motivated by Gromov-Witten theory. To any nonsingular projective variety X, we can associate a CohFT based on the cohomology ring $H^*(X,\mathbb{Q})$ together with the Poincaré pairing as the bilinear form and with the fundamental class as the unit vector (if X has cohomology in odd degree, the S_n-symmetry hypothesis of a CohFT must be replaced by appropriate skew-symmetry). Let $\overline{\mathcal{M}}_{g,n}(X)$ denote the space of stable maps to X, π the projection to $\overline{\mathcal{M}}_{g,n}$, ev_i the ith evaluation map, and $[\overline{\mathcal{M}}_{g,n}(X)]^{vir}$ the virtual fundamental class. We define

$$\Omega_{g,n}(v_1,\ldots,v_n)=\pi_*\left(\prod_{i=1}^{n}\mathrm{ev}_i^*(v_i)\cap[\overline{\mathcal{M}}_{g,n}(X)]^{vir}\right) \tag{21}$$

under convergence conditions (required for the implicit sum over curve classes $\beta\in H_2(X,\mathbb{Z})$ to be well-defined).

The tensors $\Omega_{0,3}$ and η can be used to define a product $\star$, the *quantum product*, on V via

$$\Omega_{0,3}(a,b,c)=\eta(a\star b,c), \tag{22}$$

making $(V,\eta,\star)$ a Frobenius algebra. The CohFT Ω is *semisimple* if the algebra V has no nilpotent elements, or equivalently, an orthogonal basis of idempotent elements defined over $\mathbb{C}$.

Semisimple CohFTs have been classified by Teleman [35] by generalizing a conjecture of Givental [12]. To reconstruct a CohFT Ω from the TQFT ω defined by the degree 0 part, a unique endomorphism valued matrix of power series

$$R=1+R_1z+R_2z^2+\cdots\ \in\mathrm{End}(V)[[z]]$$

must be specified. Given R, there is a concrete formula for Ω in terms of tautological classes similar to the form of Pixton's relations as in Appendix A.

4.3. **Witten's 3-spin class.** For every integer $r\geq 2$, there is a beautiful CohFT obtained from Witten's r-spin class. We review the basic properties of the construction.

Let V be an $(r-1)$-dimensional $\mathbb{Q}$-vector space with basis $e_0,\ldots,e_{r-2}$, bilinear form

$$\eta(e_a,e_b)=\delta_{a+b,r-2}\,,$$

and unit vector $\mathbf{1}=e_0$. Witten's r-spin theory provides a family of classes

$$W_{g,n}(a_1,\ldots,a_n)\in H^*(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$$

for $a_1,\ldots,a_n\in\{0,\ldots,r-2\}$. These define a CohFT on V by setting

$$W_{g,n}(e_{a_1},\ldots,e_{a_n})=W_{g,n}(a_1,\ldots,a_n)$$

and extending multilinearly.

Witten's class is homogeneous of (complex) degree

$$\frac{(r-2)(g-1)+\sum_i a_i}{r}$$

and vanishes if the degree formula fails to yield an integer.

In genus 0, the construction of Witten's class was first carried out by Witten [37] using r-spin structures (r^{th} roots of the canonical bundle). In higher genus, there are by now several constructions. Algebraic approaches have been found by Polishchuk-Vaintrob [33] (later simplified by Chiodo [5]) and Chang-Li-Li [4]. Analytic constructions by Mochizuki [25] and Fan-Jarvis-Ruan [10] are also available. The equivalence of these constructions is established in [28].

The CohFT determined by Witten's r-spin class is not semisimple. For example, for $r = 3$, the quantum product

$$e_1 \star e_1 = 0$$

vanishes. However, via a shift on the Frobenius manifold, Witten's class can be modified to be a semisimple CohFT called the *shifted r-spin Witten's class*. In the case of $r = 3$, the shift depends on one parameter ϕ and, in the new quantum-product,

$$e_1 \star e_1 = \phi e_0 \,.$$

The modification destroys the homogeneity property of Witten's class. The shifted Witten's class is supported in cohomological degrees at most the degree of Witten's class.

The R-matrix of the CohFT of Witten's 3-spin class was calculated explicitly in [28]. Here, the A and B hypergeometric series appear. When written in the basis $\{e_0, e_1\}$, we have

$$R(6z) = \begin{pmatrix} -\mathsf{B}^{even}(-z\phi^{-3/2}) & -\phi^{1/2}\mathsf{B}^{odd}(-z\phi^{-3/2}) \\ -\phi^{-1/2}\mathsf{A}^{odd}(-z\phi^{-3/2}) & \mathsf{A}^{even}(-z\phi^{-3/2}) \end{pmatrix}, \tag{23}$$

where we have used the superscripts *even* and *odd* to denote the even and odd degree part of a series.

Because the CohFT defined by Witten's class is homogeneous there is a recursive procedure explicitly by described by Givental and Teleman for calculating R. The matrix (23) satisfies the recursion and therefore is the correct R-matrix for Witten's 3-spin class. In the next Sections, we will see a direct connection between the R-matrix and the Airy differential equation.

Applying the Givental-Teleman reconstruction to the r-spin CohFT gives an alternative expression for shifted Witten's class in terms of tautological classes. The formula of the reconstructed CohFT has also terms of cohomological degree higher than (4.3). The necessary cancellation implies nontrivial relations between tautological classes. In [28], these relations for $r = 3$ are directly shown to be equivalent to Pixton's relations. For higher r, the relations are studied in [29].

4.4. **Frobenius manifolds.** An n-dimensional vector space V may be viewed as a manifold covered by a chart with coordinates $t_1, \ldots, t_n$ corresponding to a basis $e_1, \ldots, e_n$ of V. The global vector fields $\frac{\partial}{\partial t_\mu}$ may be used to identify each tangent space with V. If V is the vector space of a CohFT, all the tangent spaces of the manifold V are equipped with the (constant) metric η. The full genus 0 potential associated to Ω,

$$\Phi(\xi) = \sum_{n \geq 3} \frac{1}{n!} \int_{\overline{\mathcal{M}}_{0,n}} \Omega_{0,n}(\xi, \ldots, \xi)\,, \qquad \xi \in V\,,$$

satisfies the WDVV equations and thus determines a deformed quantum product $\star$ on the tangent space at $\xi \in V$,

$$\left.\frac{\partial^3 \Phi}{\partial t_a \partial t_b \partial t_c}\right|_\xi = \eta\left(\frac{\partial}{\partial t_a} \star \frac{\partial}{\partial t_b}, \frac{\partial}{\partial t_c}\right)\,.$$

Together, the above constructions endow V with the structure of a *Frobenius manifold*. At the origin of V with coordinates

$$t_1 = \ldots = t_n = 0\,,$$

the quantum product $\star$ agrees with the earlier definition (22). Frobenius manifolds have been introduced by Dubrovin and the full definitions can be found in his monograph [7].

The coordinates t_μ are called *flat coordinates*. If the multiplication $\star$ on the tangent space of $\xi \in V$ is semisimple, an alternative set of *canonical coordinates* u_i are defined in a neighborhood of ξ. These are defined up to additive constants and reordering by requiring that the corresponding vector fields $\frac{\partial}{\partial u_i}$ form a basis of orthogonal idempotents at each point where they are defined. Let $\mathbf{u}$ be the diagonal matrix with the functions u_i along the diagonal,

$$\mathbf{u} = \begin{pmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{pmatrix}.$$

Let Ψ be the base change matrix from the basis of vector fields $\frac{\partial}{\partial t_\mu}$ to the basis of *normalized idempotents* given by

$$\eta\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i}\right)^{-1/2} \frac{\partial}{\partial u_i}.$$

Up to constants of integration, the *R-matrix* written in the basis of normalized idempotents is uniquely determined by the property that the product

$$S = \Psi R e^{\mathbf{u}/z} \tag{24}$$

is a matrix of asymptotic fundamental solutions to the *flatness equation*

$$z\frac{\partial}{\partial t_\mu} S = \frac{\partial}{\partial t_\mu} \star S.$$

The name of the equation stems from the fact that it characterizes flat vector fields for the *Dubrovin (projective) connection* ∇_z defined by

$$\nabla_{z,X} = z\nabla_X - X\star$$

for any vector field X. Here, ∇ is the Levi-Civita connection corresponding to the metric η.

4.5. **Witten's 3-spin theory.** The 3-spin CohFT determines a 2-dimensional Frobenius manifold with potential

$$\frac{1}{2}t_0^2 t_1 + \frac{1}{72}t_1^4$$

and metric

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, $\frac{\partial}{\partial t_0}$ is the unit for the quantum product, and

$$\frac{\partial}{\partial t_1} \star \frac{\partial}{\partial t_1} = \phi \frac{\partial}{\partial t_0},$$

where $\phi = \frac{t_1}{3}$.

Let $F(x)$ be the versal deformation of the A_2-singularity $x^3 = 0$,

$$F(x) = x^3 - t_1 x + t_0.$$

We can identify the quantum product with the multiplication in the Milnor ring

$$\mathbb{C}[t_0, t_1][x]/F'(x)$$

under the identifications $\frac{\partial}{\partial t_0} \mapsto 1$ and $\frac{\partial}{\partial t_1} \mapsto -x$.

The idempotents and canonical coordinates correspond to the critical points $\pm\sqrt{\phi}$ of $F(x)$. Canonical coordinates are given by the associated critical values. Explicitly, the change of basis Ψ and the matrix $\mathbf{u}$ of canonical coordinates are given by

$$\Psi = \begin{pmatrix} \frac{-\sqrt{\phi}}{\sqrt{\Delta_+}} & \frac{\sqrt{\phi}}{\sqrt{\Delta_-}} \\ \frac{1}{\sqrt{\Delta_+}} & \frac{1}{\sqrt{\Delta_-}} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -2\phi^{3/2} + t_0 & 0 \\ 0 & 2\phi^{3/2} + t_0 \end{pmatrix},$$

where $\Delta_\pm = \mp 2\sqrt{\phi}$, and choices of roots $\sqrt{\Delta_\pm}$ have been made.

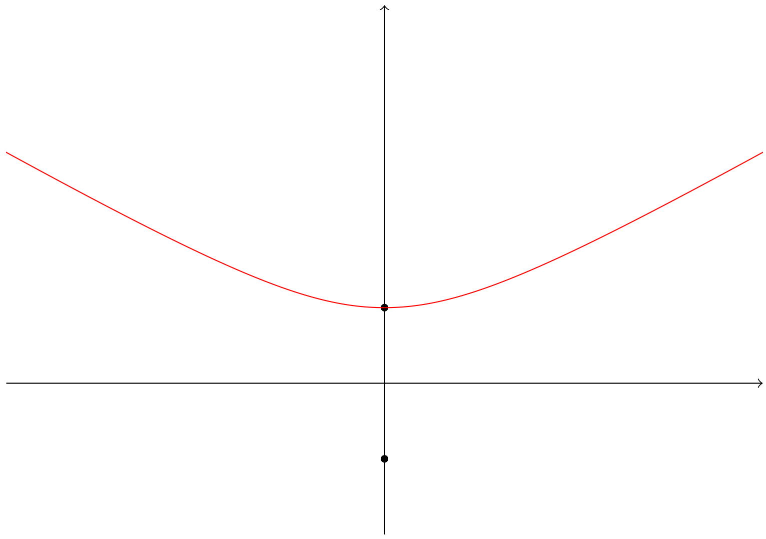

FIGURE 2. Lefschetz thimble for $i(z^3+3z)$ through critical point i.

The flatness equations may be written explicitly as the system

$$\begin{aligned} z\frac{\partial}{\partial t_0}S^0_\pm &= S^0_\pm\,, & z\frac{\partial}{\partial t_1}S^1_\pm &= S^0_\pm\,,\\ z\frac{\partial}{\partial t_0}S^1_\pm &= S^1_\pm\,, & z\frac{\partial}{\partial t_1}S^0_\pm &= \phi S^1_\pm\,, \end{aligned} \tag{25}$$

where the upper indices 0 and 1 stand for the vector components in the basis $\{e_0, e_1\}$ and the lower index distinguishes two linear independent solutions. Combining these equations we see that $S^1_\pm$ satisfies the Airy differential equation

$$\left(z\frac{\partial}{\partial t_1}\right)^2 S^1_\pm = \phi S^1_\pm\,. \tag{26}$$

Up to recaling t_1, the differential equation (26) is equivalent to (3).

The solutions $S^\mu_\pm$ are given by the asymptotic expansion for $z \to 0$ of the complex contour integrals

$$\left(\frac{2\pi z}{3}\right)^{-\frac{1}{2}} \int_{\Gamma_\pm} e^{F(x)/z}(-x)^{1-\mu}\mathrm{d}x \tag{27}$$

defined for almost all (t, z). Here, the Lefschetz thimbles $\Gamma_\pm$ correspond to the two critical points

$$p_\pm = \pm\sqrt{\phi}$$

of $F(x)/z$ and are chosen as follows (see also [38]). Suppose the critical values of $F(x)/z$ have different imaginary values. We consider $\mathrm{Re}(F/z)$ as function in the real and imaginary part of x. The cycle $\Gamma_\pm$ is the union of two integral curves of the vector field $-\nabla\,\mathrm{Re}(F/z)$ arriving at $p_\pm$ at time $-\infty$. By construction, the real part of F/z decreases fast enough when moving along $\Gamma_\pm$ from $p_\pm$ such that the contour integrals converge absolutely. By differentiating under the integral, the contour integrals for any choice of cycle are easily seen to give solutions to (25).

The Airy function arises (up to a multiplicative constant) in the case

$$\mu = 1\,,\ z = -3i\,,\ t_0 = 0\,,\ \text{and } t_1 > 0$$

for the critical point $\sqrt{\phi}$. Under these conditions, the Lefschetz thimble can be deformed into the cycle in the integral used in the defintion (2) of the Airy function.

As in Section 1, we make the integral look like a Gaussian integral by translating and scaling the integrand,

$$\left(\frac{2\pi z}{3}\right)^{-\frac{1}{2}} \int_{\Gamma_\pm} e^{F(x)/z}(-x)^{1-\mu}\mathrm{d}x = \frac{e^{u_\pm/z}}{\sqrt{2\pi}\sqrt{\Delta_\pm}} \int_{\widetilde{\Gamma}_\pm} e^{-\frac{x^2}{2}-x^3\sqrt{-z}(-3\Delta_\pm)^{-3/2}} \left(\frac{-x\sqrt{-z}}{\sqrt{-3\Delta_\pm}} \mp \sqrt{\phi}\right)^{1-\mu} \mathrm{d}x\,,$$

where $\widetilde{\Gamma}_\pm$ is the Lefschetz thimble defined by the new exponent. Because of boundeness of the integral as $z \to 0$ and the fact that in this limit the cycle $\widetilde{\Gamma}_\pm$ approaches the real line, we obtain as in Section 1 an asymptotic expansion by formally expanding the integrand in z and calculating the integrals of the individual summands. For $\mu = 1$, we obtain

$$S^1_\pm e^{-u_\pm/z} = \frac{1}{\sqrt{\Delta_\pm}} \sum_{j=0}^\infty \frac{(6j)!}{(3j)!(2j)!} \left(\frac{z}{216\Delta_\pm^3}\right)^j = \frac{1}{\sqrt{\Delta_\pm}} \mathsf{A}\left(\frac{\mp z}{6\phi^{3/2}}\right),$$

and, for $\mu = 0$,

$$S^0_\pm e^{-u_\pm/z} = \frac{\mp\sqrt{\phi}}{\sqrt{\Delta_\pm}} \sum_{j=0}^\infty \frac{(6j)!}{(3j)!(2j)!} \frac{-6j-1}{6j-1} \left(\frac{z}{216\Delta_\pm^3}\right)^j = \frac{\pm\sqrt{\phi}}{\sqrt{\Delta_\pm}} \mathsf{B}\left(\frac{\mp z}{6\phi^{3/2}}\right).$$

By (24), we obtain the R-matrix written in flat coordinates as the product

$$\Psi^{-1} S e^{-\mathbf{u}/z}.$$

By the formula

$$\Psi^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\Delta_+}} & \frac{-\sqrt{\phi}}{\sqrt{\Delta_+}} \\ \frac{1}{\sqrt{\Delta_-}} & \frac{\sqrt{\phi}}{\sqrt{\Delta_-}} \end{pmatrix},$$

we immediately arrive at (23).

5. Stable maps and quotients with target $\mathbb{CP}^1$

5.1. Antecedents. The A and B series can also be encountered in R-matrices of other CohFTs. Here, we consider the equivariant Gromov-Witten theory of $\mathbb{CP}^1$.

The analysis is motivated by the earlier study of the hypergeometric function

$$\Phi(z,q) = \sum_{d=0}^\infty \prod_{i=1}^d \frac{1}{\lambda - iz} \frac{(-1)^d}{d!} \frac{q^d}{z^d} = \sum_{d=0}^\infty q^d \prod_{i=1}^d \frac{1}{(iz-\lambda)iz}$$

which arises naturally in the geometry of the moduli of curves. Let

$$\mathbb{B}_d \to \mathcal{M}_{g,d}$$

denote the bundle with fiber $H^0(C, \mathcal{O}_C(\sum_j p_j)|_{\sum_j p_j})$ over the moduli point $[C, p_1, \ldots, p_d]$. The series Φ at $\lambda = 1$ has been used to calculate

$$\pi_* c_z^{-1}(\mathbb{B}_d),$$

where c_z^{-1} the inverse of the Chern polynomial in variable z and

$$\pi : \mathcal{C}^d_{g,n} \to \mathcal{M}_{g,n}$$

is the projection from the d-fold universal curve to $\mathcal{M}_{g,n}$. The study of Φ was first taken up by Ionel [14] to prove the nonvanishing of certain coefficients in tautological relations related to Faber's generation conjecture – the hypergeometric series A already arises in [14]. In [27], a further study of Φ was required as a part of the proof of the Faber-Zagier relations (and the series B also emerged). The series Φ plays a basic role in the proof [16] of Pixton's relations in the Chow ring.

The relation to the Gromov-Witten theory of $\mathbb{CP}^1$ is as follows. In [16, 27], the bundle $\mathbb{B}_d$ appeared as a vertex contribution in a localization calculation for stable quotients [23] to $\mathbb{CP}^1$. There is a proper map from the moduli space of of stable maps to the moduli space of stable quotients to $\mathbb{CP}^1$, compatible with localization in the sense that the vertex contributions for stable quotients are sums of localization contributions for stable maps. Hence, the localization calculation yields the same results in both spaces. Givental's conjecture on the reconstruction of CohFTs was motivated by equivariant localization: the case of the equivariant Gromov-Witten theory of toric varieties was proven in [12] using virtual localization [13].

5.2. **Frobenius manifold.** The equivariant Gromov-Witten theory $\mathbb{CP}^1$ defines a 2 dimensional CohFT by the equivariant analog of (21). The genus zero potential of the corresponding Frobenius manifold records the genus zero, equivariant primary Gromov-Witten invariants of $\mathbb{CP}^1$. It is given by

$$\frac{1}{2}t_0^2 t_1 + \frac{\lambda}{2} t_0 t_1^2 + \frac{\lambda^2}{6} t_1^3 + e^{t_1}$$

and the metric is

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & \lambda \end{pmatrix} .$$

Here, λ is the equivariant parameter. The quantum product of the Frobenius manifold coincides with the equivariant quantum product of $\mathbb{CP}^1$, determined by the equation

$$H(H-\lambda) = e^{t_1} ,$$

after we identify $\frac{\partial}{\partial t_0}$ with the fundamental class and $\frac{\partial}{\partial t_1}$ with the hyperplane class H.

Let F be the mirror curve,

$$F(x) = e^x + e^{t_1 - x} + \lambda x + t_0 .$$

Identifying H with $e^{t_1 - x}$, we can interpret the quantum cohomology ring also as

$$\mathbb{C}[t_0, e^{t_1}][e^x, e^{-x}]/F'(x) .$$

The critical points are

$$e^x = -\frac{\lambda}{2} \pm \sqrt{e^{t_1} + \frac{\lambda^2}{4}} .$$

Setting $\phi = e^{t_1} + \frac{\lambda^2}{4}$, we determine the canonical coordinates

$$u_\pm = \pm 2\sqrt{\phi} + \lambda \ln\left(-\frac{\lambda}{2} \pm \sqrt{\phi}\right) + t_0$$

and the basis change matrix

$$\Psi = \begin{pmatrix} \frac{-\frac{\lambda}{2}+\sqrt{\phi}}{\sqrt{\Delta_+}} & \frac{-\frac{\lambda}{2}-\sqrt{\phi}}{\sqrt{\Delta_-}} \\ \frac{1}{\sqrt{\Delta_+}} & \frac{1}{\sqrt{\Delta_-}} \end{pmatrix}$$

for $\Delta_\pm = \pm 2\sqrt{\phi}$.

The flatness equations can be written as

$$z\frac{\partial}{\partial t_0} S^0_\pm = S^0_\pm , \qquad z\frac{\partial}{\partial t_1} S^1_\pm = S^0_\pm + \lambda S^1_\pm ,$$
$$z\frac{\partial}{\partial t_0} S^1_\pm = S^1_\pm , \qquad z\frac{\partial}{\partial t_1} S^0_\pm = e^{t_1} S^1_\pm ,$$

in the basis $\{1, H\}$. They imply the second order differential equation

$$\left(z\frac{\partial}{\partial t_1}\right)^2 S^1_\pm - \lambda z \frac{\partial}{\partial t_1} S^1_\pm = e^{t_1} S^1_\pm$$

for $S^1_\pm$. After the variable change

$$q = e^{t_1} ,$$

the hypergeometric function Φ satisfies the same differential equation.

5.3. **Asymptotic analysis.** Asymptotic solutions to the flatness equation can again be constructed using asymptotic expansion of contour integrals

$$\frac{1}{\sqrt{-2\pi z}}\int_{\Gamma_\pm} e^{F(x,t)/z}(e^{t_1-x}-\lambda)^{1-\mu}\mathrm{d}x$$

along Lefschetz thimbles $\Gamma_\pm$ though critical points of F.

The series Φ can be identified with the oscillating integral solution for $\mu=0$ and the cycle Γ_-, up to a factor independent of $q=e^{t_1}$. To prove this claim, as both functions satisfy the same second order differential equation in q, we need only check that the functions and their first q-derivatives agree at $q=0$ up to the same factor.

We first study the limit $q\to 0$ of the integral

$$\begin{aligned}\lim_{q\to 0}\frac{1}{\sqrt{-2\pi z}}\int_{\Gamma_-} e^{F(x,t)/z}\mathrm{d}x &= \frac{1}{\sqrt{-2\pi z}}\int_{\Gamma_-} e^{(e^x+\lambda x+t_0)/z}\mathrm{d}x\\ &= \frac{(-z)^{\lambda/z}}{\sqrt{-2\pi z}}\int_{\widehat{\Gamma}_-} e^{-x+(\lambda\ln(x)+t_0)/z}\frac{\mathrm{d}x}{x},\end{aligned}$$

where the substitution $x\mapsto \ln(-xz)$ was applied in the second step. The critical point that the new Lefschetz thimble $\widehat{\Gamma}_-$ moves through is at $\frac{z}{\lambda}$ and, if we assume that this ratio is positive real, $\widehat{\Gamma}_-$ coincides with the positive real axis. So we can rewrite the limit as

$$\lim_{q\to 0}\frac{1}{\sqrt{-2\pi z}}\int_{\Gamma_-} e^{F(x,t)/z}\mathrm{d}x = \frac{(-z)^{\lambda/z}e^{t_0/z}}{\sqrt{-2\pi z}}\Gamma\left(\frac{-\lambda}{-z}\right),$$

where Γ is the Gamma function. Using Stirling's formula

$$\ln\Gamma(x) \asymp x\ln(x)-x-\frac{1}{2}\ln\left(\frac{x}{2\pi}\right)+\sum_{i=1}^{\infty}\frac{B_{2i}}{2i(2i-1)}x^{-2i},$$

where the B_{2i} are the Bernoulli numbers defined by

$$\sum_{k=0}^{\infty} B_k\frac{x^k}{k!} = \frac{x}{e^x-1},$$

we find

$$\lim_{q\to 0}\frac{1}{\sqrt{-2\pi z}}\int_{\Gamma_-} e^{F(x,t)/z}\mathrm{d}x \asymp \frac{e^{u_-|_{q=0}/z}}{\sqrt{\Delta_-|_{q=0}}}\exp\left(\sum_{i=1}^{\infty}\frac{B_{2i}}{2i(2i-1)}(z/\lambda)^{2i-1}\right), \tag{28}$$

where $u_-|_{q=0}=-\lambda+\lambda\ln(-\lambda)+t_0$ and $\Delta_-|_{q=0}=-\lambda$. Because of $\Phi(z,0)=1$, the factor we need to multiply Φ with so that it can coincide with the oscillating integral is (28). Geometrically, the Bernoulli modification is caused by the contribution of the Hodge bundle in the vertex terms of the localization.

We also need to check that at $q=0$ the first q-derivative of Φ

$$\frac{\partial\Phi}{\partial q}(z,0) = \frac{1}{(z-\lambda)z}$$

coincides with the corresponding oscillating integral up to the factor (28). Similarly to above, we compute

$$\begin{aligned}\lim_{q\to 0}\frac{\partial}{\partial q}\frac{1}{\sqrt{-2\pi z}}\int_{\Gamma_-} e^{F(x,t)/z}\mathrm{d}x &= \lim_{q\to 0}\frac{1}{\sqrt{-2\pi z}}\int_{\Gamma_-}\frac{e^{-x}}{z}e^{F(x,t)/z}\mathrm{d}x = \frac{(-z)^{\lambda/z-1}e^{t_0/z}}{z\sqrt{-2\pi z}}\Gamma\left(\frac{-\lambda}{-z}-1\right)\\ &= \frac{1}{z(z-\lambda)}\frac{(-z)^{\lambda/z}e^{t_0/z}}{\sqrt{-2\pi z}}\Gamma\left(\frac{-\lambda}{-z}\right) = \frac{1}{z(z-\lambda)}\lim_{q\to 0}\frac{1}{\sqrt{-2\pi z}}\int_{\Gamma_-} e^{F(x,t)/z}\mathrm{d}x,\end{aligned}$$

completing the proof that Φ coincides with the oscillating integral up to the factor (28).

Let us extract the A-function from a different limit of the R-matrix. To calculate the asymptotic expansion, we can proceed as in Section 4.5 by formally expanding $\exp(F/z)$ near a critical point. Let us first expand F/z as

$$\frac{1}{z}\left(u_\pm \pm 2\phi\frac{x^2}{2} - \lambda\frac{x^3}{6} \pm 2\phi\frac{x^4}{4!} - \lambda\frac{x^5}{5!} \pm 2\phi\frac{x^6}{6!} - \cdots\right),$$

which becomes

$$\frac{u_\pm}{z} - \frac{x^2}{2} + \lambda\sqrt{-z}\Delta_\pm^{-3/2}\frac{x^3}{6} - (-z)\Delta_\pm^{-1}\frac{x^4}{4!} + \lambda(-z)^{3/2}\Delta_\pm^{-5/2}\frac{x^5}{5!} - (-z)^2\Delta_\pm^{-2}\frac{x^6}{6!} + \ldots$$

after replacing $x \mapsto x\sqrt{-z}\Delta_\pm^{-1/2}$. Notice the cubic term in x essentially carries a power of $z/\phi^{3/2}$ whereas the higher order terms carry powers of z/ϕ^c for $c < \frac{3}{2}$. Therefore, if we are only interested in the coefficients of the R-matrix with lowest possible power of ϕ, we can ignore all the higher order terms of F/z. In the $\mu = 1$ integral without the higher order terms, we discover the A-series,

$$\frac{1}{\sqrt{2\pi}\sqrt{\Delta_\pm}}\int_{\tilde{\Gamma}_\pm}\exp\left(\frac{u_\pm}{z} - \frac{x^2}{2} + \lambda\sqrt{-z}\Delta_\pm^{-3/2}\frac{x^3}{6}\right)\mathrm{d}x \asymp \frac{e^{u_\pm}}{\sqrt{\Delta_\pm}}\mathsf{A}\left(\frac{\mp z\lambda^2}{8\phi^{3/2}}\right).$$

For $\mu = 0$, we similarly obtain a linear combination of the A- and B-series.

Extracting the lowest order terms can also be achieved by taking the following peculiar limit. After replacing z by z/λ^2, take the limit $\lambda \to \infty$ while keeping ϕ fixed. This limit does not make sense immediately for all data: the canonical coordinates $u_\pm$ need to be changed by additive constants and a flat basis $\{1, H - \lambda/2\}$ should be used instead of $\{1, H\}$. In the limit, the R-matrix essentially agrees with the 3-spin R-matrix. In [17], it is more generally shown how the CohFT of Witten's r-spin class arises as a limit of the equivariant Gromov-Witten theory of projective $(r-2)$-space.

5.4. **Universality.** For any Frobenius manifold obtained from a CohFT which is generically semisimple on the vector space V, relations in the tautological rings of the moduli spaces of curves can be found by studying the pole cancellations required as the limit of the Givental-Teleman formula to a non-semisimple point is taken. The first comparison result is the following

Theorem 3 (Janda)**.** *The tautological relations in $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ obtained from 3-spin and equivariant $\mathbb{CP}^1$ are equivalent.*

Since the 3-spin relations were proven to be equivalent to Pixton's relations in [28], the equivariant $\mathbb{CP}^1$ relations are also equivalent to Pixton's. A natural question, perhaps more approachable than Pixton's completeness conjecture, is the following.

Question 3. *Do all tautological relations obtained from generically semisimple CohFTs by taking limits to non-semisimple points lie in Pixton's set?*

An affirmative answer to Question 3 would imply that the hypergeometric series A and B are lurking in the structure of all generically semisimple CohFTs.[1]

Appendix A. Pixton's relations

A.1. **Strata algebra.** Let $\mathcal{S}^*_{g,n}$ be the $\mathbb{Q}$-algebra of κ and ψ classes supported on the strata $\overline{\mathcal{M}}_{g,n}$. A $\mathbb{Q}$-basis of $\mathcal{S}^*_{g,n}$ is given by isomorphism classes of pairs $[\Gamma, \gamma]$ where Γ is stable graph corresponding to a stratum of the moduli space of curves,

$$\overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n},$$

[1]Question 3 has now been answered affirmatively in [18]. The main idea is to find an appropriate set of local coordinates and vector fields near generic non-semisimple points and to use these to locally compare any given CohFT with a trivial extension of the 3-spin theory to the right dimension.

and γ is product of κ and ψ classes on $\overline{\mathcal{M}}_\Gamma$. The strata algebra $\mathcal{S}^*_{g,n}$ is graded by codimension

$$\mathcal{S}^*_{g,n} = \bigoplus_{d=0}^{3g-3+n} \mathcal{S}^d_{g,n}$$

and carries a product for which the natural push-forward map

$$\mathcal{S}^*_{g,n} \to A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \tag{29}$$

is a ring homomorphism, see [28, Section 0.3] for a detailed discussion.

The image of (29) is, by definition, the *tautological subring*

$$R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \ .$$

Hence, we have a quotient

$$\mathcal{S}^*_{g,n} \xrightarrow{q} R^*(\overline{\mathcal{M}}_{g,n}) \longrightarrow 0 \ .$$

The ideal of *tautological relations* is the kernel of q.

A.2. **Vertex, leg, and edge factors.** Pixton's relations are determined by a set

$$\mathcal{P} = \{\mathcal{R}^d_{g,A}\}$$

of elements $\mathcal{R}^d_{g,A} \in \mathcal{S}^d_{g,n}$ associated to the data

- $g, n \in \mathbb{Z}_{\geq 0}$ in the stable range $2g-2+n>0$,
- $A = (a_1, \ldots, a_n), \quad a_i \in \{0,1\}$,
- $d \in \mathbb{Z}_{\geq 0}$ satisfying $d > \frac{g-1+\sum_{i=1}^n a_i}{3}$.

The elements $\mathcal{R}^d_{g,A}$ are expressed as sums over stable graphs of genus g with n legs. Before writing the formula for $\mathcal{R}^d_{g,A}$, a few definitions are required.

The form of the hypergeometric series A and B used in Pixton's relations is following:

$$\mathsf{H}_0(T) = \mathsf{A}(-288T) = \sum_{i=0}^{\infty} \frac{(6i)!}{(2i)!(3i)!}(-T)^i = 1 - 60T + 27720T^2 - \cdots \ ,$$

$$\mathsf{H}_1(T) = -\mathsf{B}(-288T) = -\sum_{i=0}^{\infty} \frac{(6i)!}{(2i)!(3i)!}\frac{6i+1}{6i-1}(-T)^i = 1 + 84T - 32760T^2 + \cdots \ .$$

These series control the original Faber-Zagier relations and continue to play a central role in the set $\mathcal{P}$.

Let $f(T)$ be a power series with vanishing constant and linear terms,

$$f(T) \in T^2\mathbb{Q}[[T]] \ .$$

For each $\overline{\mathcal{M}}_{g,n}$, we define

$$\kappa(f) = \sum_{m\geq 0} \frac{1}{m!}\, p_{m*}\Big(f(\psi_{n+1})\cdots f(\psi_{n+m})\Big) \ \in A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}), \tag{30}$$

where p_m is the forgetful map

$$p_m : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}.$$

By the vanishing in degrees 0 and 1 of f, the sum (30) is finite.

Let $\mathsf{G}_{g,n}$ be the (finite) set of stable graphs of genus g with n legs (up to isomorphism). Let $\Gamma \in \mathsf{G}_{g,n}$. For each vertex $v \in \mathrm{V}$, we introduce an auxiliary variable ζ_v and impose the conditions

$$\zeta_v\zeta_{v'} = \zeta_{v'}\zeta_v \ , \quad \zeta_v^2 = 1 \ .$$

The variables ζ_v will be responsible for keeping track of a local parity condition at each vertex.

The formula for $\mathcal{R}^d_{g,A}$ is a sum over $\mathsf{G}_{g,n}$. The summand corresponding to $\Gamma \in \mathsf{G}_{g,n}$ is a product of vertex, leg, and edge factors:

- For $v \in \mathrm{V}$, let $\kappa_v = \kappa\big(T - T\mathsf{H}_0(\zeta_v T)\big)$.
- For $l \in \mathrm{L}$, let $\mathsf{H}_l = \zeta_{v(l)}^{a_l}\mathsf{H}_{a_l}\big(\zeta_{v(l)}\psi_l\big)$, where $v(l) \in V$ is the vertex to which the leg is assigned.
- For $e \in \mathrm{E}$, let

$$\begin{aligned}\Delta_e &= \frac{\zeta' + \zeta'' - \mathsf{H}_0(\zeta'\psi')\zeta''\mathsf{H}_1(\zeta''\psi'') - \zeta'\mathsf{H}_1(\zeta'\psi')\mathsf{H}_0(\zeta''\psi'')}{\psi' + \psi''} \\ &= (60\zeta'\zeta'' - 84) + [32760(\zeta'\psi' + \zeta''\psi'') - 27720(\zeta'\psi'' + \zeta''\psi')] + \cdots ,\end{aligned}$$

  where ζ', ζ'' are the ζ-variables assigned to the vertices adjacent to the edge e and ψ', ψ'' are the ψ-classes corresponding to the half-edges.

The numerator of Δ_e is divisible by its denominator due to the identity

$$\mathsf{H}_0(T)\mathsf{H}_1(-T) + \mathsf{H}_0(-T)\mathsf{H}_1(T) = 2.$$

Certainly, Δ_e is symmetric in the half-edges.

A.3. **Relations.** Let $A = (a_1, \dots, a_n) \in \{0,1\}^n$. We denote by $\mathcal{R}^d_{g,A} \in \mathcal{S}^d_{g,n}$ the degree d component of the strata algebra class

$$\sum_{\Gamma \in \mathsf{G}_{g,n}} \frac{1}{|\mathrm{Aut}(\Gamma)|}\,\frac{1}{2^{h^1(\Gamma)}}\,\left[\Gamma,\ \left[\prod \kappa_v \prod \mathsf{H}_l \prod \Delta_e\right]_{\prod_v \zeta_v^{\mathrm{g}(v)-1}}\right] \in \mathcal{S}_{g,n},$$

where the products are taken over all vertices, all legs, and all edges of the graph Γ. The subscript $\prod_v \zeta_v^{\mathrm{g}(v)-1}$ indicates the coefficient of the monomial $\prod_v \zeta_v^{\mathrm{g}(v)-1}$ after the product inside the brackets is expanded.

We denote by $\mathcal{P}$ the set of classes $\mathcal{R}^d_{g,d}$ for

$$d > \frac{g - 1 + \sum_{i=1}^n a_i}{3}.$$

By the following result, $\mathcal{P}$ is a set of tautological relations.

Theorem 4 (Janda)**.** *Every element $\mathcal{R}^d_{g,A} \in \mathcal{P}$ lies in the kernel of the homomorphism*

$$q : \mathcal{S}^*_{g,n} \to A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \ .$$

Pixton's relations were conjectured first in [31] and and first proven to hold in $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ in [28] using Witten's 3-spin theory and the Givental-Teleman classification of CohFTs. The Chow results of [16] are via a study of the equivariant Gromov-Witten theory of $\mathbb{CP}^1$. The application of the virtual localization formula [13] to the Gromov-Witten theory of $\mathbb{CP}^1$ bypasses the need for Teleman's cohomological results, and the proof of [16] holds in Chow.

Both Witten's 3-spin theory and the equivariant Gromov-Witten theory of $\mathbb{CP}^1$ are *Chow Field Theories*: the CohFT axioms are satisfied in the Chow rings $A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

Question 4. *Does the Givental-Teleman classification hold for semisimple Chow Field theories?*

The proof of [28] would imply the Chow vanishing of Theorem 4 if the answer to Question 4 is yes. The classification of CohFTs by Teleman uses the stable cohomology of the moduli spaces of curves. Are such stability results valid in the Chow ring?

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