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Sandpile Solitons via Smoothing of Superharmonic Functions

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Abstract: Let $F : \mathbb{Z}^2 \to \mathbb{Z}$ be the pointwise minimum of several linear functions. The theory of *smoothing* allows us to prove that under certain conditions there exists the pointwise minimal function among all integer-valued superharmonic functions coinciding with F "at infinity". We develop such a theory to prove existence of so-called *solitons* (or strings) in a sandpile model, studied by S. Caracciolo, G. Paoletti, and A. Sportiello. Thus we made a step towards understanding the phenomena of the identity in the sandpile group for planar domains where solitons appear according to experiments. We prove that sandpile states, defined using our smoothing procedure, move changeless when we apply the wave operator (that is why we call them solitons), and can interact, forming *triads* and *nodes*.

1. Introduction

Periodic patterns (Fig. 1, left) in sandpiles were studied by S. Caracciolo, G. Paoletti, and A. Sportiello in their pioneer work [2], see also Section 4.3 of [3] and Figure 3.1 in [23], Figure 9a in [28]. Experimental evidence suggests that these patterns appear in many sandpile pictures and carry a number of remarkable properties: in particular, they are self-reproducing under the action of waves. That is why we call these patterns *solitons*.

The fact that the solitons appear as "smoothings" of piece-wise linear functions was predicted by T. Sadhu and D. Dhar in [29]. We introduce a suitable definition of the *smoothing* procedure (Definition 2.5). We prove (for the first time) the existence (and uniqueness modulo translation) of solitons for all rational slopes (Theorem 1) and we prove that the "mass" (the total defect of the Laplacian, or the total difference from the maximal stable state) of the building block for the soliton of the direction (p, q) is p^2+q^2

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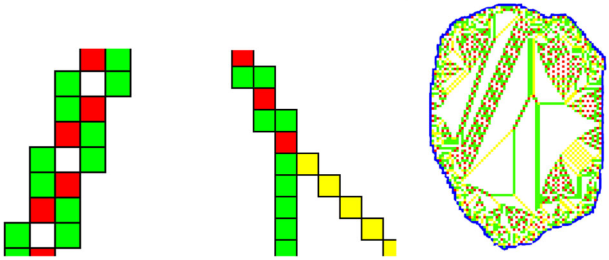

Fig. 1. These are local patterns for the soliton of direction $(1, 3)$ and the triad made by solitons of directions $(0, -1)$, $(1, -1)$, $(1, 2)$. White means three grains of sand, green—two, yellow—one, and red—zero. The rightmost picture (obtained using simulation in Golly [9]) shows the sandpile group identity for the graph inside the region with blue boundary. We see solitons of different directions near the center of the picture

(Remark 8.2). We also study local interactions of solitons: *triads* (Fig. 1, middle—three solitons meeting at a point) and *nodes* (Fig. 2). We prove that triads must satisfy a sort of balancing condition (Remark 2.15). We prove a well known (experimentally) fact that solitons move changeless under the action of waves (Corollary 2.13); triads and nodes satisfy similar property. In addition, we accurately write the theory of sandpiles on infinite domains in the absence of references, though it is absolutely parallel to the finite case. This article also contains the facts (Theorem 2, Corollary 2.13) that we need later to establish more general convergence results in sandpiles, see [14,15] for details and motivation. In short, sandpile dynamics of small perturbations of the maximal stable state is governed by a dynamic on tropical curves which also obeys power law [13].

The sandpile on $\mathbb{Z}^2$ exhibits a fractal structure; see, for example, the pictures of the identity element in the sandpile group [21] (and the rightmost picture in Fig. 1). As far as we know, only a few cases have a rigorous explanation. It was first observed in [22] that if we rescale by $\sqrt{n}$ the result of the relaxation of the state with n grains at $(0, 0)$ and zero grains elsewhere in $\mathbb{Z}^2$, it weakly converges as $n \to \infty$. Then this was studied in [20] and was finally proven in [24]. However the fractal-like pieces of the limit found their explanation later [18,19], and happen to be curiously related to Apollonian circle packing. Recently the stability of patterns was proven in [25]. In most of these fractal pictures one can find solitons which propagate along thin balanced graphs (which are called defects in [25]). We expect that the methods of this article will be used to study the fractal structure in the cases where the piece-wise linear nature of patterns is apparent (see many such examples in [27], [29], and a groundbreaking paper [4]).

We present our theory in the simplest meaningful case leaving possible generalisations (with unavoidably heavier notation) for future works.

1.1. Sandpile patterns on $\mathbb{Z}^2$. We think of $\mathbb{Z}^2$ as the vertices of the graph whose edges connect points with distance one. If $v, w \in \mathbb{Z}^2$ are neighbors we write $v \sim w$. A *state* of a sandpile is a function $\phi : \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$; we interpret $\phi(v)$ as the number of sand grains in $v \in \mathbb{Z}^2$. We can *topple* v by sending four grains from v to its neighbors, each neighbor gets one grain. If $\phi(v) \geq 4$, such a toppling is called *legal*. A *relaxation* is doing legal topplings while it is possible (for what this means for infinite graphs see Appendix A). A state ϕ is *stable* if $\phi \leq 3$ everywhere.

Definition 1.1. Let $v \in \mathbb{Z}^2$ be such that $\phi(v) = \phi(w) = 3$ where w is a neighbor of v. By *sending a wave* from v we mean making a toppling at v, following by the relaxation. We denote the obtained state by $W_v\phi$.

Note that after the first toppling the vertex v has -1 grain, and w has 4 grains, so w subsequently topples and v has a non-negative number of grains again. We are interested in states which move changeless under the action of waves, such states were previously studied experimentally in [2,3,23].

Definition 1.2. Let $(p, q) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. A state ϕ is called (p, q)-*movable*, if there exists v such that $W_v\phi(x, y) = \phi(x + p, y + q)$ for all (x, y). A state ϕ is called (p, q)-*periodic* if $\phi(x, y) = \phi(x + p, y + q)$. A state ϕ is called *line-shaped* if there exist constants p, q, c_1, c_2 such that the set $\{\phi \neq 3\}$ belongs to $\{(x, y)|c_1 \leq px + qy \leq c_2\}$.

We classify all **periodic line-shaped** movable states, we call them *solitons*. We also construct *triads*, i.e. three solitons meeting at a point, they are also movable.

In this paper we prove the following theorem.

Theorem 1 (See a proof in Sect. 8.1). *For each $p, q \in \mathbb{Z}$, $\gcd(p, q) = 1$ there exists a unique (up to a translation in $\mathbb{Z}^2$) movable (p, q)-periodic line-shaped state. Furthermore, it is (p', q')-movable, where $p', q' \in \mathbb{Z}$, $p'q - pq' = 1$.*

Moreover, a movable (p, q)-periodic line-shaped state is always $(\frac{p}{\gcd(p,q)}, \frac{q}{\gcd(p,q)})$-periodic which easily follows from our proof of this theorem (Corollary 8.5).

A state ϕ on $\mathbb{Z}^2$ is called a *background* if there exists $v \in \mathbb{Z}^2$ such that $W_v\phi = \phi$. For example, such is the state $\phi \equiv 3$ decreased at any set of vertices with pairwise distances at least two. Another example of a background is any recurrent state for a finite part of $\mathbb{Z}^2$, and 3 everywhere else. It seems to be more difficult to classify all **movable** states because many different backgrounds exist (see [23], Chapter 5). We give two conjectures, phrased in different terms.

Conjecture 1. *Any (p, q)-movable state is equal to a background plus the difference between 3 and an aforementioned soliton.*

Conjecture 2. *On a doubly periodic background, the periodic line-shaped movable states are classified by ordered pairs of tangent circles in the Apollonian packing of [19]. Given a quadratic function whose Hessian corresponds to the peak of the cone based at one such Apollonian circle, one constructs a doubly periodic background. Moving infinitesimally down the cone, in the direction of a tangent circle, should result in a line-shaped object, namely, a movable state.*

1.2. Superharmonic functions.

Definition 1.3. The *toppling function* of a relaxation is the function $\mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$ counting the number of topplings at every point during this relaxation.

It is known that the toppling function has bounded Laplacian and is minimal in a certain class of functions. Therefore, if we know the toppling function "at infinity", we can, in principle, reconstruct it. When we send n waves towards a periodic movable line-shaped state, the toppling function is zero on one side of the set $\{\phi \neq 3\}$ and is equal to n on another side. It is easy to guess (or find experimentally) that the toppling function in this case will be something like $F(x, y) = \min(px + qy, n)$ on one side of the set $\{\phi \neq 3\}$. Hence we are looking for a point-wise minimal superharmonic integer-valued function which coincide with $F(x, y)$ at infinity. However, *a priori* a pointwise minimum in such a class of functions can be $-\infty$ everywhere. We develop a theory of *smoothings* (see the definition in Sect. 2 where we state these results) to prove that the pointwise minimum is reached by slicing characteristic functions of certain sets from F, see Sect. 4. We also prove that a kind of monotonicity is preserved while doing this slicing, Sect. 5. In Sect. 6 we state well-knows facts about discrete harmonic functions. Sections 7, 8 are dedicated to a proof of Theorem 1.

To study the interaction between solitons, when several of them meet at a point, and for the needs of [14], we also study *triads* and *nodes*, to whom the last Sects. 9, 10, 11 are dedicated.

1.3. Sandpiles on infinite domains. We could not find a satisfactory reference containing the theory of sandpiles on infinite domains (in particular, the Least Action Principle for waves). We hesitated about its inclusion here, because all the statements can be proven exactly in the same way as in the finite case. Finally, for the sake of completeness, we decided to present the theory of locally-finite relaxations in Appendix A where we define and study locally-finite relaxations.

2. Smoothing, Its Relation to Waves

The discrete Laplacian Δ of a function $F : \mathbb{Z}^2 \to \mathbb{R}$ is defined as

$$\Delta F(x, y) = -4F(x, y) + F(x + 1, y) + F(x - 1, y) + F(x, y + 1) + F(x, y - 1).$$

A function F is called *harmonic* (resp., *superharmonic*) on $A \subset \mathbb{Z}^2$ if $\Delta F = 0$ (resp., $\Delta F \leq 0$) at every point in A.

Lemma 2.1. *If F, G are two superharmonic functions on $A \subset \mathbb{Z}^2$, then* $\min(F, G)$ *is a superharmonic function on A.*

Proof. Let $v \in A$. Without loss of generality, $F(v) \leq G(v)$. Then, $\Delta \min(F, G)(v) \leq \Delta F(v) \leq 0$. □

Definition 2.2. The *deviation set* $D(F)$ of a function F is the set of points where F is not harmonic, i.e.

$$D(F) = \{(x, y) \in \mathbb{Z}^2 | \Delta F(x, y) \neq 0\}.$$

Lemma 2.3. *Let $F : \mathbb{Z}^2 \to \mathbb{Z}$, $v \in \mathbb{Z}^2$, $F(v) \leq n$ and the Euclidean distance between v and the set*

$$\{\Delta F > 0\} = \{w | (\Delta F)(w) > 0\}$$

be at least $n + 2$. Let $v' \sim v$ and $F(v') > F(v)$. Then there exists a point $u \in \mathbb{Z}^2$ such that $F(u) < 0$.

Proof. Indeed, $(\Delta F)(v) \leq 0$ and $F(v') > F(v)$ imply that for some neighbor v_1 of $v = v_0$ we have $F(v_1) < F(v_0)$. Then we repeat this argument for v_1 and find its neighbor v_2 with $F(v_2) < F(v_1)$, etc. Note that all $v_0, \ldots, v_{n+1}$ does not belong to the set $\{\Delta F > 0\}$. Finally, we set $u = v_{n+1}$. We conclude by $F(v_{n+1}) \leq F(v_0) - (n + 1) \leq -1$. □

Lemma 2.4. *Let $F : \mathbb{Z}^2 \to \mathbb{Z}$, $v_0 \sim v_1 \sim \ldots \sim v_k$ be a path in $\mathbb{Z}^2$ and F be harmonic at all v_i, $0 \le i \le k-1$ and $\Delta F(v_k) < 0$. Then there exists $i \ge 0$ such that $F(v_0) = F(v_i)$ and v_i has a neighbor v' such that $F(v_i) > F(v')$.*

Proof. If $F(v_0) = F(v_k)$ then we may choose $i = k$ and such a neighbor exists since $(\Delta F)(v_k) < 0$. If not, choose the first i such that $F(v_0) = F(v_i) \neq F(v_{i+1})$ and then use the harmonicity of F at v_i. $\square$

For $A \subset \mathbb{Z}^2$, $\mathrm{C} > 0$, we denote by $B_{\mathrm{C}}(A) \subset \mathbb{Z}^2$ the set of points whose Euclidean distance to A is at most C.

Definition 2.5. For $n \in \mathbb{N}$ and a superharmonic function $F : \mathbb{Z}^2 \to \mathbb{Z}$ we define

$$\Theta_n(F) = \{G : \mathbb{Z}^2 \to \mathbb{Z} | \Delta G \le 0, F - n \le G \le F, \exists \mathrm{C} > 0, \{F \neq G\} \subset B_{\mathrm{C}}(D(F))\}.$$

In plain words, $\Theta_n(F)$ is the set of all integer-valued superharmonic functions $G \le F$, coinciding with F outside a finite neighborhood of $D(F)$, whose difference with F is at most n. Define $S_n(F) : \mathbb{Z}^2 \to \mathbb{Z}$ to be

$$S_n(F)(v) = \min\{G(v) | G \in \Theta_n(F)\}.$$

We call $S_n(F)$ *the n-smoothing of F*. Note that $S_n(F) \ge F - n$. Note that $S_n(F)$ does not necessarily belong to $\Theta_n(F)$.

Lemma 2.6. *If $F' \ge F$ for two superharmonic functions F, F', then $S_n(F') \ge S_n(F)$ for each $n \in \mathbb{Z}_{\ge 0}$.*

Proof. We automatically have $S_n(F') \ge S_n(F)$ on the set $\{F' - F \ge n\}$ and on the set $S_n(F') = F'$. Indeed, for each $G' \in \Theta_n(F')$, $G \in \Theta_n(F)$ we have $G' \ge G$ on these two sets. Thus we need to prove that $S_n(F') \ge S_n(F)$ on the set $\{F' - F < n\} \cap \{S_n(F') < F'\}$. Consider any function $G' \in \Theta_n(F')$, let the set $\{G' \neq F'\}$ belong to a C_0 neighborhood of $D(F')$.

Thus it is enough to prove that $G' \ge S_n(F)$ on the set

$$A_1 = \{F' - F < n\} \cap B_{C_0}(D(F')).$$

Consider the set

$$A_2 = \{F' - F < n\} \cap \{v | \exists v' \sim v, (F' - F)(v) > (F' - F)(v')\}.$$

Note that $F' - F$ is a superharmonic function outside of $D(F)$. It follows from Lemma 2.3 that A_2 belongs to the $(n+1)$-neighborhood of $D(F)$ since $F' - F \ge 0$.

Next we prove that $A_1 \subset B_{C_0}(A_2 \cup D(F))$. Indeed, for each point v in A_1 there exists a path of length at most C_0 to the set $D(F')$. If this path intersects $D(F)$, we are done. If not, then Lemma 2.4 asserts that for a v_i on this path for a certain $v' \sim v_i$, we have

$$n > (F' - F)(v) = (F' - F)(v_i) > (F' - F)(v')$$

and thus $v_i \in A_2$ and we proved that $A_1 \subset B_{C_0}(A_2 \cup D(F))$.

Summarising, we obtained that for each $G' \in \Theta_n(F')$, $G \in \Theta_n(F)$ we have $G' \ge G$ outside $B_{C_0}(A_2 \cup D(F)) \subset B_{C_0+n+1}(D(F))$. Thus, $\min(G, G')$ belongs to $\Theta_n(F)$, because it coincides with G outside a finite neighborhood of $D(F)$, it is superharmonic, and since $F' - G' \ge n$, $F - G \ge n$, $F' \ge F$ we have that $F - \min(G, G') \ge n$. $\square$

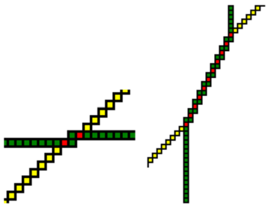

Fig. 2. Two examples of nodes. On the left we see $3 + \Delta\theta_F$ where F is $\Psi_{\text{node}} = \min(0, x - y, y, x - 1)$. On the right it is $3 + \Delta\theta_F$ with $F = \min(0, x, y - x, y - 10)$

Let us fix $p_1, p_2, q_1, q_2, c_1, c_2 \in \mathbb{Z}$ such that $p_1q_2 - p_2q_1 = 1$. Consider the following functions on $\mathbb{Z}^2$:

$$\Psi_{\text{edge}}(x, y) = \min(0, p_1x + q_1y), \tag{2.7}$$

$$\Psi_{\text{vertex}}(x, y) = \min(0, p_1x + q_1y, p_2x + q_2y + c_1), \tag{2.8}$$

$$\Psi_{\text{node}}(x, y) = \min\Big(0, p_1x + q_1y, p_2x + q_2y + c_1, (p_1 + p_2)x + (q_1 + q_2)y + c_2\Big). \tag{2.9}$$

These names correspond to the objects in the tropical world, see [14]. Namely, $\Psi_{\text{vertex}}, \Psi_{\text{edge}}, \Psi_{\text{node}}$ are local models for "smooth vertices", edges, and "nodes" of a tropical curve. To use more colorful names, the smoothing of Ψ_{vertex} is called a triad in this paper, to not confuse it with a vertex of the graph of a vertex of a polygon.

We prove the following theorem.

Theorem 2. *Let F be* (a) Ψ_{edge}, (b) Ψ_{vertex}, *or* (c) Ψ_{node}. *The sequence of n-smoothings $S_n(F)$ of F stabilises eventually as $n \to \infty$, i.e. there exists $N > 0$ such that $S_n(F) \equiv S_N(F)$ for all $n > N$. Moreover (it is not that obvious!), $S_N(F)$ coincides with F outside a finite neighborhood of $D(F)$.*

See a proof of (a) in Sect. 8 and a proof of (b,c) in Sect. 11. The problems to overcome in proofs are as follows: the deviation set $D(S_n(F))$ is infinite, and we need to prove that it "flows" only locally and can not significantly spread when we increase n. Then, even if the flow of $D(S_n(F))$ is restrained locally when n increases, the deviation locus, in principle, can encircle growing regions where $S_n(F)$ is harmonic almost everywhere. After taming these and other technicalities, the proof amounts to the fact that there exists no integer-valued linear function which is less than F on a non-empty compact set, because this linear function would correspond to a lattice point in the interior of the Newton polygon of F (the convex hull of linear parts (i, j) of functions in F), see Lemma 5.6.

Remark 2.10. A node (smoothing of Ψ_{node}) represents a deformation (controlled by calibrating c_2 in (2.9)) of two triads fusing together, see Fig. 2. From the "infinity" a node looks as two intersecting solitons, which explains the name: nodal points of tropical curves look exactly like that, see [15] for the details of this relation.

Definition 2.11. The pointwise minimal function in $\bigcup \Theta_n(F)$, which exists by Theorem 2, is called *the canonical smoothing of* F and is denoted by θ_F, see Fig. 2.

Remark 2.12. Note that $\Delta\theta_F \geq -3$ because otherwise we could decrease θ_F at a point violating this condition, preserving superharmonicity of θ_F, and this would contradict to the pointwise minimality of θ_F in $\bigcup \Theta_n(F)$.

Let F be Ψ_{edge}, Ψ_{vertex}, or Ψ_{node}, we write

$$F(x, y) = \min_{(i,j)\in A}(ix + jy + a_{ij}).$$

Consider the sandpile state $\phi = 3 + \Delta\theta_F$. By Remark 2.12, $\phi \geq 0$ and ϕ is a stable state because θ_F is superharmonic. Let $v \in \mathbb{Z}^2$ be a point far from $D(\theta_F)$. Let F be equal to $i_0x + j_0y + a_{i_0j_0}$ near v. The following corollary says, informally, that sending a wave from v increases the coefficient $a_{i_0j_0}$ by one.

Corollary 2.13. *In the above conditions, $W_v\phi = 3 + \Delta\theta_{F'}$ where W_v is the sending wave from v (Definition 1.1) and*

$$F'(x, y) = \min\big(i_0x + j_0y + a_{i_0j_0} + 1, \min(ix + jy + a_{ij}|(i, j) \in A, (i, j) \neq (i_0, j_0))\big).$$

Proof. Let H^v_ϕ (A.19) be the toppling function of the wave from v. Since

$$W_v\phi = \phi + \Delta H^v_\phi = 3 + \Delta(\theta_F + H^v_\phi),$$

we should prove that $H^v_\phi = \theta_{F'} - \theta_F$. It follows from Lemma 2.6 that $\theta_{F'} - \theta_F \geq 0$. By the Least Action Principle for waves (Proposition A.29) we have that $\theta_{F'} - \theta_F \geq H^v_\phi$ because $\theta_{F'} - \theta_F = 1$ at v, $\theta'_F - \theta_F \geq 0$ and $\phi + \Delta(\theta'_F - \theta_F) = 3 + \theta_{F'}$ is a stable state. On the other hand, the function $\theta_F + H^v_\phi$ coincides with $\theta_{F'}$ outside of a finite neighborhood of $D(F')$ and is superharmonic. Therefore, by the definition of $\theta_{F'}$, we see that $\theta_{F'} \leq \theta_F + H^v_\phi$ and this finishes the proof. □

Remark 2.14. As we will see later, all sandpile solitons are of the form $3 + \Delta\theta_{\Psi_{\text{edge}}}$.

Remark 2.15. Since a triad is a smoothing of a piece-wise linear function such as $\min(0, p_1x+q_1y, p_2x+q_2y+c_1)$ (up to adding a linear function), the direction of three solitons coming out of the center of the triad are (p_1, q_1), (p_2-p_1, q_2-q_1), $(-p_2, -q_2)$, and these directions sum up to $(0, 0)$, see Fig. 1, center.

3. Holeless Functions

We will frequently use the fact that the set $\{S_1(F) \neq F\}$ belongs to a finite neighborhood of $D(F)$ (in particular, this fact implies a pleasurable property $S_nS_k(F) = S_{n+k}(F)$). Unfortunately, this fact is not true for all superharmonic functions F, so we need to restrict the domain of functions F that we consider. Namely, we ask for the following technical property prohibiting to have arbitrary large holes in the deviation set.

Definition 3.1. We say that a function $F : \mathbb{Z}^2 \to \mathbb{Z}$ is *holeless* if there exists $\mathrm{C} > 0$ such that $B_{\mathrm{C}}(D(F))$ contains all the connected components of $\mathbb{Z}^2 \setminus D(F)$ which belong to some finite neighborhood of $D(F)$. When we want to specify the constant C we write that F is C-holeless.

Example 3.2. The functions $F = \Psi_{\text{edge}}, \Psi_{\text{vertex}}, \Psi_{\text{node}}$ (see (2.7), (2.9)) are holeless just because $\mathbb{Z}^2 \setminus D(F)$ has no components which belong to a finite neighborhood of $D(F)$.

Lemma 3.3. *If F is* C*-holeless, then for each $G \in \Theta_n(F)$ the set $\{F \neq G\}$ is contained in $B_{\max(n,\mathrm{C})}(D(F))$.*

Proof. Let $A_n = \{v \in \mathbb{Z}^2 | G(v) = F(v) - n\}$. If $v \in A_n \setminus D(F)$ then from the superharmonicity of G and harmonicity of F at v we deduce that all neighbors of v belong to A_n. Therefore the connected component of $v \in A_n$ in $\mathbb{Z}^2 \setminus D(F)$ belongs to A_n, which, in turn, belongs to a finite neighborhood of $D(F)$ because there belongs the set $\{F \neq G\}$. Thus A_n belongs to C-neighborhood of $D(F)$. By the same arguments, for $A_{n-1} = \{G = F - n + 1\}$, each point in $A_{n-1} \setminus D(F)$ is contained in the 1-neighborhood of $D(F) \cap A_n$ or, together with its connected component of $\mathbb{Z}^2 \setminus D(F)$ belongs to A_{n-1}, i.e. is contained in $B_{\mathrm{C}}(D(F))$, $A_{n-2} \setminus D(F)$ is contained in the 2-neighborhood of $D(F) \cap A_n$ or in 1-neighborhood of $D(F) \cap A_{n-1}$, or in $B_{\mathrm{C}}(D(F))$, etc. □

Corollary 3.4. *If F is* C*-holeless for some* $\mathrm{C} > 0$*, then for each $n \geq 0$ the function $S_n(F)$ belongs to $\Theta_n(F)$.*

Corollary 3.5. *Let F be one of* $\Psi_{\text{edge}}, \Psi_{\text{vertex}}, \Psi_{\text{node}}$ (*see* (2.7)*,* (2.8)*,* (2.9))*. Then for each $n \geq 1$ we have*

$$\operatorname{dist}\Big(D(F), \big\{F \neq S_n(F)\big\}\Big) \leq n,$$

where the distance is the minimum among the Euclidean distances between pairs of points $x \in D(F)$, $y \in \{F \neq S_n(F)\}$.

4. Smoothing by Steps

Let F, G be two superharmonic integer-valued functions on $\mathbb{Z}^2$. Suppose that $H = F - G$ is non-negative and bounded. Let m be the maximal value of H. Define the functions H_k, $k = 0, 1, \ldots, m$ as follows:

$$H_k(v) = \chi(H \geq k) = \begin{cases} 1, & \text{if } H(v) \geq k, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Lemma 4.2. *In the above settings, the function $F - H_m$ is superharmonic.*

Proof. Indeed, $F - H_m$ is superharmonic outside of the set $\{H = m\}$. Look at any point v such that $H(v) = m$. Then we conclude by

$$4(F - H_m)(v) = 4G(v) + 4(m-1) \geq \sum_{w \sim v} G(w) + 4(m-1) \geq \sum_{w \sim v} (F - H_m)(w).$$

□

We repeat this procedure for $F - H_m$; namely, consider $F - H_m - H_{m-1}$, $F - H_m - H_{m-1} - H_{m-2}$, etc. We have

$$H = H_m + H_{m-1} + H_{m-2} + \cdots + H_1,$$

and it follows from subsequent applications of Lemma 4.2 that all the functions $F - \sum_{n=m}^{m-k+1} H_n$ are superharmonic, for $k = 1, 2, \ldots, m$. Also, it is clear that

$$0 \leq \left(F - \sum_{n=m}^{m-k+1} H_n\right) - \left(F - \sum_{n=m}^{m-k} H_n\right) = H_{m-k} \leq 1$$

at all $v \in \Gamma$, $k = 0, \ldots, m$.

Consider a superharmonic function F. We are going to prove that two consecutive smoothings (see Definition 2.5) of F differ at most by one at every point of $\mathbb{Z}^2$.

Proposition 4.3. *For all $n \in \mathbb{N}$*

$$0 \leq S_n(F) - S_{n+1}(F) \leq 1.$$

Proof. By definition, $S_n(F) \geq S_{n+1}(F)$ at every point of $\mathbb{Z}^2$. If the inequality $S_n(F) - S_{n+1}(F) \leq 1$ doesn't hold, then the maximum M of the function $H = S_n(F) - S_{n+1}(F)$ is at least 2. We will prove that

$$S_n(F) - \chi(H \geq M) \geq F - n.$$

Namely, by Lemma 4.2 the function $S_n(F) - \chi(H \geq M)$ is superharmonic. Suppose that

$$S_n(F) - \chi(H \geq M) < F - n \quad \text{at a point} \quad v \in \mathbb{Z}^2.$$

Since the set $\{H \geq 1\}$ contains the set $\{H \geq M\}$, we arrive to a contradiction by saying that, at v,

$$F - (n+1) > S_n(F) - \chi(H \geq M) - \chi(H \geq 1) \geq S_{n+1}(F) \geq F - (n+1).$$

Therefore $S_n(F) - \chi(H \geq M) \in \Theta_n(F)$ which contradicts the minimality of $S_n(F)$.
□

Corollary 4.4. *Proposition 4.3 and Lemma 3.3 imply that for* C*-holeless F the function $S_{n+1}(F)$ can be characterized as the point-wise minimum of all superharmonic functions G such that $S_n(F) - 1 \leq G \leq S_n(F)$ and $S_n(F) - G$ vanishes outside some finite neighborhood of $D(S_n(F))$ (recall that the distance between $D(F)$, $D(S_n(F))$ is at most* $\max(\mathrm{C}, n)$*). In other words, n-smoothing $S_n(F)$ of F is the same as* 1*-smoothing of* $(n-1)$*-smoothing $S_{n-1}(F)$ of F.*

Corollary 4.5. *In the above assumptions, if $S_n(F) \neq S_{n+1}(F)$ then there exists v_0 such that $S_{n+1}(F)(v_0) = F(v_0) - (n+1)$.*

Indeed, if there is no such a point, then $S_{n+1}(F) \geq F - n$ and therefore $S_{n+1}(F) = S_n(F)$.

Remark 4.6. Let $F(x, y) = \min(x, y, 0)$ or $F(x, y) = \min(x, y, x+y, c)$ for $c \in \mathbb{Z}_{\geq 0}$. Then it is easy to check that $S_1(F)(x, y) = F(x, y)$ and therefore $\Theta_F = \{F\}$ (see Definition 2.5 for the notation).

5. Monotonicity While Smoothing

Definition 5.1. Let $e \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. We say that a function $F : \mathbb{Z}^2 \to \mathbb{Z}$ is *e-increasing* if

(a) F is a smoothing of a holeless function,
(b) $F(v) \leq F(v+e)$ holds for each $v \in \mathbb{Z}^2$,
(c) there exists a constant $\mathrm{C} > 0$ such that for each v with $F(v) = F(v-e)$, the first vertex $v - ke$ in the sequence $v, v-e, v-2e, \ldots$, satisfying $F(v-ke) < F(v-(k-1)e)$, belongs to $B_{\mathrm{C}}(D(F))$.

Example 5.2. Let $F(x, y) = \min(px + qy, 0)$ where $p, q \in \mathbb{Z}$. Note that F is (e_1, e_2)-increasing if and only if $pe_1 + qe_2 > 0$. In particular, F is $(0, 1)$-increasing if $q > 0$ and (e_1, e_2)-increasing if $p < 0 < q$ and $0 \leq e_1 \leq q-1$, $e_2 \geq |p|$.

Lemma 5.3. *If F is e-increasing, then $S_1(F)$, the 1-smoothing of F, is also e-increasing.*

Proof. Corollary 4.4 gives the property (a) of Definition 5.1, because if $F = S_n(G)$, and G is holeless, then $S_1(F) = S_{n+1}(G)$. To prove that $S_1(F)$ satisfies (b) in Definition 5.1 we argue *a contrario*. Let $H = F - S_1(F)$. Suppose that the set

$$A = \{v \in \mathbb{Z}^2 | F(v-e) = F(v), H(v-e) = 0, H(v) = 1\}$$

is not empty. Since $H|_A = 1$, we have $A \subset B_{\mathrm{C}}(D(G))$. Consider the set

$$B = \{v | H(v) = 0, \exists n \in \mathbb{Z}_{>0}, v + n \cdot e \in A, F(v) = F(v + n \cdot e)\}.$$

Consider $v \in B$. Since $v + n \cdot e \in A \subset B_C(D(F))$ and $F(v) = F(v + n \cdot e)$, then (c) in Definition 5.1 impose an absolute bound on n and therefore B belongs to a finite neighborhood of $D(F)$. Consider the following function

$$\tilde{F} = S_1(F) - \sum_{v \in B} \delta_v.$$

It is easy to verify that $\tilde{F}(v) \le \tilde{F}(v+e)$ for each v. Note that $\Delta\tilde{F}(v) \le \Delta S_1(F)(v) \le 0$ automatically for all $v \in \mathbb{Z}^2 \setminus B$. Pick any $v \in B$. Since $v + n \cdot e \in A$ for some $n \in \mathbb{Z}_{>0}$, we have

$$4\tilde{F}(v) = 4S_1(F)(v + n \cdot e) \ge \sum_{w \sim v} S_1(F)(w + n \cdot e) \ge \sum_{w \sim v} \tilde{F}(w).$$

Therefore $\tilde{F}$ is superharmonic, and satisfies $F \ge \tilde{F} \ge F - 1$ by construction, which contradicts to the minimality of $S_1(F)$ in $\Theta_1(F)$.

Finally, by Corollary 3.4 the sets $\{F \ne S_1(F)\}$ and $D(S_1(F))$ belong to $B_{\mathrm{C}}(D(F))$ for some $\mathrm{C} > 0$. Therefore the fact that $|S_1(F) - F| \le 1$ (Proposition 4.3) gives **c)** with the constant $\mathrm{C} + |e| + 1$. □

Corollary 5.4. *Let F be one of $\Psi_{\text{edge}}, \Psi_{\text{vertex}}, \Psi_{\text{node}}$ (Eqs. (2.7), (2.8), (2.9)). Let $e \in \mathbb{Z}^2 \setminus \{(0,0)\}$. If F is e-increasing, then $S_n(F)$ is also e-increasing.*

The following remark follows from the definition of smoothing.

Remark 5.5. Let $F : \mathbb{Z}^2 \to \mathbb{Z}$, $p, q, r \in \mathbb{Z}$. Let $G(x, y) = F(x, y) - px - qy - r$. Then $S_n(F)(x, y) - (px + qy + r) = S_n(G)(x, y)$.

Lemma 5.6. *Let F be Ψ_{vertex} or Ψ_{node}. Then there exists $k > 0$ such that for each $n > 0$ for each square S of size $k \times k$ inside the set $\{F \ne S_n(F)\}$ the function $S_n(F)$ is not a restriction of a linear function $m_1x + m_2y + m$, $m_1, m_2, m \in \mathbb{Z}$ on S.*

Proof. We associate each linear function $px + qy + c$ in F with the point $(p, q) \in \mathbb{Z}^2$. Then, a triangle of area $1/2$ is associated to Ψ_{vertex} and a parallelogram of area 1 is associated to Ψ_{node}. Pick any vertex of such a polygon (triangle or parallelogram) Δ. Calibrating as in Remark 5.5 we may suppose that this point is $(0, 0)$. Then, such a function F is e-monotone for all e in the dual cone for the cone at $(0, 0)$ in Δ (the dual cone is the set of vectors which have a non-negative scalar product with vectors from a given cone). By Corollary 5.4, $S_n(F)$ is also monotone in this direction. Therefore, if $S_n(F)$ is $m_1x + m_2y + m$ on S, the point (m_1, m_2) must belong to the cone at $(0, 0)$ in Δ. By doing that for each vertex of Δ we obtain that (m_1, m_2) belongs to Δ. To be able to do that we need to assume that k is big enough, namely, bigger than twice the length of each primitive vector from the edges of the dual cones considered above (this is a finite set of vectors).

Then, the fact that $(m_1, m_2) \in \Delta$ (and hence (m_1, m_2) is a vertex of Δ) contradicts the superharmonicity of $S_n(F)$. Namely, by Remark 5.5 we may assume that $(m_1, m_2) = (0, 0)$ and F is given as in (2.8) or (2.9). Therefore $S_n(F)$ is a constant $m < 0$ on S. Consider the set $\{S_n(F) = m\}$. There is a direction e such that F is e-increasing and for each $v \in \{S_n(F) = m\}$ we have that $v + le$ belongs to $\{S_n(F) = F = 0\}$ for some $l \in \mathbb{Z}_{>0}$. Then we go from the center of the set $\{S_n(F) = m\}$ in the direction e, and we find a vertex $v \in \{S_n(F) = m\}$ such that $S_n(F) \geq m$ at all neighbors of v and $S_n(F) > m$ at one of the neighbors, and this contradicts to the superharmonicity of $S_n(F)$. □

For Ψ_{edge} the similar result holds, see Lemma 7.4 (perhaps, with a more direct proof).

Remark 5.7. If the Newton polygon of F contains an integer point strictly inside, then a new bounded region will appear in $\mathbb{Z}^2 \setminus D(S_n(F))$, while smoothing. If the Newton polygon of F contains exactly k integer points in the interior of an edge, then the deviation set $D(F)$ in the dual direction will split into the $k+1$ solitons in this direction and we will see there k infinite regions in $\mathbb{Z}^2 \setminus D(S_n(F))$, while smoothing.

6. Discrete Superharmonic Integer-Valued Functions

Throughout the paper we denote all absolute constants by C, when we want to stress that C depends on other constants such as $k, \ldots$ we write it as $\mathrm{C}(k, \ldots)$ correspondingly. We will also omit writing "there is an absolute constant C with the following property...".

Lemma 6.1 ([5], Theorem 5). *Let $R > 1, v \in \mathbb{Z}^2$, and $F : B_R(v) \cap \mathbb{Z}^2 \to \mathbb{R}$ be a discrete* **non-negative** *harmonic function. Let $v' \sim v$, then*

$$|F(v') - F(v)| \leq \frac{\mathrm{C} \cdot \max_{w \in B_R(v)} F(w)}{R}.$$

Morally, this lemma provides an estimate on a derivative of a discrete harmonic function. We call $\partial_x F(x, y) = F(x+1, y) - F(x, y)$ the *discrete derivative* of F in the x-direction. The derivative ∂_y in the y-direction is defined in a similar way. We denote by $\partial_\bullet F$ the discrete derivative of a function F in any of directions x or y.

Lemma 6.2 (Integer-valued discrete harmonic functions of sublinear growth). *Let $v \in \mathbb{Z}^2$ and $\mu > 0$ be a constant. Let $R > 4\mu\mathrm{C}$. For a discrete* **integer-valued** *harmonic function $F : B_{3R}(v) \cap \mathbb{Z}^2 \to \mathbb{Z}$, the condition $|F(v')| \leq \mu R$ for all $v' \in B_{3R}(v)$ implies that F is linear in $B_R(v) \cap \mathbb{Z}^2$.*

Proof. Consider F which satisfies the hypothesis of the lemma. Note that $0 \leq F(v') + \mu R \leq 2\mu R$ for $v' \in B_{3R}(v)$ and applying Lemma 6.1 for $B_{2R}(v)$ yields

$$|\partial_\bullet F(v')| \leq \frac{\mathrm{C} \cdot 2\mu R}{R} = 2\mu\mathrm{C}, \quad \text{for all} \quad v' \in B_{2R}(v).$$

Then, applying it again for $0 \leq \partial_\bullet F(v') + 2\mu C \leq 4\mu C$ yields

$$|\partial_\bullet \partial_\bullet F(v')| \leq \frac{4\mu\mathrm{C}}{R} < 1, \quad \text{for} \quad v' \in B_R(v) \quad \text{if} \quad R > 4\mu\mathrm{C}.$$

Since F is integer-valued, all the derivatives $\partial_\bullet\partial_\bullet F$ are also integer-valued. Therefore all the second derivatives of u are identically zero in $B_R(v)$, which implies that F is linear in $B_R(v)$. $\square$

Let A be a finite subset of $\mathbb{Z}^2$, ∂A be the set of points in A which have neighbors in $\mathbb{Z}^2 \setminus A$. Let F be any function $A \to \mathbb{Z}$.

Lemma 6.3. *In the above hypothesis the following equality holds:*

$$\sum_{v \in A \setminus \partial A} \Delta F(v) = \sum_{\substack{v \in \partial A, \\ v' \in A \setminus \partial A, v \sim v'}} \big(F(v) - F(v')\big).$$

Proof. We develop left side by the definition of ΔF. All the terms $F(v)$, except for the vertices v near ∂A, cancel each other. So we conclude by a direct computation. $\square$

Definition 6.4. For $v \in \mathbb{Z}^2$ we denote by $G_v : \mathbb{Z}^2 \to \mathbb{R}$ the function with the following properties:

- $\Delta G_v(v) = 1$,
- $\Delta G_v(w) = 0$ if $w \neq v$,
- $G_v(v) = 0$,
- $G_v(w) = \frac{1}{2\pi} \log |w - v| + c + O\left(\frac{1}{|w-v|^2}\right)$ when $|w - v| \to \infty$, where c is some constant.

It is a classical fact that G_v does exist and is unique ([31], (15.12), or [17], p.104, see [8], Remark 2, for more terms in the Taylor expansion).

Corollary 6.5. *Let* $v = (0, 0)$. *By a direct calculation we conclude that*

$$|\partial_\bullet\partial_\bullet G_v(x, y)| \leq \frac{\mathrm{C}}{(x^2 + y^2 + 1)}.$$

Lemma 6.6. *The following inequality holds for all* $N \in \mathbb{Z}_{>0}$, $v \in \mathbb{Z}^2$*:*

$$\sum_{-N \leq x, y \leq N} |\partial_\bullet\partial_\bullet G_v(x, y)| \leq \mathrm{C} \ln N.$$

Proof. The maximum of this sum is attained when $v = (0, 0)$. Then the sum is estimated from above by

$$\int_{1 \leq x^2+y^2 \leq 2N^2} \frac{\mathrm{C}\, dxdy}{x^2 + y^2} + \mathrm{C} < \mathrm{C} \int_{r=1}^{2N} \frac{rdr}{r^2} + \mathrm{C} \leq \mathrm{C} \ln N.$$

$\square$

Lemma 6.7. *Let* $k, \mu \in \mathbb{N}$. *For all* $N > \mathrm{C}(k)\mu$ *the following holds. Let F be any non-negative integer-valued function on* $A = \big([0, N] \times [0, N]\big) \cap \mathbb{Z}^2$ *satisfying*

$$\max |F(v)| \leq \mu N.$$

Let $v_1, v_2, \ldots v_N$ *be points in* $\mathbb{Z}^2$ (*not necessary distinct*) *and suppose that* $G = F + \sum_{k=1}^N G_{v_k}$ (*see Definition 6.4*) *is a discrete harmonic function on A. Then there exists a square of size $k \times k$ in A such that F is linear on this square.*

Proof.

Applying Lemma 6.1 $text for\ v \in A' = \left[\frac{N}{5}, \frac{4N}{5}\right] \times \left[\frac{N}{5}, \frac{4N}{5}\right]$ we obtain $|\partial_\bullet G| \leq \frac{\mu C N}{N/5}$.

Proceeding as in Lemma 6.2, we see that in the square

$$A'' = \left[\frac{2N}{5}, \frac{3N}{5}\right] \times \left[\frac{2N}{5}, \frac{3N}{5}\right]$$

the second discrete derivatives $\partial_\bullet \partial_\bullet G$ are at most

$$\frac{C\mu}{N}$$

by the absolute value, which is less than $\frac{1}{2}$ if $N > C(k)\mu$.

Since $\sum_{w \in A} \partial_\bullet \partial_\bullet G_{v_k}(w)$ is at most $C \ln N$ (Lemma 6.6), we obtain by the direct calculation that

$$\sum_{k=1}^{N} \sum_{w \in A} \partial_\bullet \partial_\bullet G_{v_k}(w) \leq CN \ln N.$$

We cut A'' on $(\frac{N}{5k})^2$ squares of size $k \times k$. Therefore for $N > C(k)\mu$ we can find a square $A''' \subset A''$ of size $k \times k$ such that

$$\sum_{k=1}^{N} |\partial_\bullet \partial_\bullet G_{v_i}(v)| \leq 1/3 \text{ at every point } v \in A'''.$$

The estimates for $|\partial_\bullet \partial_\bullet G|$ and $\sum_{i=1}^{N} |\partial_\bullet \partial_\bullet G_{v_i}|$ imply that for all second derivatives of F we have $\partial_\bullet \partial_\bullet F(v) = 0$ for $v \in A'''$. Thus F is linear on A'''. □

7. Estimates on a Cylinder

Definition 7.1. Let $p, q \in \mathbb{Z}, q > 1$. We consider the equivalence relation $(x, y) \sim (x + q, y - p)$ on $\mathbb{Z}^2$, it respects the graph structure on $\mathbb{Z}^2$, so we define a new graph

$$\Sigma = \mathbb{Z}^2 / \sim, \quad \text{where} \quad \sim \text{ is generated by } (x, y) \sim (x + q, y - p).$$

We identify Σ with the strip $[0, q - 1] \times \mathbb{Z}$ where each vertex is connected with its neighbors and, additionally, $(0, y)$ is connected with $(q - 1, y - p)$ for all $i \in \mathbb{Z}$. The concept of discrete harmonic function easily descends to Σ.

Let G be an integer valued function on $\mathbb{Z}^2$ satisfying

$$G(x, y) = G(x + q, y - p) \quad \text{for all} \quad x, y \in \mathbb{Z} \text{ and fixed } p, q > 0.$$

The function G naturally descends to Σ. Let G be an integer valued superharmonic function on Σ. Suppose that $0 \leq G(x, y) \leq Cy$ for all $y > 0, x \in [0, q - 1]$. Suppose also that the number of points v with $\Delta G(v) < 0$ is finite and denote $\mathcal{D} = \sum_{v \in \Sigma} \Delta G(v) < 0$.

Lemma 7.2. *Let G be as above and $k > |p| + |q|$. Then, for some $m \leq C(k, |\mathcal{D}|)$ the function G is linear on*

$$\Sigma' = [0, q-1] \times [m, m+k] \subset \Sigma.$$

Proof. Choose big N. Dissect $[0, q-1] \times [0, N(|\mathcal{D}|+1)]$ on $|\mathcal{D}|+1$ parts

$$[0, q-1] \times [0, N]$$
$$[0, q-1] \times [N, 2N], \text{ etc.}$$

Then there exists a part A in this dissection where G is discrete harmonic. Note that

$$0 \leq G|_A \leq \mathrm{C} \cdot (|\mathcal{D}|+1)N.$$

Let v be the center of A. Applying Lemma 6.2 for v and $R = N/6$ we prove that derivatives $\partial_\bullet \partial_\bullet G$ are zeros in $B_{N/6}(v)$ if $N > \mathrm{C}$ and thus G is linear on $B_{N/6}(v)$. If $N/6 > 2k$ then we found desired $\Sigma' \subset B_{2k}(v)$. □

Lemma 7.3. *Let $F = \Psi_{\text{edge}}$ (see (2.7)). Then for all $n \in \mathbb{Z}_{>0}$ smoothings $S_n(F)$ are periodic in the direction $e = (q_1, -p_1)$, i.e. $S_n(F)(v) = S_n(F)(v+e)$ for all $v \in \mathbb{Z}^2$.*

Proof. Suppose, to the contrary, that $S_n(F)(v) > S_n(F)(v+e)$ for some $v \in \mathbb{Z}^2$. It follows from Lemma 2.1 that $\tilde{S}_n(F)(w) = \min(S_n(F)(w), S_n(F)(w+e))$ belongs to $\Theta_n(F)$, but $\tilde{S}_n(F)(v) < S_n(F)(v)$ which contradicts to the minimality of $S_n(F)$ in $\Theta_n(F)$. □

Lemma 7.4 (cf. Lemma 5.6). *Let Σ be from Definition 7.1, $F = \min(px+qy, 0)$, note that F descends to Σ. Let*

$$A \subset \{F \neq S_n(F)\}, A = [0, q-1] \times [m, m+|p|+|q|].$$

Suppose that $S_n(F)$ restricted to A is linear. Then $\gcd(p, q) > 1$.

Proof. Since $S_n(F)$ is periodic in the direction $(q, -p)$, we conclude that $S_n(F)(x, y)|_A = k(px+qy)+k'$ for some $k, k' \in \mathbb{Z}$. The property of $(0, 1)$-increasing implies that $k \geq 0$.

Suppose that $k = 0$, $S_n(F) = k'$ on A. Then $k' < 0$ because $S_n(F)|_A < F|_A$. Let y_0 be $\max\{y | S_n(F)(1, y) = k'\}$. Then $S_n(F)$ is not superharmonic at $(1, y)$, which is a contradiction. Therefore $k > 0$.

Consider the function $F'(x, y) = F(x, y) - px - qy$. Using Remark 5.5, we write

$$S_n(F')(x, y) = S_n(F)(x, y) - px - qy$$

and repeat verbatim all the above consideration, which gives $k < 1$.

Since $k(px+qy)$ has integer values and $0 < k < 1$ we conclude that $\gcd(p, q) > 1$. □

8. Proof of Theorem 2 for Ψ_{edge}

Proof. For the sake of notation denote $F = \Psi_{\text{edge}}$ (see (2.7)), set $p = p_1, q = q_1$, i.e. $F(x, y) = \min(0, px + qy)$. We will prove that the sequence $\{S_n(F)\}_{n=1}^{\infty}$ of n-smoothings (Definition 2.5) of F eventually stabilizes. It is easy to check that in the cases when $(p, q) = (\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)$ we have $S_1(F) = F$ (cf. Remark 4.6). Therefore, we conclude the proof of the theorem in this case by Corollary 4.4 (since $S_n(F) = S_1(F) = F$). From now on we suppose that $pq \neq 0, q > 1$ without loss of generality, and that the sequence $\{S_n(F)\}_{n=1}^{\infty}$ does not stabilize.

It follows from Lemma 7.3 that all $S_n(F)$ are periodic in the direction $(q, -p)$. Consider the quotient Σ of $\mathbb{Z}^2$ by translations by $(q, -p)$ (see Definition 7.1), Σ is a kind of infinite cylinder. Abusing notations, we think of $F, S_1(F), S_2(F), \ldots$ as functions on Σ. Note that $\mathcal{D} = \sum_{v \in \Sigma} \Delta F(v)$ is finite. Indeed, $\min(0, px + qy)$ has only finite number of points in Σ where the Laplacian is not zero.

Applying Lemma 6.3 for a big enough neighborhood of $D(F)$ we observe that $\sum_{v \in \Sigma} \Delta S_1(F)(v) = \mathcal{D}$. Similarly, we obtain $\sum_{v \in \Sigma} \Delta S_n(F)(v) = \mathcal{D}$ for all $n \in \mathbb{Z}_{>0}$ and because of superharmonicity of $S_n(F)$ we see that

$$|D(S_n(F))| = |\{v \in \Sigma | \Delta S_n(F)(v) \neq 0\}| \leq \mathcal{D}. \tag{8.1}$$

Since the sequence $\{S_n(F)\}_{n=1}^{\infty}$ does not stabilize, by Corollary 4.5 for each $n \in \mathbb{Z}_{>0}$ the set

$$A_n = \{v \in \mathbb{Z}^2 | S_n(F)(v) = F(v) - n\}$$

is not empty. Hence $A_1 \supset A_2 \supset A_3 \cdots$, and A_1 is finite because $A_1 \subset D(F)$ by Corollary 3.5. Thus we can take $v_0 \in \bigcap\limits_{n \geq 1} A_n$.

Note that F is $(0, 1)$-increasing and by Corollary 5.4 so do all $S_n(F)$. Also if $m, k \in \mathbb{Z}$ are such that $0 \leq m \leq q - 1, k > |p|$ then $pm + qk > 0$ and consequently all $S_n(F)$ are (m, k)-increasing (Corollary 5.2). The property of (m, k)-increasing gives that

$$F(v_0) - n = S_n(F)(v_0) \geq S_n(F)(v_0 - (m, k))$$

and $F(v_0) - n < F(v_0 - (m, k))$ for fixed (m, k) and $n > Ck$. Therefore $\operatorname{supp}(F - S_n(F))$ grows at least linearly in n.

For big n let

$$c = \min\{F(x, y) | (x, y) \in \operatorname{supp}(F - S_n(F))\}.$$

Applying Lemma 7.2 to the function $S_n(F) - c$ we note that $S_n(F)$ is linear on $A \subset \operatorname{supp}(F - S_n(F))$, is we choose $A = B_{q+|p|}(v)$ for some v. We conclude the proof because Lemma 7.4 implies that $\gcd(p, q) > 1$ which contradicts the definition of Ψ_{edge}. $\square$

Remark 8.2. The following equality holds in this case: $|\mathcal{D}| = p^2 + q^2$. In other words, the total defect of Laplacian (or the total difference with the state $\langle 3 \rangle$) on the building block of the soliton of the direction (p, q) is $p^2 + q^2$.

Proof. For convenience, consider a function $G(x, y) = \min(0, px - qy)$ and the lattice rectangle $R = [0, q] \times [0, p] \cap \mathbb{Z}^2$. Then

$$\mathcal{D} = \sum_{R \setminus (q, p)} \Delta G.$$

On the other hand, the sum of Laplacians over the rectangle R is reduced to the sum along its boundary (Lemma 6.3), i.e.

$$\sum_R \Delta G = \sum_{k=0}^{p}(G(0,k)-G(-1,k)) + \sum_{k=0}^{p}(G(q,k)-G(q+1,k)) \\ + \sum_{k=0}^{q}(G(k,0)-G(k,-1)) + \sum_{t=0}^{q}(G(k,p)-G(k,p+1)).$$

Since $\Delta G(q,p) = -p-q$ we have

$$-\mathcal{D} = -p-q-\sum_R \Delta G = -p-q+(p+1)p+(q+1)q.$$

□

This equality was observed earlier in [2]. Note also that p^2+q^2 is the symplectic area of an edge (p,q) in a tropical curve (see [15] for details).

Corollary 8.3. *Let* $p, p', q, q', a, a' \in \mathbb{Z}$*. Suppose that* $\gcd(p-p', q-q') = 1$*. Then there exists the canonical smoothing* $\theta_{p,q,a,p',q',a'}(x,y)$ *of* $F(x,y) = \min(px+qy+a, p'x+q'y+a')$*. Furthermore,*

$$\theta_{p,q,a,p',q',a'}(x,y) = \theta_{p-p',q-q',a-a',0,0,0}(x,y) \\ = \theta_{p-p',q-q',0,0,0,0}(x+(a-a')p'', y+(a-a')q'')$$

where $(p'',q'') \in \mathbb{Z}^2$ *satisfies* $(p-p')q''+(q-q')p'' = 1$.

Proof. The operation $f(x,y) \to f(x,y)+p'x+q'y+a'$ of adding a linear function commutes with n-smoothings and

$$\min\big((p-p')x+(q-q')y+(a-a'),0\big) \\ = \min\big((p-p')(x+(a-a')p'')+(q-q')(y+(a-a')q''),0\big).$$

□

8.1. Classification of solitons, proof of Theorem 1. Consider a movable line-shaped (p,q)-periodic state ϕ with $q > 0$. As in Sect. 7 we pass to the cylinder $\Sigma = \mathbb{Z}^2/\{(x,y) \sim (x+p, y+q)\}$. Line-shapedness of ϕ implies that $\{\phi \neq 3\} \subset \Sigma$ is contained in $[0,q-1]\times[-k,k]$ for some $k \in \mathbb{Z}$.

Lemma 8.4. *In the above setting, sending a wave from a point* $(x,y) \in \Sigma$, $x \in [0,q-1]$, $y >> 0$ *causes no topplings in the set* $\{y << 0\}$.

Proof. Suppose that there is a toppling in $\{y << 0\}$. Then the whole region $\{y < -k\}$ topples. Send n such waves where n is big enough. Then the toppling function F would be equal to n in $\{y >> 0\}$, $F = n$ in $\{y << 0\}$, $\Delta F \geq 0$ in $[0,q-1]\times[-k,k]$ and $\Delta F \leq 0$ on the set where the soliton is situated after n waves, let this be a subset of $[0,q-1]\times[\mathrm{C}n-k, \mathrm{C}n+k]$. Note that for each $v \in \Sigma$ we have $F(v) \geq n-\mathrm{C}$ because, in the above assumptions, during a wave a point does not topple only if it belongs to the deviation set of the current state, and the latter moves with some constant speed. Take a point $v_1 \in \Sigma$ with $\Delta F(v_1) < 0$. It has a neighbor v_2 with $F(v_2) < F(v_1)$ and then v_2 has a neighbor v_3 with $F(v_3) < F(v_2)$, etc. Since the distance between $\{\Delta F < 0\}$ and $\Delta F > 0$ is al least Cn we will find a point v with $F(v) < n - \mathrm{C} - 1$ which is a contradiction, cf. our proof of Lemma 2.6. □

Proof. It follows from the above lemma that the toppling function F for sending waves from a point (x, y), $y >> 0$ on ϕ is bounded from above by $\min(n, p'x + q'y + r)$ such that $pp' + qq' = \gcd(p, q)$ and $p'x + q'y + r > 0$ in $[0, q-1] \times [-k, +\infty]$. Therefore, as in the proof of Theorem 2 for Ψ_{edge} we obtain that F is linear in $[0, q-1] \times [k, k+\mathrm{C}]$ and therefore (repeating the arguments in Corollary 2.13) the soliton is $3 + \Delta\theta_{\min(p'x+q'y,0)}$ up to a translation. An application of Corollary 2.13 concludes the proof. □

Corollary 8.5. *Note that the formula* $3 + \Delta\theta_{\min(p'x+q'y,0)}$ *for the soliton depends only on the direction of* (p, q)*, therefore the soliton for* (p, q)*-periodic and for* $(\frac{p}{\gcd(p,q)}, \frac{q}{\gcd(p,q)})$*-periodic movable line-shaped states are the same.*

9. Reduction to a Smaller State

Seeking to prove the theorem for nodes and triads we apply the theorem for solitons along the "rays" of the deviation sets, and reduce the problem to a compact region. We use the notation of Theorem 2. Consider Ψ_{vertex}, (2.8). We denote by Ψ'_{vertex} the function

$$\begin{aligned}\Psi'_{\mathrm{vertex}}&(x, y)\\ &= \min\left(\theta_{\min(0, p_1x+q_1y)}(x, y), \theta_{\min(0, p_2x+q_2y+c_1)}(x, y), \theta_{\min(p_1x+q_1y, p_2x+q_2y+c_1)}(x, y)\right)\end{aligned} \tag{9.1}$$

Consider Ψ_{node}, (2.9). We denote by Ψ'_{node} the function

$$\Psi'_{\mathrm{node}}(x, y) = \min\left(\theta_{\min(0, p_1x+q_1y)}(x, y), \theta_{\min(0, p_2x+q_2y+c_1)}(x, y), \right. \tag{9.2}$$

$$\theta_{\min(p_1x+q_1y, (p_1+p_2)x+(q_1+q_2)y+c_2)}(x, y),$$

$$\left.\theta_{\min(p_2x+q_2y+c_1, (p_1+p_2)x+(q_1+q_2)y+c_2)}(x, y)\right). \tag{9.3}$$

Note that each of the functions $F = \Psi'_{\mathrm{vertex}}, \Psi'_{\mathrm{node}}$ is C-holeless for some C, because $D(\Psi'_{\mathrm{vertex}}), D(\Psi'_{\mathrm{node}})$ are periodic. Therefore by applying Corollary 3.4 we obtain the following remark.

Remark 9.4. Corollary 3.5 holds for $F = \Psi'_{\mathrm{vertex}}, \Psi'_{\mathrm{node}}$, if n is big enough.

Lemma 9.5. *Let F be Ψ_{vertex} (resp. Ψ_{node}) and F' be Ψ'_{vertex} (resp. Ψ'_{node}). The following conditions are equivalent:*

- *The sequence of n-smoothings $S_n(F)$ of F stabilizes.*
- *The sequence of n-smoothings $S_n(F')$ of F' stabilizes.*

Proof. It is enough to note that F' coincides with F outside of a finite neighborhood of $D(F)$ because we have already proven Theorem 2 for the case of Ψ_{edge}. Hence there exists n such that $|F - F'| < n$. Therefore $S_n(F) \leq F' \leq F$ and smoothings of F' can be estimated by smoothings of F and vice versa. □

We want to consider F' instead of F because of the following lemma.

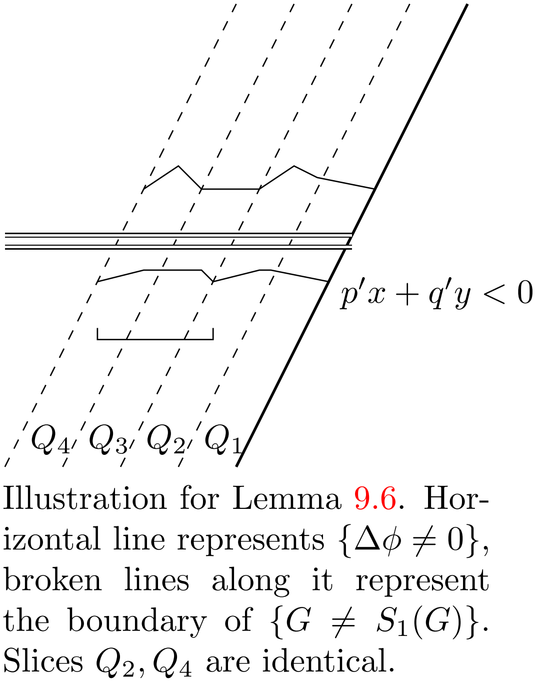

Fig. 3. Illustration for Lemma 9.6. Horizontal line represents $\{\Delta\phi \neq 0\}$, broken lines along it represent the boundary of $\{G \neq S_1(G)\}$. Slices Q_2, Q_4 are identical

Lemma 9.6. *Let $(p, q), (p', q')$ be primitive vectors such that $pq' - p'q = 1$. Denote $A = \{(x, y) | p'x + q'y \leq 0\}$. Let $G : \mathbb{Z}^2 \to \mathbb{Z}$ be equal to $\theta_{\min(0, px+qy)}$ in the region $p'x + q'y \geq 0$. Then there exists a constant* C *such that*

$$B_{\mathrm{C}}(A) \text{ contains the set } \{G \neq S_1(G)\} \setminus B_1(A).$$

Proof. We know that $\{G \neq S_1(G)\}$ is contained in the union of $B_1(A)$ and $B_1(\Delta G \neq 0)$. Therefore we need to prove that $\{G \neq S_1(G)\} \setminus B_1(A)$ (which is in 1-neighborhood of $\{\Delta G \neq 0\}$) can not be far from A. Suppose the contrary.

The function $G|_{D(G)}$ is periodic in the direction (p, q), so we can cut $D(G)$ into periodic pieces, see Fig. 3. We look at $G - S_1(G)$ on the periodic pieces and find two of them with the the same restriction of $G - S_1(G)$. Then it means that we could smooth more the initial function $\theta_{\min(0, px+qy)}$: indeed, take all the pieces in between of these two, repeat that all along as in Fig. 4, and decrease $\theta_{\min(0, px+qy)}$ according to $G - S_1(G)$ periodically. $\square$

Lemma 9.7. *For all $k \in \mathbb{Z}_{\geq 0}$ the cardinality of the set $\{F' \neq S_k(F')\}$ is finite.*

Proof. Note that $D(F')$ is made of solitons by definition of F'. Each time we apply 1-smoothing, the set $\{F' \neq S_n(F')\}$ belongs to a finite neighborhood of $D(F')$. Therefore we only need to prove that $\{S_n(F') \neq S_{n+1}(F')\}$ can not propagate far **along** a soliton, which is exactly the assertion in Lemma 9.6. $\square$

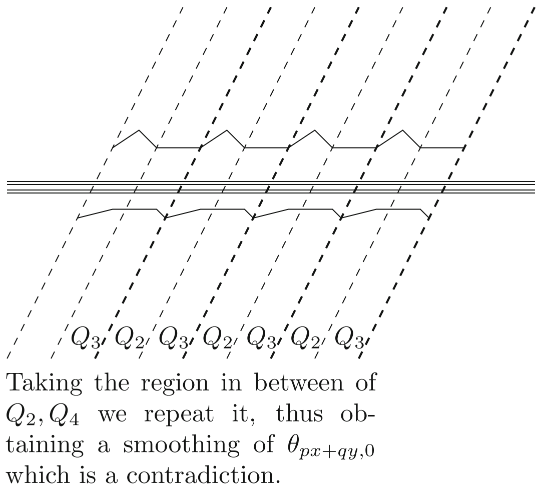

Fig. 4. Taking the region in between of Q_2, Q_4 we repeat it, thus obtaining a smoothing of $\theta_{px+qy,0}$ which is a contradiction

10. Growth of an Internal Harmonic Region

Definition 10.1. For a subset $A \subset \mathbb{Z}^2$ we define $r(A)$ as $\max_{(x,y)\in A}(\sqrt{x^2+y^2})$, i.e. the maximal distance between A and $(0, 0)$.

Lemma 10.2. *The sequence $R_n = r\Big(\{F' \neq S_n(F')\}\Big)$ grows at most linearly in n, i.e. $R_n \leq \mathrm{C}n$ for all $n \in \mathbb{Z}_{>0}$.*

Proof. It is enough to prove that $R_{n+1} \leq R_n + \mathrm{C}$ for all n. Now, look at how the support of $F' - S_{n+1}(F')$ differs from the support of $F' - S_n(F')$ outside of $B_{R_n}(O)$. It follows from Remark 9.4 that $\mathrm{supp}(F' - S_n(F'))$ belongs to the $n+1$-neighborhood of $D(F')$ for n big enough. Then we use Lemma 9.6 for each soliton-like ray in $D(F')$. $\square$

Lemma 10.3. *We suppose that the sequence $\{S_n(F)\}_{n=1}^{\infty}$ does not stabilize. Then the sequence*

$$r_n = \max\Big\{r|B_r(O) \subset \{S_n(F') \neq S_{n+1}(F')\}\Big\} \tag{10.4}$$

grows at least linearly in n, $r_n \geq \mathrm{C}n$ as long as n is big enough.

Proof. It follows from Lemma 9.5 and Corollary 4.5 that for each $n \in \mathbb{Z}_{>0}$ the set $A_n = \{v \in \mathbb{Z}^2 | S_n(F')(v) = F(v') - n\}$ is not empty. Hence $A_1 \supset A_2 \supset A_3 \cdots$, and A_1 is finite by Lemma 9.7. Thus we can take $v_0 \in \bigcap\limits_{n\geq 1} A_n$. Take any point $v \in \mathbb{Z}^2$. Consider the vector $e = v_0 - v$. By adding a suitable linear function to F we may suppose that F is e-increasing. Hence F' is e-increasing. Then $S_n(F')(v) \leq S_n(F')(v_0) = F'(v_0) - n$. There exists a constant $\mathrm{C}(F)$ (depending on the slopes of the linear parts of F) such that if $|v - v_0| < \mathrm{C}(F)n$ then $F'(v) > F'(v_0) - n$. For such v, the above formulae give $S_n(F')(v) < F'(v_0) - n < F'(v)$ which, with the fact that $|v_0|$ is a fixed finite number, concludes the lemma by taking any $\mathrm{C} > \mathrm{C}(F)$. $\square$

Lemma 10.5. *There exists a constant ρ such that the number of points v in $B_{n\mathrm{C}}(0, 0)$ with $\Delta S_n(F')(v) < 0$ is at most ρn for n big enough.*

Proof. The functions F', $S_n(F')$ coincide outside $B_{n\mathrm{C}}(O)$ and superharmonic, therefore it follows from Lemma 6.3 that

$$\sum_{v\in B_{n\mathrm{C}}(0,0)} |\Delta S_n(F')(v)| = \sum_{v\in B_{n\mathrm{C}}(0,0)} \Delta S_n(F')(v) = \sum_{v\in B_{n\mathrm{C}}(0,0)} \Delta F'(v).$$

Then, outside of a finite neighborhood of $(0, 0)$ the function $\Delta F'(v)$ coincide locally with $\Delta\theta_{\min(p'x+q'y,c)}$ in each direction, and $\sum_{v\in B_{n\mathrm{C}}(0,0)} \Delta\theta_{\min(p'x+q'y,c)}$ is linear in n for any coprime $p', q' \in \mathbb{Z}^2$. This works both for Ψ_{vertex} and Ψ_{node}. □

11. Proof of Theorem 2 for $F=\Psi_{\text{vertex}}$ and $F=\Psi_{\text{node}}$.

A geometric explanation of the proof is as follows. Since r_n (see (10.4)) grows linearly, the set $\{S_n(F') \neq F'\}$ encircles a figure with the area of order n^2. So, the set $\{\Delta S_n(F') \neq 0\}$ is of linear size, hence we can find a big part where $S_n(F')$ is harmonic and with at most linear growth, thus, it is linear (Lemma 6.7), which will contradict to Lemma 5.6.

Now we supply all the details. Suppose that the sequence $\{F_n\}$ of n-smoothings of F does not stabilize as $n \to \infty$. Therefore, by Lemma 9.5 the sequence of $\{S_n(F')\}$ of n-smoothings of F' does not stabilize. Lemma 9.7 asserts that the support of $F' - S_n(F')$ is finite, and Lemmata 10.2, 10.3 tell us that the set $\{F' \neq S_n(F')\}$ grows at most and at least linearly in n.

Remark 4.6 eliminated several simple cases. Note that, after change of coordinates $x \to \pm y$, $y \to \pm x$, if necessary, we may assume that F is $(0, 1)$-increasing (Definition 5.1) and the corner locus of F contains no vertical ray. Thus, F' and all $S_n(F')$ are $(0, 1)$-increasing by Corollary 5.4.

By Lemma 10.5 the number of points v in big disk $B_{n\mathrm{C}}(O)$ with $\Delta S_n(F')(v) < 0$ is bounded from above by ρn for some fixed ρ.

Consider $S_N(F')$ for N big enough. Then we choose C big enough and cut internal disk into squares with sides equal to $\frac{N}{\mathrm{C}}$ (later we refer to them as small squares).

Comparing the area of $\sim N^2$ of the internal disk with ρN we see that there exists a small square S which contains at most $\frac{N}{\mathrm{C}}$ points v with $\Delta S_N(F')(v) < 0$. On S, $S_N(F)$ is bounded from above and below by linear functions in N with coefficients depending on F. Then Lemma 6.7 implies that $S_n(F') - M$ should be linear on this small square $k \times k$ which is a subset of S. This, by Lemma 5.6, is a contradiction.

12. Discussion

12.1. Sand dynamic on tropical varieties, divisors. Let G be a graph and $V = \{v_1, v_2, \dots, v_k\}$ be a collection of some of its vertices. Consider the following state $\phi_V = \sum_{v\in G} v \cdot (\deg(v) - 1) - \sum_{v\in V} \delta_v$. It corresponds to the divisor $V = \sum_{v\in V} \delta_v$. Let $P = \{p_1, p_2, \dots, p_n\}$ be another collection of vertices of G. Then there exists a divisor linearly equivalent to V and containing P if and only if the relaxation of $\phi_V + \sum_{p\in P} \delta_p$ terminates.

We can produce the same type of problems for a tropical surface, if we have a sort of grid on it. For example, take a surface defined over $\mathbb{Z}$, its p-adic tropicalization has a natural grid of integral points. Tropical divisors in this case are tropical curves and we can model such curves by gluing solitons using triads and nodes.

Using relaxation in sand dynamic we can understand if there exist a divisor linearly equivalent to a given tropical divisor L, passing through prescribed set P of points

$p_1, p_2, \ldots, p_n$. For that we represent L as a collection of sand-solitons glued, then we add sand to P, and relax the obtained state. If the relaxation terminates, it produces the divisor which is linearly equivalent to L. If not, that means that such a divisor does not exist. Again, as in the one-dimensional case, this observation boils down to the question of existence of a piecewise-linear function F with $\{\Delta F > 0\} \subset L$ and $\{\Delta F < 0\} \supset P$.

12.2. Continuous models. It would be interesting to find a sandpile PDE which gives solitons in the limit. As we proved [14], straightforward passing to the limit gives piecewise linear model [15] and a tropical curve. We do not know how to argue why a priori the partial derivatives of the functions must be rational. We expect that there should be a family of deformations (e.g. such as in [30]) of the sandpile model, and the limits of deformations are the amoebas of algebraic curves, and amoebas tend to a tropical curve. If this is true, it must be a fruitful connection between sandpiles and algebraic geometry.

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Appendix A: Locally Finite Relaxations and Waves

In this section we study the relaxations and stabilizability issues. The main goal here is to establish The Least Action Principle (Proposition A.16, cf. [6]) and wave decomposition (Proposition A.26 and Corollary A.30) for **locally-finite** relaxations (Definition A.6) on infinite graphs. We also prove that given a finite upper bound on a toppling function of a state, there exists a relaxation sequence of this state which converges pointwise to a stable state (Lemma A.13).

The proofs are the same as in the finite case, but in the absence of references we give all the details here. Sandpiles on infinite graphs were previously considered, for example, in [1,7,11], but only from the distribution point of view: in their approach the relaxation (after adding a grain to a random configuration in a certain class) is locally finite almost sure with respect to a certain distribution. The ideas of this section are similar to [12].

A.1. The least action principle for locally finite relaxations, relaxability. Let Γ be a graph with at most countable set of vertices of finite degree, $\tau\colon \Gamma \to \mathbb{Z}_{>0}$ be a *threshold* function and $\gamma\colon \Gamma \to 2^{\Gamma}$ be a set-valued function such that

- $v \notin \gamma(v)$,
- if $v \in \gamma(w)$, then $w \in \gamma(v)$,
- $|\gamma(v)| \leq \tau(v)$ for all $v \in \Gamma$, where $|\gamma(v)|$ denotes the number of elements in the set $\gamma(v)$.

We interpret $\gamma(v)$ as the set of neighbors of a point $v \in \Gamma$. We write $u \sim v$ instead of $u \in \gamma(v)$, because γ induces a symmetric relation. The Laplacian Δ is the operator on the space $\mathbb{Z}^\Gamma = \{\phi\colon \Gamma \to \mathbb{Z}\}$ of *states* on Γ given by

$$\Delta\phi(v) = -\tau(v)\phi(v) + \sum_{u\sim v} \phi(u).$$

A function ϕ is called *superharmonic* if $\Delta\phi \leq 0$ everywhere.

Remark A.1. Note that $|\gamma(v)| \leq \tau(v)$ for all $v \in \Gamma$ holds if and only if the function $\phi \equiv 1$ is superharmonic.

Example A.2. In our main situation, Γ is a subset of $\mathbb{Z}^2$ and $|\gamma(v)| = \tau(v) = 4$ for all $v \in \Gamma \setminus \partial\Gamma$. In this case we obtain the standard definition of a Laplacian on $\Gamma \setminus \partial\Gamma$:

$$\Delta\phi(v) = -4\phi(v) + \sum_{u\sim v} \phi(u). \tag{A.3}$$

Definition A.4. For a point $v \in \Gamma$, we denote by T_v the *toppling* operator acting on the space of states $\mathbb{Z}^\Gamma$. It is given by

$$T_v\phi = \phi + \Delta\delta(v),$$

where $\delta(v)$ is the function on Γ taking the value 1 at v and vanishing elsewhere.

We can think that vertices v with $\tau(v) > \gamma(v)$ are connected with the stock vertex, so the system looses sand while performing topplings at such vertices.

Definition A.5. A *relaxation* $\phi_\bullet$ of a state $\phi \in \mathbb{Z}^\Gamma$ is a sequence of functions $\phi_\bullet = \{\phi_i\}_{i\in I}$, (for $I = \mathbb{Z}_{\geq 0}$ or $I = \{0, 1, \dots, n\}, n \in \mathbb{Z}_{\geq 0}$) such that $\phi_0 = \phi$ and for each $k \geq 0$ there exists $v_k \in \Gamma$ such that $\phi_k(v_k) \geq \tau(v_k)$ and $\phi_{k+1} = T_{v_k}\phi_k$. The *toppling function* $H_{\phi_\bullet}\colon \Gamma \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ of the relaxation $\phi_\bullet$ is given by

$$H_{\phi_\bullet} = \sum_{i\in I} \delta(v_i) \in \mathbb{Z}_{\geq 0} \cup \{+\infty\},$$

it counts the number of topplings at every point during this relaxation. We refer to $\{v_i\}_{i\in I}$ as a *relaxation sequence*.

Definition A.6. A relaxation $\phi_\bullet$ is called *locally-finite* if $H_{\phi_\bullet}(v)$ is finite for every $v \in \Gamma$. The *result of a locally-finite relaxation* is the state ϕ' given by the point-wise limit

$$\phi' = \phi_0 + \Delta H_{\phi_\bullet} = \lim_{k\to\infty} \phi_k.$$

Lemma A.7. *Consider a locally-finite relaxation $\phi_\bullet$ for a state ϕ and a function $F\colon \Gamma \to \mathbb{Z}_{\geq 0}$ such that $\phi + \Delta F < \tau$. Then $H_{\phi_\bullet}(v) \leq F(v)$ for all $v \in \Gamma$.*

Proof. We use the notation from Definition A.5. Consider the relaxation $\phi_\bullet$ and the corresponding sequence of functions H_n for $n = 1, \dots$ given by

$$H_n = \sum_{i=1}^{n} \delta(v_i), \tag{A.8}$$

where v_i are the points where topplings were made. Let $H_0 \equiv 0$.

It suffices to show that $H_n \le F$ for every n, and $H_0 \equiv 0 \le F$. Suppose that $n > 0$ and $H_{n-1} \le F$. Since $H_n = H_{n-1} + \delta(v_n)$, it is enough to show that $H_{n-1}(v_n) < F(v_n)$. We know that $\phi_n(v_n) \ge \tau(v_n)$ and $\phi_n(v_n) = \phi_0(v_n) + \Delta H_{n-1}(v_n)$. Therefore,

$$\begin{aligned}\tau(v_n) &\le \phi_0(v_n) - \tau(v_n)H_{n-1}(v_n) + \sum_{u\sim v_n} H_{n-1}(u)\\ &\le \phi_0(v_n) - \tau(v_n)H_{n-1}(v_n) + \sum_{u\sim v_n} F(u)\\ &= \phi_0(v_n) + \Delta F(v_n) + \tau(v_n)\big(F(v_n) - H_{n-1}(v_n)\big).\end{aligned}$$

Since $\phi_0(v_n) + \Delta F(v_n) < \tau(v_n)$ (by the hypothesis of the lemma) and $\tau(v_n) > 0$, we conclude that

$$1 \le F(v_n) - H_{n-1}(v_n).$$

□

Corollary A.9. *Consider a state ϕ. If there exists a function $F\colon \Gamma \to \mathbb{Z}_{\ge 0}$ such that $\phi + \Delta F < \tau$, then all relaxation sequences of ϕ are locally finite.*

Lemma A.10. *Consider a state ϕ and the set Ψ of all its relaxations $\psi_\bullet$. Then there exists a relaxation $\phi_\bullet$ of ϕ such that*

$$H_{\phi_\bullet}(v) = \sup_{\psi_\bullet\in\Psi} H_{\psi_\bullet}(v), \forall v \in \Gamma.$$

Proof. Consider the set $W = \{(v, k)\} \subset \Gamma \times \mathbb{Z}_{\ge 0}$ which contains all pairs (v, k) such that there exists a relaxation $\phi_\bullet^{v,k} \in \Psi$ which has k topplings at the vertex $v \in \Gamma$. Clearly, if $(v, k) \in W, k > 0$ then $(v, k-1) \in W$. The set W is at most countable, so we order it as $\{(v_n, k_n)\}_{n=1,2,\dots}$ in such a way that $(v, k-1)$ appears earlier than (v, k) for all $(v, k) \in W, k > 0$.
Take any relaxation $\phi_\bullet$. We construct relaxations $\phi_\bullet^0, \phi_\bullet^1, \dots$ in such a way that $\phi_\bullet = \phi_\bullet^0$, all $\phi_\bullet^{\ge n}$ coincide at first n topplings, and for each $n \ge 0$ the toppling function of $\phi_\bullet^n(v_n)$ is at least k_n.
Let $\phi_\bullet^{n-1}$ be already constructed, $n \ge 1$, we construct $\phi_\bullet^n$ as follows.
If the toppling function of $\phi_\bullet^{n-1}$ at v_n is at least k_n, we are done. If not, take $\phi_\bullet^{v_n,k_n}$ and consider its toppling functions $H^i_{\phi_\bullet^{v_n,k_n}}$ as in (A.8) except that we put the bottom index to the top. Take the first i such that there exists $w \in \Gamma$ such that $H_{\phi_\bullet^{n-1}}(w) < H^i_{\phi_\bullet^{v_n,k_n}}(w)$. Since it is the first such moment, for some j we have

$$H^j_{\phi_\bullet^{n-1}}(w') \ge H^i_{\phi_\bullet^{v_n,k_n}}(w')$$

for all $w' \sim w$. So we add to $\phi_\bullet^{n-1}$ the toppling at w somewhere after jth toppling, and denote the obtained relaxation sequence as $\phi_\bullet^{n-1}$ again. Note that by repeating this cycle of arguments a finite number of times, we will have that $\phi_\bullet^n(v_n) \ge k_n$. □

Definition A.11. A state ϕ is called *stable* if $\phi < \tau$ everywhere. A state ϕ is called *relaxable* if there exist a locally-finite relaxation $\phi_\bullet$ of ϕ such that ϕ'(Definition A.6) is stable. Such a relaxation $\phi_\bullet$ is called *stabilizing*.

Corollary A.12. *If ϕ is relaxable, then $H_{\phi_\bullet^1} = H_{\phi_\bullet^2}$ for any pair of stabilizing relaxations $\phi_\bullet^1$ and $\phi_\bullet^2$ of ϕ. In particular, $(\phi_\bullet^1)^\circ = (\phi_\bullet^2)^\circ$.*

Proof. Applying Lemma A.7 twice, we have $H_{\phi_\bullet^1} \le H_{\phi_\bullet^2}$ and $H_{\phi_\bullet^1} \ge H_{\phi_\bullet^2}$. □

Lemma A.13. *If all relaxations of a state ϕ are locally-finite, then ϕ is relaxable.*

Proof. Consider a point $v \in \Gamma$. We will prove that there exist $N > 0$ such that $H_{\phi_\bullet}(v) < N$ for all relaxations $\phi_\bullet$ of ϕ. Suppose the contrary. Then there exists a sequence of relaxations $\phi_\bullet^n$ such that $\lim_{n\to\infty} H_{\phi_\bullet^n}(v) = \infty$. Applying Lemma A.10 to the sequence $\phi_\bullet^n$ we see that there exists a relaxation of ϕ, that is not locally-finite.
Therefore, for any $v \in \Gamma$ there exist a relaxation $\phi_\bullet^v$ such that $H_{\phi_\bullet}(v) \le H_{\phi_\bullet^v}(v)$ for all relaxations $\phi_\bullet$ of ϕ. Applying Lemma A.10 again to the family of relaxations $\{\phi_\bullet^v\}_{v\in\Gamma}$ we find a relaxation sequence $\tilde{\phi}_\bullet$ such that $H_{\phi_\bullet}(v) \le H_{\tilde{\phi}_\bullet}(v)$ for all relaxations $\phi_\bullet$.
We claim that $\tilde{\phi}_\bullet$ is a stabilizing relaxation. Suppose that $\phi + H_{\tilde{\phi}_\bullet}$ is not stable, i.e. there exists $v \in \Gamma$ such that $\phi(v) + H_{\tilde{\phi}_\bullet}(v) \ge \tau(v)$. Therefore, we can make an additional toppling at v after the moment when all the topplings at v and its neighbors in $\tilde{\phi}_\bullet$ are already made. This contradicts to the maximality of $\tilde{\phi}_\bullet$. □

Proposition A.14. *A state ϕ is relaxable if and only if there exists a function $F\colon \Gamma \to \mathbb{Z}_{\ge 0}$ such that $\phi + \Delta F < \tau$.*

Proof. If ϕ is relaxable then we can take F to be H_ϕ. On the other hand, if such F exists, then by Lemma A.7 all the relaxations of ϕ are locally-finite. Therefore, ϕ is relaxable by Lemma A.13. □

Definition A.15. Consider a relaxable state ϕ. Denote by H_ϕ *the toppling function of* ϕ, where H_ϕ is a toppling function of some stabilizing relaxation of ϕ. Define *the relaxation of* ϕ to be the state $\phi^\circ = \phi + \Delta H_\phi$.

Proposition A.16. (The Least Action Principle, [6]) *Let ϕ be a relaxable state and $F\colon \Gamma \to \mathbb{Z}_{\ge 0}$ be a function such that $\phi + \Delta F$ is stable. Then $H_\phi \le F$. In particular, H_ϕ is the* **pointwise** *minimum of all such functions F.*

Proof. Straightforward by Lemma A.7. □

Lemma A.17. *Consider a stable state ϕ and a point $v \in \Gamma$. Then the state $T_v\phi$ is relaxable.*

Proof. Consider a function $F(z) = 1 - \delta(v)$ for every $z \in \Gamma$. Then $T_v\phi + \Delta F = \phi + \Delta\delta(v) + \Delta(1 - \Delta\delta(v)) = \phi + \Delta 1$. Applying Remark A.1 we see that $T_v\phi + \Delta F$ is stable. Thus, $T_v\phi$ is relaxable by Proposition A.14. □

A.2. Waves, their action. Sandpile waves were introduced in [10], see also [16].

Definition A.18. Let v be a point in Γ. The *wave* operator W_v, acting on the space of the stable states on Γ, is given by

$$W_v\phi = (T_v\phi)^\circ.$$

The *wave-toppling* function H_ϕ^v of ϕ at v is given by

$$H_\phi^v = \delta(v) + H_{T_v\phi}. \tag{A.19}$$

Remark A.20. Note that if v has $\tau(v)-1$ grains and has a neighbor w with $\tau(w)-1$ grains, then the result $W_v\phi$ is non-negative everywhere.

Indeed, $T_v\phi$ has -1 grain at v, but w has enough grains and will topple. So, eventually, we will have non-negative amount of sand at v.

Remark A.21. It is clear that $W_v\phi = \phi + \Delta H^v_\phi$.

Corollary A.22 ([26]). *For any $u \in \Gamma$ the value $H^v_\phi(u)$ is either* 0 *or* 1. *Furthermore,* $H^v_\phi(v) = 1$.

Proof. It follows from the proof of Lemma A.17 that $H_{T_v\phi} \le 1 - \delta(v)$. □

Lemma A.23. *Suppose that ϕ is a stable state and v a point in Γ. If $\phi+\delta(v)$ is relaxable and not stable, then the toppling function for the wave from v is less or equal than the toppling function for a relaxation of $\phi+\delta(v)$, i.e.*

$$H^v_\phi(w) \le H_{\phi+\delta(v)}(w)\ ,\ \forall w \in \Gamma.$$

Proof. It is clear that $(\phi+\delta(v))(w) = \phi(w) < \tau(w)$ for all $w \neq v$ and $(\phi+\delta(v))(v) = \tau(v)$. Therefore, T_v is the first toppling in any non-trivial relaxation sequence for $\phi+\delta(v)$ and $H_{\phi+\delta(v)}(v) \ge 1$. In particular, the function $H_{\phi+\delta(v)} - \delta(v)$ is non-negative and $H_{T_v\phi} \le H_{\phi+\delta(v)} - \delta(v)$ by Lemma A.7 since

$$\begin{aligned} T_v\phi + \Delta\big(H_{\phi+\delta(v)} - \delta(v)\big) &= \phi + \Delta\delta(v) + \Delta\big(H_{\phi+\delta(v)} - \delta(v)\big) = \phi + \Delta H_{\phi+\delta(v)} \\ &= \big(\phi+\delta(v)\big)^\circ - \delta(v) < \tau. \end{aligned}$$

□

Definition A.24. Let ϕ be a relaxable state, H_ϕ be its toppling function. Let $0 \le F \le H_\phi$. The state $\phi + \Delta F$ is called a *partial relaxation* of ϕ.

Lemma A.25. *Consider a relaxable state ϕ and an integer-valued function F on Γ such that $0 \le F \le H_\phi$. Then the state $\phi+\Delta F$ is relaxable and*

$$H_{\phi+\Delta F} = H_\phi - F.$$

Proof. By Proposition A.14 the state $\phi+\Delta F$ is relaxable because

$$\phi + \Delta F + \Delta(H_\phi - F) = \phi + \Delta H_\phi = \phi^\circ < \tau$$

and $H_\phi - F$ is non-negative. In particular, $H_\phi - F \ge H_{\phi+\Delta F}$ by Lemma A.7. On the other hand, since $H_{\phi+\Delta F} + F \ge 0$, we have

$$\phi + \Delta(H_{\phi+\Delta F} + F) = \phi + \Delta F + \Delta H_{\phi+\Delta F} = (\phi+\Delta F)^\circ < \tau.$$

Applying again Lemma A.7, we have $H_\phi \le H_{\phi+\Delta F} + F$. □

Proposition A.26. *Let ϕ be a stable state and v be a point in Γ. Suppose that $\phi+\delta(v)$ is relaxable. Then the relaxation of $\phi+\delta(v)$ can be decomposed into sending n waves from v, i.e.*

$$(\phi+\delta(v))^\circ = \delta(v) + W^n_v\phi,$$

where $n = H_{\phi+\delta(v)}(v)$ and $W^n_v(\phi) = W_v(W_v(\ldots(\phi))\ldots)$, nth power of W_v. On the level of toppling functions, this gives

$$H_{\phi+\delta(v)} = \sum_{k=0}^{n-1} H^v_{(W^k_v\phi)}.$$

Added parenthesis in the subscript are for the better readability only.

Proof. Combining Lemmata A.23 and A.25 we have

$$H_{\phi+\delta(v)} = H^v_\phi + H_{(W_v\phi+\delta(v))}.$$

If the state $W_v\phi + \delta(v)$ is not stable, then we can apply the same lemmata again. We complete the proof by iteration of this procedure and using Corollary A.22 (each wave has one toppling at v, therefore we have n waves). □

Lemma A.27. *If ϕ is a stable state and $v_1, \dots, v_m$ are vertices of Γ such that v_i is adjacent to v_{i+1} and $\phi(v_i) = \tau(v_i) - 1$ for all $i = 1, 2, \dots, m$, then $H^{v_1}_\phi = H^{v_m}_\phi$.*

Proof. It follows from the simplest case $m = 2$, for which it is just a computation. □

Definition A.28. In a given state ϕ, a *territory* is a maximal by inclusion connected component of the vertices v such that $\phi(v) = \tau(v) - 1$. Given a territory $\mathcal{T}$, we denote by $W_\mathcal{T}$ the wave which is sent from a point in $\mathcal{T}$ (by Lemma A.27 it does not matter from which one).

Basically, Corollary A.22 tells us that a wave from v increases the toppling function exactly by one in the territory to which v belongs to, and by at most one in all other vertices.

Proposition A.29. *Let ϕ be a stable stable, v be a point in Γ, and $F : \Gamma \to \mathbb{Z}_{\geq 0}$ be a function such that $F(v) \geq 1$ and $\phi + \Delta F$ is stable. Then $F \geq H^v_\phi$.*

Proof. Similar to Lemma A.7. □

Corollary A.30. (Least Action Principle for waves, cf. [6]) *Suppose that a state ϕ is stable. We send n waves from a vertex v. Let $H = \sum_{k=0}^{n-1} H^v_{(W^k\phi)}$ be the toppling function of this process. Let F be a function such that $\phi + \Delta F \geq 0$, $F(w) \geq 0$ for all w, and $F(v) \geq n$. Then $F(w) \geq H(w)$ for all w.*

Proof. We apply Proposition A.29 n times, each time decreasing F by $H^v_{W^k(\phi)}$ for $k = 0, 1, \dots, n-1$. □

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