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NONAUTONOMOUS GRADIENT-LIKE VECTOR FIELDS ON THE CIRCLE: CLASSIFICATION, STRUCTURAL STABILITY AND AUTONOMIZATION

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To Jürgen, with the gratitude and best wishes

Abstract. We study a class of scalar differential equations on the circle S^1. This class is characterized mainly by the property that any solution of such an equation possesses an exponential dichotomy both on the semi-axes $\mathbb{R}_+$ and $\mathbb{R}_+$. Also we impose some other assumptions on the structure of the foliation into integral curves for such the equation. Differential equations of this class are called gradient-like ones. As a result, we describe the global behavior of a foliation, introduce a complete invariant of the uniform equivalency, give standard models for the equations of this distinguished class. The case of almost periodic gradient-like equations is also studied, their classification is presented.

1. Introduction.

In 1973 a short note [21] was published where for nonautonomous vector fields (NVFs) given on a smooth closed manifold M a definition of uniform equivalency of two such NVFs was given and on this basis the structural stability of nonautonomous vector fields was defined. When $\dim M = 2$ a class of structurally stable nonautonomous vector fields was distinguished, the invariant determining the uniform equivalence was found, Morse type inequalities were derived connecting the topology of M and the structure of a foliation into integral curves. When these results were announced, articles on nonautonomous nonlinear dynamics were rather rare (except, of course, periodic in time systems), though they existed, see, for instance, [26, 37]. Now the interest to the nonautonomous dynamics became much more intensive, several recent books demonstrate this [16, 5, 34]. This made us return to this topic.

In this paper we present proofs of some statements of [21] for the one-dimensional situation, that is when M is the circle S^1. It is worth noting that the one-dimensional case was not discussed at all in [21].

We study a class of nonautonomous scalar differential equations of the type

$$\dot{x} = f(t, x),$$

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where a scalar function f is 1-periodic in x and defines an uniformly continuous map of $\mathbb{R}$ to the space $V^r(S^1)$ of C^rsmooth 1-periodic functions in x, $r \geq 1$. This space is endowed with C^r-norm, thus $V^r(S^1)$ becomes the Banach space. The space of maps $\mathbb{R} \to V^r(S^1)$ becomes itself a Banach space if we endow it with the norm $|v| = \sup_{t\in\mathbb{R}} ||v(t)||$ where $||\cdot||$ means the norm in $V^r(S^1)$.

An ordinary differential equation (ODE) determines a foliation of the extended phase space $\mathbb{R} \times S^1$ into integral curves (graphs of solutions). Henceforth, we consider the manifold $\mathbb{R} \times S^1$ with its uniform structure of the direct product of the standard uniform structure on $\mathbb{R}$ given by the metrics $|x - y|$ and metrics on $S^1 = \mathbb{R}/\mathbb{Z}$ induced from $\mathbb{R}$.

Two such ODEs are called *uniformly equivalent* if their related foliations are equimorphic (see below in more details). We present below four assumptions on the class of ODEs under consideration. We call the equation gradient-like if these assumptions are satisfied. For equations of this class we prove the following properties:

- A gradient-like ODE is structurally stable w.r.t. perturbations within the class of uniformly continuous maps $\mathbb{R} \to V^r(S^1)$;
- ODEs of this class are rough, i.e. a conjugating equimorphism can be chosen close to the identity map $id_{\mathbb{R}\times S^1}$ if a perturbation is small enough;
- A combinatorial type invariant is introduced which is the complete invariant of the uniform conjugacy for the gradient-like ODEs;
- for each gradient-like ODE there is an uniformly equivalent asymptotically autonomous ODE on S^1;
- if a gradient-like ODE is in addition almost periodic in t uniformly w.r.t. x, then this ODE is uniformly equivalent to an autonomous scalar ODE with simple zeroes.

Similar results can be also formulated and proved for the case of $M = I$, a segment, as well, in this case the boundary curves $\mathbb{R}\times\partial I$ are supposed to be integral curves of the differential equation to avoid some complications. The exposition in the paper is carried out for the case S^1.

The beginning of the study of nonautonomous vector fields from the viewpoint of their roughness and their structure was initiated by L. Shilnikov, our joint note [21] was the first result in this direction. The idea itself on the necessity of extending the notion of roughness (structural stability) [1] onto nonautonomous vector fields goes back to A.A. Andronov and was publicized by his wife and collaborator E.A. Leontovich-Andronova in her talk at the III All-Union USSR Mathematical Congress [17].

The first problem here was to find a proper equivalency relation that could be served for a basis of the classification for nonautonomous vector fields. By that time there existed mainly two approaches to the study of nonautonomous systems. The first traced back to Bebutov and his translation dynamical systems in the space of bounded continuous functions (see [29]). This approach was later transformed into the theory of skew product systems [37]. The second approach was the study of a nonautonomous system itself but the problem on the equivalency of two such systems and their distinguishing invariants was not formulated at that time.

2. **Nonautonomous vector fields.**

Let M be a C^∞-smooth closed manifold and $\mathcal{V}^r(M)$ be the Banach space of C^r-smooth vector fields on M endowed with

C^r-norm. A C^r*-smooth nonautonomous vector field* on M (below NVF, for brevity) is an uniformly continuous bounded map $v : \mathbb{R} \to \mathcal{V}^r(M)$. If this map v is also C^s-differentiable map, whose derivatives up to order s are uniformly continuous, we call v to be a $C^{r,s}$-smooth nonautonomous vector field. Every nonautonomous vector field v defines its solutions $x_{\tau,x_0} : \mathbb{R} \to M$, $(\tau, x_0) \in \mathbb{R} \times M$, being C^1 maps if $s = 0$, and C^{s+1} maps, if $s > 0$. As a mapping from $\mathbb{R} \times \mathbb{R} \times M \to M$, $(t; \tau, x_0) \to x(t; \tau, x_0) \in M$, this mapping is of $C^{min\{r,s\}}$. Solutions of the vector field generate a foliation F of the manifold $\mathbb{R} \times M$ (extended phase space) into its integral curves $\cup_t(x(t), t)$. Henceforth we consider the manifold $\mathbb{R} \times M$ with its standard uniform structure (see, for instance [15]). All uniformly continuous maps of $\mathbb{R} \times M$ to itself are considered with respect to this uniform structure. Recall that a homeomorphism $h : M \to M$ of a uniform space is called an equimorphism, if both mappings h, h^{-1} are uniformly continuous.

An equivalence relation for nonautonomous vector fields proposed in [21] is as follows.

Definition 2.1. Two NVFs v_1, v_1 are **uniformly equivalent** if foliations F_1, F_2 are uniformly equivalent, that is there is an equimorphism $h : \mathbb{R} \times M \to \mathbb{R} \times M$ respecting foliations (i.e. sending every integral curve γ of F_1 to an integral curve of F_2 preserving its orientation in $\mathbb{R}$).

It is clear that this equivalency relation distinguishes NVFs in which the asymptotic behavior of integral curves are different. This relation allows us to introduce the notion of structurally stable NVFs.

Definition 2.2. An NVF v is called **structurally stable**, if there is a neighborhood $\mathcal{U}$ of v in the space NV such that all NVFs in this neighborhood are uniformly equivalent.

The development of the modern theory of (autonomous) dynamical systems showed that the topological equivalency relation is in fact too rigid to get a classification of multidimensional dynamical systems: the structure of such a system can be extremely complicated [9, 10] due, in particular, to the phenomena like Newhouse ones [30, 31, 11].

Nonetheless, this equivalency relation is good enough to classify relatively simple (Morse-Smale) systems, though even in this case the classifying invariant become comparable in its complexity with the structure of a systems itself [13]. This is the achievements of the recent time, at times when this theory started, it was not known.

3. **One dimensional gradient-like NVFs.** From now on in this paper we consider NVFs on the unique closed smooth 1-dimensional manifold, the circle $S^1 = \mathbb{R}/\mathbb{Z}$. We formulate first the restrictions on NVFs which will allow one to classify these vector fields. Let x be a (1-periodic) coordinate on S^1, then a NVF defines a scalar ordinary differential equation (ODE, for brevity)

$$\dot{x} = f(t, x), \ x \in S^1 \tag{1}$$

with 1-periodic in x function f. Here f is continuous in both variables and uniformly continuous w.r.t. t, differentiable in x uniformly w.r.t. t. Hence, the usual existence and uniqueness theorem is valid for this ODE and any its solution is extended in t on the whole $\mathbb{R}$. Recall that an integral curve of the equation (below IC, for brevity) is the graph of the related solution in the extended phase space $\mathbb{R} \times S^1$.

Thus the extended phase space is foliated into integral curves. We intend to classify these equations w.r.t. the uniform equivalency relation using some combinatorial invariant to be defined later on. To do this, we impose some restrictions on the class of NVFs under consideration.

The first of these restrictions is the following

Assumption 1. *Any solution of* (1) *possesses an exponential dichotomy both on* $\mathbb{R}_+$ *and* $\mathbb{R}_-$ [27].

Recall the related definitions [27, 6, 8].

Definition 3.1. Let $x(t)$ be a solution of the equation (1). One says this solution to satisfy an exponential dichotomy of the stable type on the semi-axis $\mathbb{R}_+$, if there are positive constants C, λ such that the linear ODE $\dot{\xi} = a(t)\xi$, $a(t) = f_x(t, x(t))$, satisfies an inequality

$$\exp[\int_{\tau}^{t} a(s)ds] \le C \exp[-\lambda(t-\tau)]$$

for all $t, \tau,\ t \ge \tau \ge 0$,

Definition 3.2. Let $x(t)$ be a solution of the equation (1). One says this solution to satisfy an exponential dichotomy of the unstable type on the semi-axis $\mathbb{R}_+$, if there are positive constants C, λ such that the linear ODE $\dot{\xi} = a(t)\xi$, $a(t) = f_x(t, x(t))$, satisfies an inequality

$$\exp[\int_{t}^{\tau} a(s)ds] \le C \exp[\lambda(t-\tau)]$$

for all $t, \tau,\ 0 \le t \le \tau$.

Similar notions of exponential dichotomy of the stable and unstable types are defined for the semi-axis $t \le 0$. To be more precise, let us present these definitions as well.

Definition 3.3. Let $x(t)$ be a solution of the equation (1). One says this solution to satisfy an exponential dichotomy of stable type on the semi-axis $\mathbb{R}_-$, if there are positive constants C, λ such that for the linear ODE $\dot{\xi} = a(t)\xi$, $a(t) = f_x(t, x(t))$, an inequality

$$\exp[\int_{\tau}^{t} a(s)ds] \le C \exp[-\lambda(t-\tau)]$$

holds for all $\tau \le t \le 0$.

Definition 3.4. Let $x(t)$ be a solution of the equation (1). One says this solution to satisfy an exponential dichotomy of the unstable type on the semi-axis $\mathbb{R}_-$, if there are positive constants C, λ such that for the linearized equation $\dot{\xi} = a(t)\xi$, $a(t) = f_x(t, x(t))$, an inequality

$$\exp[\int_{t}^{\tau} a(s)ds] \le C \exp[\lambda(t-\tau)]$$

holds for all $t \le \tau \le 0$.

It follows from the Hadamard-Perron theorem [2] that for a solution $\gamma(t)$ to (1) possessing the exponential dichotomy of stable type on $\mathbb{R}_+$ there is its uniform neighborhood $U \subset \mathbb{R}_+ \times S^1$ such that all solutions of the equation (1) starting at $t = t_0 \geq 0$ within U exponentially fast tend to this IC as $t \to \infty$. Moreover, such U can be chosen in such a way that a quadratic Lyapunov function would exist in U and lateral boundary curves of this U are the level lines of the Lyapunov function. Recall how to construct such a function (see, for instance, [8]). Let us write down the equation near the integral curve using local variables $x = \gamma(t) + u$. Then the equation transforms into the form

$$\dot{u} = a(t)u + h(t,u),\ a(t) = f_x(t,\gamma(t)),\ h(t,0) = h_u(t,0) \equiv 0. \tag{2}$$

Since the map $\hat{f} : \mathbb{R} \to V^r(S^1)$, $r \geq 1$, is bounded and uniformly continuous, the following estimate holds

$$|h(t,u)| \leq n(r)|u|, \tag{3}$$

where continuous function n satisfies the conditions $n(0) = 0$, $n(r) \to 0$, as $r \to 0$.

Then a function is introduced

$$S(t,u) = \left(\int\limits_t^\infty \varphi^2(s,t)ds\right) u^2 = s^2(t)u^2,\ \varphi(t,s) = \exp[\int_s^t a(\tau)d\tau]. \tag{4}$$

The improper integral converges uniformly w.r.t. t and defines a bounded positive function $s(t)$ which is also bounded away from zero. Indeed, one has an estimate

$$\int\limits_t^\infty \varphi^2(s,t)ds \leq C^2 \int\limits_t^\infty \exp[-2\lambda(s-t)]ds = C^2/2\lambda.$$

Denote $a_0 = \sup\limits_t |a(t)| > 0$. Then the estimate from below holds

$$\int\limits_t^\infty \varphi^2(s,t)ds \geq \int\limits_t^\infty \exp[-2a_0(s-t)]ds = 1/2a_0.$$

Thus for function S the two-sided estimate is valid

$$(1/2a_0)u^2 \leq S(t,u) \leq (C^2/2\lambda)u^2. \tag{5}$$

Now let us show that S is the Lyapunov function for the equation (2) in a neighborhood of its solution $u = 0$. This means that for any initial point (t_0, u_0) for the solution $u(t)$ through (t_0, u_0) to the equation (2) the following estimate for the derivative of S takes place

$$\frac{d}{dt}S(t,u(t))|_{t_0} = \frac{d}{dt}[s^2(t)u^2(t)]|_{t_0} = 2s(t_0)s'(t_0)u_0^2 + 2s^2(t_0)u_0[a(t_0)u_0 + h(t_0,u_0)].$$

Differentiation of $s(t)$ gives $s'(t) = -1/2s(t) - a(t)s(t)$. Hence we get

$$2s(t_0)s'(t_0)u_0^2 + 2s^2(t_0)u_0[a(t_0)u_0 + h(t_0,u_0)] = -u_0^2 + 2s^2(t_0)u_0h(t_0,x_0).$$

Therefore one has

$$\frac{d}{dt}S(t,u(t))|_{t_0} = -[1 - 2s^2(t_0)h(t_0,u_0)/u_0]u_0^2.$$

Due to the estimate (3), the function within square brackets can be made greater than $1/2$ choosing $|u_0|$ small enough. Let us observe that the derivative of S along the system at the point (t_0, u_0) is the inner product (S_t, S_u) of ∇S and the tangent vector $(1, u')$ to the integral curve at the point. From the estimate obtained it follows that at a fixed $c^2 > 0$ along the curve $S(t,u) = c^2$ the inner product in bounded away from zero uniformly in t. Fixing $c > 0$ small enough gives an uniform neighborhood of the solution $u = 0$ defined as $S < c^2$.

Now we consider again the equation (1). Henceforth we denote S^1_0 the section $t = 0$ in $\mathbb{R} \times S^1$. Suppose an integral curve γ possesses exponential dichotomy of the stable type on $\mathbb{R}_+$. Then all ICs through initial points being close enough to the point $\gamma(t_0)$ at $t = t_0 > 0$ possess also the exponential dichotomy of the same type. This implies a set of traces on S^1_0 for all ICs with the exponential dichotomy of stable type on $\mathbb{R}_+$ be open and therefore to consist of a collection of disjoint intervals (the cardinality of this collection of intervals is maximum countable). This implies that the set of points on the section S^1_0, through which ICs pass with exponential dichotomy of the unstable type on $\mathbb{R}_+$, forms a closed set. The cardinality of this set is an interesting question when the Assumption 1 holds. Our further goal is to distinguish a class of NVFs that are rough and can be classified somehow. This requires of more rigid conditions on the behavior of their ICs. We shall call the set of ICs with exponential dichotomy of the stable type on $\mathbb{R}_+$ by a *stable bunch* if all of them have the same asymptotic behavior, i.e. any two solutions $x_1(t)$, $x_2(t)$ from this stable bunch satisfy inequality $|x_1(t) - x_2(t)| \to 0 \pmod 1$, as $t \to \infty$. In fact the union of all IC of the same stable bunch are a global stable manifold for any IC from this bunch. Hence, the boundaries of a stable bunch consists of one or two ICs which possess exponential dichotomy of the unstable type on $\mathbb{R}_+$.

Similar properties are valid for ICs which possess the exponential dichotomy on semi-axis $\mathbb{R}_-$. Here we define an *unstable bunch* as the union of all ICs that possess exponential dichotomy of the unstable type on semi-axis $\mathbb{R}_-$ and have the same asymptotic behavior as $t \to -\infty$. Then our next assumption is the following

Assumption 2. *There are finitely many stable bunches and finitely many unstable bunches.*

The traces on S^1_0 of those ICs which belong to one stable bunch is an interval. Extreme points of this interval correspond to ICs that possess exponential dichotomy of the unstable type on semi-axis $\mathbb{R}_+$. Similarly, the traces of ICs from one unstable bunch on S^1_0 is an interval and its extreme points correspond to those ICs which possess exponential dichotomy of the stable type on semi-axis $\mathbb{R}_-$. Thus, from this Assumption it follows the existence of finitely many ICs with exponential dichotomy of the unstable type of semi-axis $\mathbb{R}_+$ and also does for ICs with an exponential dichotomy of the stable type on semi-axis $\mathbb{R}_-$.

Our third assumption is

Assumption 3. *No a solution exists such that this solution possesses simultaneously an exponential dichotomy of the unstable type on $\mathbb{R}_+$ and an exponential dichotomy of the stable type on $\mathbb{R}_-$.*

One more property of the class of ODEs under study has to be discussed. Consider in $\mathbb{R}_+ \times S^1$ some stable bunch and let γ_1, γ_2 be its boundary solutions (left and right, respectively, in accordance with the orientation of S^1). Hence these two solutions possess an exponential dichotomy of the unstable type on $\mathbb{R}_+$. Choose some solution γ_0 of the bunch. Due to the Hadamard-Perron theorem (or merely to the existence of a Lyapunov function) one can choose an uniform neighborhood U_0 of γ_0 on $\mathbb{R}_+$ whose boundary curves are uniformly transversal to solutions of ODE through these boundary curves and such solutions enter to the neighborhood and stay there forever. Similarly, for the solution γ_1 there is its uniform neighborhood

U_1 with the property of uniform transversality of its boundary curves but all solutions passing through this neighborhood (except for γ_1) leave U_1 at some t specific for such solution. Without loss of generality we suppose U_0, U_1 be disjoint and we can choose them as thin as we wish. Now an important question arises: consider for $t \geq 0$ the semi-strip between the right boundary curve of U_1 and left boundary curve of U_0. Solutions within this semi-strip enter to it from the left and leave it from the right boundary curve. These solutions cannot stay in this semi-strip for all $t > 0$, since all of them belong to one stable bunch. Is the passing time through this semi-strip uniformly bounded for all such solutions? The answer to this question is essential for constructing equimorphism which conjugates equivalent ODEs. We shall present below an example of a nonautonomous ODE that shows the answer to this question be generally negative. Therefore this property has to be imposed to avoid a structural instability. To this end, let us choose a sufficiently thin disjoint neighborhoods $\mathcal{U}_1, \ldots, \mathcal{U}_s$ of those ICs which possess exponential dichotomy of the unstable type on $\mathbb{R}_+$. Also we select by one IC from every stable bunch and choose their thin disjoint neighborhoods $\mathcal{V}_1, \ldots, \mathcal{V}_r$. All these neighborhoods $\mathcal{U}_i, \mathcal{V}_j$ can be chosen disjoint.

Assumption 4. *For any $\varepsilon > 0$ there are neighborhoods $\mathcal{U}_1, \ldots, \mathcal{U}_s, \mathcal{V}_1, \ldots, \mathcal{V}_r$ such that the passage time of ICs from one boundary curve to another one is bounded from above.*

The same assumption is assumed to be valid on $\mathbb{R}_-$.

Henceforth, we shall call scalar ODEs on S^1 which obey Assumptions 1-4 *gradient-like*.

To demonstrate the essentiality of the Assumptions imposed, we present examples of differential equations (1) showing that violating any of these four assumptions may give an ODE being nonrough. Our first example was constructed in [19], it shows that if there is a solution that fails to possess an exponential dichotomy on some of two semi-axes $\mathbb{R}_+, \mathbb{R}_-$, then the equation may become nonrough and its structure is changed under a perturbation.

On the circle with coordinate $\varphi \in [0, 2\pi) \mod 2\pi$ consider a nonautonomous differential equation

$$\dot{\varphi} = \cos\varphi(\sin^2\varphi + e^{-t^2}). \tag{6}$$

This equation is reversible w.r.t. two involutions on the circle $L_1 : \varphi \to 2\pi - \varphi$ and $L_2 : \varphi \to \varphi + \pi$. Recall, this means that if $\varphi(t)$ is a solution to the equation, then so does $L_i\varphi(-t)$, that is $\varphi_1(t) = 2\pi - \varphi(-t)$ and $\varphi_2(t) = \varphi(-t) + \pi$.

As $t \to \pm\infty$ this equation tends to the autonomous differential equation given as $\dot{\varphi} = \cos\varphi \sin^2\varphi$ that has a nonhyperbolic equilibrium $\varphi = 0$. This limiting equation considered as nonautonomous one has the following foliation onto ICs. There are four stationary solutions $\varphi(t) = 0, \pi/2, \pi, 3\pi/2$. Two of them, $\varphi = \pi/2, 3\pi/2$ possess an exponential dichotomy on $\mathbb{R}$, the first one is asymptotically stable and the second is asymptotically unstable. More two stationary solutions do not possess an exponential dichotomy neither on R_+ no on $\mathbb{R}_-$. ICs corresponding to stationary solutions divide $\mathbb{R} \times S^1$ into four strips. All ICs in each strip go from one stationary solutions (at $-\infty$) to another one (at ∞).

Now we take into account the rapidly decaying term $\exp(-t^2)$ in the equation (6). Solutions $\varphi = \pi/2, 3\pi/2$ stay unchanged and have the same type of dichotomies. Let us consider the invariant strip $-\pi/2 \leq \varphi \leq \pi/2$ with its foliation into ICs. Note that $-\pi/2 = 3\pi/2 \pmod{2\pi}$. One can prove

Lemma 3.5. *There are two special ICs in this strip. One of them γ_1 is monotonically increasing and tends to $\varphi = 0$ as $t \to \infty$ but tends to $\varphi = -\pi/2$ as $t \to -\infty$. Another one γ_2 is also monotonically increasing and is defined by reversibility as $\varphi_1(t) = -\varphi(-t)$ where $\varphi(t)$ is the solution for γ_1.*

Two special solutions indicated in Lemma do not possess an exponential dichotomy either on $\mathbb{R}_+$ (for γ_1) or on $\mathbb{R}_-$ (for γ_2). Observe that the straight-line $\varphi = 0$ is transversal to ICs intersecting it. All ICs between two special ones γ_1, γ_2 intersect this straight-line and tend to $\varphi = \pi/2$ as $t \to \infty$ and to $\varphi = -\pi/2$ as $t \to -\infty$ and thus possess an exponential dichotomy. Other ICs lying in the strip below γ_1 or above γ_2 do not possess an exponential dichotomy on the related semi-axes since they tend in some direction in time to nonhyperbolic solution (γ_1 or γ_2). The behavior of ICs in the strip $\pi/2 \leq \varphi \leq 3\pi/2$ is similar.

This ODE under consideration changes its uniform structure under a perturbation. Indeed, consider a perturbed equation $\dot{\varphi} = \cos\varphi(\sin^2\varphi + \exp[-t^2] - \mu)$ with small positive μ. One can verify that two new bunches appear for small μ, one stable and one unstable. They arise because of the bifurcation, when nonhyperbolic equilibria on $\pm\infty$ bifurcate into two close hyperbolic equilibria. The related foliations are shown in Fig.1. The structure and bifurcation presented is similar to that observed in the Ricatti equation $\dot{x} = x^2 + \exp[-t^2] - \mu$.

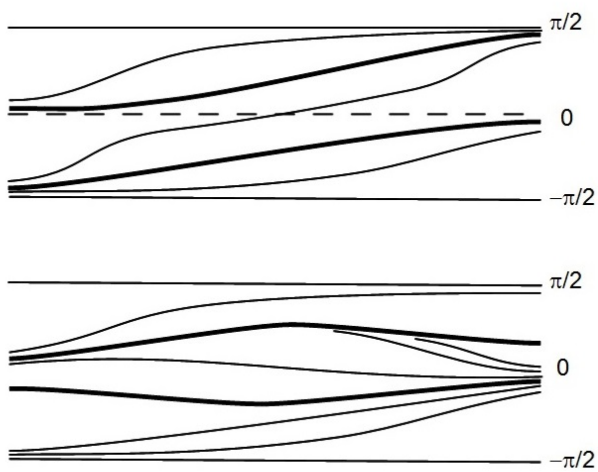

FIGURE 1. Bifurcation from infinities, $\mu = 0$, $\mu > 0$.

The second example we shall consider shows that the violation of the third assumption leads to a structurally unstable ODE. Consider an ODE that has a solution with exponential dichotomy of the unstable type on $\mathbb{R}_+$ and let a point m be its trace on the section S_0^1. Suppose the ODE under consideration is such that IC through m possesses exponential dichotomy of the stable type on $\mathbb{R}_-$. This implies that there is a sufficiently thin uniform neighborhood of this IC on $\mathbb{R}$ within which the only IC through m lies in U entirely for all t. If for this ODE first two Assumptions hold, then point m serves the boundary point of two intervals corresponding two neighboring stable bunches. Their other boundaries (to the left and to the right from m) on S_0^1 are points m_1^+, m_2^+ (these two points may coincide if the closure of two intervals make up S^1_0). Let choose by one IC from both stable bunches. Denote them as γ_1, γ_2. The minimal distance between related ICs on $t \geq 0$ is some positive number ρ. (Fig.2)

Now we perturb the ODE in such a way that the perturbed ODE would have one IC with a dichotomy of the unstable type on $\mathbb{R}_+$ close enough on $\mathbb{R}_+$ to former one through m and another IC with a dichotomy of the stable type on $\mathbb{R}_-$ being close enough on $\mathbb{R}_-$ to former IC through m. This perturbation can be chosen in two ways as shown in Fig.2. Since the uniform equivalence preserve an asymptotic behavior of ICs, these two ODEs are not uniformly equivalent.

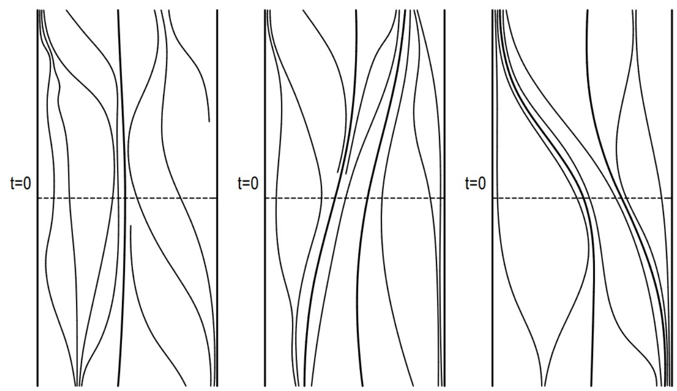

FIGURE 2. Bifurcation at the violation of Assumption 3.

To present an example that demonstrates a possibility to be unbounded for the time of passage from one boundary curve to another one in a transitory strip, we construct an equation $\dot{x} = f(t, x)$ on the segment $I = [-1, 1]$ instead of S^1. It is easy to glue two such strips to obtain an example on S^1. This shows the Assumption 4 be essential.

Consider a semi-bounded strip $\mathbb{R}_+ \times I$ with coordinates (t, x) and choose an increasing sequence of numbers $t_n \to \infty$ such that $t_0 = 0$, $t_{2k} - t_{2k-1} = 1$, $k \geq 1$, and $t_{2k+1} - t_{2k} \to \infty$, as $k \to \infty$. Within segments in t given as $[t_{2k}, t_{2k+1}]$, $k \geq 0$, we take the function $f(t, x) = g(x)$ independent on t and the same for all such segments. Function g is of class C^∞, odd and defined as follows

$$g(x) = \begin{cases} x + 1, \text{ if } x \in [-1, -2/3], \\ g(x) > 0 \text{ with the only maximum within } (-2/3, -1/3), \\ g \equiv 0, \text{ if } x \in [-1/3, 0], \\ g(x) = -g(-x) \text{ for positive } x. \end{cases}$$

We choose the function f independent in t (autonomous) for all $t \in \mathbb{R}_+$ in semi-strips $x \in [-1, -2/3]$ and $x \in [2/3, 1]$. Finally, in rectangles $[t_{2k-1}, t_{2k}] \times I$, $k \geq 1$, of the constant length we defined $f(t, x)$ in such a way that f is positive for $t \in (t_{2k-1}, t_{2k})$ with the related foliation into integral curves within such the rectangle as in Fig.3 with the smooth continuation of the foliation on the boundaries into the left and right neighboring rectangles.

Its specifics is that the solution $x = 1/3$ from the left rectangle goes up and intersects the line $x = 2/3$ between $t = t_{2k-1}$ and $t = t_{2k}$, but solution $x = -1/3$ from the left goes up and cut the line $x = 2/3$ at $t = t_{2k}$. Similarly, solutions $x = 1/3$

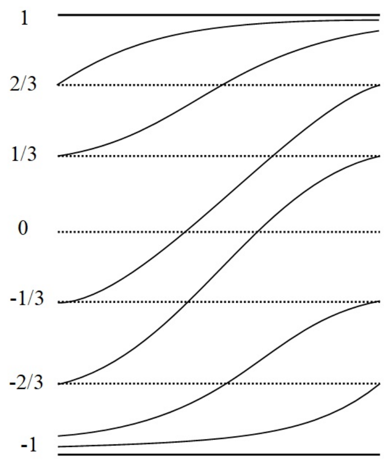

FIGURE 3. Foliation inside the rectangle.

and $x = -1/3$ from the right rectangle, as t decreases, intersect the line $x = -2/3$ respectively at $t = t_{2k-1}$ and between $t = t_{2k-1}$ and $t = t_{2k}$. All this guarantees that

- each integral curve, except $x = -1$, intersects the line $x = 2/3$ entering thus to the strip $\mathbb{R}_+ \times [2/3, 1]$ where the equation $\dot{x} = -(x-1)$ works, such solution possesses exponential dichotomy of the stable type on $\mathbb{R}_+$;
- there are countably many solutions of the full equation whose ICs coincide with the segment $x = -1/3$ on the time $[t_{2k}, t_{2k+1}]$, they have their passage time from $t = -2/3$ to $t = 2/3$ greater than $t_{2k} - t_{2k+1}$ and hence these times unbounded.

4. **Examples of gradient-like scalar ODEs.** Here we present several simple examples of scalar equations of the gradient type on S^1. The first example is trivial, it is an autonomous scalar ODE $\dot{x} = f(x)$ with a smooth 2π-periodic f having only simple zeroes ($f'(m) \neq 0$ at such a zero m) and hence there are only finitely many zeroes (equilibria). Due to compactness of S^1 all solutions to ODE tend, as $t \to \pm\infty$, to some of its equilibrium, their simplicity guarantees an exponential dichotomy of the related constant solutions on $\mathbb{R}$, as well as an exponential dichotomy on the semi-axis $\mathbb{R}_+$ or $\mathbb{R}_-$ for all solutions that tend to this constant solution as $t \to \infty$ or $t \to -\infty$, respectively.

The second example is less trivial but also well known. Consider a nonautonomous equation $\dot{\varphi} = f(t, \varphi)$ where function f is 2π-periodic in both variables and smooth (in fact, it is sufficient f to be continuous in t, φ and differentiable in φ with a continuous derivative in (t, φ)). Then the Poincaré map $P : S^1 \to S^1$ in $t = 2\pi$ is well defined. This map is a diffeomorphism on the circle (at least C^1) and hence the Poincaré rotation number is well defined. Suppose this number is rational p/q with incommensurate p, q. Then by the Poincaré theorem (see, for instance, [14]) this map has a periodic orbit of the period q and any its orbit is either periodic with the same period or tend to a periodic orbit as $n \to \infty$ (n is the number of iterations) and to (possibly another) periodic orbit as $n \to -\infty$. Suppose all these periodic orbits are hyperbolic, i.e. $DP^q \neq \pm 1$ at any periodic orbit. Then there are only finitely many periodic orbits and hence the related nonautonomous ODE is of the gradient type.

The third example is a so-called asymptotically autonomous scalar ODE [26, 40, 35]. Suppose on S^1 a differential equation $\dot{\varphi} = f(t,x)$ be given such that in the space $V^r(S^1)$, $r \geq 1$, there are limits $f \to f_+$ as $t \to \infty$ and $f \to f_-$ as $t \to -\infty$. We assume that both limiting ODEs on the circle with functions f_+, f_- have only simple zeroes as in the first example. Then any IC of the given equation asymptotically approaches as $t \to \infty$ to some φ_k^+, $f_+(\varphi_k^+) = 0$, and asymptotically approaches as $t \to -\infty$ to some φ_j^-, $f_-(\varphi_j^-) = 0$ [26]. It follows from here that any integral curve possesses an exponential dichotomy both on R_+ and R_-. In order the equation would be gradient-like, one needs to require in addition that none IC exists that asymptotically tends as $t \to -\infty$ to a stable zero of f_- and as $t \to \infty$ does to an unstable zero of f_+. Then this ODE is a nonautonomous gradient-like one. As an example of such ODE one can construct a glued differential equation. We take a rough scalar autonomous equation with the function f_+ and another rough scalar autonomous equation with the function f_-. The nonautonomous equation is glued as follows. Let for $t \leq -T$ the equation with f_- is considered, and for $t \geq T$ does the equation with f_+. In the layer between two sections $-T \leq t \leq T$ we can specify a smooth foliation in such a way that traces of ICs for the equation with f_- would join with traces of ICs to the equation with f_+ and all curves of the foliation in this layer would intersect transversely sections S_t^1, $|t| \leq T$. This makes the complete foliation smooth. Sometimes, the systems of such type are called *transitory* [28]. The derivatives in t for ICs will give gradient-like nonautomous ODE.

In the similar way examples of nonautonomous asymptotically periodic scalar gradient-like ODEs can be constructed. To that end, one needs to choose two periodic limiting ODEs as in the example 2 above (they play the role of limiting equations at $\pm\infty$) given by functions $f_-(t,x)$ and $f_+(t,x)$ being periodic in t with possibly different periods. Suppose their related foliations into ICs are disposed lower of S_{-T}^1 (for f_-) and above of S_T^1 (for f_+). Then we get on S^1 two 1-periodic in x functions $f_-(-T,x)$ and $f_+(T,x)$ being two points in the Banach space $V^1(S^1)$. Let us join these points by a smooth compact path with the parametrization $t \in [-T,T]$. Then we get a glued function $f(t,x)$ which defines the nonautonomous ODE on S^1 and coincides with $f_-(t,x)$ when $t < -T$ and does with $f_+(t,x)$ when $t > T$. The only thing we need to care about is that in a middle part of $\mathbb{R}$ traces of stable periodic solutions to negative limiting ODE (for $t < -T$) would not be connected with periodic solutions of positive limiting ODE being unstable hyperbolic for $t > T$. This is achieved by a small perturbation of the ODE in the layer.

5. The properties of gradient-like NVFs and u-invariant.

In this section we define an invariant which is able to recognize different gradient-like ODEs on S^1. This invariant is of combinatorial type, i.e. it is defined by a finite set of ingredients.

To construct the invariant we begin with some assertions necessary for the construction.

Lemma 5.1. *For a gradient-like scalar ODE at least one solution always exists which has an exponential dichotomy of the unstable type on $\mathbb{R}_+$, as well as at least one solution with an exponential dichotomy of the stable type on $\mathbb{R}_-$.*

Proof. Suppose, to the contrary, a gradient-like ODE has not any solution with exponential dichotomy of the unstable type on $\mathbb{R}_+$. This implies that there exists the unique stable bunch and the union of traces on $t = 0$ of ICs from this bunch is the whole S^1_0. Choose some IC γ from this bunch. There is an uniform neighborhood U of γ such that all ICs passing through a point $(t_0, x_0) \in U$ stay inside it for all $t \geq t_0$. Thus, one can choose such a neighborhood for any IC of the bunch. The union of these neighborhoods covers $\mathbb{R}_+ \times S^1$, thus the intersection of these neighborhoods with the section S^1_0 gives a covering of S^1_0. Compactness of S^1_0 allows one to select a finite covering and consequently to find a finite set of ICs $\gamma_1, \ldots, \gamma_n$ around which related neighborhoods $U_1, \ldots, U_n$ have been chosen. Denote related traces on S^1_0 of these ICs as points $p_1, \ldots, p_n$ enumerated in accordance with the orientation of S^1_0. Take some positive number $\varepsilon < 1/2n$. For any p_i there is $T_i > 0$ such that for $t > T_i$ the distance between traces of those ICs, which start at $t = 0$ at extreme points of $U_i \cap S^1_0$, on the section S^1_t will be lesser than ε. Thus for $t > \max\{T_1, \ldots T_n\}$ the distances between traces of any two ICs from the bunch will be lesser than $n \cdot \varepsilon < 1/2$. On the other hand, they should cover S^1_t. This contradiction proves the lemma. The similar proof holds true for the dichotomy of the stable type on R_-. □

Now we are able to define a combinatorial invariant which distinguishes two uniformly nonequivalent gradient-like ODEs on S^1. Let a gradient-like ODE be given. Take the section S^1_0. We assume that S^1 is oriented by the coordinate. On the section we consider the traces of all ICs which have an exponential dichotomy of unstable type on $\mathbb{R}_+$. Denote this set of points as $u_1, \ldots, u_n$ where the order from 1 to n is defined by the orientation of S^1_0. Similarly, we consider the traces of all ICs which have the dichotomy of stable type on $\mathbb{R}_-$. Denote them as $s_1, \ldots, s_m$ where the order from 1 to m is also defined by the orientation of S^1_0. These two sets of points on S^1_0 do not intersect in accordance with the Assumptions 1-3. Due to Lemma 5.1, integers n, m are both positive. We shall call the set of points $\{u_1, \ldots, u_n, s_1, \ldots, s_m\}$ the *equipped set of points.*

Definition 5.2. Two equipped sets on S^1 will be called equivalent, if there is a homeomorphism $h : S^1 \to S^1$ such that h sends the set $u_1, \ldots, u_n$ of the first equipped set onto the set $u'_1, \ldots, u'_n$ of the second equipped set, and the set $s_1, \ldots, s_m$ of the first equipped set to $s'_1, \ldots, s'_m$ of the second equipped set.

A class of equivalent equipped sets will be called u-invariant. The main result of the paper is the following theorem

Theorem 5.3. *Two nonautonomous scalar gradient-like ODEs on S^1 are uniformly equivalent, iff they have the same u-invariant.*

It is clear that two uniformly equivalent ODEs on S^1 have the same u-invariant. The proof of the inverse assertion is the main task.

Gradient-like ODEs possess also the property of structural stability. This notion was introduced by Andronov and Pontryagin in 1937 [1] for autonomous vector fields on 2-dimensional sphere (or on a disk with the proper assumptions on the orbit behavior on the boundary cicrle) and was studied in many details, first on sphere (Leontovich-Andronova, Maier [18]) and then on the torus (Maier). The final result for any smooth closed 2-dimensional manifold was done by Peixoto [33]. Multidimensional case was studied first by Smale [38] who defined the class of vector fields and diffeomorphisms which were called later Morse-Smale systems. Their study still continues till now [12, 13].

Leaning on Theorem 5.3 we will also prove

Theorem 5.4. *A gradient-like NVF v on S^1 is rough, that is, for any $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that all NVFs in the ε-neighborhood of v in the space $\mathbb{R} \to V^1(S^1)$ are uniformly equivalent to v and an equimorphism realizing this equivalence can be chosen δ-close to the identity mapping $id_{\mathbb{R}\times S^1}$. Moreover, all shifted NVFs $v_{t+\tau}$ are also rough with the same δ.*

Proofs of both theorems will be outlined in the next two sections.

6. **A method to construct an equimorphism.** To demonstrate the method of constructing an equimorphism for two gradient-like ODEs with the same u-invariant, we present here first the simplest possible situation when both differential equations are linear. It gives, in a sense, the simplest uniform version of the Grobman-Hartman theorem [14]. For the case of two gradient-like nonlinear ODEs the local construction of an equimorphism is similar.

Consider two scalar linear homogeneous differential equations

$$x' = a(t)x(t),\ y' = b(t)y(t). \tag{7}$$

Denote $u(t,\tau)$ the solution of (7) with the condition $u(\tau,\tau) = 1$, and let $u_1(t,\tau)$ be the similar solution for the second equation

$$u(t,\tau) = \exp[\int_\tau^t a(s)ds],\ u_1(t,\tau) = \exp[\int_\tau^t b(s)ds].$$

We assume both these linear ODEs to possess exponential dichotomy of the stable type on $\mathbb{R}_+$

$$u(t,\tau) = \exp[\int_\tau^t a(s)ds] \le M\exp[-\lambda(t-\tau)];\ t \ge \tau,\ \lambda > 0,\ M > 0, \tag{8}$$

and the same holds for the second equation with constants M_1, λ_1.

We defined above a Lyapunov function (4) for such equation. For equations under consideration we denote related functions as $S(t,x), s(t), u(t,\tau),\ S_1 t,x), s_1(t), u_1(t,$ $\tau)$.

From expressions for u, u_1 the inequalities follow

$$\frac{1}{2a_0} \le s^2(t) \le \frac{M^2}{2\lambda} \tag{9}$$

$$\frac{1}{2b_0} \le s_1^2(t) \le \frac{M_1^2}{2\lambda_1} \tag{10}$$

where $a_0 = \sup_t |a(t)|,\ b_0 = \sup_t |b(t)|$.

Observe that functions $s(t), s_1(t)$ obey the equations

$$\frac{ds}{dt} = \frac{-2a(t)s^2(t) - 1}{2s(t)}$$

$$\frac{ds_1}{dt} = \frac{-2b(t)s_1^2(t) - 1}{2s_1(t)}.$$

Our goal in this section is to prove that the map from the semi-strip D in the plane (x,t) near $x = 0$ onto the semi-strip D_1 in the plane (y,t) near $y = 0$ is an equimorphism. The strip D is defined by its boundary curves $x = 0$ and $x = C^*/s(t),\ t \ge 0$, where C^* is some positive constant. This semi-strip contains the right curve but does not contain the left one $x = 0$. Similar formulae define the strip D_1 with the change $x \to y$, $C^* \to C_1^*$, $s(t) \to s_1(t)$, $t \geq 0$.

Let us define the map $\Phi : D \to D_1$. First we change the coordinates in both semi-strips. Instead of (x, t) we take as new coordinates (C, t), similarly, we change (y, t) to (C_1, t). In accordance with the formulae for Lyapunov functions, these coordinate transformations are smooth in both variables and linear in x (respectively, y) with uniformly bounded coefficients being uniformly bounded away from zero. Then the map Φ is defined as follows. Take any point $(C, \tau) \in D$ and consider the integral curve of the first equation through this point. This curve intersects transversely the level line of the Lyapunov function S defined by C^* at some time $T(C, \tau) \leq \tau$. We fix some smooth monotone function $\alpha : \mathbb{R} \to \mathbb{R}$ given as $T_1 = \alpha(T)$ with α' positive uniformly bounded from above and from zero: $0 < r_0 \leq \alpha' \leq r_1$. For instance, as a such map one can take the identical map $\alpha(T) = T$. To facilitate the exposition, we take just this identical map. Then the point on the right boundary of D with coordinates $(C^*, T(C, \tau))$ is transformed at the point $(C_1^*, T(C, \tau))$ on the right boundary curve of D_1. On the integral curve $(t, y(t))$ of the second equation through this latter point we choose that its point whose t-coordinate is τ. Then C_1-coordinate of this point is $C_1 = y(\tau)s_1(\tau)$.

Integral curve through the point $(C = s(\tau)x_0, \tau)$ is

$$x(t) = x_0 \exp[\int_{\tau}^{t} a(u)du] = \frac{C}{s(\tau)} \exp[\int_{\tau}^{t} a(u)du],\ x_0 > 0.$$

This curve intersects the boundary level line $s(T)x = C^*$ at the point with coordinates (C^*, T) where T is found from the equation

$$Cs(T) \exp[\int_{\tau}^{T} a(u)du] - C^* s(\tau) = 0. \tag{11}$$

Denote $R(T, C, \tau)$ the left hand side of this equation. Function $T(C, \tau)$ is a solution to the equation $R(T, C, \tau) = 0$ existing by the implicit function theorem.

Integral curve $(t, y(t))$ of the second equation through the point (C_1^*, T) is given as (its second coordinate)

$$y(t) = \frac{C_1^* \exp[\int_{T}^{t} b(u)du]}{s_1(T)}.$$

For the point $\Phi(C, \tau)$ the time variable on this curve is $\tau_1 = \tau$. C_1-coordinate of this point is found from the equation

$$C_1 = \frac{C_1^* s_1(\tau) \exp[\int_{T}^{\tau} b(u)du]}{s_1(T)}. \tag{12}$$

The right hand side of the equation (12) will be denoted as $g(T, \tau)$. Thus the map Φ is given in coordinates as follows

$$C_1 = g(T(C, \tau), \tau),\ \tau_1 = \tau. \tag{13}$$

We need to prove the uniform continuity of this map along with its inverse one.

In fact, we need only to prove the uniform continuity of $g(T(C, \tau), \tau)$ in the strip $0 < C \leq C^*$. The uniform continuity of the inverse map is proved in the same way, since the inverse map is defined in the similar way. The uniform continuity of Φ, when it has been proved, allows to extend the map uniquely to the closed strip $0 \leq C \leq C^*$ by the relation $\tau_1 = \tau$.

The uniform continuity of g is proved in two steps because the derivatives of Φ is not bounded if we consider the full strip $0 < C \leq C^*$. Therefore we fix some $v > 0$ and prove that Φ-pre-image in D of the line $C_1 = v$ is a curve being the graph of a function $\varphi(\tau)$, and this function obeys estimates $0 < d_1 \leq \varphi(\tau) \leq d_2 < C^*$. Moreover, if $v \to +0$ then both $d_i(v)$ tend also to zero. Thus choosing v small enough we get that the semi-strip $0 < C \leq d_1$ is transformed on the semi-strip in D_1 which belongs to the semi-strip $0 < C_1 \leq v$. Since τ-coordinate preserves by Φ this gives uniform continuity near the line $x = 0$. In the closed semi-strip $d_1 \leq C \leq C^*$ the derivatives of Φ are uniformly bounded and this guarantees the uniform continuity for Φ there.

Now consider in D pre-images of the Lyapunov function S_1 level lines defines by constants v in D_1. First we will show that this curve is given as $C = \varphi(\tau)$ for which the inequalities hold

$$0 \leq d_1 \leq \varphi_2 \leq d_2.$$

and d_1, d_2 tend to zero, if $v \to +0$.

To this end, we solve the equations (13) w.r.t. C setting $C_1 = v$. Since the value of τ preserves, then the solutions will give a function $\varphi(\tau)$ and we need only to estimate them. These functions are given in the following way

$$\varphi_i(\tau) = \frac{C^* s(\tau) \exp[\int\limits_T^\tau a(u)du]}{s(T)},\ T = T(v,\tau).$$

Here function $T(v,\tau)$ is a solution of the equations (12) with $C_1 = v$. From this equation an estimate for the difference $\tau - T$ is derived using the exponential dichotomy

$$\tau - T \leq \frac{1}{\lambda_1} \ln \frac{M_1 C_1^* s_1(\tau)}{v s_1(T)} \leq \frac{1}{\lambda_1} \ln \frac{M_1 C_1^* \sup(s_1)}{v \inf(s_1)} = A - \frac{1}{\lambda_1} \ln(v).$$

We consider v small enough positive, so r.h.s. will tend to infinity as $v \to +0$, but it is finite, if v holds fixed. We have a similar estimate from below

$$\tau - T \geq A_0 - \frac{1}{b_0} \ln(v),\ A_0 = \frac{1}{b_0} \ln \frac{C_1^* \sup(s_1)}{\inf(s_1)}.$$

Using these estimates for $\tau - T$ we come to the following estimates for function $\varphi(\tau)$

$$B_0(v)^{a_0/b_0} \leq \varphi(\tau) \leq B_1(v)^{\lambda/\lambda_1}. \tag{14}$$

These estimates say that the pre-image of any strip $v \leq C_1 \leq C_1^*$ in D_1 is a strip in D lying in the the strip $0 < d \leq C \leq C^*$. This implies that for proving uniform continuity one may use estimates for the derivatives of g. These derivatives are as follows

$$\frac{\partial g}{\partial C} = \frac{\partial g}{\partial T} \cdot \frac{\partial T}{\partial C},$$

and using (12) we get

$$\frac{\partial g}{\partial T} = \frac{C_1^* s_1(\tau) \exp[\int\limits_T^\tau b(s)ds]}{s_1^3(T)} = \frac{C_1}{s_1^2(T)},\ \frac{\partial T}{\partial C} = \frac{2s^2(T)}{C}. \tag{15}$$

Therefore, the estimate for $\dfrac{\partial g}{\partial C}$ in the strip $C \geq d$ is given as

$$|\frac{\partial g}{\partial T} \cdot \frac{\partial T}{\partial C}| \leq \frac{2C_1^* \sup(s^2)}{d \inf(s_1^2)} = R_1. \tag{16}$$

Now the calculation of derivative

$$\frac{Dg}{D\tau} = \frac{\partial g}{\partial T}\frac{\partial T}{\partial \tau} + \frac{\partial g}{\partial \tau}$$

gives the formulae

$$\frac{\partial g}{\partial T} = \frac{C_1}{s_1^2(T)},\ \frac{\partial g}{\partial \tau} = -\frac{C_1}{2s_1^2(\tau)},\ \frac{\partial T}{\partial \tau} = \frac{C^* s^2(T)}{s^2(\tau)}.$$

Thus, we come to the estimate

$$|\frac{Dg}{D\tau}| \leq |\frac{\partial g}{\partial T}||\frac{\partial T}{\partial \tau}| + |\frac{\partial g}{\partial \tau}| \leq C_1^*(C^* \frac{(\sup(s^2))}{\inf(s_1^2)\inf(s^2)} + \frac{1}{2\inf(s_1^2)}) = R_2. \tag{17}$$

These estimates for derivatives show the uniform finiteness of derivatives.

Now we are able to complete the proof of the uniform continuity of Φ. Choose the metrics in both strips:

$$\rho((C,\tau),(C',\tau')) = \max\{|C - C'|, |\tau - \tau'|\},$$
$$\rho_1((C_1,\tau_1),(C_1',\tau_1')) = \max\{|C_1 - C_1'|, |\tau_1 - \tau_1'|\}.$$

Let a positive ε be given. Put $v_1 = \varepsilon/2$ in (14) and take

$$\delta(\varepsilon) = \min\{\varepsilon/2, B_0(\varepsilon/2)^{a_0/b_0}, \varepsilon/2R_1, \varepsilon/R_2\}$$

Take two arbitrary points $P = (C,\tau)$, $P' = (C',\tau')$ in D at the distance $\rho(P,P') < \delta$. Three cases are possible: i) both P, P' belong to the sub-strip $0 < C \leq \delta$; ii) both points belong to the sub-strip $\delta \leq C \leq C^*$; iii) one point, say P, belongs to first sub-strip but P' belongs to the second sub-strip. In the first case both points $\Phi(P), \Phi(P')$ belong to the strip $0 < C_1 \leq \varepsilon/2$. Since $\rho(P,P') = \max\{|C - C'|, |\tau - \tau'|\} \leq \varepsilon/2$ and Φ preserves τ-coordinate, then $\rho_1(\Phi(P),\Phi(P')) \leq \varepsilon$. The same estimate holds for the third case due to estimates for derivatives of g. The second case is reduced to the combination of the first and third ones if we joint points P, P' by the segment on the right boundary of the semi-strip $0 < C \leq \delta$.

The construction gives an equimorphism only in semi-strips $x > 0$ and $y > 0$. In order to get an equimorphism of the whole strip around $x = 0$ and $y = 0$, respectively, one needs to conform them along the straight-lines $x = 0$ and $y = 0$, respectively. In the construction presented it is done automatically since the section $t = \tau$ is transform to the section $t = \tau$. For the general case one needs to care about it.

7. **Uniform equivalence and structural stability.** If two nonautonomous gradient-like ODEs on S^1 are uniformly equivalent, then there is an equimorphism $h : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$ which transforms the foliation into ICs of the first ODE to the foliation of the second ODE. Restriction of h on S_0^1 sends this circle onto some closed curve homotopic to S_0^1 in $\mathbb{R} \times S^1$. Since S_0^1 is the cross-section to ICs of the first ODE, then the closed curve is the curve such that ICs of the second ODE intersect it topologically transversally. Hence, traces of ICs from the skeleton of the first ODE are transformed by h to points on the closed curve through which pass some ICs of the second ODE. If IC γ of the first ODE corresponds to an u-point, then γ possesses an exponential dichotomy of the unstable type on $\mathbb{R}$. In particular,

in some its uniform neighborhood U thin enough no other ICs can lie wholly for all $\mathbb{R}$. This implies that the related IC $\gamma' = h(\gamma)$ of the second ODE also has such neighborhood U'. This means that γ' can correspond only to u-point of the second ODE (not its s-point), due to gradient-likeness of the second ODE and the uniform equivalence of foliations.

The same holds true for ICs through s-points. Let us calla *skeleton* of the ODE the set of those ICs whose traces on S^1_0 make up the equipped set. Thus, the skeleton of the first ODE transforms by h to that of the second ODE. Therefore a shift in time of the curve $h(S^1_0)$ along ICs of the second ODE allows to move this closed curve to the section S^1_0 and to get a homeomorphism $S^1_0 \to S^1_0$ that transforms the equipped set of the first ODE to the equipped set of the second ODE. Thus, the uniform equivalence of two nonautonomous gradient-like ODEs implies they to have the same u-invariant.

Thus, to prove Theorem 5.3, we need, having the same u-invariant for two nonautonomous gradient-like ODEs, to construct an equimorphism $h : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$ which transforms the foliation into ICs of the first gradient-like ODE to the foliation of the second ODE. The idea of constructing equimorphism is close, in a sense, to that performed in the linear case (see above) but with some modifications exploiting the structure of ODEs. At the first step one needs to construct an equimorphism in some special neighborhoods of their skeletons and after that to extend this equimorphism to the whole manifold $\mathbb{R} \times S^1$. For a gradient-like ODE its skeleton depends continuously on a perturbations in the class of uniformly continuous bounded maps $\mathbb{R} \to V^r(S^1)$. This allows us to prove that a conjugating equimorphism can be found to be close to the identity map on $\mathbb{R} \times S^1$. In the general case of two gradient-like ODEs with the same u-invariant the construction of a conjugating equimorphism is geometrical and requires a detailed description. We present here only some details of this construction. A complete exposition will be done elsewhere.

We start with constructing a sufficiently thin neighborhood of the skeleton for the ODE. Its u-points $(u_1, u_2, \dots, u_n)$ on S^1_0 are distributed on m intervals (s_1, s_2), $(s_2, s_3), \dots, (s_m, s_1)$ in S^1_0 formed by the traces of s-points (Assumption 3). ICs through u-points which belong to the same interval (s_i, s_{i+1}) are in the same unstable bunch. Next, it is evident from Assumption 3, that all ICs through u-points possess an exponential dichotomy of the unstable type on $\mathbb{R}$ (we call such IC to be globally unstable). Similarly, s-points are distributed on n intervals (u_1, u_2), $(u_2, u_3), \dots, (u_n, u_1)$, ICs passing through s-points of one interval belong to the same stable bunch. It will be shown below that $m = n$. s-points possess an exponential dichotomy of the stable type on $\mathbb{R}$ (we call such IC to be globally stable).

Choose IC γ_1 through the point u_1. This IC belongs to some unstable bunch. The intersection of this bunch with S^1_0 defines some interval whose boundary points are s_i, s_{i+1} (in according to the chosen orientation of S^1). Recall that a sufficiently thin neighborhood of γ_1 does not contain wholly (i.e. for all $t \in \mathbb{R}$) any IC distinct from γ_1 and boundary curves of this neighborhood are uniformly transversal to all ICs passing through points of these curves. There is an alternative: either no other points exist from the set of u-points which belong to the same interval (s_i, s_{i+1}) or there are such ICs. If the first case is realized, the second ODE also has a corresponding IC γ'_1 which passes through the point u'_1 of its equipped set and there is some its uniform isolating neighborhood U'_1. We can construct an equimorphism from U_1 to U'_1 by the method similar to as in the preceding section.

In the second case we gather all ICs from the same unstable bunch and they are asymptotic to each other as $t \to -\infty$ and therefore cannot be divided by s-points. Thus we have to construct a neighborhood $\mathcal{U}_1$ for all this collection of ICs from the bunch. All these ICs are globally unstable ones. We first choose some isolating uniform neighborhood U_1 of γ_1. ICs $\gamma_2, \ldots, \gamma_s$, which belong to the same bunch, intersect the only boundary curve of U_1, these intersections are transversely and give a successive series of points at times $\tau_s < \cdots < \tau_2 < 0$. Thus, one can choose isolating sufficiently thin neighborhoods $U_2, \ldots, U_s$ such that they do not intersect for $t \geq 0$ each other and U_1, but their boundary curves intersect the same boundary curve of U_1 transversely at points close to traces of $\gamma_2, \ldots, \gamma_s$. Moreover, these intersection points separate disjoint segments on the related boundary curve of U_1. Thus, we construct a tree-shape neighborhood $\mathcal{U}_1$ for all collection from one unstable bunch through (s_i, s_{i+1}) (see Fig.4).

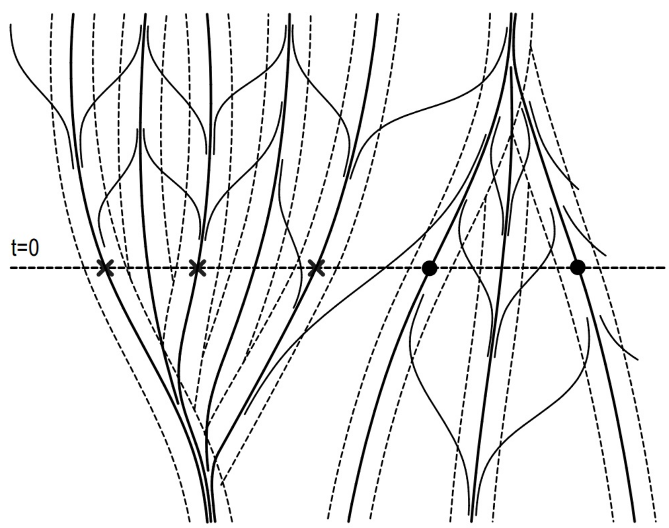

FIGURE 4. Neighborhoods: crosses correspond to u-solutions, bold points correspond to s-solutions.

Now we proceed in the similar way constructing neighborhoods $\mathcal{U}_2, \ldots, \mathcal{U}_s$ involving points $u_{k_1+1}, \ldots, u_{k_1+k_2}$, related with another unstable bunch, and so forth, till we exhaust all collection of u-points in the equipped set. It is clear that the neighborhoods can be constructed not intersecting and situated at a finite distance from each other.

So far, the construction did not involve s-points $s_1, \ldots, s_m$. Now we continue the construction proceeding in a similar way but starting with ICs through points $s_1, \ldots, s_m$ on S_0^1 and using stable bunches with traces distributed on intervals $(u_1, u_2), (u_2, u_3), \ldots, (u_n, u_1)$,. Thus, we construct other neighborhoods $\mathcal{V}_1, \ldots, \mathcal{V}_r$. These neighborhoods are also chosen sufficiently thin in order to be separated from each other and from neighborhoods $\mathcal{U}_i$.

The $\mathcal{U}$-neighborhoods and $\mathcal{V}$-neighborhoods we constructed have the following characterization. They contain wholly only those ICs which make up the skeleton. All other ICs either leave the related neighborhood when t increasing at some finite time (for $\mathcal{U}_i$) or when t decreasing at some finite time (for $\mathcal{V}_j$).

Nevertheless, we need to control the time passage from one neighborhood to another one and between different semi-strips composing the same $\mathcal{U}_i$. This is impossible so far, since ICs may exist which leave some $\mathcal{U}_i$, enter into V-shaped region bounded by two neighboring semi-strips from $\mathcal{U}_i$ and go to ∞. Similarly for $\mathcal{V}_j$, when ICs exist which go from $-\infty$ entering into its V-shaped region. To control the passage time, we choose for every fixed $\mathcal{U}_i$ by one IC passing in each V-shaped region between two neighboring semi-strips of $\mathcal{U}_i$ and construct sufficiently thin neighborhoods of these ICs. Such ICs will be called *separating*. A separating IC possesses an exponential dichotomy of the stable type on $\mathbb{R}_+$, hence all ICs leaving $\mathcal{U}_i$ and entering into the V-shaped region tend as $t \to \infty$ to the separating IC and a time passage from the boundary of $\mathcal{U}_i$ to the neighborhood of a separating IC is finite due to Assumption 4. We do the same for $\mathcal{V}_j$ neighborhoods. After these constructions any IC either in finite time enters to some $\mathcal{V}_j$ through its lateral boundary curve (leftmost or rightmost one) and this time is uniformly bounded from above for all such ICs (Assumption 4).

7.1. **Conjugating equimorphism.** Suppose we have two nonautonomous gradient-like ODEs on S^1 with the same u-invariant. Recall that objects related with the second ODE we denote by the same letters as for the first ODE but with primes. At the first step we define this equimorphism inside of the constructed neighborhoods $\mathcal{U}_1, \dots, \mathcal{U}_s$, $\mathcal{V}_1, \dots, \mathcal{V}_r$. At the second step we extend the equimorphism outside of the union of these neighborhoods defining it wholly in $\mathbb{R} \times S^1$. Since the construction is geometrical, it requires more room than we have here. So, we present only some details, the complete exposition will be given elsewhere. The key observation here is that any neighborhood $\mathcal{U}_i$ corresponds to a unique unstable bunch. Similarly, a neighborhood $\mathcal{V}_j$ also corresponds to a unique stable bunch. Thus, ICs leaving $\mathcal{U}_i$ through their leftmost boundary curve L_i should enter to some $\mathcal{V}_j$ through its rightmost boundary curve R_j and the same is true for the rightmost boundary curves R_i and L_k for $\mathcal{V}_k$, where k can be distinct from j. This implies a statement

Proposition 1. *The number n of neighborhoods $\mathcal{U}_i$ is equal to the number m of $\mathcal{V}_j$.*

In order to demonstrate the main points of the construction and how to conform them in order to guarantee that any IC of the first ODE would transform by an equimorphism to some IC of the second ODE we discuss two points.

The first point is a construction an equimorphism h within a fixed neighborhood $\mathcal{U}_i$. This neighborhood contains all ICs of the skeleton which pass through u-points of the equipped set and belong to the same unstable bunch. To simplify notations, we omit the index i. We suppose the orientation of S^1 is fixed. We understand the movement in a right direction on the circle when it is conformed with this orientation. In particular, u-points corresponding to $\mathcal{U}_i$ are distributed from the left to the right: $u_1, u_2, \dots, u_s$. For the second ODE we have the same set of u'-points $u_1, u_2, \dots, u_s$ and a homeomorphism $\tau : S^1_0 \to S^1_0$ transforming the first set to the second one. This homeomorphism can either preserve orientation on the circle for the second manifold $\mathbb{R} \times S^1$ or reverse it. The first task is to construct an equimorphism inside the region on the first cylinder $\mathbb{R} \times S^1$ bounded by two extreme ICs γ_1 and γ_s and containing inside other ICs $\gamma_2, \dots, \gamma_{s-1}$.

We start with the leftmost boundary curve L of the neighborhood $\mathcal{U}$. On this curve we define some equimorphism to that boundary curve of the neighborhood $\mathcal{U}'$ which is defined by the map τ. A construction similar to that as in Section 6 allows one to extend this equimorphism into the strip bounded by L and γ_1. The same is done for the rightmost boundary curve R for $\mathcal{U}$ and IC γ_s. Using unform semi-strips of separating ICs we construct an equimorphism inside uniform neighborhoods of these ICs for some $T > 0$, then extend this equimorphism till the boundary curves of V-shaped regions for $\mathcal{U}$ and after that to other part of the region between γ_1 and γ_s. This gives an equimorphism of this region to the region on the second cylinder $\mathbb{R} \times S^1$.

The equimorphism on boundary curves L and R for $\mathcal{U}_1$ defines equimorphisms on boundary curves for two neighboring neighborhoods $\mathcal{V}_i$ and $\mathcal{V}_{i+1}$. Then we proceed extending this equimorphism inside of these neighborhoods, and so forth. The main difficulty at these constructions is to care about the right continuation of the maps in such a way that they would be coordinated at the points of ICs from the skeleton, since these maps are extended to these ICs by continuity using the uniform continuation.

8. **Autonomization.** In this section we introduce another invariant of uniform equivalency which shows that a gradient-like ODE on S^1 is uniformly equivalent to an asymptotically autonomous ODE. Let a gradient-like ODE be given. Then a definite u-invariant corresponds to this ODE. Consider the circle S^1_0 and related equipped set $(u_1, \dots, u_n, s_1, \dots s_m)$. The numeration in each group ("u" and "s") is done in accordance with the orientation of S^1. The section S^1_0 is divided into n intervals by points u_i : (u_1, u_2), $(u_2, u_3), \dots,$ (u_n, u_1). Each interval (u_i, u_{i+1}) contains some number k_i of s-points arranged in the orientation order, $k_1 + k_2 + \cdots + k_n = m$. Observe that some k_i may be zeroes.

Next we choose some smooth function f_+ on S^1 whose all zeroes are simple. In particular, this function has exactly n zeroes with positive derivatives at the points $(u_1, \dots, u_n)$ and no other such zeroes. Hence, more n simple zeroes s'_i, $i = 1, \dots, n$, with negative derivatives should exist at some points within each interval (u_i, u_{i+1}). The collection of points $(u_1, \dots, u_n; s'_1, \dots, s'_n)$ makes up a complete set of zeroes for f_+.

Similarly, there is a smooth function f_- with only simple zeroes, and its zeroes with negative derivatives are just points $(s_1, \dots, s_m)$. Then there are exactly m simple zeroes (u'_1, u'_m) with positive derivatives at some points on every interval (s_j, s_{j+1}). Again, we assume these be all zeroes for f_-. Functions f_+, f_- define each autonomous ODEs at $\pm\infty$. The scheme of the asymptotically autonomous ODE under construction is the following. Take an annulus (cylinder) K on the plane $\mathbb{R}^2$ with polar coordinates $(r, \varphi) : 1 \le r \le 2$. We mark on the outer boundary $r = 2$ zeroes of the function f_+ in accordance with the agreement that $\varphi/2\pi$ corresponds to the coordinate on S^1. Similarly, we mark zeroes of f_- at the inner boundary $r = 1$ with the same agreement. Now we join points $(u_1, \dots, u_n)$ on the circle $r = 2$ by the disjoint paths with points $(s'_1, \dots s'_m)$ on the circle $r = 1$ in accordance with the equipped set and the points $(s_1, \dots s_m)$ on the circle $r = 1$ with the points $(u'_1, \dots, u'_m)$ on the circle $r = 2$.

Namely, take the point u_1 of the equipped set. There is the unique IC with an exponential dichotomy of the unstable type on $\mathbb{R}_+$ corresponding to u_1. This IC belongs to the unique unstable bunch of ICs. This bunch defines two boundary ICs with a dichotomy of the stable type on $\mathbb{R}_-$, that is, two neighboring points s_j, s_{j+1} (j is taken by mod m) from the equipped set. Within the interval on the circle $r = 1$ between its related zeroes s_j, s_{j+1} with negative derivatives there is only one zero u'_j with the positive derivative. We joint by a simple path point u_1 on the circle

$r = 2$ with u'_j on the circle $r = 1$. Let us cut the annulus along this path, we get a curvilinear rectangle which lateral sides are the doubled path, its upper side is the former circle $r = 2$ and its lower side is the former circle $r = 1$. The construction proceeds in this rectangle.

Take now u_2. If the equipped set for ODE does not contain s-points between u_1 and u_2 (recall that u_2 is the next along the orientation), then both u_1, u_2 belong to the same interval of S^1_0 defined by s-points. This means, due to the construction, that both ICs corresponding to u_1, u_2 belong to the same unstable bunch. This means that as $t \to -\infty$ both these ICs are asymptotically approach to each other. Then we need to join with a path point u_2 on $r = 2$ with the same point u'_j as for u_1, this path should not intersect the previous path, that is, it should belong to the rectangle. This second path distinguishes a curvilinear triangle with boundaries composed from two paths and that arc on the circle $r = 2$ with extreme points u_1, u_2 which contains a unique s'-zero. We cut off mentally this triangle from the rectangle and get new rectangle.

Suppose now that on S^1_0 between u_1 and u_2 (in accordance with the orientation of S^1_0) there is at least one s-point of the equipped set, say s_1 (one can always regard this renumbering, if necessary, the set of s-points). On the interval in S^1_0 defined by two neighboring points (u_1, u_2) we get the ordered set of s-points $s_1, s_2, \dots, s_{k_1}$. The next following by orientation s-point, s_{k_1+1}, (it may coincide with s_1, if $k_1 = m$) lies out of (u_1, u_2). Now we choose an interval on the bottom circle $r = 1$ between two zeroes with negative derivatives of the function f_- corresponding s_{k_1}, s_{k_1+1}. Due to the choice of f_-, between these two zeroes there is a unique zero with the positive derivative u'_{k_1}. Join by a path points u_2 on $r = 2$ and u'_{k_1} on $r = 1$ in such a way that two paths constructed and two arcs of the circle $r = 2$ and $r = 1$ would give a rectangle in the annulus. To complete the first step of the induction we connect within the sub-rectangle obtained points $s_1, s_2, \dots, s_{k_1}$ by paths with the unique point s'_1 on $r = 2$ corresponding to a zero of f_+ with the negative derivative on the interval (u_1, u_2).

Thus we have made one step of the induction. Repeating this procedure gives the annulus with all paths (see Fig.5). This construction gives the asymptotically autonomous gradient-like ODE with its limiting ODEs defined by functions f_+, f_-. The following theorem states this.

Theorem 8.1. *For any gradient-like nonautonomous ODE v on S^1 there is an asymptotically autonomous gradient-like ODE being uniformly equivalent to v.*

Proof. To construct such an ODE we take on the cylinder $\mathbb{R} \times S^1$ the foliations generated by $\dot{x} = f_-$ for $t < -T$ and the foliation for $\dot{x} = f_+$ when $t > T$. In the annulus $t = -T$ and $t = T$ we construct smooth foliation as in the example above in accordance with behavior of paths constructed. This gives the needed asymptotically autonomous ODE. □

9. **Almost periodic gradient-like equations on S^1.** In this section we apply the theory developed in previous sections to the classical nonautonomous case – almost periodic in time scalar differential equations on the circle $S^1 = \mathbb{R}/\mathbb{Z}$

$$\dot{x} = f(t, x),\ f(t, x+1) \equiv f(t, x). \tag{18}$$

We assume the equation, as above, be gradient-like one. The equation (18) is generated by the map $\hat{f} : \mathbb{R} \to V^r(S^1)$, $r \geq 1$, being almost periodic and therefore uniformly continuous.

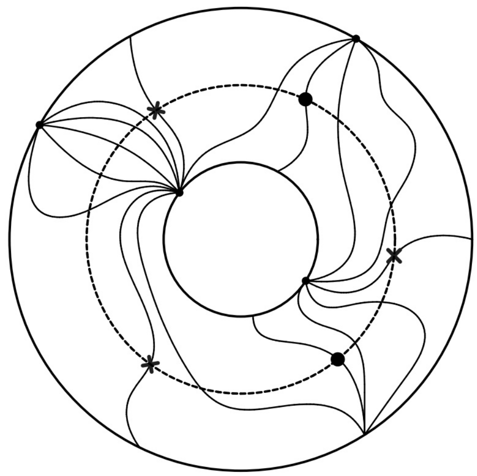

FIGURE 5. Construction of asymptotically autonomous ODE: crosses correspond to u-solutions, bold points correspond to s-solutions.

Let us recall some notions from the theory of almost periodic functions [24, 7, 25]. A continuous map $h : \mathbb{R} \to B$ into Banach space B is called *almost periodic*, if for any $\varepsilon > 0$ there exists a relatively dense set $L(\varepsilon) \subset \mathbb{R}$ of ε-almost periods such that for any $l \in L(\varepsilon)$ the following inequality holds

$$\sup_{t\in\mathbb{R}} ||h(t+l) - h(t)|| < \varepsilon.$$

Recall that a subset $L \subset \mathbb{R}$ is relatively dense, if there is a positive number $T \in \mathbb{R}$ such that any interval $(a, a+T)$, $a \in \mathbb{R}$, contains at least one number $l \in L$.

There is another definition of an almost periodic function introduced by Bochner. It relies on the following theorem by Bochner [3]. For a bounded continuous function $h : \mathbb{R} \to B$ consider the sequence of shifted functions $\{h(t+t_n)\}$.

Definition 9.1. Function h is called normal, if for any sequence of its shifts $\{h(t+t_n)\}$ there is a subsequence $\{h(t+t_{n_k})\}$ such that this subsequence converges in the topology of uniform convergence on $\mathbb{R}$.

Theorem 9.2 (Bochner). *A continuous function $h : \mathbb{R} \to B$ is almost periodic iff it is normal.*

Trivial examples of gradient-like almost periodic equations on S^1 are the following ones. The first is an autonomous differential equation on S^1, $\dot{x} = f(x)$, when a C^1-smooth 1-periodic function has only simple zeroes (where $f' \neq 0$). For the second equation consider the case when $f(t,x)$, $f(t,x+1) \equiv f(t,x)$, is T-periodic in t and the related Poincaré map on S^1 in the period T is a diffeomorphism of S^1 with a rational rotation number. Then, as is known, all solutions are either nT-periodic or tend to nT-periodic solutions as $t \to \pm\infty$. The integer $n \in \mathbb{N}$ is the same for all periodic solutions. If all periodic solutions have multipliers distinct of unity (i.e. the related periodic points of the Poincaré map are hyperbolic), then this periodic ODE is gradient-like one. A somewhat less trivial example is a small smooth almost periodic perturbation of any of these two examples.

The second example shows that an almost periodic solution can go around S^1 infinitely many times when t increases (decreases). For instance, any function of the form $x(t) = \alpha t + u(t)$ (mod 1) with any $\alpha \in \mathbb{R}$ and arbitrary almost periodic function u is almost periodic on S^1. In fact, in this case we have a particular example of an almost periodic mapping which takes its values in some metric space (X, d) with the metrics d. Recall that a continuous mapping $g : \mathbb{R} \to X$ is almost periodic, if for any $\varepsilon > 0$ there is a relatively dense set $L(\varepsilon) \subset \mathbb{R}$ such that for any $l \in L(\varepsilon)$ the following inequality holds

$$d(g(t + l), g(t)) \leq \varepsilon \text{ for any } t \in \mathbb{R}.$$

Here we have $X = S^1 = \mathbb{R}/\mathbb{Z}$ with the standard metrics $d(x, y) = \min\{|x - y|, 1 - |x - y|\}$, $0 \leq x, y \leq 1$.

Another equivalent definition of a.p. mapping $g : \mathbb{R} \to X$ is due to Bochner: any sequence of shifts $g(t+\tau_n)$ is precompact in the topology of the uniform convergence of mappings $\mathbb{R} \to X$.

For the case of S^1 a function $x(t)$ be an almost periodic function with values on S^1 if it is almost periodic as a function $\mathbb{R} \to \mathbb{C}$ understanding S^1 as being embedded into $\mathbb{C}$ as $|z| = 1$.

In this section we prove the following theorem

Theorem 9.3. *Suppose a differential equation $\dot{x} = f(t, x)$ on S^1 is gradient-like and almost periodic in t. Then each its solution which possesses an exponential dichotomy of the unstable type on $\mathbb{R}_+$ is almost periodic and in fact possesses an exponential dichotomy of the unstable type on $\mathbb{R}$, and each its solution with an exponential dichotomy of the stable type on $\mathbb{R}_-$ is almost periodic and possesses an exponential dichotomy of the stable type on $\mathbb{R}$. These a.p. solutions can be united into finitely many $n \in \mathbb{N}$ pairs – one stable and one unstable – bounded a closed strip in $\mathbb{R} \times S^1$. The union of these strips makes up the whole $\mathbb{R} \times S^1$. Any solution within such a strip, distinct from the boundary ones, tend to the boundary stable almost periodic solution as $t \to \infty$ and to the boundary unstable a.p. solution as $t \to -\infty$. This ODE is uniformly equivalent to the autonomous scalar differential equation $\dot{x} = f(x)$ with a smooth function f having $2n$ simple zeroes on S^1.*

To begin with we formulate several auxiliary statements.

Lemma 9.4. *Suppose γ be an integral curve possessing an exponential dichotomy of the unstable type on $\mathbb{R}_+$. Then there is an uniform neighborhood of γ in $\mathbb{R}_+ \times S^1$ such that any IC intersecting this neighborhood belongs to the neighborhood only during a finite time on $\mathbb{R}_+$. The lateral boundary curves of this neighborhood are uniformly transversal to ICs intersecting them.*

Such a neighborhood will be called an *isolating* neighborhood of such integral curve. As such a neighborhood of the IC related to γ a neighborhood defined by a level of the Lyapunov function corresponding to the solution $x_0(t)$ defining γ can be chosen (see above).

Lemma 9.5. *Suppose some IC of the unstable type on $\mathbb{R}_+$ for ODE (18) is given. There is $\delta > 0$ such that if $||f(t, x) - g(t, x)||_{V^1(S^1)} < \delta$, then the boundary curves of an isolating neighborhood U for this IC remain uniformly transversal for IC of the perturbed equation $\dot{x} = g(t, x)$ and this neighborhood contains the only IC $x_g(t)$ which stay wholly in U for all $t \in \mathbb{R}_+$. This IC $x_g(t)$ depends continuously on $||f(t, x) - g(t, x)||_{V^1(S^1)}$ at the "point" f.*

Proofs of these lemmata can be obtained using standard results of the theory of exponential dichotomy, see [8].

Consider some solution $x_0(t)$ of the differential equation (18) which possesses an exponential dichotomy of the unstable type on $\mathbb{R}_+$. Since this equation is gradient-like one, such solution exists and, in virtue of Assumption 3, it also possesses exponential dichotomy of the unstable type on $\mathbb{R}_-$. This implies this solution to possess the exponential dichotomy of unstable type on the whole $\mathbb{R}$. Choose some its isolating neighborhood U on $\mathbb{R}$ and consider any sequence of shifts $\tau_n \to \infty$ as $n \to \infty$. We need to prove that the sequence of shifted functions $x_0(t+\tau_n)$ is precompact, i.e. a subsequence $n_k \to \infty$ exists such that the subsequence $x_0(t+\tau_{n_k})$ converges in the topology of uniform convergence of maps $\mathbb{R} \to S^1$.

Since ODE (18) is almost periodic, there exists some subsequence $n_k \to \infty$ as $k \to \infty$ such that the sequence of shifted functions $f(t+\tau_{n_k}, x)$ converges in the topology of uniform convergence in the space of continuous maps $\mathbb{R} \to V^r(S^1)$. Thus, for any $\varepsilon > 0$ there is an integer $K(\varepsilon) \in \mathbb{N}$ such that for any $k, k' > K(\varepsilon)$ the inequality holds for all $t \in \mathbb{R}$: $||f(t+\tau_{n_k}, x) - f(t+\tau_{n_{k'}}, x)|| < \varepsilon$, where the norm is taken in the space $V^r(S^1)$. Differential equation $\dot{x} = f(t+\tau_{n_k}, x)$ evidently has the solution $x_0(t+\tau_{n_k})$. If U is an isolating neighborhood for IC corresponding to $x_0(t)$, then its shift on t_{n_k} gives the isolating neighborhood of shifted solution. For ε small enough ODEs with r.h.s. $f(t+\tau_{n_k}, x)$ and $f(t+\tau_{n'_k}, x)$ are close enough and since f is rough, then all shifted equations are also rough with the same radius of roughness. In particular, for ε small enough an equimorphism realizing the uniform equivalence of the foliation for ODE $\dot{x} = f(t+\tau_{n_k}, x)$ and the foliation for ODE $\dot{x} = f(t+\tau_{n_{k'}}, x)$ is close to the identity map $id_{\mathbb{R}\times S^1}$. This implies that in the isolating neighborhood of the solution $x_0(t+\tau_{n_k})$ can lie wholly only one solution of ODE with r.h.s. $f(t+\tau_{n_{k'}}, x)$ and this solution has an exponential dichotomy of the unstable type on $\mathbb{R}$. Thus, we get a unique solution of the ODE $\dot{x} = f(t+\tau_{n_{k'}}, x)$ with an exponential dichotomy of the unstable type on $\mathbb{R}$. Shifting it back in time on $-\tau_{n_{k'}}$ allows us to find a solution $x_1(t)$ of the same type for the initial ODE (18). We can do the same procedure for $x_0(t)$ changing the sequence τ_n and subsequences n_k. In such a way, we shall find several solutions with the same type of dichotomy of the initial ODE which are connected by this limiting procedure with $x_0(t)$. Since there are finitely many solutions with the dichotomy of unstable type on $\mathbb{R}$, we break all such solutions into several groups. It is clear that if, using this procedure, we get a solution $x_1(t)$ from the solution $x_0(t)$ and next do it from $x_1(t)$ to $x_2(t)$, then combining a proper sequence of shifts we get $x_2(t)$ from $x_0(t)$, as well. Thus, solutions from different groups cannot be obtained from each other by the limit. But all solutions from the same group can be obtained from each other by the given procedure. In particular, this means that starting from solution $x_0(t)$ we can get $x_0(t)$ itself using some sequences of shifts. In a sense, one group of solutions corresponds to one periodic orbit and its shifts on the periods if one considers a periodic gradient-like ODE. Observe, that solutions from one group are separated from each other: these solutions are separated by their uniform isolating neighborhoods.

Thus we proved that shifts of a given solutions $x_0(t)$ with the exponential dichotomy of unstable type on $\mathbb{R}$ is precompact: its limit sets consists of solutions of one group. Therefore, this solution is almost periodic.

The same considerations work if we start from a solution of ODE (18) with the exponential dichotomy of stable type on $\mathbb{R}_-$. Then we conclude, due to gradient-likeness of (18), that this solution possesses by the exponential dichotomy of stable type on $\mathbb{R}$. After that we again break solutions of such type into groups, and so forth. Solutions from different groups of stability cannot coincide and therefore all extended phase manifold $\mathbb{R} \times S^1$ is divided into several strips whose boundary ICs are one stable and one unstable ones.

As an immediate application of this theorem, let us consider an almost periodic perturbation of a periodic gradient-like ODE on S^1. For the unperturbed ODE its Poincaré map in the period is rough, that is, it has a rational rotation number and all its periodic points are hyperbolic. Under a sufficiently small a.p. perturbation near each hyperbolic periodic IC in $\mathbb{R} \times S^1$ a unique a.p. IC arises. The type of exponential dichotomy for the a.p. solutions on $\mathbb{R}$ will be the same as was for the related periodic solution of the unperturbed equation. All other solutions will tend to a.p. solutions as $t \to \pm\infty$. Thus we get

Corollary 1. *A sufficiently small a.p. perturbation of a periodic gradient-like ODE is uniformly equivalent to an autonomous ODE with the same number of its simple zeroes as for the number of periodic orbits for the Poincaré map.*

It is worth remarking that if an almost periodic ODE fails to be rough then it may not have an almost periodic solutions at all. For instance, a periodic ODE on S^1 having an irrational rotation number and realizing the Denjoy case for its Poincaré map (for C^1-smooth f) has not any almost periodic solutions. The first such example for an almost periodic case was constructed in [32] (see its more detailed consideration in [23]).

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