THE GEOMETRY OF QUADRANGULAR CONVEX PYRAMIDS

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Abstract. A convex quadrangular pyramid $ABCDE$, where $ABCD$ is the base and E — the apex, is called *strongly flexible*, if it belongs to a continuous family of pairwise non-congruent quadrangular pyramids that have the same lengths of corresponding edges. $ABCDE$ is called *strongly rigid*, if such family does not exist. We prove the strong rigidity of convex quadrangular pyramids and prove that strong rigidity fails in the self-intersecting case. Let $L = \{l_1, \ldots, l_8\}$ be a set of positive numbers, then a *realization* of L is a convex quadrangular pyramid $ABCDE$ such, that $|AB| = l_1$, $|BC| = l_2$, $|CD| = l_3$, $|DA| = l_4$, $|EA| = l_5$, $|EB| = l_6$, $|EC| = l_7$, $|ED| = l_8$. We prove that the number of pairwise non-congruent realizations is $\leqslant 4$ and give an example of a set L with three pairwise non-congruent realizations.

1. Introduction

A polyhedron M in the three dimensional space $\mathbb{R}^3$ is called *flexible* (see [2], [4]), if there exists a continuous family of polyhedra M_t, $0 \leqslant t$, where

(1) $M_0 = M_t$;
(2) polyhedra M_t have the same combinatorial structure, as M;
(3) corresponding faces of M and M_t are congruent;
(4) angles between (some) faces of M and corresponding faces of M_t are different.

A not flexible polyhedron is called *rigid*. The Cauchy Rigidity Theorem states that a convex polyhedron is rigid (see [2], [4]). However, a non-convex polyhedron can be flexible [1].

We introduce a notion of the *strong flexibility* and the *strong rigidity*.

Definition 1.1. A polyhedron M in the three dimensional space $\mathbb{R}^3$ is *strongly flexible*, if there exists a continuous family of polyhedra M_t, $0 \leqslant t$, where

(1) $M_0 = M$;
(2) polyhedra M_t have the same combinatorial structure, as M;
(3) corresponding edges of M and M_t are equal;
(4) some face(s) of M and the corresponding face(s) of M_t are not congruent.

A not strongly flexible polyhedron is called *strongly rigid*.

Remark 1.1. A cube is rigid, but strongly flexible. A triangular pyramid is, of course, rigid and strongly rigid.

A convex quadrangular pyramid is the simplest polyhedron (after triangular pyramid). We will prove the following statement.

Theorem 3.1. *A convex quadrangular pyramid is strongly rigid.*

A non-convex quadrangular pyramid is also strongly rigid (**Consequence 3.1.**), but strong rigidity fails in the self-intersecting case (**Example 3.1.**).

Our quadrangular pyramids will be labelled, i.e. A, B, C, D will be vertices of base in order of going around it and E will be the apex. For a given set L of positive numbers $L = \{l_1, \ldots, l_8\}$ we ask about the existence of a labelled quadrangular pyramid $ABCDE$ such that $|AB| = l_1$, $|BC| = l_2$, $|CD| = l_3$, $|DA| = l_4$, $|EA| = l_5$, $|EB| = l_6$, $|EC| = l_7$ and $|ED| = l_8$. Such pyramid will be called a *realization* of the set L.

Theorem 4.1. *The number of pairwise non-congruent realizations of a set L is $\leqslant 4$.*

We give an example (**Example 4.1.**) of the set with three pairwise non-congruent realizations.

2. Strong flexibility

Theorem 2.1. *A generic polyhedron in $\mathbb{R}^3$ is strongly rigid.*

Proof. In what follows by k-face of a polyhedron we will understand a face with k vertices. Let the number of k-faces of a polyhedron M be n_k, $k = 3, 4, \dots, m$. Then it has $e = \frac{1}{2}\sum_{i=3}^m i \cdot n_i$ edges and

$$v = r + 2 - \sum_{i=3}^m n_i = \frac{\sum_{i=3}^m (i-2)\cdot n_i}{2} + 2$$

vertices. Let us assume that some m-face rigidly belongs to xy-plane and some edge of this face is rigidly fixed. Then vertices of this face have $2(m-2)$ degrees of freedom and all other vertices have $3(v-m)$ degrees of freedom. Thus, all vertices have in sum

$$2(m-2) + 3(v-m) = \frac{3\cdot\sum_{i=3}^m (i-2)\cdot n_i}{2} - m + 2$$

degrees of freedom. But we have relations also:

- lengths of all edges are fixed — $(r-1)$ relations;
- vertices of each face are contained in one plane — $(i-3)$ relations for each i-face.

Thus, the number of relations is

$$r - 1 + \sum_{i=3}^m n_i\cdot(i-3) - (m-3) = \frac{3\cdot\sum_{i=3}^m (i-2)\cdot n_i}{2} - m + 2.$$

We see, that the number of relations equals the number of degrees of freedom, thus, M is strongly rigid. □

Remark 2.1. Only polyhedra with symmetries can be strongly flexible.

3. Strong rigidity of a convex quadrangular pyramid

Theorem 3.1. *A convex quadrangular pyramid is strongly rigid.*

Proof. We will assume that the base $ABCD$ of a quadrangular pyramid $ABCDE$ belongs the the xy-plane, vertex A is at origin, vertex B has coordinates $(1,0)$, the quadrangle $ABCD$ belongs to the upper half-plane and the apex E belongs to the upper half-space. Let coordinates of the vertex D be (a_1, b_1), of the vertex C — (a_2, b_2) and of the vertex E — (a_3, b_3, c_3). Let us assume that $ABCDE$ is strongly flexible and there exists a continuous deformation $A'B'C'D'E'$, where

$$A' = (0,0),\ B' = (1,0),\ C' = (a_2 + x_2, b_2 + y_2),\ D' = (a_1 + x_1, b_1 + y_1),\ E' = (a_3 + x_3, b_3 + y_3, c_3 + z_3)$$

and the following system holds:

$$\begin{cases} (a_1 + x_1)^2 + (b_1 + y_1)^2 = a_1^2 + b_1^2 \\ (a_2 + x_2 - 1)^2 + (b_2 + y_2)^2 = (a_2 - 1)^2 + b_2^2 \\ (a_2 + x_2 - a_1 - x_1)^2 + (b_2 + y_2 - b_1 - y_1)^2 = (a_2 - a_1)^2 + (b_2 - b_1)^2 \\ (a_3 + x_3)^2 + (b_3 + y_3)^2 + (c_3 + z_3)^2 = a_3^2 + b_3^2 + c_3^2 \\ (a_3 + x_3 - 1)^2 + (b_3 + y_3)^2 + c_3 + z_3)^2 = (a_3 - 1)^2 + b_3^2 + c_3^2 \\ (a_3 + x_3 - a_1 - x_1)^2 + (b_3 + y_3 - b_1 - y_1)^2 + (c_3 + z_3)^2 = (a_3 - a_1)^2 + (b_3 - b_1)^2 + c_3^2 \\ (a_3 + x_3 - a_2 - x_2)^2 + (b_3 + y_3 - b_2 - y_2)^2 + (c_3 + z_3)^2 = (a_3 - a_2)^2 + (b_3 - b_2)^2 + c_3^2 \end{cases}$$

The elimination of variables (see [3]) x_3, y_3, z_3, x_2, y_2 and y_1 from this system gives us a polynomial $R(x_1, a_1, b_1, a_2, b_2, a_3, b_3, c_3)$ of degree 3 in variable x_1.

Thus, we have a new system

$$\begin{cases} r_0(a_1,b_1,a_2,b_2,a_3,b_3,c_3)=0 \\ r_1(a_1,b_1,a_2,b_2,a_3,b_3,c_3)=0 \\ r_2(a_1,b_1,a_2,b_2,a_3,b_3,c_3)=0 \\ r_3(a_1,b_1,a_2,b_2,a_3,b_3,c_3)=0 \end{cases}$$

where r_0, r_1, r_2, r_3 are coefficients of the polynomial R, as polynomial in x_1. The elimination of variables b_1, a_3, b_3, c_3 from this system gives us two solutions:

$$a_2 = a_1 + 1 \text{ and } a_1 = \frac{a_2^3 - a_2^2 + a_2 b_2^2 + b_2^2}{a_2^2 + b_2^2}.$$

The second solution gives

$$b_1 = \frac{b_2 \cdot (a_2^2 - 2a_2 + b_2^2)}{a_2^2 + b_2^2} \Rightarrow \begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} = -b_2 < 0\,.$$

Thus, we have a clockwise rotation from the vector $\overline{OC}$ to the vector $\overline{OD}$, i.e. the quadrangle $ABCD$ is not convex.

If $a_2 = a_1 + 1$, then it is easy to obtain, that $b_2 = b_1$, $b_3 = \frac{1}{2} b_1$ and $a_3 = \frac{1}{2} \cdot (a_1 + 1)$, i.e. the base is a parallelogram and the apex is just above its center O. Thus, $|EA| = |EC|$ and $|EB| = |ED|$.

Let $ABCDE$ be strongly flexible and $A_1B_1C_1D_1E_1$ be a member of our family. Then $A_1B_1C_1D_1$ is also a parallelogram with the same lengths of edges. As $|E_1A_1| = |E_1C_1|$ and $|E_1B_1| = |E_1D_1|$, then apex E_1 is just above the center O_1 of the base. Let $|A_1O_1| > |AO|$, then $|E_1O_1| < |EO|$ (because $|E_1A_1| = |EA|$). But then $|B_1O_1| < |BO|$, thus $|E_1B_1| < |EB|$. Contradiction. ☐

Consequence 3.1. *A non-convex quadrangular pyramid is strongly rigid.*

Proof. Using rotations, shifts and scalings we can assume, that non-convex quadrangle $ABCD$ is in the upper half-plane, $A = (0,0)$ and $B = (1,0)$.

If this pyramid is strongly flexible, then we are in the scope of the second solution of the previous theorem. We know that the rotation from the vector $\overline{AB}$ to the vector $\overline{AC}$ is counter clockwise, but the rotation from the vector $\overline{AC}$ to the vector $\overline{AD}$ is clockwise.

As

$$b_1 = \frac{b_2 \cdot (a_2^2 - 2a_2 + b_2^2)}{a^2 + b^2} > 0,$$

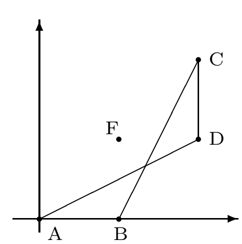
then $a_2^2 - 2a_2 + b_2^2 > 0$. The line BC has the equation $(a_2 - 1)y - b_2 x + b_2 = 0$. As $b_2 > 0$ and

$$(a_2 - 1) \cdot \frac{b_2(a_2^2 - 2a_2 + b_2^2)}{a_2^2 + b_2^2} - b_2 \cdot \frac{a_2^3 - a_2^2 + a_2 b_2^2 + b_2^2}{a_2^2 + b_2^2} + b_2 = -\frac{b_2 \cdot (a_2^2 - 2a_2 + b_2^2)}{a_2^2 + b_2^2} < 0,$$

then segments AD and BC intersect. ☐

Example 3.1. A self-intersecting quadrangular pyramid can be strongly flexible. Here is an example.

Let us consider the self-intersecting pyramid $ABCDE$: $A = (0,0)$, $B = (1,0)$, $C = (2,2)$, $D = (2,1)$, $E = (1,1,1)$.



Here F is the projection of the apex E to xy-plane. We will prove that this pyramid belongs to a continuous family that realizes strong flexibility.

Let $A'B'C'D'E'$ be a member of this family: $A'=(0,0)$, $B'=(1,0)$, $C'=(x_2,y_2)$, $D'=(x_1,y_1)$, $E'=(x_3,y_3,z_3)$. Then

$$\left\{\begin{array}{l} x_1^2+y_1^2=5 \\ (x_2-1)^2+y_2^2=5 \\ (x_2-x_1)^2+(y_2-y_1)^2=1 \\ x_3^2+y_3^2+z_3^2=3 \\ (x_3-1)^2+y_3^2+z_3^2=2 \\ (x_3-x_1)^2+(y_3-y_1)^2+z_3^2=2 \\ (x_3-x_2)^2+(y_3-y_2)^2+z_3^2=3 \end{array}\right. \Rightarrow \left\{\begin{array}{l} x_1^2+y_1^2=5 \\ (x_2-1)^2+y_2^2=5 \\ x_1x_2+y_1y_2-x_2=4 \\ x_3=1 \\ y_3^2+z_3^2=2 \\ x_1+y_1y_3=3 \\ y_2y_3=2 \end{array}\right.$$

Actually equations of this system are not independent — all variables are functions of y_1:

$$y_1^2y_3^2-6y_1y_3+y_1^2+4=0,\ y_2y_3=2,\ x_1+y_1y3=3,\ x_1x_2+y_1y_2-x_2=4,\ y_3^2+z_3^2=2.$$

As

$$(y_1^2y_3^2-6y_1y_3+y_1^2+4)'_{y_1}(y_1=1,y_3=1)\neq 0 \text{ and } (y_1^2y_3^2-6y_1y_3+y_1^2+4)'_{y_3}(y_1=1,y_3=1)\neq 0,$$

then we have continuous family of quadrangular self-intersecting pyramids whose edges have fixed lengths.

4. Realizations

Let lengths of all edges of a labelled quadrangular pyramid $ABCDE$ are given. As there cannot exist a continuous family of such pyramids, we can ask about the number of them (pairwise non congruent).

Definition 4.1. Let L be a set of eight positive numbers $L=\{l_1,\dots,l_8\}$. A *realization* of this set is a convex quadrangular pyramid $ABCDE$, $ABCD$ — the base, E — the apex, such that

$$|AB|=l_1,\ |BC|=l_2,\ |CD|=l_3,\ |DA|=l_4,\ |EA|=l_5,\ |EB|=l_6,\ |EC|=l_7,\ |ED|=l_8.$$

We will assume that $l_1=1$.

Theorem 4.1. *The number of realizations of a set L is $\leqslant 4$.*

Proof. Let a convex quadrangular pyramid $ABCDE$ be in the standard position. Using the notation of the previous section, we obtain the system

$$\left\{\begin{array}{l} x_1^2+y_1^2=l_4 \\ (x_2-1)^2+y_2^2=l_2 \\ (x_2-x_1)^2+(y_2-y_1)^2=l_3 \\ x_3^2+y_3^2+z_3^2=l_5 \\ (x_3-1)^2+y_3^2+z_3^2=l_6 \\ (x_3-x_2)^2+(y_3-y_2)^2+z_3^2=l_7 \\ (x_3-x_1)^2+(y_3-y_1)^2+z_3^2=l_8 \end{array}\right.$$

The elimination of variables x_3,y_3,z_3,x_2,y_2,y_1 gives a polynomial of the forth degree in x_1. □

Example 4.1. We can give an example of the set L, which has three realizations.

Let $ABCDE$ be a convex quadrangular pyramid in standard position, where $|BC|=2$, $|CD|=\sqrt{2}$, $|DA|=1$, $|EA|=\sqrt{2}$, $|EB|=\sqrt{5}$, $|ED|=\sqrt{3}$ and the length of the edge EC we will define later. Using notation of the section 3, we can write the system

$$\begin{cases} x_1^2+x_2^2=1 \\ (x_2-1)^2+y_2^2=4 \\ (x_2-x_1)^2+(y_2-y_1)^2=2 \\ x_3^2+y_3^2+z_3^2=2 \\ (x_3-1)^2+y_3^2+z_3^2=5 \\ (x_3-x_1)^2+(y_3-y_1)^2+z_3^2=3 \end{cases} \Rightarrow \begin{cases} x_1^2+y_1^2=1 \\ x_2^2+y_2^2-2x_2=3 \\ x_1x_2+y_1y_2-x_2=1 \\ x_3=-1 \\ y_3^2+z_3^2=1 \\ x_1-y_1y_3=0 \end{cases}$$

The value of the angle $\angle A=\alpha$ uniquely defines the quadrangle $ABCD$ and also uniquely defines the position of the apex E. Thus, $|EC|^2$ is the function of α.

the value of α is changed from the minimal value $\alpha_0\approx 0.9449$ (here points A, C and D are on one line and $|EC|^2\approx 7.8284$) to the maximal value $\alpha_1=3\pi/4$ (here $y_3=-1$, $z_3=0$ and $|EC|^2\approx 9.3067$).

$|EC|^2$ increases on the interval $(\alpha_0,\pi/2)$. The point $\pi/2$ is the local maximum: $|EC|^2=9$. Then $|EC|^2$ decreases on the interval $(\pi/2,\approx 1.9404)$ and in the end of this interval it has the local minimum ≈ 8.9555. After that $|EC|^2$ increases on the interval $(\approx 1.9404, 3\pi/4)$. It means that the set $L=\{1,2,\sqrt{2},1,\sqrt{2},\sqrt{5},r,\sqrt{3}\}$, where $8.9555<r<9$, has three pairwise non-congruent realizations.

References

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