



Lie algebras of vertical derivations on semiaffine varieties with torus actions



Ivan Arzhantsev^{a,1}, Alvaro Liendo^{b,*,2}, Taras Stasyuk^c

^a National Research University Higher School of Economics, Faculty of Computer Science, Pokrovsky Boulevard 11, Moscow, 109028, Russia

^b Instituto de Matemática y Física, Universidad de Talca, Casilla 721, Talca, Chile

^c Department of Mathematics and Mechanics, Lomonosov Moscow State University, Leninskie Gory 1, Moscow, 119991, Russia

ARTICLE INFO

Article history:

Received 5 February 2019

Received in revised form 12 June 2020

Available online 15 July 2020

Communicated by S. Kovács

MSC:

Primary 13N15; 14L30; secondary 14M25; 17B66

Keywords:

Torus action

Divisorial fan

Additive group action

Demazure root

Derivation

Lie algebra

ABSTRACT

Let X be a normal variety endowed with an algebraic torus action. An additive group action α on X is called vertical if a general orbit of α is contained in the closure of an orbit of the torus action and the image of the torus normalizes the image of α in $\text{Aut}(X)$. Our first result in this paper is a classification of vertical additive group actions on X under the assumption that X is proper over an affine variety. Then we establish a criterion as to when the infinitesimal generators of a finite collection of additive group actions on X generate a finite-dimensional Lie algebra inside the Lie algebra of derivations of X .

© 2020 Elsevier B.V. All rights reserved.

1. Introduction

Let \mathbf{k} be a field of characteristic zero. The n -dimensional algebraic torus \mathbb{T} over \mathbf{k} is the algebraic group \mathbb{G}_m^n , where \mathbb{G}_m is the multiplicative group, i.e., the set \mathbf{k}^* of non-zero elements in the base field endowed with its natural structure of algebraic group under multiplication. A \mathbb{T} -variety is a normal variety endowed with a faithful action of the algebraic torus. The complexity of a \mathbb{T} -variety is the codimension of a general orbit and since the action is faithful the complexity equals $\dim X - \dim \mathbb{T}$. There are well known

* Corresponding author.

E-mail addresses: arjantsev@hse.ru (I. Arzhantsev), aliendo@inst-mat.atalca.cl (A. Liendo), taras4834@gmail.com (T. Stasyuk).

¹ The first author was supported by the grant RSF 19-11-00172.

² The second author was partially supported by the projects FONDECYT Regular 1160864 and 1200502.

combinatorial descriptions of \mathbb{T} -varieties. The \mathbb{T} -varieties of complexity zero are called toric varieties and were first introduced by Demazure in [7], see the textbooks [15,11,6] for a modern account on the subject. They are described by certain collections of rational polyhedral cones called fans. For higher complexity, several partial classifications were given until a full classification of \mathbb{T} -varieties was achieved in [1,2], see [3] and the references therein for a historical account.

Let now \mathbb{G}_a be the additive group, i.e., the base field \mathbf{k} endowed with its natural structure of algebraic group under addition. Let X be a variety and $\alpha : \mathbb{G}_a \times X \rightarrow X$ be a \mathbb{G}_a -action on X . The *infinitesimal generator* of the action is the derivation $D : \mathcal{O}_X \rightarrow \mathcal{O}_X$ of the structure sheaf of X given by

$$\Gamma(U, D) : \Gamma(U, \mathcal{O}_X) \rightarrow \Gamma(U, \mathcal{O}_X), \quad f \mapsto \left[\frac{d}{dt}(\alpha^* f) \right]_{t=0}.$$

An additive group action on a \mathbb{T} -variety X is called *compatible* if the image of \mathbb{T} in $\text{Aut}(X)$ is contained in the normalizer of the image of the subgroup \mathbb{G}_a . A compatible \mathbb{G}_a -action is called *vertical* if, moreover, a general \mathbb{G}_a -orbit is contained in the closure of a \mathbb{T} -orbit. Letting $\mathbf{k}(X)$ be the field of rational functions on X , we have that a compatible \mathbb{G}_a -action is vertical if and only if $\mathbf{k}(X)^{\mathbb{T}} \subseteq \mathbf{k}(X)^{\mathbb{G}_a}$. In [7] a description of vertical \mathbb{G}_a -actions on toric varieties was given and in [13,14] vertical \mathbb{G}_a -actions on affine \mathbb{T} -varieties were described. Recall that an algebraic variety X is called *semiaffine* if the morphism $X \rightarrow \text{Spec } \Gamma(X, \mathcal{O}_X)$ induced by the global sections functor is proper [12]. In particular, affine and complete varieties are semiaffine. Our first main result contained in Theorem 3.9 is a generalization of such descriptions to vertical \mathbb{G}_a -actions on semiaffine varieties. Our classification is given in terms of the combinatorial description of \mathbb{T} -varieties in [1,2]. The infinitesimal generators of vertical \mathbb{G}_a -actions are in correspondence with certain triples $(\phi, \lambda_\rho, \chi^e)$ where ϕ is a rational function in $\mathbf{k}(X)^{\mathbb{T}}$, λ_ρ is a 1-parameter subgroup of \mathbb{T} corresponding to a ray of the fan Σ of the normalization of a general \mathbb{T} -orbit closure and χ^e is a character of \mathbb{T} such that $\langle \rho, e \rangle = -1$ where $\langle \cdot, \cdot \rangle$ is the canonical pairing realizing the 1-parameter subgroup lattice of a torus as the dual of the character lattice. We denote the infinitesimal generator of the \mathbb{G}_a -action by $D_{\phi, \rho, e}$ and we also call them *root derivations* since the image of the \mathbb{G}_a -action in $\text{Aut}(X)$ is a root subgroup.

Let now $\mathbb{D} = \{D_{\phi_i, \rho_i, e_i}\}_{i=1}^m$ be a finite set of root derivations. The set \mathbb{D} is called *cyclic* if $\langle \rho_{i+1}, e_i \rangle > 0$ for all $i = 1, \dots, m$, where we set $\rho_{m+1} := \rho_1$. Furthermore, a set \mathbb{D} is called *simple* if $\prod_{i=1}^m \phi_i \in \mathbf{k}$ and $\langle \rho_i, e_i \rangle = -1$, $\langle \rho_{i+1}, e_i \rangle = 1$ and $\langle \rho, e_i \rangle = 0$ for all i and for all rays ρ of the fan Σ different from ρ_i and ρ_{i+1} . Our second main result in this paper is the following theorem, see Theorem 5.1 for a more precise statement.

Theorem. *Let $\mathbb{D} = \{D_{\phi_i, \rho_i, e_i}\}_{i=1}^m$ be a finite set of root derivations. Then the Lie algebra generated by \mathbb{D} over \mathbf{k} is finite dimensional if and only if every cyclic subset of \mathbb{D} is simple.*

As an application, we extend to the non-complete case and to arbitrary complexity previous results in [7] and [4]. We say that a linear algebraic group G acts on a \mathbb{T} -variety X *vertically* if the action is faithful, the image of G in $\text{Aut}(X)$ is normalized by \mathbb{T} , and $\mathbf{k}(X)^{\mathbb{T}} \subset \mathbf{k}(X)^G$. In Theorem 6.3 we show that if a linear algebraic group G acts on a \mathbb{T} -variety X vertically, then G is a *group of type A*, i.e., a maximal semisimple subgroup of G is isomorphic to a factor group of the direct product $\text{SL}_{r_1}(\mathbf{k}) \times \dots \times \text{SL}_{r_s}(\mathbf{k})$ for some positive integers r_1, \dots, r_s by a finite central subgroup.

The content of the paper is as follows. In Section 2 we introduce the combinatorial description of \mathbb{T} -varieties due to Altmann, Hausen and Süß that we use in the paper. In Section 3 we provide the announced classification of vertical additive group actions on semiaffine \mathbb{T} -varieties. In Section 4 we show that a simple set \mathbb{D} such that every cyclic subset is also simple generates a Lie algebra isomorphic to $\mathfrak{sl}_r(\mathbf{k})$ for some $r \in \mathbb{Z}_{\geq 2}$. The theorem stated above is proved in Section 5. Finally, in Section 6 we prove the application to linear algebraic groups acting on a \mathbb{T} -variety vertically.

Acknowledgements

The authors would like to warmly thank the anonymous referee for useful comments and corrections.

2. Combinatorial description of \mathbb{T} -varieties

In this chapter we briefly recall the combinatorial description of \mathbb{T} -varieties used in this paper. For more details, see [15,11,6] for toric varieties and [1–3] for general \mathbb{T} -varieties.

2.1. Toric varieties

In this subsection \mathbf{k} is a field of characteristic zero not necessarily algebraically closed. Let M be a lattice of rank n and $N = \text{Hom}(M, \mathbb{Z})$ be its dual lattice. We let $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$, $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\langle \cdot, \cdot \rangle : N_{\mathbb{Q}} \times M_{\mathbb{Q}} \rightarrow \mathbb{Q}$ be the corresponding duality that we also denote by $\langle v, u \rangle = v(u)$. Let $\mathbb{T} = \text{Spec } \mathbf{k}[M]$ be the algebraic torus whose character lattice is M . The torus \mathbb{T} is an algebraic group isomorphic to \mathbb{G}_m^n , where \mathbb{G}_m is the multiplicative group of the base field \mathbf{k} .

Recall that a toric variety X is a normal variety endowed with a faithful regular action $\mathbb{T} \times X \rightarrow X$ of the algebraic torus \mathbb{T} having an open orbit. A fan Σ in $N_{\mathbb{Q}}$ is a finite collection of strictly convex polyhedral cones such that every face of $\sigma \in \Sigma$ is contained in Σ and for all $\sigma, \sigma' \in \Sigma$ the intersection $\sigma \cap \sigma'$ is a face in both cones σ and σ' . A toric variety $X(\Sigma)$ is built from Σ in the following way. For every $\sigma \in \Sigma$, we define an affine toric variety $X(\sigma) = \text{Spec } \mathbf{k}[\sigma^{\vee} \cap M]$, where $\sigma^{\vee} \subseteq M_{\mathbb{Q}}$ is the dual cone of σ and $\mathbf{k}[\sigma^{\vee} \cap M]$ is the semigroup algebra of $\sigma^{\vee} \cap M$, i.e.,

$$\mathbf{k}[\sigma^{\vee} \cap M] = \bigoplus_{u \in \sigma^{\vee} \cap M} \mathbf{k} \cdot \chi^u, \quad \text{with } \chi^0 = 1, \text{ and } \chi^u \cdot \chi^{u'} = \chi^{u+u'}, \forall u, u' \in \sigma^{\vee} \cap M.$$

Furthermore, if $\tau \subseteq \sigma$ is a face of σ , then the inclusion of algebras $\mathbf{k}[\sigma^{\vee} \cap M] \hookrightarrow \mathbf{k}[\tau^{\vee} \cap M]$ induces a \mathbb{T} -equivariant open embedding $X(\tau) \hookrightarrow X(\sigma)$ of affine \mathbb{T} -varieties. The toric variety $X(\Sigma)$ associated to the fan Σ is then defined as the variety obtained by gluing the family $\{X(\sigma) \mid \sigma \in \Sigma\}$ along the open embeddings $X(\sigma) \hookrightarrow X(\sigma \cap \sigma') \hookrightarrow X(\sigma')$ for all $\sigma, \sigma' \in \Sigma$.

Following [12], an algebraic variety X is called semiaffine if the morphism $X \rightarrow \text{Spec } \Gamma(X, \mathcal{O}_X)$ induced by the global sections functor is proper. If, moreover, this morphism is projective, we say that X is semiprojective. In both cases, $\Gamma(X, \mathcal{O}_X)$ is finitely generated and so $\text{Spec } \Gamma(X, \mathcal{O}_X)$ is an affine variety [12, Corollary 3.6]. For instance, complete or affine varieties are semiaffine, while projective or affine varieties are semiprojective. Furthermore, any blow-up of a semiaffine (resp. semiprojective) variety is also semiaffine (resp. semiprojective).

A toric variety $X(\Sigma)$ is semiprojective if and only if the fan Σ is the normal fan of a (non-necessarily bounded) polyhedron in $M_{\mathbb{Q}}$, see [6, Proposition 7.2.9]. Furthermore, $X(\Sigma)$ is semiaffine if and only if the support of the fan $\text{Supp}(\Sigma) = \bigcup_{\sigma \in \Sigma} \sigma$ is a convex set, see [6, Exercise 15.1.7].

2.2. Affine \mathbb{T} -varieties and p -divisors

In the remaining of this section \mathbf{k} is an algebraically closed field of characteristic zero. Let σ be a pointed polyhedral cone in $N_{\mathbb{Q}}$. We define $\text{Pol}_{\sigma}(N_{\mathbb{Q}})$ to be the set of all polyhedra with tail cone σ . We also include the empty set $\emptyset \in \text{Pol}_{\sigma}(N_{\mathbb{Q}})$. The set $\text{Pol}_{\sigma}(N_{\mathbb{Q}})$ with Minkowski sum has the structure of abelian semigroup with identity, where the addition rule for \emptyset is defined as $\emptyset + \Delta = \emptyset$ for all $\Delta \in \text{Pol}_{\sigma}(N_{\mathbb{Q}})$.

Let Y be a normal projective variety. A polyhedral divisor on Y is a formal sum $\mathcal{D} = \sum_Z \mathcal{D}_Z \cdot Z$, where $\mathcal{D}_Z \in \text{Pol}_{\sigma}(N_{\mathbb{Q}})$, $Z \subseteq Y$ is a prime divisor and $\mathcal{D}_Z = \sigma$ for all but finitely many Z . We say that σ is the tail

cone of \mathcal{D} denoted by $\text{tail } \mathcal{D}$. For every $y \in Y$ we define the slice at y by $\mathcal{D}_y = \sum_{Z \ni y} \mathcal{D}_Z$. The locus of \mathcal{D} is $\text{Loc } \mathcal{D} = \{y \in Y \mid \mathcal{D}_y \neq \emptyset\}$. For every $u \in \text{tail } \mathcal{D}^\vee$ we can evaluate \mathcal{D} in u by letting $\mathcal{D}(u)$ be the \mathbb{Q} -divisor in $\text{Loc } \mathcal{D}$ given by

$$\mathcal{D}(u) = \sum_{Z \subseteq \text{Loc}(\mathcal{D})} \min\langle \mathcal{D}_Z, u \rangle \cdot Z,$$

where $Z \subseteq \text{Loc}(\mathcal{D})$ runs through all prime divisors in $\text{Loc}(\mathcal{D})$.

Consider a rational Cartier divisor D on a normal variety Y . The divisor D is semiample if it admits a basepoint-free multiple, i.e. for some $n \in \mathbb{Z}_{>0}$ the sets $Y_f := Y \setminus \text{Supp}(\text{div}(f) + nD)$, where $f \in \Gamma(Y, \mathcal{O}(nD))$, cover Y . Further, D is big if for some $n \in \mathbb{Z}_{>0}$ there is a section $f \in \Gamma(Y, \mathcal{O}(nD))$ with an affine locus Y_f .

A polyhedral divisor \mathcal{D} on Y is called a p -divisor if $\text{Loc } \mathcal{D}$ is semiprojective and

- (i) for every $u \in \text{tail } \mathcal{D}^\vee$ the evaluation $\mathcal{D}(u)$ is semiample;
- (ii) for every $u \in \text{rel.int}(\text{tail } \mathcal{D}^\vee)$ the evaluation $\mathcal{D}(u)$ is big.

Let us recall that the complexity of a faithful algebraic torus action $\mathbb{T} \times X \rightarrow X$ on an algebraic variety X is the difference $\dim X - \dim \mathbb{T}$. In particular, toric varieties are precisely normal algebraic varieties with a torus action of complexity zero.

The main classification result for affine \mathbb{T} -varieties in [1] is the following.

Theorem 2.1. *To any p -divisor \mathcal{D} on a normal projective variety Y one can associate a normal affine \mathbb{T} -variety $X(\mathcal{D}) = \text{Spec } A(\mathcal{D})$ of dimension $\text{rank } N + \dim Y$ and complexity $\dim Y$ given by*

$$A(\mathcal{D}) = \bigoplus_{u \in \text{tail } \mathcal{D}^\vee \cap M} A_u X^u, \quad \text{where } A_u = \Gamma(\text{Loc } \mathcal{D}, \mathcal{O}(\mathcal{D}(u))) \subseteq \mathbf{k}(Y).$$

Conversely, any normal affine \mathbb{T} -variety is equivariantly isomorphic to $X(\mathcal{D})$ for some p -divisor \mathcal{D} on some normal projective variety Y .

2.3. Arbitrary \mathbb{T} -varieties and divisorial fans

We recall the main facts of the description of not necessarily affine \mathbb{T} -varieties given in [2] in terms of divisorial fans. To describe non-affine \mathbb{T} -varieties, we need to describe first \mathbb{T} -invariant open embeddings.

Let Δ be a σ -polyhedron in $N_{\mathbb{Q}}$ and let $e \in \sigma^\vee$. The face of Δ defined by e is the set

$$\text{face}(\Delta, e) = \{p \in \Delta \mid p(e) = \min\langle \Delta, e \rangle\}.$$

Let now $\mathcal{D} = \sum_Z \mathcal{D}_Z \cdot Z$ and $\mathcal{D}' = \sum_Z \mathcal{D}'_Z \cdot Z$ be two p -divisors on Y . We define the support of \mathcal{D} as the union of divisors $Z \in Y$ such that $\mathcal{D}_Z \neq \emptyset$ or $\mathcal{D}_Z \neq \text{tail}(\mathcal{D})$. We say that \mathcal{D}' is a face of \mathcal{D} if $\mathcal{D}'_Z \subseteq \mathcal{D}_Z$ holds for all prime divisors $Z \subseteq Y$, and for any $y \in \text{Loc}(\mathcal{D}')$ there are $e_y \in \sigma^\vee \cap M$ and D_y in the linear system $|\mathcal{D}(e_y)|$ such that

- (i) $y \notin \text{Supp}(D_y)$;
- (ii) $\mathcal{D}'_y = \text{face}(\mathcal{D}_y, e_y)$;
- (iii) $\text{face}(\mathcal{D}'_y, e_y) = \text{face}(\mathcal{D}_{\tilde{y}}, e_y)$ for every $\tilde{y} \in Y \setminus \text{Supp}(D_y)$.

Let \mathcal{D}' be a face of \mathcal{D} . Then $A(\mathcal{D}) \subseteq A(\mathcal{D}')$ and this inclusion is the comorphism of a \mathbb{T} -equivariant open embedding $X(\mathcal{D}') \hookrightarrow X(\mathcal{D})$ of affine \mathbb{T} -varieties [2, Proposition 3.4 and Definition 5.1]. We define the intersection of two p -divisors as the polyhedral divisor

$$\mathcal{D} \cap \mathcal{D}' = \sum_{Z \subseteq Y} (\mathcal{D}_Z \cap \mathcal{D}'_Z) \cdot Z.$$

Let Y be a normal projective variety. A divisorial fan \mathcal{S} on Y is a finite collection of \mathbb{p} -divisors such that for every two $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$ the intersection $\mathcal{D} \cap \mathcal{D}'$ is a face of each and belongs to \mathcal{S} . The set of tail cones $\text{tail } \mathcal{D}$ over all \mathbb{p} -divisors $\mathcal{D} \in \mathcal{S}$ is a fan called the tail fan $\text{tail } \mathcal{S}$ of the divisorial fan \mathcal{S} .

For every divisorial fan \mathcal{S} on Y we can define a \mathbb{T} -scheme $X(\mathcal{S})$ by gluing the family of affine \mathbb{T} -varieties $\{X(\mathcal{D}) \mid \mathcal{D} \in \mathcal{S}\}$ along the \mathbb{T} -invariant affine open sets $\{X(\mathcal{D} \cap \mathcal{D}') \mid \mathcal{D}, \mathcal{D}' \in \mathcal{S}\}$. The main result in [2] is the following.

Theorem 2.2. *Every normal \mathbb{T} -variety is equivariantly isomorphic to $X(\mathcal{S})$ for some divisorial fan \mathcal{S} on a normal projective variety Y .*

Furthermore, in [2, Section 7], conditions for $X(\mathcal{S})$ to be separated and/or complete are given by proving a \mathbb{T} -equivariant version of the valuative criterion for separateness and properness. A straightforward application of these results provides a criterion for a morphism of \mathbb{T} -varieties to be proper.

For the following section, we need a technical lemma that we introduce below. Let \mathcal{S} be a divisorial fan on a normal projective variety Y and \mathcal{D} be a \mathbb{p} -divisor in \mathcal{S} . The support $\text{Supp } \mathcal{S}$ is defined as the union of the supports of all $\mathcal{D} \in \mathcal{S}$. We define the following \mathbb{p} -divisors with affine locus.

- (i) For every pair (Z, v) , where $Z \subseteq Y$ is a prime divisor and $v \in \mathcal{D}_Z$ is a vertex such that $\mathcal{O}(\mathcal{D}(u))|_Z$ is big for $u \in \text{rel.int}(\text{cone}(\mathcal{D}_Z - v)^\vee)$, we let $\mathcal{D}_{Z,v} = v \cdot Z + \emptyset \cdot (\text{Supp } \mathcal{S} \setminus Z) + \emptyset \cdot (Y \setminus Y_Z)$, where Y_Z is an affine open set of $Y \setminus (\text{Supp } \mathcal{S} \setminus Z)$ such that $Z \cap Y_Z \neq \emptyset$.
- (ii) For every ray $\rho \in \text{tail}(\mathcal{D})$ such that $\mathcal{D}(u)$ is big for $u \in \text{rel.int}(\rho^\perp \cap \sigma^\vee)$, we let $\mathcal{D}_\rho = \emptyset \cdot \text{Supp}(\mathcal{S}) + \emptyset \cdot (Y \setminus Y_0)$ with tail cone $\text{tail } \mathcal{D}_\rho = \rho$, where Y_0 is an affine open set of $Y \setminus \text{Supp } \mathcal{S}$.

Let \mathcal{C} be the union of all the \mathbb{p} -divisors $\mathcal{D}_{Z,v}$ and \mathcal{D}_ρ defined above for all $\mathcal{D} \in \mathcal{S}$. By [16, Proposition 3.13], these \mathbb{p} -divisors are in bijective correspondence with \mathbb{T} -invariant prime Weil divisors in $X(\mathcal{S})$. It is easy to deduce from the proof that every $\mathcal{D}' \in \mathcal{C}$ is a face of some $\mathcal{D} \in \mathcal{S}$. In particular, we have \mathbb{T} -equivariant open embeddings $X(\mathcal{D}') \hookrightarrow X(\mathcal{D}) \hookrightarrow X(\mathcal{S})$.

Lemma 2.3. *The union U of the family of open sets $\{X(\mathcal{D}') \subseteq X(\mathcal{S}) \mid \mathcal{D}' \in \mathcal{C}\}$ is a big open set in $X(\mathcal{S})$, i.e., $X(\mathcal{S}) \setminus U$ has codimension at least two in $X(\mathcal{S})$.*

Proof. This is a local statement, so we can assume $\mathcal{S} = \mathcal{D}$. By [16, Proposition 3.13], the \mathbb{p} -divisors on $\mathcal{D}' \in \mathcal{C}$ are in bijection with prime \mathbb{T} -invariant divisors on $X(\mathcal{D})$ and the union U of \mathbb{T} -invariant open sets $X(\mathcal{D}')$ with $\mathcal{D}' \in \mathcal{C}$ intersects all \mathbb{T} -invariant divisors. This yields that the complement of the \mathbb{T} -invariant open set U contains no prime Weil divisor and so $X(\mathcal{D}) \setminus U$ has codimension at least two. \square

3. Vertical additive group actions on normal \mathbb{T} -varieties

In this section we provide a classification of additive group actions on semiaffine \mathbb{T} -varieties. Such a classification was known before in the particular case of affine \mathbb{T} -varieties.

3.1. Preliminaries on additive group actions

Let \mathbb{G}_a be the additive group of the base field \mathbf{k} of characteristic zero but not necessarily algebraically closed. Regular additive group actions on an affine variety X are classified by certain derivations on its ring of regular functions $\mathbf{k}[X]$. We briefly introduce this well known correspondence in this section. For more details, see [10].

A derivation of a \mathbf{k} -algebra A is a linear map $D : A \rightarrow A$ satisfying the Leibniz rule, i.e., $D(fg) = fD(g) + gD(f)$ for every $f, g \in A$. For an affine variety X , a derivation of X is just a derivation of its structure ring $\mathbf{k}[X]$.

A derivation $D : \mathbf{k}[X] \rightarrow \mathbf{k}[X]$ on an affine variety X is called a locally nilpotent derivation (LND) if for every $f \in \mathbf{k}[X]$ there exists $i \in \mathbb{Z}_{>0}$ such that $D^i(f) = 0$, where D^i denote the i -th composition of D with itself.

The main classifying result for additive group actions on affine varieties is the following.

Theorem 3.1. *Let X be an affine variety and $D : \mathbf{k}[X] \rightarrow \mathbf{k}[X]$ be a derivation on X . If D is locally nilpotent then the comorphism $\alpha : \mathbb{G}_a \times X \rightarrow X$ of the exponential map*

$$\mathbf{k}[X] \rightarrow \mathbf{k}[t] \otimes_{\mathbf{k}} \mathbf{k}[X] \quad \text{given by} \quad f \mapsto \sum_i \frac{t^i D^i(f)}{i!}$$

defines a \mathbb{G}_a -action on X . Conversely, every \mathbb{G}_a -action on X arises in this way.

Let now X be an arbitrary variety and \mathcal{F} be a sheaf of \mathbf{k} -algebras on X . A derivation of \mathcal{F} is a map $D : \mathcal{F} \rightarrow \mathcal{F}$ such that $\Gamma(U, D) : \Gamma(U, \mathcal{F}) \rightarrow \Gamma(U, \mathcal{F})$ is a derivation for every open affine set $U \subseteq X$. A derivation $D : \mathcal{O}_X \rightarrow \mathcal{O}_X$ of the structure sheaf of X is simply called a derivation on X . If X is affine, it is easy to check that every derivation $D : \mathbf{k}[X] \rightarrow \mathbf{k}[X]$ defines a derivation of the structure sheaf \mathcal{O}_X . By a well known construction coming from differential geometry, derivations on X correspond to vector fields on X . The set of all derivations $D : \mathcal{O}_X \rightarrow \mathcal{O}_X$ is denoted by $\text{Lie}(X)$ and has a natural structure of Lie algebra with Lie bracket given by the commutator $[D, D'] = D \circ D' - D' \circ D$.

Let now X be a variety and $\alpha : \mathbb{G}_a \times X \rightarrow X$ be a \mathbb{G}_a -action on X . In this case we obtain a derivation $D : \mathcal{O}_X \rightarrow \mathcal{O}_X$ of the structure sheaf of X from the \mathbb{G}_a -action in the following way. Let $U \subseteq X$ be an affine open set. Hence $\alpha^{-1}(U)$ is an open set containing $\{0\} \times U \subset \mathbb{G}_a \times X$. We define D via

$$\Gamma(U, D) : \Gamma(U, \mathcal{O}_X) \rightarrow \Gamma(U, \mathcal{O}_X), \quad f \mapsto \left[\frac{d}{dt} (\alpha^* f) \right]_{t=0}.$$

By abuse of notation, when the affine open set U is clear from the context we will denote $\Gamma(U, D)$ simply by D . The derivation D is called the infinitesimal generator of the \mathbb{G}_a -action.

The following result taken from [8, Corollary 2.2] provides a classification of \mathbb{G}_a -actions on semiaffine varieties in terms of derivations of the structure sheaf generalizing the one for affine case.

Theorem 3.2. *Let X be a semiaffine variety and $D : \mathcal{O}_X \rightarrow \mathcal{O}_X$ be a derivation of the structure sheaf. Then D defines a regular \mathbb{G}_a -action α on X if and only if there exists a nonempty affine open set $U \subseteq X$ such that $\Gamma(U, D) : \Gamma(U, \mathcal{O}_X) \rightarrow \Gamma(U, \mathcal{O}_X)$ is locally nilpotent.*

A derivation $D : \mathcal{O}_X \rightarrow \mathcal{O}_X$ corresponding to a \mathbb{G}_a -action on X is called a *regularly integrable derivation*.

Any derivation $D : \mathcal{O}_X \rightarrow \mathcal{O}_X$ on a normal variety X can be extended to a derivation $D : \mathbf{k}(X) \rightarrow \mathbf{k}(X)$ by the Leibniz rule. Indeed, taking any affine open set $U \subseteq X$, we have $\mathbf{k}(X) = \mathbf{k}(U)$ and for $f, g \in \mathbf{k}[U]$ we define

$$D\left(\frac{f}{g}\right) = D\left(f \cdot \frac{1}{g}\right) = D(f) \cdot \frac{1}{g} - \frac{f}{g^2} \cdot D(g) = \frac{gD(f) - fD(g)}{g^2}.$$

In particular, $D : \mathcal{O}_X \rightarrow \mathcal{O}_X$ is completely determined by its action on any affine open set $U \subseteq X$ or by its action on rational functions $\mathbf{k}(X)$. Furthermore, a derivation $D : \mathbf{k}(X) \rightarrow \mathbf{k}(X)$ of the field of rational functions of X is called regular if it defines a derivation of \mathcal{O}_X , i.e., if for every affine open set $U \subseteq X$ we have $D(\mathbf{k}[U]) \subseteq \mathbf{k}[U]$.

3.2. Compatible \mathbb{G}_a -actions on \mathbb{T} -varieties

Let M be a lattice of finite rank and X be a \mathbb{T} -variety, where $\mathbb{T} = \text{Spec } \mathbf{k}[M]$. Let also $D : \mathcal{O}_X \rightarrow \mathcal{O}_X$ be a derivation on X . We say that D is *homogeneous* with respect to M if $\Gamma(U, D)$ is homogeneous as a linear map for every \mathbb{T} -invariant open set $U \subseteq X$, i.e., if $\Gamma(U, D)$ sends homogeneous elements to homogeneous elements. If D is non-zero, we define the degree of D by $\deg D = \deg D(g) - \deg g$ for any homogeneous $g \notin \ker D$ in any \mathbb{T} -invariant affine open set U .

Definition 3.3. Let $\mathbb{T} \times X \rightarrow X$ be a \mathbb{T} -action on X . In analogy with the notions of root and root subgroup of an algebraic group, we say that an action $\alpha : \mathbb{G}_a \times X \rightarrow X$ on X is *compatible* if the normalizer of the image of \mathbb{G}_a in $\text{Aut}(X)$ contains the image of \mathbb{T} . If D is the regularly integrable derivation corresponding to α , then α is compatible if and only if there exists a character χ of \mathbb{T} such that for every \mathbb{T} -invariant affine open set $U \subseteq X$ we have

$$\gamma \circ D \circ \gamma^{-1} = \chi(\gamma) \cdot D, \quad \text{for all } \gamma \in \mathbb{T}.$$

In the next lemma we characterize regularly integrable vector fields that give rise to compatible \mathbb{G}_a -actions.

Lemma 3.4. *A \mathbb{G}_a -action α is compatible if and only if the corresponding regularly integrable derivation $D : \mathcal{O}_X \rightarrow \mathcal{O}_X$ is homogeneous.*

Proof. Let $U \subseteq X$ be a \mathbb{T} -invariant affine open set and let χ^e , $e \in M$, be the character in the definition of compatible \mathbb{G}_a -action. We have $D = \chi^{-e}(\gamma) \cdot \gamma \circ D \circ \gamma^{-1}$, $\forall \gamma \in \mathbb{T}$. Take a homogeneous element $f \in \mathbf{k}[U]$ of degree u' and let $D(f) = g = \sum_{u \in M} g_u \chi^u$. Now a routine computation yields

$$\begin{aligned} \sum_{u \in M} g_u \chi^u &= D(f) = \chi^{-e}(\gamma) \cdot \gamma \circ D \circ \gamma^{-1}(f) \\ &= \chi^{-e-u'}(\gamma) \cdot \sum_{u \in M} \chi^u(\gamma) \cdot g_u \chi^u, \quad \forall \gamma \in \mathbb{T}. \end{aligned}$$

This equality holds for all $\gamma \in \mathbb{T}$ if and only if $g_u = 0$ for $u \neq e + u'$. In particular, D is homogeneous. \square

Remark that the above proof shows that the character χ in the definition of compatible \mathbb{G}_a -action equals χ^e , where e is the degree of D . We need the following refinement of Theorem 3.2 for compatible \mathbb{G}_a -actions on \mathbb{T} -varieties.

Proposition 3.5. *Let X be a semiaffine \mathbb{T} -variety and $D : \mathcal{O}_X \rightarrow \mathcal{O}_X$ be a derivation of the structure sheaf. Then D defines a compatible regular \mathbb{G}_a -action α on X if and only if D is homogeneous and there exists a nonempty \mathbb{T} -invariant affine open set $U \subseteq X$ such that $\Gamma(U, D) : \Gamma(U, \mathcal{O}_X) \rightarrow \Gamma(U, \mathcal{O}_X)$ is locally nilpotent.*

Proof. Assume first that α is compatible and let χ be the character in the definition of compatible \mathbb{G}_a -action. By Theorem 3.2, there exists an affine open set $U \subseteq X$ where D is locally nilpotent. We will show that we can take this open set U to be \mathbb{T} -invariant. Indeed, if D defines a compatible \mathbb{G}_a -action, then the \mathbb{T} -action and the \mathbb{G}_a -action span an action of the solvable group $G = \mathbb{G}_a \rtimes \mathbb{T}$ where the homomorphism for the semidirect product $\mathbb{T} \rightarrow \text{Aut}(\mathbb{G}_a)$ is given by $\gamma \mapsto (t \mapsto \chi(\gamma) \cdot t)$. By [19], there is a G -invariant open subset $U \subseteq X$ where a geometric quotient $U \mapsto U/G$ exists. By [18, Theorem 10], restricting the open set U we can assume $U \simeq Y \times U/G$, where Y is a general G -orbit in X . Being a homogeneous space of a solvable group, Y is an affine variety. Finally, we can further restrict U so that U/G is affine, proving that U can

indeed be taken as an affine open subset; see also [17, Theorem 3] for a direct argument on the existence of such affine open subset U . In particular, D is a homogeneous LND on the \mathbb{T} -invariant affine open set U .

For the converse, if there is a \mathbb{T} -invariant affine open set U where D is an LND, then Theorem 3.2 shows that D defines a \mathbb{G}_a -action on X and such action is compatible by Lemma 3.4 since D is homogeneous. \square

Definition 3.6. Let now X be a \mathbb{T} -variety and $\mathbf{k}(X)^{\mathbb{T}}$ be the field of \mathbb{T} -invariant rational functions. A homogeneous derivation $D : \mathcal{O}_X \rightarrow \mathcal{O}_X$ is called *vertical* if $D(\mathbf{k}(X)^{\mathbb{T}}) = 0$, and *horizontal* otherwise. In case D is the derivation associated to a \mathbb{G}_a -action, the corresponding \mathbb{G}_a -action is called vertical or horizontal, respectively.

In geometric terms, the condition $D(\mathbf{k}(X)^{\mathbb{T}}) = 0$ means that a general orbit of the \mathbb{G}_a -action on X is contained in a \mathbb{T} -orbit closure.

3.3. Additive group actions on toric varieties

In this section, we recall the characterization of compatible regular \mathbb{G}_a -actions on a toric variety X in terms of the corresponding regularly integrable derivations. For the results in this subsection, the base field \mathbf{k} is not necessarily algebraically closed. These results were first given in [7] in an implicit way. Demazure's approach was generalized and simplified in [13,14,8].

Let $X = X(\Sigma)$ be a toric variety. Let $\rho \in N$ and $e \in M$. We define the linear map $D_{\rho,e} : \mathbf{k}[M] \rightarrow \mathbf{k}[M]$, $\chi^u \mapsto \rho(u)\chi^{u+e}$. It is a routine verification that $D_{\rho,e}$ satisfies the Leibniz rule and thus it defines a homogeneous derivation on $\mathbf{k}[M]$. Moreover, every homogeneous derivation on $\mathbf{k}[M]$ is a multiple of $D_{\rho,e}$ for some $\rho \in N$ and some $e \in M$. Since $D_{a\rho,e} = aD_{\rho,e}$, without loss of generality we may and will assume in the sequel that ρ is primitive, i.e. $\rho/n \notin N$ for all $n > 1$. Let $\Sigma(1)$ denote the set of primitive vectors on the rays of Σ . The next theorem taken from [8, Proposition 3.8] gives a classification of compatible regular \mathbb{G}_a -actions on $X(\Sigma)$.

Theorem 3.7. *Let $X(\Sigma)$ be a semiaffine toric variety. Then D is the derivation of a compatible regular action $\alpha : \mathbb{G}_a \times X(\Sigma) \rightarrow X(\Sigma)$ if and only if D is a scalar multiple of $D_{\rho,e}$ for some $\rho \in \Sigma(1)$ and $e \in M$ such that $\rho(e) = -1$, and $\rho'(e) \geq 0$ for all $\rho' \in \Sigma(1) \setminus \{\rho\}$.*

3.4. Higher complexity \mathbb{T} -varieties

We now describe vertical \mathbb{G}_a -actions on higher complexity \mathbb{T} -varieties. In this subsection the field \mathbf{k} needs to be algebraically closed. First, we need the following technical lemma.

Lemma 3.8. *Let X be a normal variety and $D : \mathbf{k}(X) \rightarrow \mathbf{k}(X)$ be a \mathbf{k} -derivation of the field of rational functions on X . Assume that D is regular on a big open set, i.e., D restricts to a derivation $D_U : \mathcal{O}_U \rightarrow \mathcal{O}_U$ on an open set $U \subseteq X$ with complement of codimension at least 2. Then D is regular on X .*

Proof. The statement is local, so without loss of generality, we assume X is affine. Now, since $D|_U$ is regular, we have $D|_U : \mathcal{O}_U \rightarrow \mathcal{O}_U$. Hence, we can take global sections obtaining $\Gamma(U, D) : \Gamma(U, \mathcal{O}_X) \rightarrow \Gamma(U, \mathcal{O}_X)$. Since X is normal, we have $\Gamma(U, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$ [9, Corollary 11.4]. This yields that D is regular. \square

Let Y be a normal projective variety and $X = X(\mathcal{S})$ where \mathcal{S} is a divisorial fan on Y . Let α be a vertical \mathbb{G}_a -action on X with corresponding regularly integrable derivation $D : \mathcal{O}_X \rightarrow \mathcal{O}_X$ and $K_0 = \mathbf{k}(X)^{\mathbb{T}} = \mathbf{k}(Y)$ be the field of \mathbb{T} -invariant rational functions. The normalization \overline{X} of the base extension $X \times_{\text{Spec } \mathbf{k}} \text{Spec } K_0$ is a toric variety over the field K_0 with fan $\Sigma = \text{tail } \mathcal{S}$.

Since the \mathbb{G}_a -action is vertical, the field of \mathbb{G}_a -invariant functions $\mathbf{k}(X)^{\mathbb{G}_a}$ contains K_0 . This yields that the \mathbb{G}_a -action lifts to \overline{X} and since $\mathbf{k}(X) = K_0(\overline{X})$, we have that both, the \mathbb{G}_a -action on X and the \mathbb{G}_a -action on \overline{X} , are given by the same derivation $D : \mathbf{k}(X) \rightarrow \mathbf{k}(X)$. Now Theorem 3.7 yields $D = \phi D_{\rho,e} := D_{\phi,\rho,e}$ where $\phi \in K_0 = \mathbf{k}(Y)$, $\rho \in \Sigma(1)$, and $e \in M$.

Let $P(0)$ denote the set of vertices of a polyhedron P . To state our classification of regular vertical \mathbb{G}_a -actions on $X(\mathcal{S})$ we need to define the following \mathbb{Q} -divisor in analogy with [14, Theorem 2.4]:

$$D_e := \sum_{Z \subseteq \text{Loc } \mathcal{S}} \min\{v(e) \mid v \in \mathcal{D}_Z(0), \mathcal{D} \in \mathcal{S}\} \cdot Z,$$

where $\text{Loc } \mathcal{S}$ is the union of the $\text{Loc } \mathcal{D}$ for all $\mathcal{D} \in \mathcal{S}$, see also [5, Theorem 1.7].

Theorem 3.9. *Let $X(\mathcal{S})$ be a semiaffine \mathbb{T} -variety and $\Sigma = \text{tail } \mathcal{S}$. Then a derivation D of the field $\mathbf{k}(X(\mathcal{S}))$ is the derivation of a vertical compatible regular action $\alpha : \mathbb{G}_a \times X(\mathcal{S}) \rightarrow X(\mathcal{S})$ if and only if $D = D_{\phi,\rho,e}$, where $e \in M$, $\phi \in \Gamma(\text{Loc } \mathcal{S}, \mathcal{O}(D_e))$, and $\rho \in \Sigma(1)$ such that $\rho(e) = -1$, and $\rho'(e) \geq 0$ for all $\rho' \in \Sigma(1) \setminus \{\rho\}$.*

Proof. Let $D = D_{\phi,\rho,e}$. The fact that D is locally nilpotent on any affine open set $X(\mathcal{D}) \subseteq X(\mathcal{S})$, where $\mathcal{D} \in \mathcal{S}$ is such that ρ_e is a ray in $\text{tail } \mathcal{D}$ is proven in [14, Theorem 2.4], see also [5, Theorem 1.7]. Hence, by Proposition 3.5 we only need to prove that D is regular. Moreover, by Lemma 3.8 it is enough to verify that D is regular on a big open set and thus by Lemma 2.3 it suffices to verify that D restricts to a regular derivation on $X(\mathcal{D})$ for all \mathcal{D} in the family \mathcal{C} of p-divisors therein.

Let $Z \subset X(\mathcal{S})$ be a prime divisor and v be a vertex in some polyhedral coefficient \mathcal{D}'_Z of a p-divisor \mathcal{D}' in \mathcal{S} . Assume further that $\mathcal{D} = \mathcal{D}_{Z,v}$ is contained in the family \mathcal{C} . Recall that the tail cone $\text{tail } \mathcal{D}$ is $\{0\} \subseteq N_{\mathbb{Q}}$ and for every $u \in M$, the element $f\chi^u \in A(\mathcal{D})$ if and only if $\text{ord}_Z(f) + v(u) \geq 0$. Hence for D to leave $A(\mathcal{D})$ invariant we need that for all $f\chi^u \in A(\mathcal{D})$ the element $\phi f\chi^{u+e} \in A(\mathcal{D})$. This yields

$$\text{ord}_Z \phi + \text{ord}_Z f + v(u + e) = (\text{ord}_Z \phi + v(e)) + (\text{ord}_Z f + v(u)) \geq 0.$$

The last inequality holds for all $u \in M$ if and only if $\text{ord}_Z \phi + v(e) \geq 0$. Indeed, recall that the evaluations $\mathcal{D}(u)$ are all \mathbb{Q} -Cartier. Replacing u by a multiple, we can assume that $\mathcal{D}(u)$ is Cartier which implies that it is locally principal. Hence, there exists $f \in \mathbf{k}(Y)$ satisfying $\text{ord}_Z(f) + \lfloor v(u) \rfloor = 0$. This yields that the rational function ϕ belongs to $\Gamma(\text{Loc } \mathcal{S}, \mathcal{O}(D_e))$. The remaining conditions of the theorem follow from Theorem 3.7 since D defines a regular \mathbb{G}_a -action on the toric $\mathbf{k}(Y)$ -variety \overline{X} given by the fan Σ . \square

4. Lie algebras generated by root derivations

In this and next sections we study Lie algebras generated by a finite collection of derivations of a field. The study is motivated by the results of the previous section. On the other hand, the objects we are dealing with are elementary, and for convenience of the reader we re-introduce all necessary notions and definitions.

Let \mathbf{k} be an algebraically closed field of characteristic zero, Y be an algebraic variety over \mathbf{k} , and $\mathbf{k} \subseteq K_0 := \mathbf{k}(Y)$ be the corresponding field extension. Let M be a lattice of rank n and $K_0(M)$ be the quotient field of the group algebra $K_0[M]$.

Let us introduce a class of K_0 -derivations of the field $K_0(M)$. Let $N = \text{Hom}(M, \mathbb{Z})$ be the dual lattice and let $N \times M \rightarrow \mathbb{Z}$, $(v, u) \mapsto \langle v, u \rangle = v(u)$ be the corresponding duality pairing. Given a fan Σ in $N_{\mathbb{Q}}$, we say that a vector $e \in M$ is a *Demazure root* of the fan Σ if there is ray $\rho \in \Sigma(1)$ such that $\langle \rho, e \rangle = -1$ and $\langle \rho', e \rangle \geq 0$ for all $\rho' \in \Sigma(1)$, $\rho' \neq \rho$. We say that the ray ρ is *associated* with the root e .

Let us define a *root derivation* $D_{\phi,\rho,e}$ of the field $K_0(M)$, where $\phi \in K_0$ and e is a Demazure root of the fan Σ with the associated ray ρ , by the formula

$$D_{\phi,\rho,e}(\chi^u) = \phi \cdot \rho(u) \cdot \chi^{u+e}.$$

Since the field extension $K_0 \subseteq K_0(M)$ is generated by the elements χ^u , $u \in M$, and ρ is a linear function on M , this formula defines a K_0 -derivation of the field $K_0(M)$.

Remark 4.1. By Theorem 3.9, the derivation corresponding to a vertical \mathbb{G}_a -action on a semiaffine \mathbb{T} -variety $X(\mathcal{S})$ is a root derivation of the field $\mathbf{k}(X) = \mathbf{k}(Y)(M)$ with respect to the tail fan Σ .

Let $m \in \mathbb{Z}_{\geq 2}$ and $\mathbb{D} = \{D_{\phi_i,\rho_i,e_i}\}_{i=1}^m$ be a set of root derivations of the field $K_0(M)$. We let

$$D_i = D_{\phi_i,\rho_i,e_i}, \quad \phi(D_i) = \phi_i, \quad \rho(D_i) = \rho_i, \quad \text{and} \quad e(D_i) = e_i.$$

Let us further assume that the derivations D_i are pairwise non-proportional over the field K_0 .

Definition 4.2. A set \mathbb{D} is called *cyclic* if $\langle \rho_{i+1}, e_i \rangle > 0$ for all $i = 1, \dots, m$, where we set $\rho_{m+1} := \rho_1$. A set \mathbb{D} is *almost simple* if

$$\langle \rho, e_i \rangle = \begin{cases} -1, & \text{if } \rho = \rho_i; \\ 1, & \text{if } \rho = \rho_{i+1}; \\ 0, & \text{if } \rho \in \Sigma(1) \setminus \{\rho_i, \rho_{i+1}\}. \end{cases}$$

An almost simple set \mathbb{D} is *simple* if $\prod_{i=1}^m \phi_i \in \mathbf{k}$. A simple set is *very simple* if any of its cyclic subsets is simple as well.

Remark 4.3. Without loss of generality we may assume in the definition of a very simple set that for every cyclic subset the product of corresponding functions ϕ_i equals 1. Indeed, we take a point $y_0 \in Y$ such that all functions ϕ_i are defined and not equal zero at y_0 , and replace each function ϕ_i by $\frac{1}{\phi_i(y_0)}\phi_i$. This changes every derivation D_i by a scalar multiple.

We say that a root e associated with a ray ρ is *elementary*, if $\langle \rho, e \rangle = -1$, $\langle \rho', e \rangle = 1$ for some $\rho' \in \Sigma(1)$, and $\langle \rho'', e \rangle = 0$ for all $\rho'' \in \Sigma(1) \setminus \{\rho, \rho'\}$. A root derivation D is called *elementary* if $e(D) = e$ with some elementary root e . Clearly, a cyclic set \mathbb{D} is almost simple if and only if \mathbb{D} consists of elementary derivations.

Consider the Lie algebra $\text{Der}_{K_0}(K_0(M))$ of all K_0 -derivations of the field $K_0(M)$. Let $\text{Lie}_{K_0}(\mathbb{D})$ be the Lie algebra over the field K_0 generated by \mathbb{D} in $\text{Der}_{K_0}(K_0(M))$. Further, let $\text{Lie}(\mathbb{D})$ be the Lie algebra over the field \mathbf{k} generated by \mathbb{D} in the Lie algebra $\text{Der}(K_0(M))$ of all \mathbf{k} -derivations of the field $K_0(M)$. We denote by $r = r(\mathbb{D})$ the number of pairwise distinct elements of the set $\{\rho_1, \dots, \rho_m\}$.

Proposition 4.4.

- (1) If the set \mathbb{D} is almost simple, then the Lie algebra $\text{Lie}_{K_0}(\mathbb{D})$ is isomorphic to $\mathfrak{sl}_r(K_0)$.
- (2) If the set \mathbb{D} is very simple, then the Lie algebra $\text{Lie}(\mathbb{D})$ is isomorphic to $\mathfrak{sl}_r(\mathbf{k})$.

In order to prove this proposition, we need an auxiliary result from graph theory proven below in Lemma 4.5. Let K_0 be a field and Γ_r be a complete oriented graph on the set of vertices $V = \{1, \dots, r\}$. Assume that we are given a function $\phi: E \rightarrow K_0^*$, where E is the set of edges of Γ_r . For any oriented path P in Γ_r we let $\phi(P)$ be the product of $\phi(e)$ over all edges e in P . By a *marking* on Γ we mean a function ϕ such that $\phi(C) = 1$ for every oriented cycle C in Γ_r .

We say that a function $\psi: V \rightarrow K_0^*$ is a *potential* of a function ϕ if $\phi([jk]) = \psi(k)\psi(j)^{-1}$ for every edge $[jk] \in E$. It is easy to see that a function ϕ admits a potential if and only if ϕ is a marking.

Let E' be a subset in E such that there exists an oriented cycle in Γ_r passing through all vertices in Γ_r whose set of edged is contained in E' . A function $\phi': E' \rightarrow K_0^*$ such that $\phi'(C') = 1$ for every oriented cycle C' on E' is said to be a *partial marking* of Γ_r .

Lemma 4.5. *Every partial marking ϕ' of the graph Γ_r can be extended to a marking ϕ of Γ_r .*

Proof. For every edge $u = [jk] \in E$ we can find an oriented path P_1 from j to k in E' and we let $\phi(u) = \phi(P_1)$. This function is well defined. Indeed, let P_2 be another oriented path from j to k in E' and P_3 be an oriented path from k to j in E' . Then

$$\phi(P_1) = \phi(P_3)^{-1} = \phi(P_2).$$

Let us check that ϕ is a marking. Let C be an oriented cycle in Γ_r . We replace every edge $[jk]$ of C which is not contained in E' by an oriented path P from j to k in E' . We obtain an oriented cycle C' with $\phi(C) = \phi(C')$ and all edges of C' are in E' . By definition of a partial marking we have $\phi(C') = 1$. \square

Proof of Proposition 4.4. We begin with the proof of (1). For our purposes we may assume that all functions ϕ_i are constants 1. Let $\kappa_1, \dots, \kappa_r$ be all pairwise distinct elements in the set $\{\rho_1, \dots, \rho_m\}$. Assume first that the collection $\kappa_1, \dots, \kappa_r$ is linearly independent in $N_{\mathbb{Q}}$. Let us extend the collection $\kappa_1, \dots, \kappa_r$ to a basis $\beta = \{\kappa_1, \dots, \kappa_r, \tau_{r+1}, \dots, \tau_n\}$ of a sublattice N' of finite index in N . The dual lattice $M' := \text{Hom}(N', \mathbb{Z})$ contains M as a sublattice of finite index. Every D_i extends to a K_0 -derivation of the field $K_0(M')$. Using our basis β and its dual basis β^* , we identify $K_0(M')$ with the field of rational functions $K_0(x_1, \dots, x_r, y_{r+1}, \dots, y_n)$ where $x_i = \chi^{\kappa_i^*}$ and $y_i = \chi^{\tau_i^*}$ with κ_i^* and τ_i^* the respective basis element dual to κ_i and τ_i in β^* . Under this identification, the derivation D_s from \mathbb{D} coincides with the derivation $\partial_{j_i} := x_j \frac{\partial}{\partial x_i}$, where $\kappa_i = \rho_s$ and $\kappa_j = \rho_{s+1}$. This shows that the algebra $\text{Lie}_{K_0}(\mathbb{D})$ acts on the subalgebra $K_0[x_1, \dots, x_r]$ via the standard representation of the algebra $\mathfrak{sl}_r(K_0)$. In particular, $\text{Lie}_{K_0}(\mathbb{D})$ is isomorphic to $\mathfrak{sl}_r(K_0)$.

Assume now that the collection $\kappa_1, \dots, \kappa_r$ is linearly dependent in $N_{\mathbb{Q}}$. Then let $\lambda_1 \kappa_1 + \dots + \lambda_r \kappa_r = 0$ be a linear relation with $\lambda_i \in \mathbb{Q}$. Computing the value of all e_i on this combination, we obtain $\lambda_1 = \dots = \lambda_r$. Hence, up to a constant factor, the only linear relation is $\kappa_1 + \dots + \kappa_r = 0$.

In particular, $\kappa_1, \dots, \kappa_{r-1}$ are linearly independent. Let us extend the collection $\kappa_1, \dots, \kappa_{r-1}$ to a basis $\beta = \{\kappa_1, \dots, \kappa_{r-1}, \tau_r, \dots, \tau_n\}$ of a sublattice N' of finite index in N . The dual lattice $M' := \text{Hom}(N', \mathbb{Z})$ contains M as a sublattice of finite index. Every D_i extends to a K_0 -derivation of the field $K_0(M')$. Using our basis β and its dual basis β^* , we identify $K_0(M')$ with the field of rational functions $K_0(x_1, \dots, x_{r-1}, y_{r+1}, \dots, y_{n+1})$ where $x_i = \chi^{\kappa_i^*}$ and $y_i = \chi^{\tau_i^*}$ with κ_i^* and τ_i^* the respective basis element dual to κ_i and τ_i in β^* . Let us embed the field $K_0(x_1, \dots, x_{r-1}, y_{r+1}, \dots, y_{n+1})$ into the field $K_0(z_1, \dots, z_{r-1}, z_r, y_{r+1}, \dots, y_{n+1})$ sending x_i to $\frac{z_i}{z_r}$. Then the derivation D_s from \mathbb{D} coincides with the restriction to $K_0(x_1, \dots, x_{r-1}, y_{r+1}, \dots, y_{n+1})$ of the derivation $z_j \frac{\partial}{\partial z_i}$ of the field $K_0(z_1, \dots, z_r, y_{r+1}, \dots, y_{n+1})$, where $\kappa_i = \rho_s$ and $\kappa_j = \rho_{s+1}$. This shows that the algebra $\text{Lie}_{K_0}(\mathbb{D})$ acts on the subalgebra $K_0[z_1, \dots, z_r]$ via the standard representation of the algebra $\mathfrak{sl}_r(K_0)$. In particular, $\text{Lie}_{K_0}(\mathbb{D})$ is isomorphic to $\mathfrak{sl}_r(K_0)$.

Now we come to (2). This time we realize the D_s as the derivation $\phi_s \partial_{ij}$ of field $K_0(x_1, \dots, x_r)$. It follows from the definition of an almost simple set that the indices i, j determine the index s uniquely. Let us denote the function ϕ_s as ϕ_{ij} .

Consider a complete oriented graph Γ_r on the set of vertices $V = \{1, \dots, r\}$. Let E' be the set of m edges of Γ_r corresponding to pairs $[ij]$ defined by the derivations D_1, \dots, D_m . Consider a function

$$\phi': E' \rightarrow K_0^*, \quad \phi'([ij]) = \phi_{ij}.$$

By definition of a very simple set, this function is a partial marking on the graph Γ_r . By Lemma 4.5, such a partial marking can be extended to a marking $\phi: E \rightarrow K_0^*$. In turn, this marking admits a potential $\psi: V \rightarrow K_0^*$.

This shows that the derivations $\phi_{ij}\partial_{ij}$ are sent to the derivations ∂_{ij} via an automorphism of the field $K_0(x_1, \dots, x_r)$ given by $x_l \mapsto \psi(l)x_l, l = 1, \dots, r$. Thus the algebra $\text{Lie}(\mathbb{D})$ is isomorphic to the \mathbf{k} -subalgebra of $\text{Der}(K_0(x_1, \dots, x_r))$ generated by $\partial_{ij}, 1 \leq i \neq j \leq r$. The latter algebra is obviously isomorphic to $\mathfrak{sl}_r(\mathbf{k})$. This completes the proof of Proposition 4.4. \square

Example 4.6. Consider the fan Σ in the lattice \mathbb{Z}^2 consisting of three two-dimensional cones generated by primitive vectors $\kappa_1 = (1, 0), \kappa_2 = (0, 1), \kappa_3 = (-1, -1)$ with all their faces. We define a very simple set $\mathbb{D} = \{D_1, \dots, D_6\}$ by letting $\phi_1 = \dots = \phi_6 = 1$,

$$e_1 = (-1, 1), \quad e_2 = (0, -1), \quad e_3 = (1, 0), \quad e_4 = (-1, 0), \quad e_5 = (0, 1), \quad e_6 = (1, -1),$$

and

$$\rho_1 = \kappa_1, \quad \rho_2 = \kappa_2, \quad \rho_3 = \kappa_3, \quad \rho_4 = \kappa_1, \quad \rho_5 = \kappa_3, \quad \rho_6 = \kappa_2.$$

The corresponding derivations of the field $K_0(x_1, x_2)$ have the form

$$x_2 \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad -x_1^2 \frac{\partial}{\partial x_1} - x_1 x_2 \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_1}, \quad -x_1 x_2 \frac{\partial}{\partial x_1} - x_2^2 \frac{\partial}{\partial x_2}, \quad x_1 \frac{\partial}{\partial x_2}.$$

If we embed the field $K_0(x_1, x_2)$ into the field $K_0(z_1, z_2, z_3)$ by sending x_i to $\frac{z_i}{z_3}, i = 1, 2$, these derivations coincide with restrictions of the derivations

$$z_2 \frac{\partial}{\partial z_1}, \quad z_3 \frac{\partial}{\partial z_2}, \quad z_1 \frac{\partial}{\partial z_3}, \quad z_3 \frac{\partial}{\partial z_1}, \quad z_2 \frac{\partial}{\partial z_3}, \quad z_1 \frac{\partial}{\partial z_2}.$$

The algebra $\text{Lie}_{K_0}(\mathbb{D})$ is isomorphic to $\mathfrak{sl}_3(K_0)$. Moreover, we have constructed a representation of the Lie algebra $\mathfrak{sl}_3(K_0)$ into $\text{Der}(K_0[x_1, x_2])$.

5. A criterion for finite-dimensionality

In this section we give a criterion for a Lie algebra generated by a finite set of root derivations to be finite dimensional.

Theorem 5.1. *Let $\mathbb{D} = \{D_{\phi_i, \rho_i, e_i}\}_{i=1}^m$ be a set of root derivations of the field $K_0(M)$.*

- (1) *The Lie algebra $\text{Lie}_{K_0}(\mathbb{D})$ is finite dimensional if and only if every cyclic subset of \mathbb{D} is almost simple.*
- (2) *The Lie algebra $\text{Lie}(\mathbb{D})$ is finite dimensional if and only if every cyclic subset of \mathbb{D} is simple.*

In the remaining of this section we prove Theorem 5.1 in several steps. We begin with a commutator formula. The proof of the following lemma is straightforward.

Lemma 5.2. *Let $D_i = D_{\phi_i, \rho_i, e_i}$ and $D_j = D_{\phi_j, \rho_j, e_j}$ be root derivations. Then the commutator is given by*

$$[D_i, D_j](\chi^u) = \phi_i \phi_j (\langle \rho_i, e_j \rangle \langle \rho_j, u \rangle - \langle \rho_i, u \rangle \langle \rho_j, e_i \rangle) \chi^{u+e_i+e_j} \quad \text{for all } u \in M.$$

Let us start with the “only if” direction in (1). We will prove that if \mathbb{D} contains a cyclic subset that is not almost simple then $\text{Lie}_{K_0}(\mathbb{D})$ is infinite dimensional. Without loss of generality we may assume that

- (a) the set \mathbb{D} is cyclic but not almost simple;
- (b) the set \mathbb{D} contains no proper cyclic subset.

Condition (b) implies that $\langle \rho_i, e_j \rangle = 0$ for all $j \neq i - 1, i$. In particular, the rays $\rho(D), D \in \mathbb{D}$, are pairwise distinct. Let us denote these rays by ρ_1, \dots, ρ_m . Moreover, we have $\langle \rho_i, \sum_{j=1}^m e_j \rangle = \langle \rho_i, e_{i-1} \rangle - 1$. Hence, under condition (b) we have that

$$\text{a set } \mathbb{D} \text{ is almost simple if and only if } \mathbb{D} \text{ is cyclic and } \langle \rho_i, \sum_{j=1}^m e_j \rangle = 0, \text{ for all } i \in \{1, \dots, m\}. \quad (1)$$

We proceed by induction on m . Let $m = 2$ and $\mathbb{D} = \{D_1, D_2\}$. We put $a = \langle \rho_2, e_1 \rangle$. Since \mathbb{D} is not almost simple, we may assume up to renumbering that either $a \geq 2$ or $a = 1 = \langle \rho_1, e_2 \rangle$ and $\langle \rho_3, e_2 \rangle > 0$ for some $\rho_3 \in \Sigma(1), \rho_3 \neq \rho_1$. Consider the derivation

$$D'_2 = \underbrace{[\dots [[D_1, D_2], D_2] \dots D_2]}_{a+1}.$$

Claim 1. *The derivation D'_2 is non-zero.*

Proof. Take a vector $u \in M$ with $\langle \rho_2, u \rangle = 1$. We claim that $D'_2(\chi^u) \neq 0$. Note that $D_2^2(\chi^u) = 0$ and we have

$$\begin{aligned} (-1)^{a+1} D'_2(\chi^u) &= D_2^{a+1} D_1(\chi^u) - (a+1) D_2^a D_1 D_2(\chi^u) \\ &= \phi_1 \phi_2 (\langle \rho_1, u \rangle \langle \rho_2, u + e_1 \rangle - (a+1) \langle \rho_2, u \rangle \langle \rho_1, u + e_2 \rangle) D_2^a(\chi^{u+e_1+e_2}) \\ &= \phi_1 \phi_2 (\langle \rho_1, u \rangle (1+a) - (a+1) (\langle \rho_1, u \rangle + \langle \rho_1, e_2 \rangle)) D_2^a(\chi^{u+e_1+e_2}) \\ &= -\phi_1 \phi_2 (a+1) \langle \rho_1, e_2 \rangle D_2^a(\chi^{u+e_1+e_2}). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} D_2^a(\chi^{u+e_1+e_2}) &= \phi_2^a \langle \rho_2, u + e_1 + e_2 \rangle \langle \rho_2, u + e_1 + 2e_2 \rangle \dots \langle \rho_2, u + e_1 + ae_2 \rangle \chi^{u+e_1+(a+1)e_2} \\ &= \phi_2^a (1+a-1)(1+a-2) \dots (1+a-a) \chi^{u+e_1+(a+1)e_2} \neq 0. \end{aligned}$$

This shows $D'_2(\chi^u) \neq 0$. \square

Claim 2. *The derivation D'_2 is a root derivation associated with the ray ρ_2 .*

Proof. The derivation D'_2 is homogeneous of degree $e_1 + (a+1)e_2$. We have

$$\begin{aligned} \langle \rho_1, e_1 + (a+1)e_2 \rangle &= -1 + (a+1) \langle \rho_1, e_2 \rangle \geq \langle \rho_1, e_2 \rangle, & \langle \rho_2, e_1 + (a+1)e_2 \rangle &= -1, \quad \text{and} \\ \langle \rho', e_1 + (a+1)e_2 \rangle &= \langle \rho', e_1 \rangle + (a+1) \langle \rho', e_2 \rangle \geq \langle \rho', e_2 \rangle \end{aligned}$$

for all $\rho' \in \Sigma(1) \setminus \{\rho_1, \rho_2\}$. Thus the vector $e_1 + (a+1)e_2$ is a Demazure root of the fan Σ associated with the ray ρ_2 .

Take a vector $w \in M$ with $\langle \rho_2, w \rangle = 0$. Then $D_2(\chi^w) = 0$ and the condition $\langle \rho_2, w + e_1 + ae_2 \rangle = a - a = 0$ implies $D_2(D_2^a D_1(\chi^w)) = 0$. We conclude that $D'_2(\chi^w) = 0$ and thus

$$D'_2(\chi^u) = \phi \langle \rho_2, u \rangle \chi^{u+e_1+(a+1)e_2}$$

for all $u \in M$ with some $\phi \in K_0$. \square

Claim 3. We have $\langle \bar{\rho}, \deg D'_2 \rangle > \langle \bar{\rho}, \deg D_2 \rangle$, where $\bar{\rho} := \sum_{\rho \in \Sigma(1)} \rho$.

Proof. Take $\rho \in \Sigma(1)$. The inequality

$$\langle \rho, e_1 + (a + 1)e_2 \rangle = \langle \rho, e_1 \rangle + a\langle \rho, e_2 \rangle + \langle \rho, e_2 \rangle \geq \langle \rho, e_2 \rangle$$

is strict for $\rho = \rho_1$ if $a > 1$ and is strict for $\rho = \rho_3$ if $a = 1$. \square

Claim 4. The set $\mathbb{D}' := \{D_1, D'_2\}$ is again cyclic, but not almost simple.

Proof. For the first assertion, we have

$$\langle \rho_1, e_1 + (a + 1)e_2 \rangle = -1 + (a + 1)\langle \rho_1, e_2 \rangle > 0.$$

If $a > 1$ then $\langle \rho_2, e_1 \rangle > 1$ and \mathbb{D}' is not almost simple. If $a = 1$ and $\langle \rho_3, e_2 \rangle > 0$ then

$$\langle \rho_3, e_1 + (a + 1)e_2 \rangle > \langle \rho_3, e_2 \rangle > 0$$

and again \mathbb{D}' is not almost simple. \square

Repeating this procedure with the pair $\{D_1, D'_2\}$ and so on, we obtain infinitely many root derivations having different degrees by Claim 3. This proves that the Lie algebra $\text{Lie}_{K_0}(\mathbb{D})$ has infinite dimension.

Now assume that $m > 2$. Replace the set $\mathbb{D} = \{D_1, \dots, D_m\}$ by the set

$$\widehat{\mathbb{D}} = \{D_1, \dots, D_{m-2}, \widehat{D}_{m-1}\},$$

where $\widehat{D}_{m-1} = [D_{m-1}, D_m]$. We claim that $\widehat{\mathbb{D}}$ is again cyclic and not almost simple. Indeed, we have $\langle \rho_{m-1}, e_{m-1} + e_m \rangle = \langle \rho_{m-1}, e_{m-1} \rangle = -1$. Take $u \in M$ with $\langle \rho_m, u \rangle = 0$ and $\langle \rho_{m-1}, u \rangle > 0$. By Lemma 5.2, we have

$$\widehat{D}_{m-1}(\chi^u) = -\phi_1 \phi_2 \langle \rho_{m-1}, u \rangle \langle \rho_m, e_{m-1} \rangle \chi^{u+e_{m-1}+e_m} \neq 0.$$

Moreover, the derivation \widehat{D}_{m-1} annihilates all functions χ^u with $\langle \rho_{m-1}, u \rangle = 0$, and thus it is a non-zero root derivation of degree $e_{m-1} + e_m$. Since $\langle \rho_1, e_{m-1} + e_m \rangle = \langle \rho_1, e_m \rangle > 0$ we have that $\widehat{\mathbb{D}}$ is cyclic. By Lemma 5.2, we obtain that $\rho(\widehat{D}_{m-1}) = \rho(D_{m-1})$ and since the sum of degrees of derivations from \mathbb{D} and $\widehat{\mathbb{D}}$ are equal, we conclude that $\widehat{\mathbb{D}}$ is not almost simple by (1). By the induction hypothesis, the Lie algebra generated by $\widehat{\mathbb{D}}$ is infinite dimensional. Thus the Lie algebra $\text{Lie}_{K_0}(\mathbb{D})$ is infinite dimensional as well.

We proceed with the “only if” direction in (2). Assume that the algebra $\text{Lie}(\mathbb{D})$ is finite dimensional. We already know that every cyclic subset in \mathbb{D} is almost simple. Suppose that there is an almost simple subset $\{D_1, \dots, D_m\}$ in \mathbb{D} which is not simple. Since all derivations D_i are elementary, we can take a finite extension $M \subseteq M'$ of lattices such that $K_0(M') \cong K_0(x_1, \dots, x_n)$ and every derivation D_i acts on this field as $\phi_i \partial_{j_i}$, where $\partial_{j_i} = x_j \frac{\partial}{\partial x_i}$; see the proof of Proposition 4.4. Consider the derivation

$$D_{1,1} := [\dots [[D_1, D_2], D_3] \dots D_m], D_1] = (\phi_1 \dots \phi_m) D_1.$$

Similarly, the derivation $D_{1,s}$ defined recursively via $D_{1,s} := [\dots [[D_{1,s-1}, D_2], D_3] \dots D_m], D_1]$ equals $(\phi_1 \dots \phi_m)^s D_1$ for any positive integer s . Hence if the function $\phi_1 \dots \phi_m$ is non-constant then the algebra $\text{Lie}(\mathbb{D})$ has infinite dimension, a contradiction.

Now we come to the “if” direction in (1) and (2).

Proposition 5.3. *Assume that the set \mathbb{D} contains no cyclic subset. Then the Lie algebras $\text{Lie}_{K_0}(\mathbb{D})$ and $\text{Lie}(\mathbb{D})$ are finite-dimensional and nilpotent.*

Proof. We subdivide the proof into several lemmas.

Lemma 5.4. *Let $D_i, D_j \in \mathbb{D}$. Then the commutator $[D_i, D_j]$ is either zero or a root derivation. More precisely, if $\langle \rho_i, e_j \rangle > 0$ then $[D_i, D_j] = D_{\psi, \rho_j, e_i + e_j}$, with $\psi = \langle \rho_i, e_j \rangle \phi_i \phi_j$.*

Proof. By Lemma 5.2, if either $\rho_i = \rho_j$ or we have $\langle \rho_i, e_j \rangle = \langle \rho_j, e_i \rangle = 0$, then $[D_i, D_j] = 0$. If $\langle \rho_i, e_j \rangle > 0$ then $\langle \rho_j, e_i \rangle = 0$; otherwise $\{D_i, D_j\}$ is a cyclic subset. Now the assertion follows from Lemma 5.2. \square

Let now $\kappa_1, \dots, \kappa_r$ be pairwise distinct elements of the set $\{\rho_1, \dots, \rho_m\}$. Then every ρ_j coincides with a unique κ_s , and we set $\alpha(j) = s$. This defines a map $\alpha: \{1, \dots, m\} \rightarrow \{1, \dots, r\}$.

Lemma 5.5. *One can reorder the set $\{1, \dots, r\}$ in such a way that the condition $\langle \rho_i, e_j \rangle > 0$ implies $\alpha(i) > \alpha(j)$.*

Proof. If for every ρ_i there is an element e_j with $\langle \rho_i, e_j \rangle > 0$, then we find a cyclic subset in \mathbb{D} . Hence there is ρ_i such that $\langle \rho_i, e_j \rangle \leq 0$ for every j . Then we set $\rho_i = \kappa_1$ and proceed by induction on r . \square

From now on we assume that the condition $\langle \rho_i, e_j \rangle > 0$ implies $\alpha(i) > \alpha(j)$.

Lemma 5.6. *Let D_i, D_j, D_s be elements in \mathbb{D} . If $\langle \rho_i, e_j \rangle > 0$ and $\langle \rho_s, e_i + e_j \rangle > 0$, then $\alpha(s) > \alpha(j)$. In particular, the set $\mathbb{D} \cup \{[D_i, D_j]\}$ contains no cyclic set.*

Proof. If $\langle \rho_s, e_j \rangle > 0$ then $\alpha(s) > \alpha(j)$. If $\langle \rho_s, e_i \rangle > 0$ then $\alpha(s) > \alpha(i) > \alpha(j)$. The second assertion follows from the first one and Lemma 5.4. \square

With any vector $e \in M$ we associate a vector $v(e) \in \mathbb{Z}^r$, $v(e) := (\langle \kappa_1, e \rangle, \dots, \langle \kappa_r, e \rangle)$. Let us say that a vector $v \in \mathbb{Z}^r$ is *appropriate* if there is an index i such that $v_j = 0$ for $j < i$, $v_i = -1$, and $v_j \geq 0$ for $j > i$.

Clearly, the vector $v(e_i)$ is appropriate for every root e_i corresponding to a derivation in \mathbb{D} . Moreover, by Lemma 5.6 every multiple commutator $[\dots, [D_{i_1}, D_{i_2}], \dots, D_{i_k}]$ with $D_{i_1}, D_{i_2}, \dots, D_{i_k} \in \mathbb{D}$ is either zero or the vector $v(e_{i_1} + e_{i_2} + \dots + e_{i_k})$ is appropriate.

The proof of the following lemma is elementary and left to the reader.

Lemma 5.7. *Consider a finite collection of appropriate vectors in \mathbb{Z}^r . Then only finitely many linear combinations of these vectors with non-negative integer coefficients are appropriate vectors.*

Since $v(e_{i_1} + e_{i_2} + \dots + e_{i_k}) = v(e_{i_1}) + v(e_{i_2}) + \dots + v(e_{i_k})$, we conclude from Lemma 5.7 that only finitely many multiple commutators of the derivations D_1, \dots, D_m can be non-zero. So the Lie algebras $\text{Lie}_{K_0}(\mathbb{D})$ and $\text{Lie}(\mathbb{D})$ are finite-dimensional and nilpotent. This completes the proof of Proposition 5.3. \square

Proposition 5.8. *Assume that every element of \mathbb{D} is contained in a cyclic subset and every cyclic subset is almost simple. Then $\mathbb{D} = \mathbb{D}_1 \sqcup \dots \sqcup \mathbb{D}_s$, where $\mathbb{D}_1, \dots, \mathbb{D}_s$ are maximal cyclic subsets. The Lie algebra $\text{Lie}_{K_0}(\mathbb{D})$ is isomorphic to $\mathfrak{sl}_{r_1}(K_0) \oplus \dots \oplus \mathfrak{sl}_{r_s}(K_0)$ for some positive integers r_1, \dots, r_s . Moreover, if every cyclic subset of \mathbb{D} is simple then the Lie algebra $\text{Lie}(\mathbb{D})$ is isomorphic to $\mathfrak{sl}_{r_1}(\mathbf{k}) \oplus \dots \oplus \mathfrak{sl}_{r_s}(\mathbf{k})$.*

Proof. Let \mathbb{D}_1 and \mathbb{D}_2 be two different maximal cyclic subsets of \mathbb{D} .

Lemma 5.9. *We have $\langle \rho_i, e_j \rangle = \langle \rho_j, e_i \rangle = 0$ for any $D_i \in \mathbb{D}_1, D_j \in \mathbb{D}_2$. In particular, the sets $\{\rho(D), D \in \mathbb{D}_1\}$ and $\{\rho(D), D \in \mathbb{D}_2\}$ are disjoint.*

Proof. First assume that $\rho_i = \rho_j$. Putting the cyclic set \mathbb{D}_2 into the cyclic set \mathbb{D}_1 just before the element D_i and starting from D_j , we obtain a bigger cyclic set, a contradiction. This proves that the sets $\{\rho(D), D \in \mathbb{D}_1\}$ and $\{\rho(D), D \in \mathbb{D}_2\}$ are disjoint.

Assume that $\langle \rho_i, e_j \rangle > 0$. Since the root e_j is elementary, the vector ρ_i is defined uniquely by this property. But D_j is contained in the cyclic set \mathbb{D}_2 , and thus we should have $\rho_i \in \{\rho(D), D \in \mathbb{D}_2\}$, a contradiction. \square

By Proposition 4.4, we have $\text{Lie}_{K_0}(\mathbb{D}_i) \cong \mathfrak{sl}_{r_i}(K_0)$ and $\text{Lie}(\mathbb{D}_i) \cong \mathfrak{sl}_{r_i}(\mathbf{k})$, where r_i is the number of pairwise distinct elements in the set $\{\rho(D), D \in \mathbb{D}_i\}$. By Lemma 5.2 and Lemma 5.9, the elements from \mathbb{D}_i and \mathbb{D}_j with $i \neq j$ commute. Since the Lie algebra \mathfrak{sl}_r is simple, we obtain the assertions of Proposition 5.8. \square

Proposition 5.10. *Assume that every cyclic subset in \mathbb{D} is almost simple. Denote by \mathbb{D}' the subset of elements of \mathbb{D} that are contained in cyclic subsets. Then we have a semidirect product structure*

$$\text{Lie}_{K_0}(\mathbb{D}) \cong \text{Lie}_{K_0}(\mathbb{D}') \ltimes \mathfrak{N},$$

where \mathfrak{N} is a finite-dimensional nilpotent ideal in $\text{Lie}_{K_0}(\mathbb{D})$. If every cyclic subset of \mathbb{D} is simple, we have

$$\text{Lie}(\mathbb{D}) \cong \text{Lie}(\mathbb{D}') \ltimes \mathfrak{n},$$

where \mathfrak{n} is a finite-dimensional nilpotent ideal in $\text{Lie}(\mathbb{D})$.

Proof. We begin with a computational lemma.

Lemma 5.11. *Take $D_i \in \mathbb{D}', D_j \in \mathbb{D} \setminus \mathbb{D}'$ and let $\widehat{D} = [D_i, D_j]$. Then either $\widehat{D} = 0$ or \widehat{D} is a root derivation and \widehat{D} is not contained in a cyclic subset of $\mathbb{D} \cup \{\widehat{D}\}$.*

Proof. We subdivide the proof into four cases.

Case 1. If $\rho_i = \rho_j$ or $\langle \rho_i, e_j \rangle = \langle \rho_j, e_i \rangle = 0$, then $\widehat{D} = 0$.

Case 2. If $\langle \rho_i, e_j \rangle > 0$ and $\langle \rho_j, e_i \rangle > 0$, then $\{D_i, D_j\}$ is a cyclic set, a contradiction with the choice of D_j .

Case 3. Assume that $\langle \rho_i, e_j \rangle > 0$ and $\langle \rho_j, e_i \rangle = 0$. By Lemma 5.2, we have $\widehat{D} = D_{\psi, \rho_j, e_i + e_j}$. Assume that \widehat{D} is contained in a cyclic subset, say $\dots, D_k, \widehat{D}, D_s \dots$. Then $\langle \rho_j, e_k \rangle > 0$ and $\langle \rho_s, e_i + e_j \rangle > 0$. If $\langle \rho_s, e_j \rangle > 0$ then we replace \widehat{D} by D_j in the cyclic subset and obtain a cyclic subset, a contradiction with the choice of D_j . If $\langle \rho_s, e_j \rangle = 0$ and $\langle \rho_s, e_i \rangle > 0$ then we replace \widehat{D} by the elements D_j, D_i in the cyclic subset and obtain a cyclic subset containing D_j , again a contradiction.

Case 4. Assume that $\langle \rho_i, e_j \rangle = 0$ and $\langle \rho_j, e_i \rangle > 0$. Since the root e_i is elementary, we have $\rho_j \in \rho(\mathbb{D}')$. By Lemma 5.2, we obtain $\widehat{D} = D_{\psi, \rho_i, e_i + e_j}$. Assume that \widehat{D} is contained in a cyclic subset, say $\dots, D_k, \widehat{D}, D_s \dots$. Then $\langle \rho_i, e_k \rangle > 0$ and $\langle \rho_s, e_i + e_j \rangle > 0$. By assumptions of Proposition 5.10, the root e_i is elementary. It implies $\langle \rho_j, e_i \rangle = 1$. If $\langle \rho_s, e_j \rangle > 0$ then $\dots, D_k, D_i, D_j, D_s \dots$ is a cyclic subset, a contradiction. If $\langle \rho_s, e_j \rangle \leq 0$ and $\langle \rho_s, e_i \rangle > 0$ then $\rho_s = \rho_j$ (e_i is elementary) and

$$\langle \rho_s, e_i + e_j \rangle = \langle \rho_j, e_i \rangle + \langle \rho_j, e_j \rangle = 1 - 1 = 0,$$

a contradiction. This concludes the proof of Lemma 5.11. \square

If $\widehat{D} \neq 0$, we add \widehat{D} to \mathbb{D} and continue the process. Let us prove that for any $e_j \in e(\mathbb{D} \setminus \mathbb{D}')$ the number of roots of the form $e_j + \sum_{e \in e(\mathbb{D}')} \lambda_e e$ with non-negative integer coefficients λ_e is finite.

Since such a linear expression may be not unique, we consider only expressions, where the sum $\sum \lambda_e$ is minimal. Consider the subset $E \subset e(\mathbb{D}')$ consisting of all elementary roots e such that $\lambda_e > 0$. Then the subset E contains no cycle, because the sum of roots in a cycle is zero. We claim that the number of roots $e_j + \sum_{e \in E} \lambda_e e$ with given e_j and E is finite.

Since E contains no cycle, we can order the rays $(\kappa_1, \dots, \kappa_p)$ associated with roots in E in such a way that for any $e \in E$ the vector $v(e) = (\langle \kappa_1, e \rangle, \dots, \langle \kappa_p, e \rangle)$ is appropriate. A slight generalization of Lemma 5.7 shows that only finitely many vectors of the form $v(e_j) + \sum_{e \in E} \lambda_e v(e)$ have all coordinates at least -1 . This implies the claim.

Thus the number of non-zero commutators is also finite. This shows that the process stops in finitely many steps with a set \mathbb{D}'' . In particular, the set $\mathbb{D}'' \setminus \mathbb{D}'$ is stable under taking commutators with elements of \mathbb{D}' .

By Proposition 5.3, the Lie algebra generated by $\mathbb{D}'' \setminus \mathbb{D}'$ is finite dimensional and nilpotent. Since its generating set is stable under taking commutators with elements of \mathbb{D}' , the same holds for the whole algebra. We conclude that $\text{Lie}_{\mathbb{K}_0}(\mathbb{D}) \cong \text{Lie}_{\mathbb{K}_0}(\mathbb{D}') \ltimes \text{Lie}_{\mathbb{K}_0}(\mathbb{D}'' \setminus \mathbb{D}')$ and $\text{Lie}(\mathbb{D}) \cong \text{Lie}(\mathbb{D}') \ltimes \text{Lie}(\mathbb{D}'' \setminus \mathbb{D}')$. This concludes the proof of Proposition 5.10. \square

Finally, Proposition 5.8 and Proposition 5.10 prove the “if” direction in (1) and (2). This completes the proof of Theorem 5.1.

6. A Demazure type theorem

Definition 6.1. An affine algebraic group G over the ground field \mathbf{k} is said to be a *group of type A* if a maximal semisimple subgroup of G is isomorphic to a factor group of $\text{SL}_{r_1}(\mathbf{k}) \times \dots \times \text{SL}_{r_s}(\mathbf{k})$ for some positive integers r_1, \dots, r_s by a finite central subgroup.

Definition 6.2. Let X be an algebraic variety with an action of an algebraic torus \mathbb{T} . A faithful action $G \times X \rightarrow X$ of an algebraic group G is said to be *vertical*, if the image of G in the group $\text{Aut}(X)$ is normalized by \mathbb{T} and general G -orbits on X are contained in closures of \mathbb{T} -orbits.

The following theorem is a generalization of a classical result due to Demazure: the automorphism group of a complete toric variety is an affine algebraic group of type A, see [7, Proposition 3.3].

Theorem 6.3. *Let an affine algebraic group G admit a vertical action on a \mathbb{T} -variety X . Then G is a group of type A.*

Proof. By [20, Theorem 3] we can embed X equivariantly into a completion X' so that now \mathbb{T} and G act on the variety X' . In particular, the variety X' is semiaffine. Let us also replace the group G by its subgroup G' generated by all \mathbb{G}_a -subgroups in G . It is well known that G' is a semidirect product of a maximal semisimple subgroup of G and the unipotent radical of G . The torus \mathbb{T} acts on G by automorphisms and thus preserves the subgroup G' . The \mathbb{T} -action on the tangent algebra \mathfrak{g} of G' is diagonalizable and the group G' is generated by \mathbb{T} -normalized \mathbb{G}_a -subgroups. They correspond to vertical regularly integrable derivations of X' and the Lie algebra \mathfrak{g} is generated by such derivations. Now, Theorem 5.1 and Propositions 5.8 and 5.10 complete the proof. \square

Remark that taking X to be a complete toric variety we recover Demazure’s result since any faithful action $G \times X \rightarrow X$ normalized by the acting torus \mathbb{T} is vertical.

References

- [1] Klaus Altmann, Jürgen Hausen, Polyhedral divisors and algebraic torus actions, *Math. Ann.* 334 (3) (2006) 557–607.
- [2] Klaus Altmann, Jürgen Hausen, Hendrik Süß, Gluing affine torus actions via divisorial fans, *Transform. Groups* 13 (2) (2008) 215–242.
- [3] Klaus Altmann, Nathan Owen Ilten, Lars Petersen, Hendrik Süß, Robert Vollmert, The geometry of T -varieties, in: *Contributions to Algebraic Geometry*, in: EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, pp. 17–69.
- [4] Ivan Arzhantsev, Jürgen Hausen, Elaine Herppich, Alvaro Liendo, The automorphism group of a variety with torus action of complexity one, *Mosc. Math. J.* 14 (3) (2014) 429–471.
- [5] Ivan Arzhantsev, Alvaro Liendo, Polyhedral divisors and SL_2 -actions on affine T -varieties, *Mich. Math. J.* 61 (4) (2012) 731–762.
- [6] David Cox, John Little, Henry Schenck, *Toric Varieties*, Graduate Studies in Math., vol. 124, American Mathematical Society, Providence, RI, 2011.
- [7] Michel Demazure, Sous-groupes algébriques de rang maximum du groupe de Cremona, *Ann. Sci. Éc. Norm. Supér.* 3 (1970) 507–588.
- [8] Adrien Dubouloz, Alvaro Liendo, On rational additive group actions, *Int. J. Math.* 27 (8) (2016) 1650060 (19 pages).
- [9] David Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995.
- [10] Gene Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*, Encyclopaedia Math. Sci., vol. 136, Springer, Berlin, 2006.
- [11] William Fulton, *Introduction to Toric Varieties*, Annals of Math. Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993.
- [12] Jacob Eli Goodman, Alan Landman, Varieties proper over affine schemes, *Invent. Math.* 20 (1973) 267–312.
- [13] Alvaro Liendo, Affine T -varieties of complexity one and locally nilpotent derivations, *Transform. Groups* 15 (2) (2010) 389–425.
- [14] Alvaro Liendo, \mathbb{G}_a -actions of fiber type on affine T -varieties, *J. Algebra* 324 (12) (2010) 3653–3665.
- [15] Tadao Oda, *Convex Bodies and Algebraic Geometry: an Introduction to Toric Varieties*, A Series of Modern Surveys in Math., vol. 15, Springer Verlag, Berlin, 1988.
- [16] Lars Petersen, Hendrik Süß, Torus invariant divisors, *Isr. J. Math.* 182 (2011) 481–504.
- [17] Vladimir Popov, Birational splitting and algebraic group actions, *Eur. J. Math.* 2 (1) (2016) 283–290.
- [18] Maxwell Rosenlicht, Some basic theorems on algebraic groups, *Am. J. Math.* 78 (1956) 401–443.
- [19] Maxwell Rosenlicht, A remark on quotient spaces, *An. Acad. Bras. Ciênc.* 35 (1963) 487–489.
- [20] Hideyasu Sumihiro, Equivariant completion, *J. Math. Kyoto Univ.* 14 (1974) 1–28.