SLIDE POLYNOMIALS AND SUBWORD COMPLEXES

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Abstract. Subword complexes were defined by A. Knutson and E. Miller in 2004 for describing Gröbner degenerations of matrix Schubert varieties. The facets of such a complex are indexed by pipe dreams, or, equivalently, by the monomials in the corresponding Schubert polynomial. In 2017 S. Assaf and D. Searles defined a basis of slide polynomials, generalizing Stanley symmetric functions, and described a combinatorial rule for expanding Schubert polynomials in this basis. We describe a decomposition of subword complexes into strata called slide complexes, that correspond to slide polynomials. The slide complexes are shown to be homeomorphic to balls or spheres.

1. Introduction

1.1. **Schubert polynomials and pipe dreams.** Schubert polynomials $\mathfrak{S}_w \in \mathbb{Z}[x_1, x_2, \dots]$ were defined by I. N. Bernstein, I. M. Gelfand and S. I. Gelfand [BGG73] and by A. Lascoux and M.-P. Schützenberger [LS82]. They can be viewed as "especially nice" polynomial representatives of classes of Schubert varieties $[X_w] \in H^*(G/B)$, where $G = \mathrm{GL}_n(\mathbb{C})$ is a general linear group, B is a Borel subgroup in G, and G/B is a full flag variety. It is well known that their coefficients are nonnegative, and there exists a manifestly positive combinatorial rule for computing these coefficients.

One can also be interested in the K-theory $K_0(G/B)$. Instead of Schubert classes $[X_w] \in H^*(G/B)$, one would consider the classes of their structure sheaves $[\mathcal{O}_w] \in K_0(G/B)$. These classes also have a nice presentation, known as Grothendieck polynomials $\mathfrak{G}_w^{(\beta)} \in \mathbb{Z}[\beta, x_1, x_2, \dots]$, depending on an additional parameter β. They also have integer nonnegative coefficients, but, as opposed to Schubert polynomials, they are not homogeneous in the usual sense; however, they become homogeneous if we set $\deg \beta = -1$. They can be viewed as "deformations" of the Schubert polynomials $\mathfrak{S}_w$: evaluating $\mathfrak{G}_w^{(\beta)}$ at $\beta = 0$, we recover the corresponding Schubert polynomial $\mathfrak{S}_w = \mathfrak{G}_w^{(0)}$.

Schubert and Grothendieck polynomials can be described combinatorially by means of diagrams called *pipe dreams*, or *rc-graphs*. These diagrams are configurations of pseudolines associated to a permutation; to each such diagram one can assign a monomial. A pipe dream is said to be *reduced* if each pair of pseudolines intersects at most once. The Schubert (resp. Grothendieck) polynomial for a permutation w is obtained as the sum of the corresponding monomials for reduced (resp. not necessarily reduced) pipe dreams associated to w. This theorem, due to S. Billey and N. Bergeron [BB93] and to S. Fomin and An. Kirillov [FK96], is an analogue of Littlewood's presentation of Schur polynomials as sums over Young tableaux. In particular, this implies positivity of the coefficients of Schubert and Grothendieck polynomials. A brief recall about pipe dreams is given in § 2.3.

In [KM05], A. Knutson and E. Miller proposed a geometric interpretation of pipe dreams for a permutation w: they correspond to the irreducible components of a "deep" Gröbner degeneration of the corresponding matrix Schubert variety $\overline{X_w}$ to a union of affine subspaces. A combinatorial structure of this union of subspaces is encoded by a certain simplicial complex, known as the *pipe dream complex* for w. From this one can deduce that the multidegree of $\overline{X_w}$ with respect to the maximal torus $T \subset B \subset G$ equals the Schubert polynomial $\mathfrak{S}_w$.

In the subsequent paper [KM04] the same authors put the notion of a pipe dream complex into a more general context, defining *subword complexes* for an arbitrary Coxeter system, and prove that such complexes are shellable and, moreover, homeomorphic to balls or, in certain "rare"

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cases, to spheres. This implies many interesting results about the geometry of the corresponding Schubert varieties, both matrix and usual ones, including new proofs for normality and Cohen–Macaulayness of Schubert varieties in a full flag variety.

1.2. **Slide and glide polynomials.** Recently, S. Assaf and D. Searles [AS17] defined *slide polynomials* $\mathfrak{F}_Q$. This is another family of polynomials with properties similar to Schubert polynomials: in particular, they form a basis in the ring of polynomials in countably many variables and enjoy a *manifestly positive* Littlewood–Richardson rule. They are indexed by pipe dreams Q with an extra combinatorial condition, usually called the *quasi-Yamanouchi pipe dreams*. This condition is similar to the Yamanouchi condition for skew Young tableaux; the precise definitions are in § 2.4

Moreover, there exist combinatorial positive formulas for expressing Schubert polynomials in the slide basis: each Schubert polynomial is expressed as a linear combination of slide polynomials with coefficients 0 or 1.

Slide polynomials also have a K-theoretic counterpart: *glide polynomials* $\mathcal{G}_Q^{(\beta)}$, defined by O. Pechenik and D. Searles in [PS19] (note that the names "Schubert" and "Grothendieck" also start with S and G, respectively). Similarly, there are explicit expressions of Grothendieck polynomials via glide polynomials.

1.3. **Slide complexes.** The main objects defined in this paper are analogues of subword complexes corresponding to slide polynomials. We call them *slide complexes*. Each subword complex can be subdivided into slide complexes. We show that these complexes are shellable (Theorem 4.2). Our main result, Theorem 4.7, states that each slide complex is homeomorphic to a ball or a sphere.

In the case of pipe dream complexes, from a slide complex we can recover the corresponding slide and glide polynomials: the slide (resp. glide) polynomial is obtained as the sum of monomials corresponding to facets (resp. all interior faces) of the corresponding complex. This provides us a topological interpretation of the combinatorial expression for $\mathfrak{S}_w$ via $\mathfrak{F}_Q$ and of $\mathfrak{G}_w^{(\beta)}$ via $\mathcal{G}_Q^{(\beta)}$ (Corollaries 4.10 and 4.11).

1.4. **Possible relation with degenerations of matrix Schubert varieties.** In this paper we are dealing only with combinatorial constructions and do not address the geometric picture. It would be interesting to explore the relation of slide polynomials with degenerations of matrix Schubert varieties. A natural question is as follows: for a matrix Schubert variety $\overline{X_w}$, does there exist an "intermediate degeneration" $\overline{X_w} \to \bigcup \overline{Y_{w,Q}}$, with the irreducible components indexed by quasi-Yamanouchi pipe dreams of shape w, such that the multidegree of each irreducible component $\overline{Y_{w,Q}}$ is equal to the slide polynomial $\mathfrak{F}_Q$? This would, in particular, provide a geometric interpretation of the Littlewood–Richardson coefficients for slide polynomials, studied by S. Assaf and D. Searles in [AS17].

1.5. **Structure of the paper.** This text is organized as follows. In Sec. 2 we recall the definitions of Schubert and Grothendieck polynomials, provide their description using pipe dreams, and describe slide and glide polynomials in terms of pipe dreams. Sec. 3 contains the definition of a subword complex for an arbitrary Coxeter system. We also recall the proof of its shellability and show that it can be homeomorphic to a ball or a sphere. Then we focus on the most important particular case of pipe dream complexes. The main results of this paper are contained in Sec. 4: in § 4.1 we define slide complexes for an arbitrary Coxeter system and show that they are shellable and homeomorphic to balls or spheres. In § 4.2 we show that the decomposition of a pipe dream complex into slide complexes corresponds to the presentation of the corresponding Schubert (resp. Grothendieck) polynomial as the sum of slide (resp. glide) polynomials.

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2. Schubert, Grothendieck, slide and glide polynomials

2.1. The symmetric group. We will denote by $\mathcal{S}_n$ the symmetric group on n letters, i.e. the group of bijective maps from $\{1,\dots,n\}$ onto itself. It is generated by the simple transpositions $s_i=(i\leftrightarrow i+1)$ for $1\le i\le n-1$, modulo the Coxeter relations:

- $s_i^2=e$;
- $s_is_j=s_js_i$ for $|i-j|\ge 2$ (far commutativity);
- $s_is_{i+1}s_i=s_{i+1}s_is_{i+1}$ for each $i=1,\dots,n-2$ (braid relation).

We will use the one-line notation for permutations: for example, $w=\overline{1423}$ brings 1 to 1, 2 to 4, 3 to 2, and 4 to 3.

Each permutation $w\in\mathcal{S}_n$ can be expressed as a product $w=s_{i_1}\dots s_{i_k}$ of simple transpositions. We will say that w is presented by the *word* $(s_{i_1},\dots,s_{i_k})$. The minimal length of a word presenting w is called the *length* of w and denoted by $\ell(w)$; in this case the word is said to be *reduced*. It is well known that $\ell(w)$ is equal to the number of inversions in w, i.e.

$$\ell(w)=\{\#(i,j)\mid 1\le i<j\le n, w(i)>w(j)\}.$$

The longest permutation in $\mathcal{S}_n$ will be denoted by w_0. This is the permutation which brings i into $n+1-i$ for each i; clearly, $\ell(w_0)=\binom{n}{2}=\frac{n(n-1)}{2}$.

This permutation has several reduced presentations; later we will need the following one:

$$w_0=(s_{n-1}\dots s_3s_2s_1)(s_{n-1}\dots s_3s_2)(s_{n-1}\dots s_3)\dots(s_{n-1}s_{n-2})(s_{n-1}).$$

2.2. Schubert and Grothendieck polynomials. Denote the set of variables $x_1,\dots,x_n$ by $\mathbf{x}$ and consider the polynomial ring $\mathbb{Z}[\mathbf{x}]$. The group $\mathcal{S}_n$ acts on this ring by interchanging variables:

$$w\circ f(x_1,\dots,x_n)=f(x_{w(1)},\dots,x_{w(n)}).$$

Definition 2.1. For $i=1,\dots,n-1$ we define the *divided difference operators* $\partial_i:\mathbb{Z}[\mathbf{x}]\to\mathbb{Z}[\mathbf{x}]$ as follows:

$$\partial_i f(\mathbf{x})=\frac{f(\mathbf{x})-s_i\circ f(\mathbf{x})}{x_i-x_{i+1}}.$$

Since the numerator is antisymmetric with respect to x_i and x_{i+1}, it is divisible by the denominator, so the ratio is indeed a polynomial with integer coefficients.

The divided difference operators satisfy the Coxeter relations:

- $\partial_i^2=0$,
- $\partial_i\partial_j=\partial_j\partial_i$ if $|i-j|\ge 2$,
- $\partial_i\partial_{i+1}\partial_i=\partial_{i+1}\partial_i\partial_{i+1}$ for each $i=1,\dots,n-2$.

Definition 2.2. *Schubert polynomials* $\mathfrak{S}_w$ are defined as the elements of $\mathbb{Z}[\mathbf{x}]$ indexed by permutations $w\in\mathcal{S}_n$ and satisfying the relations

$$\mathfrak{S}_{id}=1,$$

$$\partial_i\mathfrak{S}_w=\begin{cases}\mathfrak{S}_{ws_i}, & \text{if } \ell(ws_i)<\ell(w),\\ 0 & \text{otherwise}\end{cases}$$

for each $i=1,\dots,n-1$.

A. Lascoux and M.-P. Schützenberger [LS82] have shown that the Schubert polynomials are uniquely determined by these relations. Equivalently, they can be constructed using the recurrence relation

$$\mathfrak{S}_{ws_i}(\mathbf{x})=\partial_i\mathfrak{S}_w(\mathbf{x}),\ \text{if } \ell(ws_i)<\ell(w)$$

with the initial condition

$$\mathfrak{S}_{w_0}(\mathbf{x})=x_1^{n-1}x_2^{n-2}\dots x_{n-2}^2x_{n-1}.$$

This recurrence relation can be written as follows: if $s_{i_k}\dots s_{i_1}$ is a reduced word for a permutation w_0w, then

$$\mathfrak{S}_w=\partial_{i_1}\dots\partial_{i_k}\mathfrak{S}_{w_0}.$$

Since the divided difference operators satisfy the far commutativity and braid relations, and every reduced word for w_0w can be transformed into any other reduced word just by these two operations, $\mathfrak{S}_w$ is well defined (i.e. does not depend upon a choice of the reduced word).

Grothendieck polynomials, introduced by A. Lascoux in [Las07], can be viewed as a deformation of Schubert polynomials. Their definition is similar, but instead of ∂_i we need to use the *isobaric divided difference operators* $\pi_i^{(\beta)}$.

Definition 2.3. Let β be a formal parameter. For $i = 1, \ldots, n-1$ define the *β-isobaric divided difference operators* $\pi_i^{(\beta)} : \mathbb{Z}[\beta, \mathbf{x}] \to \mathbb{Z}[\beta, \mathbf{x}]$:

$$\pi_i^{(\beta)} f(\mathbf{x}) = \frac{(1+\beta x_{i+1})f(\mathbf{x}) - (1+\beta x_i)s_i \circ f(\mathbf{x})}{x_i - x_{i+1}}.$$

Just like the divided difference operators, their isobaric counterparts also satisfy the Coxeter relations:

- $\pi_i^{(\beta)}\pi_j^{(\beta)} = \pi_j^{(\beta)}\pi_i^{(\beta)}$ for $|i-j| \geq 2$,
- $\pi_i^{(\beta)}\pi_{i+1}^{(\beta)}\pi_i^{(\beta)} = \pi_{i+1}^{(\beta)}\pi_i^{(\beta)}\pi_{i+1}^{(\beta)}$ for each $i = 1, \ldots, n-2$.

Definition 2.4. Define *β-Grothendieck polynomials* $\mathfrak{G}_w^{(\beta)}$ using the initial condition

$$\mathfrak{G}_{w_0}^{(\beta)}(\mathbf{x}) = x_1^{n-1}x_2^{n-2}\ldots x_{n-2}^2 x_{n-1}$$

and the recurrence relation

$$\mathfrak{G}_w^{(\beta)} = \pi_{i_1}^{(\beta)}\ldots\pi_{i_k}^{(\beta)}\mathfrak{G}_{w_0}^{(\beta)},$$

where $s_{i_k}\ldots s_{i_1}$ is a reduced word for the permutation w_0w.

Since the operators $\pi_i^{(\beta)}$ satisfy the Coxeter relations, these polynomials are also well-defined. One can immediately see that, since $\pi_i^{(0)} = \partial_i$ and $\mathfrak{G}_{w_0}^{(\beta)} = \mathfrak{S}_{w_0}$, we have $\mathfrak{G}_w^{(0)} = \mathfrak{S}_w$ for each $w \in \mathcal{S}_n$. So setting in $\mathfrak{G}^{(\beta)}$ the parameter $\beta = 0$, we recover the Schubert polynomials.

2.3. **Pipe dreams.** In this subsection we discuss pipe dreams: the main combinatorial tool for dealing with Schubert and Grothendieck polynomials.

Definition 2.5. Consider an $(n \times n)$-square and fill it with the elements of two types: *crosses* and *elbows* in such a way that all the crosses are situated strictly above the antidiagonal. We will omit the elbows situated below the antidiagonal. This diagram is called a *pipe dream*, or an *rc-graph* ("RC" stands for "reduced compatible").

Each pipe dream can be viewed as a configuration of n strands joining the left edge of the square with the top edge. Let us index the initial and terminal points of these strands by the numbers from 1 to n, going from top to bottom and from the left to the right.

Pipe dream is said to be *reduced* if every pair of strands crosses at most once and *nonreduced* otherwise.

For the sake of better visibility we will usually draw the crosses in red.

The following figure provides an example of reduced and non-reduced pipe dreams.

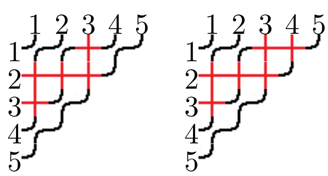


FIGURE 1. A reduced and a non-reduced pipe dream

Definition 2.6. Each pipe dream can be viewed as a bijective map from the set of the initial points of the strands to the set of its terminal points. Let us assign to each *reduced* pipe dream P the corresponding permutation $w(P) \in \mathcal{S}_n$. It will be called the *shape* of P. Moreover, to each pipe dream P we associate the set D_P of the coordinates of its crosses (the first and the second coordinates stand for the row and column numbers, respectively).

For example, the shape of the left pipe dream P at Fig. 1 equals $w(P) = \overline{15423}$, and the set D_P is equal to $D_P = \{(1,3),(2,1),(2,2),(2,3),(3,1)\}$.

Definition 2.7. Define the *reduction* operation reduct on the set of pipe dreams as follows: we read the rows of a pipe dream from top to bottom, reading each row from the right to the left. Each time we find a crossing of two strands that have already crossed before (i.e. above), we replace this cross by an elbow.

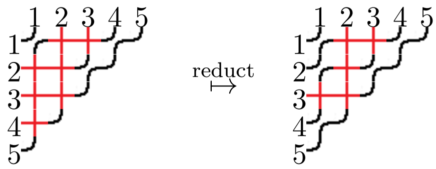


FIGURE 2. The reduction operation reduct applied to a non-reduced pipe dream

Obviously, reduct(P) is reduced for each P, and the operation acts trivially on reduced pipe dreams: reduct$(P) = P$. We say that the *shape* of a nonreduced pipe dream P is the shape $w(\mathrm{reduct}(P))$ of its reduction.

The set of all pipe dreams of a given shape $w \in \mathcal{S}_n$ will be denoted by $\mathrm{PD}(w)$. We will also denote the subset of reduced pipe dreams of this shape by $\mathrm{PD}_0(w) \subset \mathrm{PD}(w)$.

The following theorem was proven by S. Billey and N. Bergeron and independently by S. Fomin and An. Kirillov.

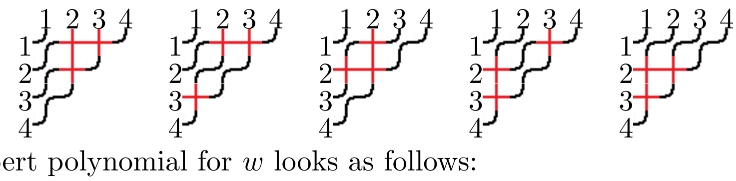
Theorem 2.8. [BB93, FK96] *The Schubert polynomials satisfy the equality*

$$\mathfrak{S}_w = \sum_{P \in \mathrm{PD}_0(w)} \mathbf{x}^P,$$

where

$$\mathbf{x}^P := \prod_{(i,j) \in D_P} x_i.$$

Example 2.9. Consider the permutation $w = \overline{1432}$. There are five reduced pipe dreams of shape w:



Hence the Schubert polynomial for w looks as follows:

$$\mathfrak{S}_{\overline{1432}}(x_1,x_2,x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_2^2x_3.$$

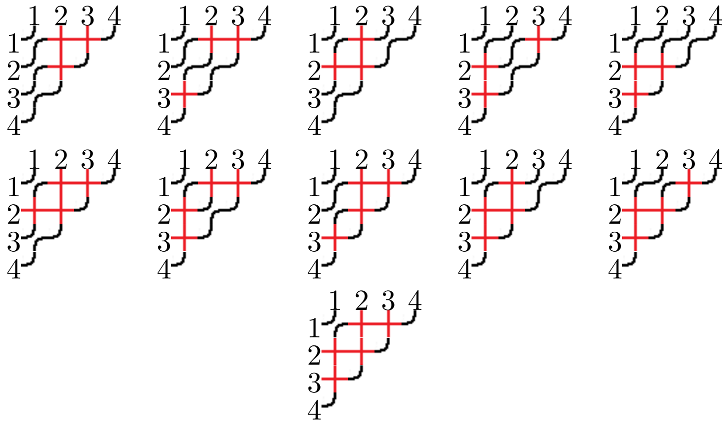
This theorem has the following modification for Grothendieck polynomials.

Definition 2.10. Denote by $\mathrm{ex}(P)$ the *excess* of P, i.e. the number of "redundant" crosses in a (non-reduced) pipe dream P. Namely, set $\mathrm{ex}(P) = \#\left(D_P \smallsetminus D_{\mathrm{reduct}}(P)\right)$.

Theorem 2.11. [FK93, FK94] *The Grothendieck polynomials satisfy the following equality:*

$$\mathfrak{G}_w^{(\beta)} = \sum_{P \in \mathrm{PD}(w)} \beta^{\mathrm{ex}(P)} \mathbf{x}^P.$$

Example 2.12. To continue Example 2.9, let us compute the Grothendieck polynomial for $w = \overline{1432}$. The set $\mathrm{PD}(w)$ consists of 11 pipe dreams, 5 of them being reduced and 6 non-reduced.

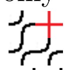


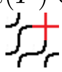
The corresponding Grothendieck polynomial is equal to

$$\mathfrak{G}^{(\beta)}_{(1432)} = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_2^2x_3 + \beta x_1^2x_2^2 + 2\beta x_1^2x_2x_3 + 2\beta x_1x_2^2x_3 + \beta^2x_1^2x_2^2x_3.$$

Since $\mathrm{ex}(P) = 0$ if and only if P is reduced, we have $\mathfrak{G}^{(\beta)}_w = \mathfrak{S}_w + \beta(\ldots)$. This implies the equality $\mathfrak{G}^{(0)}_w = \mathfrak{S}_w$ we mentioned above.

2.4. **Slide and glide polynomials.** S. Assaf and D. Searles [AS17] introduced another basis in the ring of polynomials: the *slide polynomials*. One of their main features is that each Schubert polynomial can be represented as a sum of slide polynomials with the coefficients 0 or 1. The subsequent paper [PS19] provides a similar construction for Grothendieck polynomials. Let us recall these constructions here.

Definition 2.13. Let P be a (possibly non-reduced) pipe dream. Denote a *slide move* S_i as follows. Suppose that the leftmost cross in the i-th row of P is located strictly to the right of the rightmost cross in the $(i+1)$-st row (in particular, the row $i+1$ can contain only elbows). In this case, the leftmost cross in the i-th row can be shifted one step southwest:  $\mapsto$ . If the initial pipe dream was non-reduced, this move can look as follows:  $\mapsto$ . If the leftmost cross in the i-th row is either in the first column or weakly left of a cross in the $(i+1)$-st row, we will say that S_i acts on P identically.

Note that a slide move preserves the shape of a pipe dream. Indeed, $\mathrm{reduct}(S_i(P))$ and $\mathrm{reduct}(P)$ either coincide or are obtained one from the other by a shape-preserving move of one cross:  $\mapsto$ .

Definition 2.14. If all slide moves act on P identically: i.e., for each i we either have the i-th row starting with a cross, or the leftmost cross in the i-th row is located weakly left of a cross from the $(i+1)$-st row, then P is said to be *quasi-Yamanouchi.*

Denote the set of all quasi-Yamanouchi pipe dreams of shape w by $\mathrm{QPD}(w) \subset \mathrm{PD}(w)$, and the subset of all reduced quasi-Yamanouchi pipe dreams by $\mathrm{QPD}_0(w) = \mathrm{PD}_0(w) \cap \mathrm{QPD}(w)$.

Definition 2.15. The *destandardization* operations $\mathrm{dst}\colon \mathrm{PD}(w) \to \mathrm{QPD}(w)$ and $\mathrm{dst}_0\colon \mathrm{PD}_0(w) \to \mathrm{QPD}_0(w)$ are defined as repeated applications of slide moves to a pipe dream until it becomes quasi-Yamanouchi.

A straightforward check shows that the resulting quasi-Yamanouchi pipe dream does not depend on the order of slide moves and hence is well-defined.

Definition 2.16. Let $Q \in \mathrm{QPD}_0(w)$ be a reduced quasi-Yamanouchi pipe dream. The set $\mathrm{dst}_0^{-1}(Q)$ is called the *slide group* of Q. If $Q \in \mathrm{QPD}(w)$ is a not necessarily reduced quasi-Yamanouchi pipe dream, then $\mathrm{dst}^{-1}(Q)$ is called the *glide group* of Q.

Definition 2.17. For $Q \in \mathrm{QPD}_0(w)$, the *slide polynomial* $\mathfrak{F}_Q$ is defined as the sum of monomials over the corresponding slide group of pipe dreams:

$$\mathfrak{F}_Q = \sum_{P \in \mathrm{dst}_0^{-1}(Q)} \mathbf{x}^P.$$

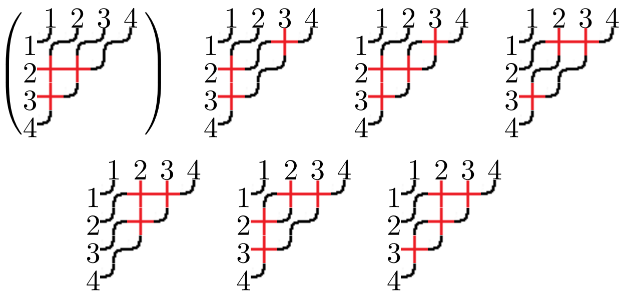
For $Q \in \mathrm{QPD}(w)$, the *glide polynomial* $\mathcal{G}_Q^{(\beta)}$ is defined as the sum of monomials over the glide group of pipe dreams:

$$\mathcal{G}_Q^{(\beta)} = \sum_{P \in \mathrm{dst}^{-1}(Q)} \beta^{\mathrm{ex}(P)-\mathrm{ex}(Q)} \mathbf{x}^P.$$

This definition together with Theorems 2.8 and 2.11 implies that

$$\mathfrak{S}_w = \sum_{Q \in \mathrm{QPD}_0(w)} \mathfrak{F}_Q \qquad \text{and} \qquad \mathfrak{G}_w^{(\beta)} = \sum_{Q \in \mathrm{QPD}(w)} \beta^{\mathrm{ex}(Q)} \mathcal{G}_Q^{(\beta)}.$$

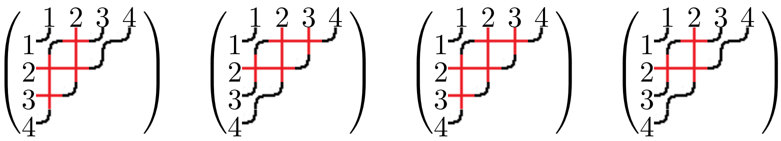
Example 2.18. There are five quasi-Yamanouchi pipe dreams of shape $w = \overline{1432}$. This means that $\mathrm{PD}(w)$ splits into 5 glide groups. One of them consists of seven pipe dreams (the quasi-Yamanouchi pipe dream is shown in parenthesis):



Consequently, the corresponding glide polynomial equals

$$\mathcal{G}_{s_3 s_2 s_3}^{(\beta)} = 2\beta x_1^2 x_2 x_3 + \beta x_1 x_2^2 x_3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3.$$

Each of the remaining four glide groups consists of one pipe dream:



Hence each of the corresponding glide polynomials has only one term:

$$\mathcal{G}_{s_2 s_3 s_2 s_3}^{(\beta)} = x_1 x_2^2 x_3; \qquad \mathcal{G}_{s_3 s_2 s_3 s_2}^{(\beta)} = x_1^2 x_2^2; \qquad \mathcal{G}_{s_3 s_2 s_3 s_2 s_3}^{(\beta)} = x_1^2 x_2^2 x_3; \qquad \mathcal{G}_{s_2 s_3 s_2}^{(\beta)} = x_1 x_2^2.$$

3. Subword complexes and pipe dream complexes

3.1. **Subword complexes.** Consider an arbitrary Coxeter system (Π, Σ), where Π is a Coxeter group, and Σ a system of simple reflections minimally generating Π. We will be particularly interested in the situation where $\Pi = \mathcal{S}_n$ is a symmetric group and $\Sigma = \{s_1, \dots, s_{n-1}\}$ is the set of simple transpositions.

Definition 3.1. A *word* of length m is a sequence $Q = (\sigma_1, \dots, \sigma_m)$ of simple reflections. A subsequence $P = (\sigma_{i_1}, \sigma_{i_2}, \dots, \sigma_{i_k})$ where $1 \leqslant i_1 < i_2 < \dots < i_k \leqslant m$ is a *subword* of Q.

We say that P *represents* $\pi \in \Pi$ if $\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_k}$ is a reduced decomposition of π. If some subword of P represents π, we will say that P *contains* π.

The *subword complex* $\Delta(Q, \pi)$ is a set of nonempty subwords $Q \setminus P$ whose complements P contain π. It is a simplicial complex; one of its faces belongs to the boundary of another if and only if the first of the corresponding subwords is a subset of the second one.

All reduced subwords for $\pi \in \Pi$ have the same length. So the complex $\Delta(Q, \pi)$ is pure of dimension $m - \ell(\pi)$. The words $Q \setminus P$ such that P represents π are their facets.

Example 3.2. Let $\Pi = S_4, \pi = \overline{1432}, Q = s_3s_2s_1s_3s_2s_3$. The permutation π has two reduced decompositions $s_2s_3s_2$ and $s_3s_2s_3$. Let us label the center of a pentagon with s_1 and its vertices with reflections s_3, s_2, s_3, s_2, s_3 in the cyclic order. Then the facets of $\Delta(Q, \pi)$ are the triples that consist of two adjacent vertices and the center of the pentagon.

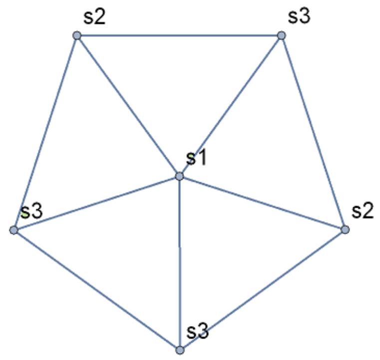


FIGURE 3. Subword complex $\Delta(s_3s_2s_1s_3s_2s_3, s_2s_3s_2)$.

Definition 3.3. Let Δ be a simplicial complex and $F \in \Delta$ a face. The *deletion* of F from Δ is a complex

$$\mathrm{del}(F, \Delta) = \{G \in \Delta \mid G \cap F = \varnothing\}.$$

The **link** of F in Δ is a complex

$$\mathrm{link}(F, \Delta) = \{G \in \Delta \mid G \cap F = \varnothing, G \cup F \in \Delta\}.$$

Definition 3.4. An n-dimensional complex Δ is said to be *vertex decomposable* if it is pure and satisfies one of the following properties:

- Δ is an n-dimensional simplex; or
- there exists a vertex $v \in \Delta$ such that $\mathrm{del}(v, \Delta)$ is a vertex decomposable n-dimensional complex, while $\mathrm{link}(v, \Delta)$ is an $(n-1)$-dimensional vertex decomposable complex.

Definition 3.5. A *shelling* of a simplicial complex Δ is a total order on the set of its facets with the following property: for each i, j, such that $1 \leqslant i < j \leqslant t$, there exist a k, where $1 \leqslant k < j$, and a vertex $v \in F_j$ such that $F_i \cap F_j \subseteq F_k \cap F_j = F_j \smallsetminus \{v\}$.

A complex admitting a shelling is said to be *shellable.*

The definition of shelling can be restated in the following way: for each j such that $2 \leqslant j \leqslant t$ the complex $(\bigcup_{i<j} F_i) \cap F_j$ is pure of dimension $\dim F_j - 1$.

The notion of vertex decomposability was introduced in [BP79]; in the same paper it is shown that it implies shellability.

Proposition 3.6 ([BP79])**.** *Vertex decomposable complexes are shellable.*

The following statement was proven in [KM04, Thm 2.5].

Theorem 3.7. *Subword complexes are vertex decomposable, and hence shellable.*

Definition 3.8. The *Demazure product* $\delta(Q) \in \Pi$ of a word Q is defined by induction as follows: $\delta(\sigma) = \sigma$ for $\sigma \in \Sigma$, and

$$\delta(Q, \sigma) = \begin{cases} \delta(Q)\sigma & \text{if } \ell(\delta(Q)\sigma) > \ell(\delta(Q)), \\ \delta(Q) & \text{otherwise.} \end{cases}$$

In other words, we multiply the elements in Q from left to right, omitting the letters that decrease the length of the product obtained at each step. One can also think about the Demazure product as the product in the monoid generated by Σ subject to the relations of the Coxeter group with the relation $s_i^2 = e$ being replaced by $s_i^2 = s_i$, cf. [KM04, Def. 3.1].

Now let us recall the main results from [KM04]

Theorem 3.9 ([KM04, Thm 3.7]). *The complex $\Delta(Q,\pi)$ is either a ball or a sphere. A face $Q \setminus P$ is contained in its boundary iff $\delta(P) \neq \pi$.*

Corollary 3.10 ([KM04, Cor. 3.8]). *$\Delta(Q,\pi)$ is a sphere if $\delta(Q) = \pi$, and a ball otherwise.*

3.2. **Pipe dream complexes.** Pipe dreams are closely related to subword complexes of a certain form. Let $\Pi = \mathcal{S}_n$, and let us fix the following word for the longest permutation:

$$Q_{0,n} = (s_{n-1}s_{n-2}\dots s_3s_2s_1)(s_{n-1}s_{n-2}\dots s_3s_2)(s_{n-1}s_{n-2}\dots s_3)\dots(s_{n-1}s_{n-2})(s_{n-1}).$$

This word is obtained by reading the table

$$\begin{array}{cccccc} s_1 & s_2 & s_3 & \dots & s_{n-2} & s_{n-1} \\ s_2 & s_3 & \dots & s_{n-2} & s_{n-1} & \\ s_3 & \dots & s_{n-2} & s_{n-1} & & \\ \vdots & \vdots & \iddots & & & \\ s_{n-2} & s_{n-1} & & & & \\ s_{n-1} & & & & & \end{array}$$

from right to left, from top to bottom. Let $P \in \mathrm{PD}_0(w)$ be a reduced pipe dream of shape $w \in \mathcal{S}_n$. For each of the crosses occuring in P let us take the simple transposition from the corresponding cell of the table. We obtain a subword word(P) in $Q_{0,n}$. It is clear that word(P) represents the permutation w. The converse is also true: if T is a subword in $Q_{0,n}$ representing w, then the pipe dream with crosses corresponding to the letters of T is reduced and has the shape w.

We obtain a bijection betwen the elements of $\mathrm{PD}_0(w)$ and facets of the complex $\Delta(Q_{0,n}, w)$.

Definition 3.11. The complex $\Delta(Q_{0,n}, w)$ is called a *pipe dream complex.*

The reduction of a pipe dream corresponds naturally to the Demazure product of the related subword: for each pipe dream P we have $w(\mathrm{reduct}(P)) = \delta(\mathrm{word}(P))$. Moreover, if P is a nonreduced pipe dream of shape w, then word(P) contains word(reduct(P)) as a subword and hence contains the permutation w.

Using these facts and Theorem 3.9, we see that the pipe dreams from $\mathrm{PD}(w)$ bijectively correspond to the interior faces of the pipe dream complex $\Delta(Q_{0,n}, w)$.

Corollary 3.12. *The Grothendieck polynomial $\mathfrak{G}_w$ is obtained as the sum of monomials corresponding to the interior faces of the corresponding pipe dream complex. Namely, for $w \in \mathcal{S}_n$ we have*

$$\mathfrak{G}_w^{(\beta)} = \sum_{P \in \mathrm{int}(\Delta(Q_{0,n},w))} \beta^{\mathrm{codim}(P)} \mathbf{x}^P$$

(By $\mathbf{x}^P$ we denote the monomial of the pipe dream corresponding to a face P).

Example 3.13. Fig. 4 represents the pipe dream complex for $w = \overline{1432}$. The pipe dreams are split into groups corresponding to their shapes; each pipe dream is indexed by the corresponding monomial $\beta^{\mathrm{ex}(P)}\mathbf{x}^P$ occuring in the Grothendieck polynomial.

For $k \in \mathbb{Z}_{\geq 0}$ we introduce the following notation: $\mathrm{PD}_k(w) = \{P \in \mathrm{PD}(w) \mid \mathrm{ex}(P) = k\}$.

Corollary 3.14. *For each permutation $w \in \mathcal{S}_n$ we have*

$$\sum_{k=0}^{n(n-1)/2} (-1)^k |\,\mathrm{PD}_k(w)| = 1.$$

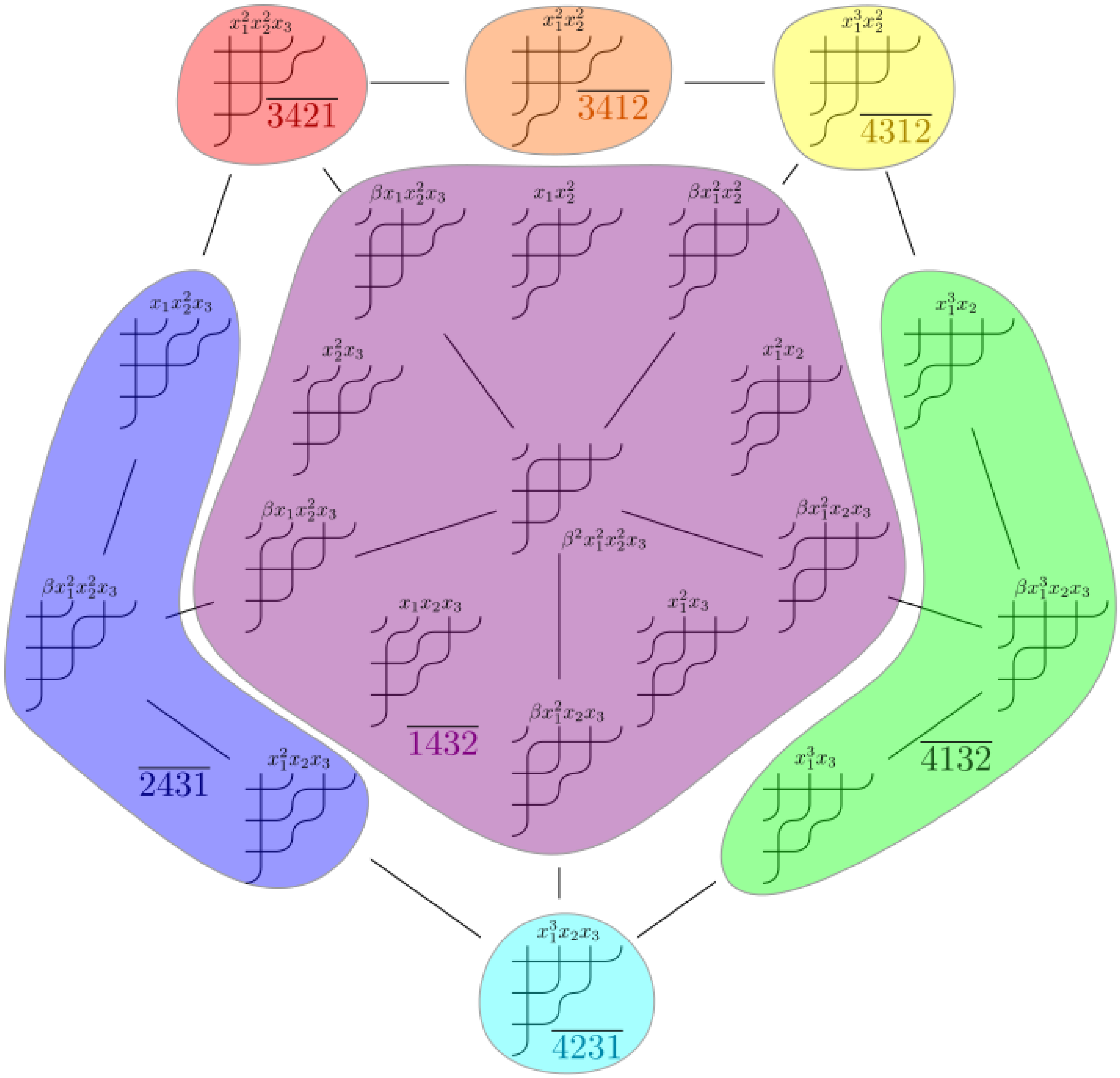


FIGURE 4. The pipe dream complex for $w = \overline{1432}$

Proof. Specializing the Grothendieck polynomial at $\mathbf{x} = (1, 1, \dots, 1)$ and $\beta = -1$, we obtain the following relation:

$$\mathfrak{G}_w^{(-1)}(1,1,\dots,1) = \sum_{P\in \mathrm{PD}(w)} (-1)^{\mathrm{ex}(P)} = \sum_{k=0}^{n(n-1)/2} (-1)^k |\,\mathrm{PD}_k(w)| = \sum_{P\in \mathrm{int}(\Delta(Q_{0,n},w))} (-1)^{\mathrm{codim}(P)} =$$
$$= \sum_{P\in\Delta(Q_{0,n},w)} (-1)^{\mathrm{codim}(P)} + \sum_{P\in\partial\Delta(Q_{0,n},w)} (-1)^{\mathrm{codim}(P)} = (-1)^d\chi_{\Delta(Q_{0,n},w)} + (-1)^{d-1}\chi_{\partial\Delta(Q_{0,n},w)}.$$

Here $d = n(n-1)/2 - \ell(w)$ is the dimension of the pipe dream complex, and χ_Δ stands for the Euler characteristic of Δ.

For each $w \neq \delta(Q_{0,n}) = w_0$ the pipe dream complex $\Delta(Q_{0,n}, w)$ is homeomorphic to a d-dimensional ball, and its boundary is homeomorphic to a $(d-1)$-dimensional sphere (for $w = w_0$

the corresponding pipe dream complex is a point). This means that

$$\chi_{\Delta(Q_{0,n},w)} = 1,$$
$$\chi_{\partial\Delta(Q_{0,n},w)} = 1 - (-1)^d,$$

and hence

$$\mathfrak{G}^{(-1)}(1,1,\ldots,1) = \sum_{k=0}^{n(n-1)/2} (-1)^k |\operatorname{PD}_k(w)| = (-1)^d \Big(1 - (1 - (-1)^d)\Big) = (-1)^{2d} = 1.$$

□

4. Slide complexes

In this section we provide the main construction of this paper: we define the stratification of subword complexes into strata corresponding to slide (or glide) groups. These strata are called *slide complexes*. We show that, just like the subword complexes, the slide complexes are homeomorphic to balls or spheres.

4.1. **Slide complexes in general.**

As before, let (Π, Σ) be a Coxeter system.

Definition 4.1. Let Q, S be two words in the alphabet Σ. By *slide complex of subwords* $\widetilde{\Delta}(Q,S)$ we denote the set of subwords $Q \setminus P$, such that their complements P contain S as a subword. Similarly to the case of subword complexes, this set of subwords has a natural structure of a simplicial complex.

The following theorem is similar to Theorem 3.7

Theorem 4.2. *Slide complexes are vertex decomposable and hence shellable.*

Proof. It is clear that slide complexes are pure.

Let $Q = (\sigma, \sigma_2, \ldots, \sigma_m), S = (s_{j_1}, s_{j_2}, \ldots, s_{j_l})$ be two words in the alphabet Π. Let $Q' = (\sigma_2, \ldots, \sigma_m)$ and $S' = (s_{j_2}, \ldots, s_{j_l})$. Then $\operatorname{link}(\sigma, \widetilde{\Delta}(Q,S)) = \widetilde{\Delta}(Q', S)$. If the word S starts with the letter σ, then $\operatorname{del}(\sigma, \widetilde{\Delta}(Q,S)) = \widetilde{\Delta}(Q', S')$. Otherwise we have $\operatorname{del}(\sigma, \widetilde{\Delta}(Q,S)) = \operatorname{link}(\sigma, \widetilde{\Delta}(Q,S)) = \widetilde{\Delta}(Q', S)$.

This means that for a vertex σ the result of its deletion and its link in $\widetilde{\Delta}(Q,S)$ are slide complexes. Then we use the induction by the length of Q. □

Let us introduce an analogue of the Demazure product for subwords.

Definition 4.3. Denote by $\widetilde{\delta}(Q)$ the word obtained from Q by replacing each maximal group of consecutive identical letters $s_i \ldots s_i$ by one letter s_i.

For example, $\widetilde{\delta}(s_1 s_1 s_2 s_1 s_2 s_2 s_2) = s_1 s_2 s_1 s_2$.

Lemma 4.4 ([BLVS+99]). *Let Δ be a pure shellable simplicial complex. If each of its faces of codimension 1 belongs to at most two facets, then Δ is either a ball or a sphere. The boundary of Δ consists of the codimension 1 faces that belong to exactly one facet.*

Let $Q = (\sigma_1, \ldots, \sigma_m)$. We will denote the word $(\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_m)$ by $Q \setminus \sigma_i$.

Lemma 4.5. *Let T and S be two words in the alphabet* Π*, such that $\widetilde{\delta}(S) = S$, $|S| = |T| - 1$, and T contains S as a subword. Then:*

1. *there exist at most two letters $\sigma \in T$, such that $T \setminus \sigma$ coincides with S;*
2. *if $\widetilde{\delta}(T) = S$, there exist exactly two such letters σ;*
3. *if $\widetilde{\delta}(T) \neq S$, such a letter σ is unique.*

Proof. Let $T = \sigma_1 \ldots \sigma_m$. If $S = T \setminus \sigma_i = T \setminus \sigma_{i+k}$, it is obvious that $\sigma_i = \sigma_{i+1} = \ldots = \sigma_{i+k}$. Hence if $k \neq 1$, there exist two consecutive identical letters in S, and this means that $\widetilde{\delta}(S) \neq S$. So $k = 1$, and there are at most two letters satisfying this condition.

Now we distinguish between the two cases. If there are two consecutive identical letters in T, we can remove any of them, obtaining $\widetilde{\delta}(T) = S$. If T has no pair of consecutive identical leters, then $\widetilde{\delta}(T) = T \neq S$, and we can remove only one letter. □

Lemma 4.6. *Let P and S be two words in the alphabet Π, and $\widetilde{\delta}(S) = S$.*

(1) *Let $\widetilde{\delta}(P) = S$ and let T be a subword in P containing S. Then we have $\widetilde{\delta}(T) = S$.*
(2) *Let P contain S as a subword, and let $\widetilde{\delta}(P) \neq S$. Then there exists a subword T in P such that T contains S, differs from it by one letter: $|T| = |S| + 1$, and $\widetilde{\delta}(T) \neq S$.*

Proof. (1) Obvious.
(2) Since $\widetilde{\delta}(P) \neq S = \widetilde{\delta}(S)$, there exists a letter σ occuring in $\widetilde{\delta}(P)$, but not in S. It corresponds to a sequence of consecutive identical letters $\sigma \dots \sigma$ in P. The word P contains S as a subword. If we add one letter σ to this subword, we will obtain the required word T.

□

The following result is the main theorem of this paper. It is analogous to Thm 3.9 due to Knutson and Miller.

Theorem 4.7. *Let Q and S be two words in the alphabet Π, and let $\widetilde{\delta}(S) = S$. Then the slide complex $\widetilde{\Delta}(Q, S)$ is homeomorphic to a sphere if $\widetilde{\delta}(Q) = S$ and to a ball otherwise. A face $Q \setminus P$ belongs to the boundary of this complex if and only if $\widetilde{\delta}(P) \neq S$.*

Proof. Theorem 4.2 implies that the complex $\widetilde{\Delta}(Q, S)$ is pure and shellable. Lemma 4.5 (1) states that any codimension 1 face of a slide complex belongs to at most two facets. Thus Lemma 4.4 implies that slide complexes are homeomorphic either to balls or to spheres.

From Lemma 4.5, (2) and (3), we see that the boundary of a slide complex consists of the faces $Q \setminus T$ of codimension 1 satisfying the property $\widetilde{\delta}(T) \neq S$. Lemma 4.6 implies that a face $Q \setminus P$ belongs to a boundary face of codimension 1 (and hence belongs to the boundary) if and only if $\widetilde{\delta}(P) \neq S$. The same lemma also means that a slide complex has empty boundary (and hence is homeomorphic to a sphere) if and only if $\widetilde{\delta}(Q) = S$. □

Remark 4.8. If $\widetilde{\delta}(S) \neq S$, the slide complex is not necessarily homeomorphic to a ball or sphere. For instance, if $Q = s_1s_1s_1s_1$ and $S = s_1s_1$, the complex $\widetilde{\Delta}(Q, S)$ is the 2-skeleton of a tetrahedron.

4.2. **Slide complexes in pipe dream complexes.** In this subsection we study the relation between the slide and glide groups of pipe dreams and the slide complexes.

As we have seen, the pipe dreams of shape $w \in \mathcal{S}_n$ (both reduced and non-reduced) bijectively correspond to the internal faces of the pipe dream complex $\Delta(Q_{0,n}, w)$ (this complex is homeomorphic to a ball unless $w = w_0$).

Proposition 4.9. *The interior part of the pipe dream complex* $\operatorname{int}(\Delta(Q_{0,n}, w))$ *can be decomposed into the disjoint union of the interior parts of slide complexes:*

$$\operatorname{int}(\Delta(Q_{0,n}, w)) = \bigsqcup_{\substack{S \text{ word in } \Pi \\ \widetilde{\delta}(S)=S \\ \delta(S)=w}} \operatorname{int}\left(\widetilde{\Delta}(Q_{0,n}, S)\right). \tag{1}$$

This decomposition is consistent with the decomposition of the set $\mathrm{PD}(w)$ *into glide groups: the pipe dreams from each glide group bijectively correspond to the pipe dreams from the interior part of the corresponding slide complex.*

Proof. First let us show that the formula (1) holds. Indeed, let $Q_{0,n} \setminus P$ be an internal face of the pipe dream complex of w. This means that $\delta(P) = w$. Let $S = \widetilde{\delta}(P)$. It is clear that $\delta(S) = \delta(P) = w$, $\widetilde{\delta}(S) = S$, and hence $Q \setminus P$ belongs to the interior of the slide complex $\widetilde{\Delta}(Q_{0,n}, S)$.

Let us show the converse. If $Q_{0,n} \setminus P$ is an interior face for the slide complex $\widetilde{\Delta}(Q_{0,n}, S)$ (with $\widetilde{\delta}(S) = S$ and $\delta(S) = w$), then $\widetilde{\delta}(P) = S$, and hence $\delta(P) = \delta(\widetilde{\delta}(P)) = \delta(S) = w$ and $Q_{0,n} \setminus P$ is an interior face of the pipe dream complex $\Delta(Q_{0,n}, w)$.

Now let $P \in \mathrm{PD}(w)$ be a pipe dream corresponding to the subword $\mathrm{word}(P)$ in $Q_{0,n} = (\sigma_1, \sigma_2, \ldots, \sigma_{n(n-1)/2})$. Suppose that the action of the slide move S_i on P is not identical: it moves a cross (i,j) southwest to the position $(i+1, j-1)$. Let σ_k and σ_{k+m} be two identical letters corresponding to the old and the new positions of this cross in $Q_{0,n}$. Since the pipe dream P has no crosses in the i-th row to the left of the j-th column, and the row $i+1$ does not contain crosses to the right of the $(j-1)$-st column, this means that the letters $\sigma_{k+1}, \ldots, \sigma_{k+m-1}$ do not occur in the subword $\mathrm{word}(P)$. The slide move S_i acts on $\mathrm{word}(P)$ as follows: $\mathrm{word}(P)$ contains either both letters σ_k and σ_{k+m}, or only σ_k. At the meantime, $\mathrm{word}(S_i(P))$ contains only the letter σ_{k+m}, but not σ_k.

This implies that $\widetilde{\delta}(\mathrm{word}(P)) = \widetilde{\delta}(\mathrm{word}(S_i(P)))$, so for any two pipe dreams from the same glide group the corresponding faces belong to the interior of the same slide complex.

The converse is also true. Indeed, let $P \in \mathrm{PD}(w)$. Suppose that σ_k and σ_{k+m} are two identical letters in $Q_{0,n}$, such that

- the subword $\mathrm{word}(P)$ contains either σ_k or both of them;
- the interval $\sigma_{k+1}, \ldots, \sigma_{k+m-1}$ does not contain letters equal to $\sigma_k = \sigma_{k+m}$;
- the letters $\sigma_{k+1}, \ldots, \sigma_{k+m-1}$ do not occur in $\mathrm{word}(P)$.

Then replacing σ_k or the pair of letters (σ_k, σ_{k+m}) in $\mathrm{word}(P)$ by the letter σ_{k+m} corresponds to a slide move applied to P. This operation preserves $\widetilde{\delta}(\mathrm{word}(P))$. Moreover, if for $P_1, P_2 \in \mathrm{PD}(w)$ we have $\widetilde{\delta}(\mathrm{word}(P_1)) = \widetilde{\delta}(\mathrm{word}(P_2)) = S$, then we can use such operations to reduce both subwords $\mathrm{word}(P_1)$ and $\mathrm{word}(P_2)$ to the same "rightmost" subword T in $Q_{0,n}$ satisfying $\widetilde{\delta}(T) = S$. Thus, if $\mathrm{word}(P_1)$ and $\mathrm{word}(P_2)$ belong to the interior of the same slide complex, then P_1 and P_2 belong to the same glide group. □

This implies the following corollary, which is similar to Corollary 3.12: it states that a glide polynomial is obtained as the sum of monomials over the interior faces of the corresponding slide complex.

Corollary 4.10. *Let $Q \in \mathrm{QPD}(w)$. Then*

$$\mathcal{G}_Q^{(\beta)} = \sum_{P \in \mathrm{int}(\widetilde{\Delta}(Q_{0,n}, \mathrm{word}(Q)))} \beta^{\mathrm{codim}(P)} \mathbf{x}^P.$$

Here by $\mathbf{x}^P$ we denote the monomial for the pipe dream corresponding to the face P.

Specializing $\beta = 0$, we recover a similar statement for slide polynomials.

Corollary 4.11. *Let $Q \in \mathrm{QPD}_0(w)$. Then*

$$\mathfrak{F}_Q = \sum_P \mathbf{x}^P,$$

where the sum is taken over all facets P of the complex $\widetilde{\Delta}(Q_{0,n}, \mathrm{word}(Q)))$.

Example 4.12. Fig. 5 represents the pipe dream complex for the permutation $w = \overline{1432}$ as a disjoint union of the interiors of slide complexes. The quasi-Yamanouchi pipe dreams are depicted in blue. For each pipe dream we indicate the monomial $\beta^{\mathrm{ex}(P)-\mathrm{ex}(Q)}\mathbf{x}^P$ from the corresponding glide polynomial.

For $k \in \mathbb{Z}_{\geq 0}$ let $\mathrm{QPD}_k(w) = \{Q \in \mathrm{QPD}(w) \mid \mathrm{ex}(Q) = k\}$. The following corollary states that the alternating sum of the numbers of quasi-Yamanouchi pipe dreams with a given excess is 1.

Corollary 4.13. *For each permutation $w \in \mathcal{S}_n$ we have the following equality:*

$$\sum_{k=0}^{n(n-1)/2} (-1)^k |\mathrm{QPD}_k(w)| = 1.$$

Proof. Since the slide complexes are homeomorphic to balls, and the Euler characteristic of a ball is equal to 1, we obtain similarly to Corollary 3.14 that

$$\mathcal{G}_Q^{(-1)}(1, \ldots, 1) = 1$$

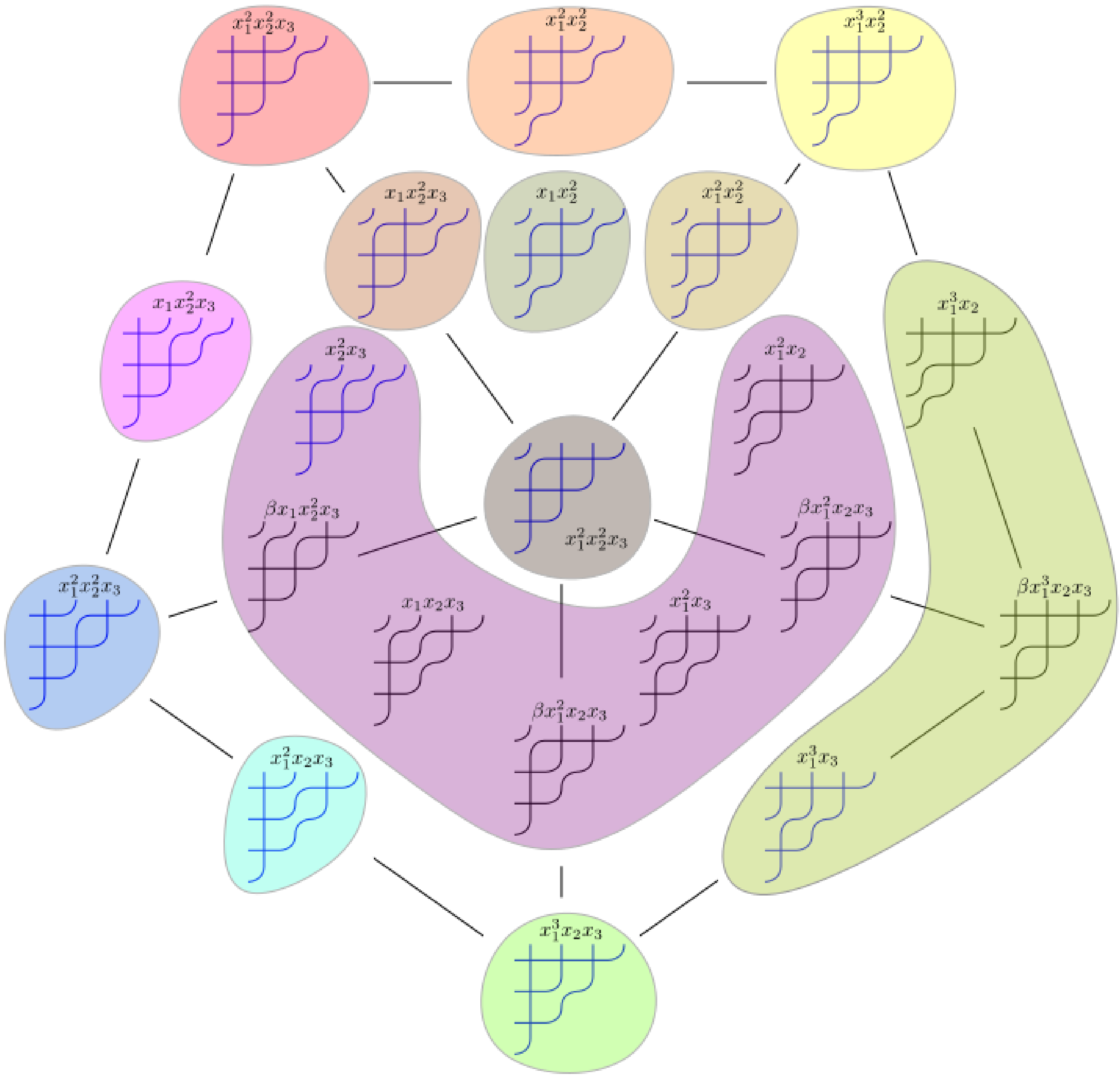


FIGURE 5. Pipe dream complex $w = \overline{1432}$ represented as a union of the interiors of slide complexes

for each $Q \in \mathrm{QPD}(w)$. Let us specialize the equality

$$\mathfrak{S}_w^{(\beta)}(\mathbf{x}) = \sum_{Q\in\mathrm{QPD}(w)} \beta^{\mathrm{ex}(Q)}\mathcal{G}_Q^{(\beta)}(\mathbf{x})$$

at $\beta = -1, \mathbf{x} = (1,\dots,1)$, and use the fact that $\mathfrak{S}_w^{(-1)}(1,\dots,1) = \mathcal{G}_Q^{(-1)}(1,\dots,1) = 1$ for each $w \in \mathcal{S}_n, Q \in \mathrm{QPD}(w)$. We obtain the desired formula:

$$1 = \sum_{Q\in\mathrm{QPD}(w)} (-1)^{\mathrm{ex}(Q)} = \sum_{k=0}^{n(n-1)/2} (-1)^k |\,\mathrm{QPD}_k(w)|.$$

$\square$

References

[AS17] Sami Assaf and Dominic Searles. Schubert polynomials, slide polynomials, Stanley symmetric functions and quasi-Yamanouchi pipe dreams. *Adv. Math.*, 306:89–122, 2017.

[BB93] Nantel Bergeron and Sara Billey. RC-graphs and Schubert polynomials. *Experiment. Math.*, 2(4):257–269, 1993.

[BGG73] I. N. Bernšteĭn, I. M. Gel′fand, and S. I. Gel′fand. Schubert cells, and the cohomology of the spaces G/P. *Uspehi Mat. Nauk*, 28(3(171)):3–26, 1973.

[BLVS+99] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Gunter M. Ziegler. *Oriented Matroids*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2 edition, 1999.

[BP79] Louis J. Billera and J. Scott Provan. A decomposition property for simplicial complexes and its relation to diameters and shellings. *Annals of the New York Academy of Sciences*, 319(1):82–85, 1979.

[FK93] Sergey Fomin and Anatol N. Kirillov. Yang-baxter equation, symmetric functions and grothendieck polynomials, 1993.

[FK94] Sergey Fomin and Anatol N Kirillov. Grothendieck polynomials and the yang-baxter equation. In *Proc. Formal Power Series and Alg. Comb*, pages 183–190, 1994.

[FK96] Sergey Fomin and Anatol N. Kirillov. The Yang-Baxter equation, symmetric functions, and Schubert polynomials. In *Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993)*, volume 153, pages 123–143, 1996.

[KM04] Allen Knutson and Ezra Miller. Subword complexes in Coxeter groups. *Adv. Math.*, 184(1):161–176, 2004.

[KM05] Allen Knutson and Ezra Miller. Gröbner geometry of Schubert polynomials. *Ann. of Math. (2)*, 161(3):1245–1318, 2005.

[Las07] Alain Lascoux. Anneau de grothendieck de la variété de drapeaux. In *The Grothendieck Festschrift*, pages 1–34. Springer, 2007.

[LS82] Alain Lascoux and Marcel-Paul Schützenberger. Polynômes de Schubert. *C. R. Acad. Sci. Paris Sér. I Math.*, 294(13):447–450, 1982.

[PS19] Oliver Pechenik and Dominic Searles. Decompositions of Grothendieck polynomials. *Int. Math. Res. Not. IMRN*, (10):3214–3241, 2019.

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