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On the König Graphs for a 5-Path and Its Spanning Supergraphs

D. B. Mokeev[1,2*] and D. S. Malyshev[2]**

[1]*Lobachevsky State University of Nizhny Novgorod, pr. Gagarina 23, Nizhny Novgorod, 603950 Russia*

[2]*National Research University "Higher School of Economics", ul. Bolshaya Pecherskaya 25/12, Nizhny Novgorod, 603155 Russia*

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Abstract—We describe the hereditary class of graphs whose every subgraph has the property that the maximum number of disjoint 5-paths (paths on 5 vertices) is equal to the minimum size of the sets of vertices having nonempty intersection with the vertex set of each 5-path. We describe this class in terms of the "forbidden subgraphs" and give an alternative description, using some operations on pseudographs.

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INTRODUCTION

Let $\mathcal{X}$ be a set of some graphs. An arbitrary set of some pairwise vertex disjoint induced subgraphs of a graph G isomorphic to graphs in $\mathcal{X}$ (in what follows, $\mathcal{X}$-subgraphs) is called an *$\mathcal{X}$-packing* of G. An arbitrary set of vertices in a graph G having nonempty intersection with the vertex set of each induced $\mathcal{X}$-subgraph in G is called a *vertex cover of G with respect to $\mathcal{X}$* (in what follows, we abbreviate it to an *$\mathcal{X}$-cover*).

A graph is called *König for $\mathcal{X}$* if, in its every induced subgraph, the maximum size of an $\mathcal{X}$-packing is equal to the minimum size of an $\mathcal{X}$-cover (see [1]). The class of all König graphs for a set $\mathcal{X}$ is denoted by $\mathcal{K}(\mathcal{X})$.

The class $\mathcal{K}(\mathcal{X})$ is *hereditary* for every $\mathcal{X}$; i.e., $\mathcal{X}$ is closed under vertex removal. It is known that each hereditary class can be characterized by the set of minimal forbidden subgraphs, i.e. inclusion minimal vertices of the graphs not belonging to the class [2].

Note that, in the literature, an $\mathcal{X}$-cover also often means a set of vertices in a graph G covering all (not only induced) subgraphs of G isomorphic to graphs in $\mathcal{X}$ (see, for example, [3, 4]). However, the vertex set of each (not only induced) $\mathcal{X}$-subgraph G induces a graph that is a spanning supergraph of a graph in $\mathcal{X}$.

Denote by $\langle\mathcal{X}\rangle$ the set of all spanning supergraphs over graphs in $\mathcal{X}$, i.e. the set of graphs containing all graphs in $\mathcal{X}$ and all graphs obtained from them by adding edges. Then every "noninduced" $\mathcal{X}$-cover of a graph G coincides with its $\langle\mathcal{X}\rangle$-cover. An $\langle\mathcal{X}\rangle$-packing corresponds in turn to a set of arbitrary (and not only generated) $\mathcal{X}$-subgraphs pairwise not containing common vertices.

Henceforth, we will use the notation $\langle H\rangle$ (instead of $\langle\{H\}\rangle$) for the set of all spanning supergraphs of a graph H.

Many works are devoted of the $\mathcal{X}$-packing and $\mathcal{X}$-cover problems (for the "induced" and "noninduced" cases), where $\mathcal{X}$ consists of P_k (a simple path on k vertices), and their algorithmic aspects. It is known in particular that the $\langle P_k\rangle$-cover problem is NP-hard in the general case for $k \geq 2$ (see [5, 6]), and the $\langle P_k\rangle$-packing problem is polynomially solvable for $k = 2$ [7], NP-hard for $k \geq 3$ [8, 9], and APX-hard for $k \geq 4$ (see [10]).

*E-mail: mokeevdb@gmail.com
**E-mail: dsmalyshev@rambler.ru

It is known however that the P_k- and $\langle P_k\rangle$-packing problems as well as P_k- and $\langle P_k\rangle$-cover problems are solved in linear time in the class of forests for any k (see [5, 9]). Moreover, some description of more complex graph classes is known on which these problems are solved for various k in polynomial time (see [11–13]), including classes of König graphs and their subclasses (see [14–16]).

The present article is devoted to describing the class of König graphs for $\langle P_5\rangle$ and continues a series of investigations carried out earlier for the graph classes $\mathcal{K}(\langle P_3\rangle)$ (see [15]) and $\mathcal{K}(\langle P_4\rangle)$ (see [14]). For both classes, some complete description is obtained in terms of forbidden subgraphs (all minimal forbidden subgraphs are described), as well as a "constructive" description, — a procedure was described that makes it possible to construct any graph of a given class.

In the present article, the class $\mathcal{K}(\langle P_5\rangle)$ is also completely described in both of the above-mentioned ways: In Section 2, we describe all graphs that are forbidden for this class. In Sections 3 and 4, we describe the class of ST_5-graphs obtained from pseudographs by applying edge subdivision, replacing vertices by empty graphs, and adding the so-called "terminal subgraphs", and prove that the class coincides with $\mathcal{K}(\langle P_5\rangle)$.

1. DEFINITIONS AND NOTATIONS

In the article, we use the notations K_n, P_n, and C_n for complete graphs, simple paths, and simple cycles on n vertices respectively. We denote by S_n a tree on $n+1$ vertices with n vertices that are leaves and designate as $S_{n,m}$ a tree on $n+m+2$ vertices two of which are central and the remaining are leaves, where n leaves are adjacent to one central vertex and m is adjacent to the other. Denote by $S_n + e$ a graph obtained from S_n by adding one edge, denote by $K_n - e$ a graph obtained from K_n by removing one edge, and designate as n-fan a graph obtained from P_{n+1} by adding a vertex adjacent to each of its vertices.

The greatest number of elements in a $\langle P_5\rangle$-packing of a graph G will be denoted by $\mu_{\langle P_5\rangle}(G)$; and the least number of vertices in its $\langle P_5\rangle$-cover, by $\beta_{\langle P_5\rangle}(G)$.

A subgraph isomorphic to one of the graphs of $\langle P_5\rangle$ will be called a *quintet*. Denote the quintet consisting of v_1, v_2, v_3, v_4, and v_5 by $(v_1, v_2, v_3, v_4, v_5)$.

Denote by $V(G)$ the set of vertices in G. The set of vertices adjacent to a vertex v will be called the *neighborhood* of v and denoted by $N(v)$.

Let G be a graph and let $A \subseteq V(G)$. Denote by $G[A]$ the subgraph induced by A. Denote by $G \setminus A$ the graph obtained from G by removing all vertices of A. We use the notation $\mathrm{Free}(\mathcal{X})$ for the class of graphs not containing induced subgraphs from a set $\mathcal{X}$.

Refer as a 5-*class* in a cycle C_{5n} to an inclusion maximal set of vertices the distance between which pairwise divides by 5. It is not hard to see that the number of 5-classes in every such cycle is equal to 5. Obviously, since each quintet consists of 5 vertices, every graph G satisfies the inequality

$$5\mu_{\langle P_5\rangle}(G) \le |V(G)|.$$

Consider a vertex-inclusion-minimal graph F not belonging to the class $\mathcal{K}(\langle P_5\rangle)$ in which $|V(F)| \le 5\mu_{\langle P_5\rangle}(F) + 4$; i.e., the value $\mu_{\langle P_5\rangle}(F)$ is maximal for the given number of vertices. Obviously, in its every spanning supergraph F', each set of vertices inducing a $\langle P_5\rangle$-subgraph of F generates a $\langle P_5\rangle$-subgraph of F'. Therefore,

$$\beta_{\langle P_5\rangle}(F) \le \beta_{\langle P_5\rangle}(F'), \qquad \mu_{\langle P_5\rangle}(F) \le \mu_{\langle P_5\rangle}(F').$$

But $\mu_{\langle P_5\rangle}(F)$ is maximal for this number of vertices. Hence,

$$\mu_{\langle P_5\rangle}(F') = \mu_{\langle P_5\rangle}(F) < \beta_{\langle P_5\rangle}(F) \le \beta_{\langle P_5\rangle}(F');$$

i.e., $F' \notin \mathcal{K}(\langle P_5\rangle)$.

Let F be some subgraph of G. Then G includes an induced subgraph isomorphic to F or one of its spanning supergraphs F'; i.e., G is not König for $\langle P_5\rangle$.

If a graph F in which $|V(F)| \le 5\mu_{\langle P_5\rangle}(F) + 4$ is a vertex-and-edge-inclusion-minimal graph not belonging to $\mathcal{K}(\langle P_5\rangle)$ then we will refer to it as a *minimal forbidden graph* of this class. In what

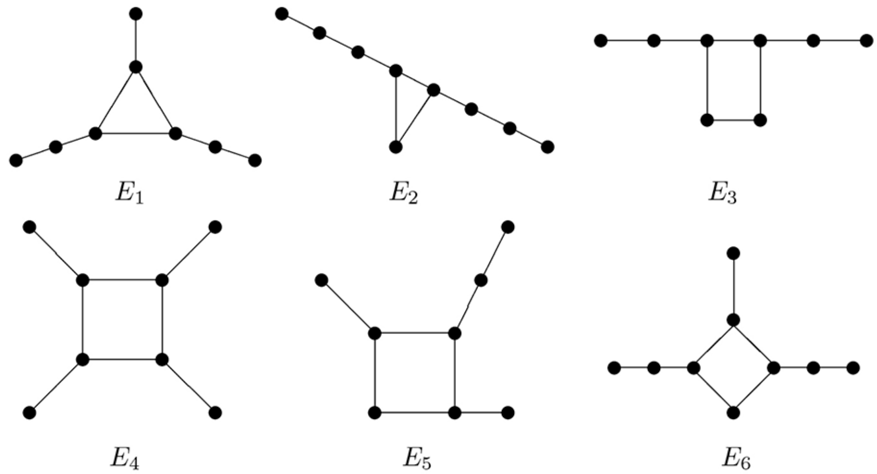

Fig. 1. Forbidden graphs $E_1, \ldots, E_6$ for $\mathcal{K}(\langle P_5 \rangle)$.

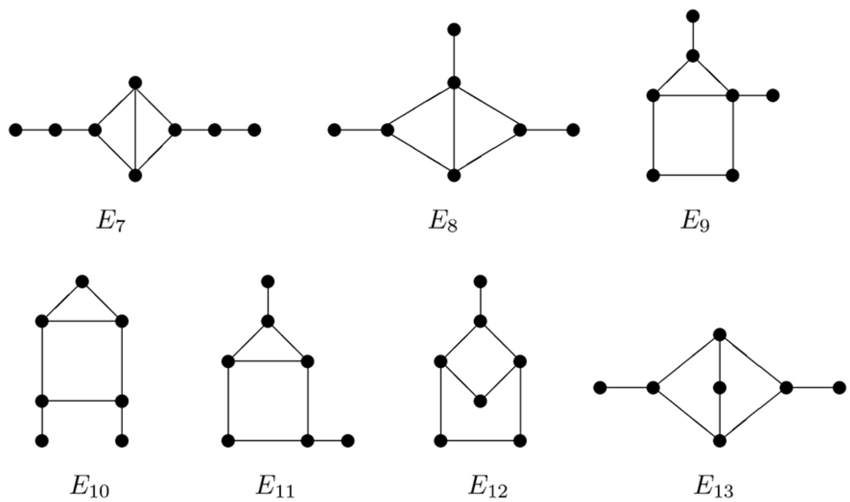

Fig. 2. Forbidden graphs $E_7, \ldots, E_{13}$ for $\mathcal{K}(\langle P_5 \rangle)$.

follows, we will prove that the class $\mathcal{K}(\langle P_5 \rangle)$ is completely defined by such forbidden graphs and hence is *monotone*, i.e., closed under edge and vertex removal.

Note that, in every graph G, we have

$$\mu_{\langle P_5 \rangle}(G) \leq \beta_{\langle P_5 \rangle}(G);$$

therefore, for proving the equality of these quantities in a graph, it suffices to find a $\langle P_5 \rangle$-packing and a $\langle P_5 \rangle$-cover of the graph G of the same size.

2. FORBIDDEN SUBGRAPHS

It is easy to prove by a straightforward check that the graphs $E_1, E_2, \ldots, E_{13}$ depicted in Figs. 1 and 2 do not belong to $\mathcal{K}(\langle P_5 \rangle)$. For each of them, $\mu_{\langle P_5 \rangle}(G) = 1$ and $\beta_{\langle P_5 \rangle}(G) = 2$; moreover, each of them consists of at most 9 vertices. Furthermore, each proper subgraph of each of them is König for $\langle P_5 \rangle$. Thus, we have

Lemma 1. *$E_1, E_2, \ldots, E_{13}$ are minimal forbidden subgraphs for $\mathcal{K}(\langle P_5 \rangle)$.*

It was proved in [17] that, for each $q \geq 4$ (and hence, for $q = 5$), every forest is a König graph with respect to P_q (this follows from Theorem 1 [17]). Since a forest has no other subgraphs isomorphic to graphs in $\langle P_5 \rangle$ than P_5, we have

Lemma 2. *Each forest is a König graph for* $\langle P_5 \rangle$.

Now, consider the infinite series of forbidden subgraphs for $\mathcal{K}(\langle P_5 \rangle)$. Obviously, for every i in $\{0, 1, 2, 3, 4\}$ and $k \geq 2$, we have

$$\mu_{\langle P_5 \rangle}(C_{5k+i}) = \beta_{\langle P_5 \rangle}(C_{5k-i}) = k, \qquad \mu_{\langle P_5 \rangle}(C_{5+i}) = \beta_{\langle P_5 \rangle}(C_5) = 1.$$

Therefore and by Lemma 2, we have

Lemma 3. *The cycle* C_n *belongs to* $\mathcal{K}(\langle P_5 \rangle)$ *if* $n \leq 4$ *or* n *divides by* 5, *and it is a minimal forbidden subgraph for* $\mathcal{K}(\langle P_5 \rangle)$ *if* $n > 5$ *and* n *does not divide by* 5.

Consider the family of connected graphs obtained from a simple cycle by adding a set of subgraphs isomorphic to K_1 and K_2 so that exactly one vertex of each added subgraph is adjacent exactly to one vertex of the cycle. We will refer to such graphs as *hedgehogs*. The vertices of degree greater than 2 in hedgehogs will be called *knots*. Note that the number of knots is at most the number of added subgraphs.

Denote by $A_1(n, k)$ the hedgehog obtained from C_n by adding one subgraph K_1 and one subgraph K_2; and designate as $A_2(n, k)$ the hedgehog obtained from C_n by adding 2 subgraphs K_2. In both cases, k is the distance between the knots of the graph and $k = 0$ if the graph has a single knot.

Denote by $A_3(k_1, k_2, k_3)$ the hedgehog obtained from $C_{k_1+k_2+k_3}$ by adding three subgraphs K_1. Here k_1, k_2, and k_3 are the lengths of the paths into which the knots of the graph divide the cycle (if the graph has less than 3 knots then one or two parameters are assumed equal 0).

Let us formulate and prove some conditions for the graphs $A_1(n, k)$, $A_2(n, k)$, and $A_3(k_1, k_2, k_3)$ to be minimal forbidden subgraphs for $\mathcal{K}(\langle P_5 \rangle)$:

Lemma 4. $A_1(5t, 5k + 1)$ *and* $A_1(5t, 5k + 4)$ *are minimal forbidden subgraphs for* $\mathcal{K}(\langle P_5 \rangle)$ *for every* $t \geq 1$ *and* $0 \leq k < t/2$.

Proof. Let $n = 5t$. Suppose that the graph G is isomorphic to one of the graphs $A_1(n, 5k + 1)$ and $A_1(n, 5k + 4)$. It is not hard to see that $|V(G)| = 5t + 3$ and $\mu_{\langle P_5 \rangle}(G) = t$. Consequently,

$$|V(G)| \leq 5\mu_{\langle P_5 \rangle}(G) + 4. \tag{1}$$

Suppose that $\beta_{\langle P_5 \rangle}(G) = t$. Every $\langle P_5 \rangle$-cover of G includes a $\langle P_5 \rangle$-cover of the cycle C_n in this graph but every least $\langle P_5 \rangle$-cover of this cycle is its 5-class and contains t vertices.

Prove that no 5-class of a cycle C_n is a $\langle P_5 \rangle$-cover of G.

Let C be a 5-class of C_n. The two cases are possible:

- One of the knots in G is adjacent to a vertex in C. Then the distance from it to the other closest vertex in C is equal to 4, i.e., the knot, the adjacent vertex not adjacent to the cycle, and three more vertices of the cycle induce a quintet not covered by C.

- None of the knots is adjacent to a vertex in C. Since the distance between knots is equal to $5k + 1$ or $5k + 4$, none of the knots is contained in C. Consider a knot of G adjacent to a noncyclic vertex of degree 2. It is evident that the distance from this knot to one of the vertices of C is equal to 3. Then the knot, the two vertices not belonging to the cycle, and two more vertices of the cycle induce a quintet that is not covered by C.

Thus, no least $\langle P_5\rangle$-cover of a cycle in G is a $\langle P_5\rangle$-cover of G. Therefore,

$$\beta_{\langle P_5\rangle}(G) > t, \qquad G \notin \mathcal{K}(\langle P_5\rangle).$$

Consider a subgraph X of G obtained by removing a vertex of degree 1 not adjacent to a knot or an edge incident to it. The connected component of X which is not a separated vertex is the hedgehog obtained from a cycle C_n by adding the two subgraphs K_1. Obviously, $\mu_{\langle P_5\rangle}(X) = t$. In turn, the 5-class of a cycle not containing vertices adjacent to knots is a $\langle P_5\rangle$-cover of X, i.e., $\beta_{\langle P_5\rangle}(X) = t$. Thus, X is the König graph for $\langle P_5\rangle$.

Consider the subgraph X of G obtained by removing either a vertex of the added subgraph adjacent to a vertex or an edge joining it to the corresponding knot. Its connected component having more than two connected components is the hedgehog obtained from C_n by adding one subgraph K_1 or K_2. It is not hard to see that $\mu_{\langle P_5\rangle}(X) = \beta_{\langle P_5\rangle}(X) = t$. Therefore, X is the König graph for $\langle P_5\rangle$.

Other subgraphs in G are forests or isomorphic to the cycle C_{5t}. By Lemmas 2 and 3, they all are König for $\langle P_5\rangle$.

Thus, G is a vertex-and-edge-inclusion- minimal graph for which $\mu_{\langle P_5\rangle}(G) < \beta_{\langle P_5\rangle}(G)$. This and inequality (1) imply that G is a minimal forbidden subgraph for $\mathcal{K}(\langle P_5\rangle)$.

Lemma 4 is proved. □

Lemma 5. *$A_2(5t, 5k+2)$ and $A_2(5t, 5k+3)$ are minimal forbidden subgraphs for the class $\mathcal{K}(\langle P_5\rangle)$ for every $t \ge 1$ and $0 \le k < t/2$.*

Proof. Let $n = 5t$. Suppose that a graph G is isomorphic to $A_2(n, 5k+2)$ or $A_2(n, 5k+3)$. It is not hard to see that $|V(G)| = 5t + 4$ and $\mu_{\langle P_5\rangle}(G) = t$. Consequently,

$$|V(G)| \le 5\mu_{\langle P_5\rangle}(G) + 4. \tag{2}$$

As in the proof of Lemma 4, for checking the inequality $\beta_{\langle P_5\rangle}(G) > \mu_{\langle P_5\rangle}(G)$ it suffices to verify that no 5-class of C_n is a $\langle P_5\rangle$-cover of G.

Let C be a 5-class of the cycle C_n. Since $k \equiv 2 \pmod 5$ or $k \equiv 3 \pmod 5$, the knots of G belong to different 5-classes of its cycle. Consider the knot not belonging to C. Since the distances between the closest vertices of every 5-class is equal to 5, there exists a path from the given knot to some vertex of C of length 3 or 4. Then the knot, the two vertices not belonging to the cycle, and two vertices of this path induce a quintet that is not covered by C.

Thus, no least $\langle P_5\rangle$-covering cycle of G is a $\langle P_5\rangle$-cover of G. Consequently,

$$\beta_{\langle P_5\rangle}(G) > t, \qquad G \notin \mathcal{K}(\langle P_5\rangle).$$

Consider a proper subgraph X of G containing a cycle. It has the connected component X' with the cycle and, possibly, connected components of 1 or 2 vertices. It is easy to see that $\mu_{\langle P_5\rangle}(X) = t$. If X' has 2 knots then one of them is adjacent to a terminal vertex. Denote by v the other knot or the unique knot of X' or an arbitrary vertex of the cycle if X' does not contain knots. It is not hard to see that the 5-class of the cycle containing v is a $\langle P_5\rangle$-cover of X; i.e., $\beta_{\langle P_5\rangle}(X) = t$; and so, X is the König graph for $\langle P_5\rangle$.

All other proper subgraphs of G are forests. Consequently, by Lemma 2, every proper subgraph of G is König for $\langle P_5\rangle$.

Thus, G is a vertex-and-edge-inclusion-minimal graph for which $\mu_{\langle P_5\rangle}(G) < \beta_{\langle P_5\rangle}(G)$. By (1), this implies that G is the minimal forbidden subgraph for $\mathcal{K}(\langle P_5\rangle)$.

Lemma 5 is proved. □

Lemma 6. *For all $k_1, k_2, k_3 \ge 0$, the graphs*

$$A_3(5k_1+1, 5k_2+1, 5k_3+3), \qquad A_3(5k_1+4, 5k_2+4, 5k_3+2)$$

are minimal forbidden subgraphs for $\mathcal{K}(\langle P_5\rangle)$.

Proof. Suppose that G is isomorphic to one of the graphs $A_3(5k_1+1, 5k_2+1, 5k_3+3)$ and $A_3(5k_1+4, 5k_2+4, 5k_3+2)$. Put $t = k_1+k_2+k_3+1$ in the first case and $t = k_1+k_2+k_3+2$ in the second. It is easy that $|V(G)| = 5t+3$ and $\mu_{\langle P_5\rangle}(G) = t$. Consequently,

$$|V(G)| \le 5\mu_{\langle P_5\rangle}(G) + 4. \tag{3}$$

As in the proof of Lemma 4, for checking the inequality $\beta_{\langle P_5\rangle}(G) > \mu_{\langle P_5\rangle}(G)$ it suffices to verify that no 5-class of the cycle C_n is a $\langle P_5\rangle$-cover of G.

Let C be a 5-class of the cycle C_{5t}. Since the two paths between the knots have lengths $5k_1+1$ and $5k_2+1$ or lengths $5k_1+4$ and $5k_2+4$; therefore, G contains three knots and they belong to three consecutive 5-classes of its cycle. Then there exists a vertex in C adjacent to a knot of G. Consequently, there exists a path of length 4 from the given knot to some vertex of the set C. Then the knot, the vertex not belonging to the cycle and adjacent to it, and three vertices of this path induce a quintet not covered by C.

Thus, no least $\langle P_5\rangle$-cover of a cycle of G is a $\langle P_5\rangle$-covering of G. Hence,

$$\beta_{\langle P_5\rangle}(G) > t, \qquad G \notin \mathcal{K}(\langle P_5\rangle).$$

All proper subgraphs of G are also proper subgraphs of

$$A_1(5t, 5k_1+1), \qquad A_1(5t, 5k_2+1), \qquad A_2(5t, 5k_3+3),$$
$$A_1(5t, 5k_1+4), \qquad A_1(5t, 5k_2+4), \qquad A_2(5t, 5k_3+2)$$

or forests. Hence, by Lemma 2 and the proofs of Lemmas 4 and 5, every proper subgraph in G is König with respect to $\langle P_5\rangle$.

Thus, G is a vertex-and-edge-inclusion-minimal graph not belonging to $\mathcal{K}(\langle P_5\rangle)$. By inequality (3), this implies that G is a minimal forbidden subgraph for $\mathcal{K}(\langle P_5\rangle)$.

Lemma 6 is proved. □

Choose (not necessarily different) vertices x and y in a cycle C_n. Add to C_n two vertices one of which is adjacent to x and the other is adjacent to both adjacent vertices of y. Denote the so-obtained graph by $B(n, k)$, where k is the distance between x and y. All vertices of degree greater than 2 will be called *knots*. Note that the number of knots is always 3 if $k \ne 1$, and the distance between two knots is always equal to 2. Such knots will be called *coupled*; and the third knot, *separated*.

Lemma 7. *For every $t \ge 1$ and $1 \le k \le t/2$, the graph $B(5t, 5k)$ is a minimal forbidden subgraph for $\mathcal{K}(\langle P_5\rangle)$.*

Proof. Suppose that the graph G is isomorphic to $B(5t, 5k)$. It is not hard to see that $|V(G)| = 5t+2$ and $\mu_{\langle P_5\rangle}(G) = t$. Consequently, we have

$$|V(G)| \le 5\mu_{\langle P_5\rangle}(G) + 4. \tag{4}$$

As in the proof of Lemma 4, for checking the inequality $\beta_{\langle P_5\rangle}(G) > \mu_{\langle P_5\rangle}(G)$, it suffices to verify that no 5-class of a cycle C_n is a $\langle P_5\rangle$-covering of G.

Let C be a 5-class of the cycle C_{5t}. It is not hard to see that the 5-class of the cycle containing a separated knot also contains a common adjacent vertex of coupled knots. Thus, all knots in G belong to three consecutive 5-classes of its cycle. Then there exists a vertex in C adjacent to a knot of the graph. Consequently, there exists a path of length 4 from this knot to some vertex in C. Then the knot, the added vertex adjacent to it, and three vertices of this path induce a quintet not covered by C.

Thus, no least $\langle P_5\rangle$-cover of a cycle of G is a $\langle P_5\rangle$-cover of G. Therefore,

$$\beta_{\langle P_5\rangle}(G) > t, \qquad G \notin \mathcal{K}(\langle P_5\rangle).$$

Consider a proper subgraph X of G containing some cycle C_4. It includes the connected component X' with the cycle and, possibly, connected components that are trees. Lemma 2 implies that $X \in \mathcal{K}(\langle P_5\rangle)$ if and only if $X' \in \mathcal{K}(\langle P_5\rangle)$.

If the component X' contains also a cycle C_{5t} then it is not hard to see that $\mu_{\langle P_5\rangle}(X) = t$, and the 5-class of the large cycle containing one of the knots is a $\langle P_5\rangle$-cover of X; i.e., $\beta_{\langle P_5\rangle}(X) = t$.

Suppose that X' contains no cycle C_{5t}. Then it can be obtained from P_n by adding a vertex y adjacent to two vertices of the path at distance 2, where $n \le 5t$, and, possibly, adding a terminal vertex to the vertex of the path lying at distance $5k$ from y. Prove that $X' \in \mathcal{K}(\langle P_5\rangle)$.

Carry out the proof by induction on $|V(X')|$. If $|V(X')| \le 5$; then, obviously, $\mu_{\langle P_5\rangle}(X') = \beta_{\langle P_5\rangle}(X')$ and both quantities are equal to 0 or 1.

Furthermore, suppose that $|V(X')| \ge 6$ and every proper subgraph H of X' satisfies $\mu_{\langle P_5\rangle}(H) = \beta_{\langle P_5\rangle}(H)$. If $n = 9$ and the added vertex is central in X' then $\mu_{\langle P_5\rangle}(X') = 2$ and the two central vertices of the graph constitute a $\langle P_5\rangle$-cover of the graph X'; i.e., $\beta_{\langle P_5\rangle}(X') = 2$. In the remaining cases, there exists a quintet $(v_1, v_2, v_3, v_4, v_5)$ such that the degree of v_1 is equal to 1 in the initial path, and the vertex y is not adjacent to v_4 or is adjacent to v_4 and v_2.

If y is adjacent to v_1 and v_3 then put $Q = (y, v_1, v_2, v_3, v_4)$ and $z = v_4$. If y is adjacent to v_2 and v_4 then $Q = (v_1, v_2, v_3, v_4, y)$ and $z = v_4$. If there is a terminal vertex u adjacent to v_4 then $Q = (v_1, v_2, v_3, v_4, u)$ and $z = v_4$. Otherwise, put $Q = (v_1, v_2, v_3, v_4, v_5)$ and $z = v_5$.

Consider the graph H obtained from X' by removing the vertices of the quintet Q. Let M be a greatest $\langle P_5\rangle$-cover and let C be a least $\langle P_5\rangle$-cover of H. By the induction assumption, $|M| = |C|$. Adding the quintet Q to M, we obtain a $\langle P_5\rangle$-packing of size $|M| + 1$. Adding a vertex z to C, we obtain a $\langle P_5\rangle$-cover of X' of the same size. Consequently,

$$\mu_{\langle P_5\rangle}(X') = \beta_{\langle P_5\rangle}(X'), \qquad X' \in \mathcal{K}(\langle P_5\rangle).$$

All remaining subgraphs of G are also proper subgraphs of $A_3(5k-1, 5(t-k)-1, 2)$ or forest, and hence, by Lemma 2 and the proof of Lemma 6, every subgraph of G is König for $\langle P_5\rangle$.

Thus, G is a vertex-and edge-inclusion- minimal graph not belonging to $\mathcal{K}(\langle P_5\rangle)$. Hence, (4) implies that G is a minimal forbidden subgraph for the class $\mathcal{K}(\langle P_5\rangle)$.

The proof of Lemma 7 is complete. □

3. ST_5-GRAPHS

Describe the procedure of ST_5-extension and the class of ST_5-graphs and also prove that the ST_5-extension of pseudographs always gives König graphs for $\langle P_5\rangle$.

Definition 1. Refer to a connected subgraph H of some graph G as *terminal* if there exists a vertex $c \in V(G \setminus H)$ adjacent to at least one vertex in H such that H is a connected component of the graph $G \setminus \{c\}$. The vertex c will be called a *contact vertex* of the terminal subgraph of H.

Definition 2 [18]. The operation of *replacing a vertex x with a graph H* consists in the following:

(1) x is removed from H;

(2) several new vertices are added to the graph that are joined to each other so that they induce a subgraph isomorphic to H;

(3) each new vertex is joined by an edge to all vertices in the set $N(x)$ in the initial graph.

Consider the graph class Free($\langle P_5\rangle$). Each connected component of such a graph contains no quintets; i.e., is a graph with the number of vertices at most 4; or is isomorphic to one of the graphs S_k, $S_k + e$, or $S_{k,m}$, where k and m are arbitrary naturals. Note that $K_2 = S_1$, $P_3 = S_2$, $K_3 = S_2 + e$, and $P_4 = S_{1,1}$ up to isomorphism. Refer as an F_5-*graph* to arbitrary graph in the set

$$\{K_1,\ C_4,\ K_4 - e,\ K_4,\ S_k,\ S_k + e,\ S_{k,m},\ \ k, m \in \mathbb{N}\}.$$

Thus, each connected component of a graph of class Free($\langle P_5\rangle$) is isomorphic to one of the F_5-graphs.

Definition 3. Let H be a pseudograph (a graph in which loops and multiple edges are admitted). The procedure of ST_5-*extension* of H consists in the following:

Step 1. Partition each cyclic edge (each edge belonging to a cycle, including multiple edges and loops) of the pseudograph H by four vertices. All vertices but those added in subdividing will be called *old*. The vertices added in subdividing will be called *close* if they are adjacent to old vertices and *distant* otherwise.

Step 2. Replace some close vertices and some vertices of degree 2 that are not cyclic by empty graphs of arbitrary sizes. No vertices being replaced must be adjacent to each other.

Step 3. Add several vertices to the graph by joining each of them by an edge with some distant vertex.

Step 4. Add to the graph several terminal F_5-subgraphs so that the contact vertex of each of them is old and is not replaced by an empty graph at Step 2.

Step 5. Add to the graph several subgraphs isomorphic to S_m, where m are arbitrary natural numbers, and join a central vertex of each of them (or, possibly, both vertices of the graph for $m = 1$) to one central vertex of some terminal subgraph isomorphic to S_k or K_1 that is added at the previous step. Moreover, if $k = 1$ then the vertex of the subgraph S_1 the central vertices of the subgraphs S_m are joined to must be the same.

We refer to the so-obtained graph as the ST_5-*extension* of the pseudograph H. Refer as an ST_5-*graph* to a graph that is the ST_5-extension of an arbitrary pseudograph.

Lemma 8. *Each subgraph of some ST_5-graph is an ST_5-graph.*

Proof. Suppose that an ST_5-graph G is obtained by the ST_5-extension of a pseudograph H. Each subgraph of G can be obtained from it by removing some edges and removing some separated vertices from the obtained graph.

Obviously, removing separated vertices from an ST_5-graph also induces an ST_5-graph. Thus, it suffices to show that the graph obtained by removing an edge from an ST_5-graph is also an ST_5-graph.

Let G' be a subgraph of some graph G obtained from it by removing one edge e.

If e is a bridge in G then either e is a noncyclic edge of the pseudograph H or belongs to the terminal subgraph added at Step 3, 4, or 5 or joins a vertex of such a terminal graph to its contact vertex. In the first case, G' can be obtained by the ST_5-extension of the pseudograph obtained from H by removing the edge e. Otherwise, G' consists of the subgraph that is the ST_5-extension of H and the graph that is either an F_5-graph and can be obtained from the ST_5-extension of K_1 or obtained from a graph isomorphic to S_k by adding to it some terminal F_5-subgraphs, and hence is an ST_5-graph too. In other words, G' can be obtained by the ST_5-extension of the pseudograph $H \cup K_1$ or $H \cup S_k$ respectively.

If e is not a bridge in G then the following cases are possible:

(1) The edge e belongs to the terminal subgraph T added at Step 4 or 5 or joins the vertex T with its contact vertex. Then the subgraph obtained from T by removing e is also a terminal F_5-subgraph of G' or two terminal F_5-subgraphs if e is a bridge in T. Consequently, G' can be obtained by the ST_5-extension of the pseudograph H.

(2) The edge e belongs to a cycle of 4 vertices not included in a terminal F_5-subgraph. Then its removal turns one of the vertices obtained in replacing a vertex by an independent set (Step 2) into a terminal vertex. Such a vertex is adjacent to an old vertex or a distant vertex, i.e., can be added at Step 4 or 3. Thus, G' can be obtained by the ST_5-extension of the pseudograph H.

(3) The edge e does not belong to a cycle of 3 or 4 vertices. Then at least one of the vertices incident to it is added in subdividing the cyclic edge at Step 1. Denote by e_0 this edge of the pseudograph H. Removing e destroys at least one of the cycles in G. Denote by A the set of vertices in G and G' included in a cycle of 5 or more vertices in G and not included in one such cycle in G'. Remove from A the vertices added at Step 2 and denote the obtained set by A_0.

It is not hard to see that if all vertices of A_0 in G' are declared old then all operations of adding vertices and terminal subgraphs at Steps 2, 4, or 5 remain well defined, and the operation of adding terminal vertices at Step 3 can now be performed at Step 4.

Denote by H' the pseudograph obtained from H by the edge e_0 and all edges that cease to become cyclic after removing e_0 and adding to it the set of vertices A_0 and all edges joining in G' the vertices of A_0 between each other and to the old vertices of G'. Then G' can be obtained by the ST_5-extension of the pseudograph H'.

Lemma 8 is proved. □

Theorem 1. *Each ST_5-graph is a König graph for $\langle P_5\rangle$.*

Proof. Let G be the graph obtained by the ST_5-extension of an arbitrary pseudograph H. Prove that

$$\mu_{\langle P_5\rangle}(G) = \beta_{\langle P_5\rangle}(G).$$

Carry out the proof by induction on the number of edges in G. If each of its connected components is an F_5-graph then G has no quintets and $\mu_{\langle P_5\rangle}(G) = \beta_{\langle P_5\rangle}(G) = 0$.

Suppose that G contains at least one quintet and, for every subgraph G' of G with fewer edges, we have $\mu_{\langle P_5\rangle}(G') = \beta_{\langle P_5\rangle}(G')$. We can assume that G is connected. Since G contains at least one quintet, it satisfies one of the following conditions:

(1) G has a terminal F_5-subgraph T with contact vertex y such that $G[V(T) \cup \{y\}]$ contains a quintet or a pair of terminal F_5-subgraphs T_1 and T_2 with common contact vertex y such that $G[V(T_1) \cup V(T_2) \cup \{y\}]$ contains a quintet. This is possible in the following cases:

- T is isomorphic to C_4, $K_4 - e$, or K_4;
- T is isomorphic to $S_{k,m}$, where $k, m \in \mathbb{N}$, and y is adjacent to at least one of the leafs of T or to both of its central vertices;
- T is isomorphic to $S_k + e$, where $k \geq 3$, and y is adjacent to at least one of the vertices in T that is not central;
- T is isomorphic to S_k, where $k \geq 3$, and y is adjacent to at least two leaves of T;
- both T_1 and T_2 are different from K_1;
- one of the subgraphs of T_1 and T_2 is isomorphic to K_1 and the other contains at least three vertices; moreover, if it is isomorphic to S_k then y is adjacent to one of its leaves.

Note that every quintet of $G[V(T) \cup \{y\}]$ or $G[V(T_1) \cup V(T_2) \cup \{y\}]$ contains the vertex y. Let Q be one of these quintets. Consider the graph G' that is obtained from G by removing the vertices of Q. Let M be the greatest $\langle P_5\rangle$-packing and let C be the least $\langle P_5\rangle$-cover of G'. By the induction assumption, $|M| = |C|$. Adding to M the quintet Q, we obtain a $\langle P_5\rangle$-packing of size $|M| + 1$. Adding the vertex y to C, we arrive to a $\langle P_5\rangle$-cover of G of the same size.

(2) The graph G contains no terminal F_5-subgraphs in (1) but contains a cycle of 5 or more vertices. Denote by D the set of vertices of G belonging to such cycles and designate as C_0 the set of all cyclic vertices in H (they are all old in G and are contained in D). To each vertex in C_0 there corresponds at least one quintet containing this vertex, two close vertex adjacent to it, and two distant vertices at distance 2 from it. Denote by M_0 the set of such quintets. All quintets in M_0 pairwise do not contain common vertices and lie in D. Obviously, $|C_0| = |M_0|$.

Consider the graph $G' = G \setminus D$. Let M be a greatest $\langle P_5\rangle$-packing and let C be a least $\langle P_5\rangle$-cover of G'. By the induction assumption, $|M| = |C|$. The set $M \cup M_0$ is a $\langle P_5\rangle$-packing of G.

Show that $C \cup C_0$ is a $\langle P_5\rangle$-covering of G. Consider two vertices $x, y \in C_0$ adjacent in the pseudograph H (x and y can coincide if (x, x) is a loop in H). It is not hard to see that the procedure of ST_5-extension turns the edge (x, y) of the pseudograph H into a subgraph of G isomorphic to the F_5-graph $S_{k+1,m+1}$, where k and m are the numbers of vertices added to the given subgraph at Steps 2 and 3.

Observe also that if a vertex $v \in D$ is adjacent in G to a vertex of G' not added at Step 3 then $v \in C_0$. Thus, $G \setminus C_0$ is the union of G' and a set of graphs $S_{k,m}$ and contains the same number of quintets as G'. In other words, C_0 covers all quintets of G not covered by C; i.e., $C \cup C_0$ is a $\langle P_5\rangle$-cover of G.

Since $|C_0| = |M_0|$, we have

$$\mu_{\langle P_5\rangle}(G) = \beta_{\langle P_5\rangle}(G).$$

(3) The graph G contains no cycles of 5 or more vertices, and no terminal F_5-subgraphs together with a contact vertex contain a quintet. In this case, the pseudograph H is a tree.

Note that the cycles of three vertices in G can be formed only by the vertices from terminal F_5-subgraphs and their contact vertices, while the cycles of four vertices can be constituted alongside this by the vertices replaced with independent sets at Step 2. Since these vertices are not adjacent to each other, G contains a terminal F_5-subgraph: this is either the subgraph added at Step 4 or 5 or the subgraph induced by old vertices and, possibly, vertices added at Steps 2, 4, and 5.

We can assume without loss of generality that all vertex-inclusion-maximal terminal F_5-subgraphs of G are added at Step 4. In this case, either the tree H consists of a single vertex (but then it is easy to see that $\mu_{\langle P_5\rangle}(G) = \beta_{\langle P_5\rangle}(G) = 0$; otherwise, we have case (1)), or at least one terminal F_5-subgraph is added to each of its leaves at Step 4.

Let y be one of the leaves of H, let T be the union of terminal F_5-subgraphs of G with contact vertex y, and let x be a neighbor of y in the tree H. The graph $G[V(T) \cup \{y\}]$ is an F_5-graph but not a terminal subgraph of G; otherwise, each component of T is not an inclusion-maximal terminal F_5-subgraph. Consequently, the vertex x was replaced by an independent set at Step 2. In other words, in G, the vertex y is adjacent to pairwise disjoint vertices $x_1, x_2, \dots x_m$, where $m \geq 2$, each of which is adjacent exactly to one vertex more.

Then $G[V(T) \cup \{y, x_1, x_2, \dots x_m\}]$ is a terminal subgraph of G and hence contains a quintet; otherwise, each component of T is not an inclusion maximal terminal F_5-subgraph. Such a quintet necessarily passes through y and one of the vertices $x_1, x_2, \dots x_m$. Thus, let (x_1, y, z_1, z_2, z_3) be a quintet in G and, moreover, $\{z_1, z_2, z_3\} \subseteq V(T)$.

Consider the graph $G' = G \setminus \{z_1, z_2, z_3, y\}$. Let M be a greatest $\langle P_5\rangle$-packing and let C be a least $\langle P_5\rangle$-cover of the graph G'. By the induction assumption, $|M| = |C|$. If one of the quintets M contains x_1, replace x_1 therein with x_2; i.e., assume without loss of generality that M contains no quintets with vertex x_1. Then, adding the quintet (x_1, y, z_1, z_2, z_3) to M, we obtain a $\langle P_5\rangle$-packing of size $|M| + 1$. Adding y to C, we obtain a $\langle P_5\rangle$-cover of G of the same size.

Thus, $\mu_{\langle P_5\rangle}(G) = \beta_{\langle P_5\rangle}(G)$. By Lemma 8, every subgraph of G is also an ST_5-graph. Consequently, each ST_5-graph is König for $\langle P_5\rangle$.

The proof of Theorem 1 is complete. □

4. COMPLETE DESCRIPTION OF THE GRAPHS IN $\mathcal{K}(\langle P_5\rangle)$

Show that every König graph for $\langle P_5\rangle$ is an ST_5-extension of some pseudograph and also prove that the forbidden subgraphs described in Section 2 completely describe this graph class.

Introduce the notation $\mathcal{F}$ for the set of forbidden graphs described in Lemmas 1 and 3–7:

$$\begin{aligned}\mathcal{F} = \{E_1, E_2, \dots, E_{13}\} &\cup \{C_n \mid n > 5 \text{ and } n \text{ does not divide by } 5\} \\ &\cup \left\{A_1(5t, 5k+1), A_1(5t, 5k+4) \mid t \geq 1,\ 0 \leq k < \frac{t}{2}\right\} \\ &\cup \left\{A_2(5t, 5k+2), A_2(5t, 5k+3) \mid t \geq 1,\ 0 \leq k < \frac{t}{2}\right\} \\ &\cup \left\{A_3(5p+1, 5q+1, 5r+3), A_3(5p+4, 5q+4, 5r+2) \mid p, q, r \geq 0\right\} \\ &\cup \left\{B(5t, 5k) \mid t \geq 1,\ 1 \leq k \leq \frac{t}{2}\right\}.\end{aligned}$$

Theorem 2. *The following are equivalent for a graph G:*

(1) *G is a König graph for $\langle P_5 \rangle$;*

(2) *G contains no $\mathcal{F}$-subgraphs;*

(3) *G is the ST_5-extension of some pseudograph.*

Proof. Lemmas 1, 3–7 and the remark at the end of Section 1 imply that (1) yields (2). Theorem 1 gives that (3) implies (1). Show that (2) implies (3).

Let G be a connected graph not containing $\mathcal{F}$-subgraphs. Let $T_1, T_2, \dots, T_n$ be the terminal subgraphs of G each of which is isomorphic to S_k, where k are various naturals, and let x be their common contact vertex. Moreover, let x be adjacent only to the central vertices of the subgraphs $T_1, T_2, \dots, T_n$ and be not a contact vertex for other terminal F_5-subgraphs but possibly the subgraphs K_1. Denote by X the set of all such vertices x in G.

Remove from G a maximal set of the vertex disjoint terminal maximal F_5-subgraphs. If the obtained graph contains some terminal subgraphs isomorphic to K_1 or S_k, where $k \in \mathbb{N}$, exactly one vertex of which belongs to X then remove also them. Denote the obtained graph by G_0 and show that it contains no triangles. Suppose that G_0 contains a clique of size 4 and the vertices x, y, z, and u are pairwise adjacent in this graph. Consider the following cases:

Case 1. The graph G contains a vertex v with two vertices in $\{x, y, z, u\}$. Assume without loss of generality that v is adjacent to x and y. Since none of the 4-vertex subgraphs induced by the vertices of the set $\{x, y, z, u, v\}$ is a terminal subgraph of G; therefore, at least two of these vertices are adjacent to other vertices of this graph. Denote these vertices by a and b.

If a and b coincide and have at least one neighbor among the vertices z, u, and v then G contains a subgraph C_6. Otherwise, if one of the vertices a and b is adjacent to z or u and the other, to v; then G contains a subgraph E_{11}. In all remaining cases, if one of a and b is adjacent to one of the vertices z, u, and v then G contains a subgraph E_8.

Thus, there are no vertices adjacent to z, u, and v. But then there exists a vertex a adjacent to x and b that is adjacent to y. Since the subgraph $S_k + e$ containing the vertices x, v, z, u, and a is not a terminal subgraph of G (here and below, by this we mean the subgraph consisting of the vertices x, v, z, u, a, and the other neighbors of x but y if such neighbors exist); therefore, a is adjacent to the vertex $c \in V(G) \setminus \{x, y, z, u, v\}$ (in fact, some of the neighbors of x is adjacent to c but, without loss of generality, we assume that this neighbor is denoted by a).

If $a = b$ then G contains a subgraph E_{12}. Otherwise, repeating the above arguments, we can show that b is adjacent to a vertex $d \in V(G) \setminus \{x, y, z, u, v\}$. If $\{c, d\} \cap \{a, b\} \neq \varnothing$ or $c = d$ then G has a subgraph C_6. Otherwise, G has a subgraph E_3.

Case 2. The graph G contains no vertex adjacent to two vertices in $\{x, y, z, u\}$. Since the subgraphs K_4 and C_3 induced by the vertices of x, y, z, u are not terminal subgraphs of G, at least two of these vertices are adjacent to other different vertices of this graph. Suppose that a is adjacent to x and b is adjacent to y in G; moreover, that a and b do not coincide and are not adjacent to other vertices of this clique. Note also that a and b are not adjacent to each other and have no common neighbors; otherwise, G has a subgraph C_6.

Since the subgraph $S_k + e$ containing the vertices a, x, z, and u is not a terminal subgraph of G, at least one of the vertices a, z, and u is adjacent to a vertex of the graph different from y. By analogy, at least one of the vertices b, u, and z is adjacent to a vertex of the graph different from x. The following cases are possible up to symmetry:

- There is a vertex c adjacent to a and b. Then G has a subgraph C_7.

- There is a vertex c adjacent to z. Then G has a subgraph E_8.

- There is a vertex c adjacent to a and a vertex d adjacent to b. Then G has a subgraph E_3.

Thus, G_0 does not include a subgraph K_4.

Suppose that G_0 contains a subgraph 3-fan. Let x, y, z, u, and v be vertices of G_0 and let the graph contain all edges between them but (z, x), (y, u), and, possibly, (x, u).

Since a graph isomorphic to $S_{k,m}$ with central vertices y and z and the cycle constituted by the vertices x, y, z, and u (if G_0 contains the edge (x, u)) are not terminal subgraphs with contact vertex v in G, one of the following conditions is fulfilled up to symmetry:

- There exists a vertex a different from y, u, and v and adjacent to x. Note that a is not adjacent to the vertices y, z, u, and v; otherwise, G has a subgraph C_6.

  Since the subgraph induced by the vertices y, z, u, and v is not a terminal subgraph with contact vertex x in G; therefore, at least one of the vertices y, z, u, and v is adjacent to other vertices of this graph that are different from x. But then G includes one of the subgrpaphs E_9 or E_{11}.

  Observe that if G contains the edge (x, u) then all vertices x, y, z, and u are symmetric; therefore, none of them is adjacent to other vertices but x, y, z, u, and v. In this case, the cycle (x, y, z, u, x) is a terminal F_5-subgraph of G; a contradiction. Thus, we can assume that the vertices x and u are not adjacent.

- There are no vertices different from y, z, and v and adjacent to x and u. Then there is a vertex a adjacent to y and a vertex b adjacent to a. Note that a and b are not adjacent to the vertices z, v and the neighbors of v; and also x is not adjacent to u; otherwise, G has a subgraph C_6 or a subgraph analogous to the previous case.

  Since the graph isomorphic to $S_k + e$ with central vertex v is not a terminal subgraph with contact vertex y in G; therefore, there exists a vertex $c \notin \{x, y, z, u, v, a, b\}$ adjacent to z or a pair of adjacent vertices c and d different from a and b, one of which is adjacent to v. In the first case, G has a subgraph $A_1(5, 1)$; and in the second, a subgraph $A_2(5, 2)$.

Thus, the graph G_0 has no subgraph 3-fan.

Suppose that G_0 includes a subgraph $K_4 - e$. Suppose that x, y, z, and u are the vertives of G_0 and this graph contains all edges between them but (z, u). Suppose also that all pairs of adjacent vertices but x and y have no common neighbors.

Suppose that none of the vertices of $N(x) \cap N(y)$ is adjacent in G to other vertices of this graph but x and y. Then in G there is a path x, a_1, a_2, a_3, where a_1 and a_2 do not coincide with y; otherwise, $x \in X$ and x is a central vertex of a terminal subgraph isomorphic to S_k.

Assume that $a_3 = y$. Then there is no vertex adjacent simultaneously to two vertices from x, y, z, u, a_1, and a_2 constituting an edge except the pair x, y; otherwise, G has a subgraph C_6.

Since the graphs isomorphic to $S_{k,m}$ with central vertices x, a_1 and y, a_2 are not terminal subgraphs, one of the following four conditions is fulfilled for G:

(1) There exist vertices b and c different from x, y, z, u, a_1, and a_2 and adjacent in G to a_1 and a_2 respectively. Then G has a subgraph E_4.

(2) There exist vertices b and c different from x, y, z, u, a_1, and a_2 and adjacent to each other in G; moreover, one of them is adjacent to a_1 or to a_2. Then G has a subgraph $A_1(5, 1)$.

(3) G has a path x, b_1, b_2, where $b_1 \neq a_1$, $b_2 \neq a_2$, and a vertex c adjacent to a_1. Then G has a subgraph $A_1(5, 1)$. The following symmetric case is considered similarly: the path y, b_1, b_2 and a vertex c adjacent to a_2.

(4) G contains paths x, b_1, b_2 and y, c_1, c_2, where $b_1 \neq c_1$, $b_2 \notin \{a_2, y\}$, and $c_2 \notin \{a_1, x\}$. Then, if, among the vertices b_1, b_2, c_1, and c_2, there are coinciding vertices, then G has a subgraph C_6; otherwise, G has a subgraph $A_2(5, 2)$.

Thus, $a_3 \neq y$; i.e., G contains a path of length 3 joining x and y. Similarly, G contains a path y, b_1, b_2, b_3; moreover, $x \notin \{b_1, b_2, b_3\}$, $a_1 \notin \{b_1, b_2\}$, and $b_1 \notin \{a_1, a_2\}$. If a_3 coincides with one of the vertices b_1, b_2, and b_3 or $a_2 = b_2$, then G contains a cycle of 6 or 7 vertices. Thus, all vertices a_1, a_2, a_3, b_1, b_2, and b_3 are pairwise distinct; but then G has a subgraph E_2.

Thus, without loss of generality, we assume that the graph G contains a vertex v adjacent to u and not adjacent to x and y.

Suppose that there exists a vertex $a \notin \{y, z, u, v\}$ adjacent to x. Then there are no vertices different from x, y, and v and adjacent to z; otherwise, G contains a subgraph E_8. Moreover, a is not adjacent to v, u, and z; otherwise, G has a subgraph C_6 or 3-fan.

Suppose that the vertex v is adjacent to z. Then, repeating the above arguments, we conclude that there are no vertices different from x, y, and v and adjacent to u. Since the graph induced by the vertices y, u, v, and z is not a terminal subgraph with contact vertex x in G; therefore, there exists a vertex $b \notin \{x, y, z, u, v, a\}$ adjacent in G to one of the vertices y and v. But then G has a subgraph E_{13}. Thus, v is not adjacent to z.

Since the graph isomorphic to $S_{k,m}$ with central vertices x and u is not a terminal subgraph with contact vertex y in G, there is a vertex $b \notin \{x, y, z, u, v, a\}$ adjacent in G to one of the vertices a and v. The following cases are possible:

(1) The vertex b is adjacent to v. Then y has no other neighbors but x, u, z, and a; otherwise, G has a subgraph E_5 or C_6. Note that if y is adjacent to a then, analogously, x has no other neighbors but y, u, z, and a.

Since the graph isomorphic to $S_k + e$ (or $K_4 - e$ if y is adjacent to a) with central vertex x is not a terminal subgraph with contact vertex u in G; therefore, there exists a vertex $c \notin \{x, y, z, u\}$ adjacent to a in G. If $c = b$ then G has a subgraph C_6; otherwise, G has a subgraph E_3.

(2) There are no vertices but u adjacent to v. Then b is adjacent to a. Since the graph isomorphic to $S_{k,m}$ with central vertices y and u is not a terminal subgraph with contact vertex x in G; there exists a path of three vertices with beginning at y; i.e., there exist vertices $c \notin \{x, y, z, u, v\}$ adjacent to y in G and $d \notin \{x, y, z, u, v, c\}$ adjacent to c in G. If $c = a$ then G has a subgraph E_{13}. If $c = b$ and $d = a$ then G has a subgraph E_{12}. If $d = b$ then G has a subgraph C_6. Otherwise, G has a subgraph E_6.

Thus, there are no vertices but y, z and u adjacent to x. By analogy, there are no vertices but x, z, and u, adjacent to y. Since the graphs isomorphic to $S_k + e$ with central vertices u and z are not terminal subgraphs of G, there exist paths of three vertices with beginnings at u and z. Without loss of generality, we may assume that G has some vertices a adjacent to v, b adjacent to z, and c adjacent to b, where $\{a, b, c\} \cap \{x, y, z, u\} = \varnothing$.

Assume that $b = v$. Then, since the subgraph induced by the vertices x, y, z, and u is not a terminal subgraph of G, there is a vertex $d \notin \{x, y, z, u, v\}$ adjacent to one of z and u. If $d = a$ then G has a subgraph C_6. Otherwise, G has a subgraph E_{10}. Thus, $b \neq v$. Then, if $a = b$, $a = c$, or $c = v$ then G has a subgraph C_6. Otherwise, G has a subgraph E_7.

Thus, the graph G_0 does not include a subgraph $K_4 - e$.

Suppose that G_0 has a triangle. Denote by x, y, and z the vertices of some triangle in G_0.

Since a graph isomorphic to S_1 and containing two of the vertices x, y, and z is not a terminal subgraph of G, at least two of these vertices are adjacent to other vertices of this graph. Let u be adjacent to x and v be adjacent to y in G. Note that $u \neq v$; otherwise, G_0 has a subgraph $K_4 - e$.

Suppose that u and v are adjacent. Since the subgraph of G consisting of x, y, u, and v, is not terminal, at least one of these vertices is adjacent to another vertex of this graph different from z. Consider the following cases up to symmetry:

(1) The graph G includes a vertex a adjacent to u. Since the subgraph isomorphic to $S_{k,m}$ with central vertices x and u is not a terminal subgraph with contact vertex y in G, there exists a vertex b adjacent to each of the vertices a, z, and v or a path of three vertices not passing through y with beginning at x, i.e., b is adjacent to x and c is adjacent to b. Note that, in any case, $b \neq a$ and $c \neq a$; otherwise, G has a subgraph C_6. The following cases are possible:

- b is adjacent to z. Then G has a subgraph E_{11}.

- b is adjacent to v. Then G has a subgraph E_{10}.
- b is adjacent to x, c is adjacent to b. If $c \in \{a, u\}$ then G has a subgraph C_6. Otherwise, G has a subgraph $A_1(5, 1)$.
- The graph G has no vertices but x, y, and u adjacent to z and v, but G has a vertex $b \notin \{x, y, z, u, v\}$ adjacent to a. Since the graph isomorphic to $S_k + e$ with central vertex y is not a terminal subgraph with contact vertex u in G, there exists a vertex adjacent to x or a path of three vertices not passing through u with beginning at y. In the first case, G has a subgraph $A_1(5, 1)$, and in the second, one of the subgraphs E_6, E_{10}, and C_6.

(2) The graph G has no vertices adjacent to u and v except x and y but has a vertex a adjacent to x. Since the subgraph isomorphic to $S_{k,m}$ with central vertices y and v is not a terminal subgraph with contact vertex x in G, there exists a vertex $b \notin \{x, y, u, v\}$ adjacent to z or a path of three vertices not passing through x with beginning at y. In the first case, if $b = a$ then G has a subgraph C_6; otherwise, G has a subgraph E_9. Consider the second case in more detail.

So, suppose that G has no vertices except x and y adjacent to z, but G has a vertex $b \notin \{x, z, u, v, a\}$ adjacent to y and a vertex $c \notin \{x, y, z, u, v, b\}$ adjacent to b. Since the subgraph isomorphic to $S_{k,m}$ with central vertices x and u is not a terminal subgraph of G, there exists a vertex $d \notin \{x, y, z, u, v\}$ adjacent to a. If $d \in \{b, c\}$ or $a = c$ then G has a subgraph C_6; otherwise, G has a subgraph E_3.

Thus, the vertices u and v are not adjacent; i.e., no vertex adjacent to one of x, y, and z, is adjacent to neighbors of two other vertices.

Suppose that every path of five vertices in G passing through x and y has z. If G has a quintet (a, b, x, c, d) then the vertices a and d are terminal, each of the vertices b and c is adjacent only with x and the terminal vertices, and y is adjacent only to x and z. It is not hard to see that x is a contact vertex for some subgraphs isomorphic to S_k; moreover, it is adjacent only to their centers. Therefore, $x \in X$. After the removal of the corresponding subgraphs S_k, x is a central vertex of the terminal subgraph isomorphic to S_k containing y with contact vertex z. If G has a quintet (a, b, y, c, d) then we make a similar conclusion about y. Otherwise, z is a contact vertex of a terminal F_5-subgraph containing the vertices x and y.

All listed cases contradict the definition of G_0. Thus, there exists a path of five vertices passing through x and y and not containing z. By analogy, there exists a path of five vertices passing through each pair of vertices of the triangle and not passing through its third vertex. The three cases are possible up to symmetry:

(1) There exist vertices u, v, and w adjacent to x, y, and z respectively. Then, without loss of generality, we assume that there exist a vertex a adjacent to u and a vertex b adjacent to v. If $a = b$ then G has a subgraph C_6; otherwise, G has a subgraph E_1.

(2) There are no vertices except x and y adjacent to z. Then there exist paths of four vertices with beginnings at x and y. If the vertices of these paths are different then G has a subgraph E_2. Otherwise, G has a cycle with the number of vertices from 6 to 8.

Thus, G_0 has no triangles.

Consider some cycle of 4 vertices (x, y, z, u) in G_0. Suppose that G has a vertex a adjacent to x and a vertex b adjacent to y. The vertices a and b are not adjacent; otherwise, G has a subgraph C_6. The following cases are possible up to symmetry:

(1) G contains vertices c and d different from x, y, z, and u and adjacent to z and u respectively. If $c = a$ and $d = b$ then G has a subgraph C_6; if $c = a$ but $d \neq b$ then G has a subgraph E_{13}; otherwise, G has a subgraph E_4.

(2) Only x and z are adjacent to u in G but there exists $c \notin \{y, u, a\}$ adjacent to z. Then there is no vertex different from y and adjacent to b; otherwise, G has a subgraph E_5. Since the graphs isomorphic to $S_{k,m}$ with pairs of central vertices x, y and y, z are not terminal subgraphs of G, there exist vertices v and w not coinciding with x and z and adjacent to a and c respectively. If $v = w$ then G has a subgraph C_6; if $v = c$ or $w = a$ then G has a subgraph E_{12}; otherwise, G has a subgraph E_6.

(3) In G, only x and z are adjacent to u and a is adjacent to z. Then there are no other vertices but x and z adjacent to a; otherwise, G has a subgraph E_{13}. Since the subgraph induced by the vertices x, a, z, and u is not a terminal subgraph with contact vertex y in G, there exists a vertex different from y, u, and a adjacent to one of the vertices x and z. Assume without loss of generality that there exists a vertex $c \notin \{y, u, a\}$ adjacent to z. Then the consideration of this is reduced to the previous case.

(4) In the graph G, $N(u) = \{x, z\}$ and $N(z) = \{y, u\}$. Since the graphs isomorphic to $S_{k,m}$ with pairs of central vertices x, u and y, z are not terminal subgraphs of G, there exist vertices c and d adjacent to a and b respectively. If $c = d$ then G has a subgraph C_7; otherwise, G has a subgraph E_3.

Thus, in every cycle of four vertices in G_0, two nonadjacent vertices in G have degree 2. Moreover, for each such pair of vertices u and v, we have $N(u) = N(v)$. Call such vertices *similar*. Taking an arbitrary 4-vertex cycle in G_0, remove one of the two similar vertices and repeat this operation while such cycles exist. Denote the so-obtained graph by G_1. Note that if G_1 has a cycle then G_1 contains at least five vertices.

Suppose that G_1 has a cycle. Let Z be its block of size greater than 2. Each vertex in Z is cyclic. Since G has no subgraphs from $\mathcal{F}$, Lemma 3 implies that the length of every cycle in Z divides by 5.

Denote by U the set of the vertices in Z adjacent in G to at most three vertices pairwise not similar to each other of degree greater than 1. Consider first the case of $U \neq \varnothing$. Let $|U| \geq 2$. Prove that every two vertices in U are at distance in Z dividing by 5.

Suppose that this fails. Consider two vertices x and y in U at the minimal distance not dividing by 5 in Z. Consider a cycle W of minimal length containing x and y. It is not hard that G has paths x, a, b and y, c, d such that a, b, c, d are pairwise distinct and do not belong to W (this follows from the absence in Z of cycles of length not dividing by 5 and the minimality of the length of the cycle W).

Suppose that the length of W is equal to $5t$ and the distance between x and y in Z is equal to $5k + r$, where $0 \leq k \leq t/2$ and $1 \leq r \leq 4$. If there exists a shortest path between x and y lying completely in W then, for $r \in \{1, 4\}$, G has a forbidden subgraph $A_1(5t, 5k + r)$; and for $r \in \{2, 3\}$, a forbidden subgraph $A_2(5t, 5k + r)$; a contradiction.

Consider the case when every shortest path between x and y has some vertices not lying in W. On one of them, choose vertices s and t lying in W such that all vertices of this path between s and t are not contained in W. Note that $\{s, t\} \subseteq U$. All distances between the vertices x and s, s and t, t and y divide by 5 (because all these distances are less than the distance between x and y). Then the distance between x and y divides by 5; a contradiction.

Thus, if $|U| \geq 2$ then every two vertices in U are at distance dividing by 5 in Z. Denote by Y the set of vertices in Z at distance $5k$ from an arbitrary vertex $x \in U$, where $k \in \mathbb{N}$. By the previous argument, $U \subseteq Y$. It is also easy to see that the vertices of the set Y divide Z into paths of length 5. In other words, if the shortest path between two vertices in Y or a cycle beginning and ending at the same vertex in Y has no other vertices in Y then it passes exactly through fore vertices each of which has degree 2 in G_1.

The vertices of the block Z not belonging to Y and analogous vertices of other analogous blocks will be called *passage* vertices in the graph G_1.

All vertices of the block Z adjacent to vertices of Y have degree 2 in G; otherwise, G has a forbidden subgraph of type $A_1(n, k)$, while the remaining passage vertices in Z constitute pairs, and each of them is adjacent in G only to the other vertex of the pair, to vertices of degree 2 constituting exactly one similarity class, and to vertices of degree 1; otherwise, G has a forbidden subgraph of type $A_2(n, k)$.

If $U = \varnothing$ then all vertices in Z are adjacent in G exactly to 2 similarity classes of degree greater than 1; i.e., Z is a cycle. Then the vertices of at least two of its 5-classes have degree 2 in G; moreover, the vertices of these 5-classes are pairwise nonadjacent; otherwise, G has a forbidden subgraph of one of the types $A_3(k_1, k_2, k_3)$, $B(n, k)$, or one of the subgraphs E_6 and E_{12}. Denote by x a vertex adjacent to vertices of both of these 5-classes. All vertices in Z the distance to which from x does not divide by 5 (and analogous vertices of other such blocks) will also be assumed passage vertices in G_1.

Thus, if a passage vertex is adjacent to a nonpassage vertex in G_1 then it has degree 2 in G and there can exist similar vertices; otherwise, it is adjacent to vertices of degree 2 constituting exactly one similarity class and exactly to one analogous vertex and also can be adjacent to vertices of degree 1.

Construct a pseudograph H by replacing in all cycles of the graph G_1 each path of four passage vertices with one edge joining the vertices adjacent to its ends (if G_1 has no cycles then $H = G_1$). The graph G_1 is obtained from H by subdividing each cyclic edge by four vertices (Step 1), the graph G_0 is obtained from G_1 by replacing vertices with empty graphs (Step 2), and the graph G is obtained from G_0 by applying Steps 3–5, i.e., is the ST_5-extension of the pseudograph H.

Theorem 2 is proved. □

A graph class is called *monotone* if it is closed under not only vertex removal but also under edge removal. It is known that each monotone class can be characterized by the set of minimal forbidden subgraphs, i.e., by vertex-and-edge-inclusion-minimal graphs not belonging to this class.

Formulate a corollary to Theorem 2 that is confirmed also by Lemma 8:

Corollary. *The class $\mathcal{K}(\langle P_5\rangle)$ is monotone and is completely defined by the set of minimal forbidden subgraphs $\mathcal{F}$.*

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