OPEN SAITO THEORY FOR A AND D SINGULARITIES

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Abstract. A well-known construction of B. Dubrovin and K. Saito endows the parameter space of a universal unfolding of a simple singularity with a Frobenius manifold structure. In our paper we present a generalization of this construction for the singularities of types A and D, that gives a solution of the open WDVV equations. For the A-singularity the resulting solution describes the intersection numbers on the moduli space of r-spin disks, introduced recently in a work of the second author, E. Clader and R. Tessler. In the second part of the paper we describe the space of homogeneous polynomial solutions of the open WDVV equations associated to the Frobenius manifolds of finite irreducible Coxeter groups.

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1. Introduction

Frobenius manifolds, introduced by B. Dubrovin in the early 90s, gave a geometric approach to study solutions of the *WDVV equations*

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu}\eta^{\mu\nu}\frac{\partial^3 F}{\partial t^\nu \partial t^\gamma \partial t^\delta} = \frac{\partial^3 F}{\partial t^\delta \partial t^\beta \partial t^\mu}\eta^{\mu\nu}\frac{\partial^3 F}{\partial t^\nu \partial t^\gamma \partial t^\alpha}, \quad 1 \le \alpha, \beta, \gamma, \delta \le N, \tag{1.1}$$

where $F = F(t^1, \dots, t^N)$ is an analytic function defined on some open subset $M \subset \mathbb{C}^N$, $\eta = (\eta_{\alpha\beta})$ is an $N \times N$ symmetric non-degenerate matrix with complex coefficients, $(\eta^{\alpha\beta}) := \eta^{-1}$ and we use the convention of sum over repeated Greek indices. The WDVV equations appear in many areas of mathematics, including singularity theory and curve counting theories in algebraic geometry. In Gromov–Witten theory the WDVV equations describe the structure of primary Gromov–Witten invariants in genus 0 and naturally come from a certain relation in the cohomology of the moduli space of stable curves.

Suppose that a function F satisfies the WDVV equations together with the additional assumption

$$\frac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta}. \tag{1.2}$$

The function F defines a commutative product $\circ$ on each tangent space T_pM by

$$\frac{\partial}{\partial t^\alpha} \circ \frac{\partial}{\partial t^\beta} := \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}\eta^{\gamma\delta}\frac{\partial}{\partial t^\delta}, \quad 1 \le \alpha, \beta \le N.$$

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One can immediately see that the WDVV equations are equivalent to the associativity of this product and property (1.2) means that the vector field $\frac{\partial}{\partial t^1}$ is the unit. One can go in the opposite direction and consider a manifold with a commutative, associative algebra structure and a symmetric, non-degenerate bilinear form on each tangent space. Under certain conditions such a manifold in special local coordinates, called the *flat coordinates*, can be described by a solution F of the WDVV equations, satisfying property (1.2). The conditions, needed for the existence of a function F, were systematically studied by B. Dubrovin [Dub96, Dub99], who called manifolds, satisfying these conditions, *Frobenius manifolds*. The function F is then called a *Frobenius manifold potential*. The bilinear form is traditionally called a *metric*.

In his fundamental works [Sai82, Sai83] K. Saito constructed a flat metric on the parameter space of a universal unfolding of any simple singularity. B. Dubrovin [Dub98] then proved that together with a certain commutative, associative algebra structure on each tangent space this metric defines a Frobenius manifold structure on the parameter space of the universal unfolding. These Frobenius manifolds are often called the *Saito Frobenius manifolds*. Remarkably, the same Frobenius manifolds appear in the study of the geometry of the moduli spaces of algebraic curves with certain additional structures, the so-called Fan–Jarvis–Ruan–Witten (FJRW) theory [FJR13]. This is one of the manifestations of mirror symmetry.

In the same way, as the WDVV equations appeared in Gromov–Witten theory, another system of non-linear PDEs, called the *open WDVV equations*, appeared more recently in open Gromov–Witten theory [HS12, Theorem 2.7] (see also [PST14, BCT18, BCT19]). Let $F = F(t^1, \dots, t^N)$ be a solution of the WDVV equations (1.1), satisfying condition (1.2). The open WDVV equations associated to F are the following PDEs for a function $F^o = F^o(t^1, \dots, t^N, s)$, depending on an additional variable s:

(1.3)
$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu}\eta^{\mu\nu}\frac{\partial^2 F^o}{\partial t^\nu \partial t^\gamma} + \frac{\partial^2 F^o}{\partial t^\alpha \partial t^\beta}\frac{\partial^2 F^o}{\partial s \partial t^\gamma} = \frac{\partial^3 F}{\partial t^\gamma \partial t^\beta \partial t^\mu}\eta^{\mu\nu}\frac{\partial^2 F^o}{\partial t^\nu \partial t^\alpha} + \frac{\partial^2 F^o}{\partial t^\gamma \partial t^\beta}\frac{\partial^2 F^o}{\partial s \partial t^\alpha}, \quad 1 \le \alpha, \beta, \gamma \le N,$$

(1.4)
$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu}\eta^{\mu\nu}\frac{\partial^2 F^o}{\partial t^\nu \partial s} + \frac{\partial^2 F^o}{\partial t^\alpha \partial t^\beta}\frac{\partial^2 F^o}{\partial s^2} = \frac{\partial^2 F^o}{\partial s \partial t^\beta}\frac{\partial^2 F^o}{\partial s \partial t^\alpha}, \qquad 1 \le \alpha, \beta \le N.$$

Solutions of equations (1.3), (1.4), relevant in open Gromov-Witten theory and also in the works [PST14, BCT18, BCT19], satisfy the additional condition

$$\frac{\partial^2 F^o}{\partial t^1 \partial t^\alpha} = 0, \qquad \frac{\partial^2 F^o}{\partial t^1 \partial s} = 1. \tag{1.5}$$

The solutions of the open WDVV equations from the works [BCT18, BCT19] are associated to the Saito Frobenius manifold of the A-singularity and they were constructed using ideas of FJRW theory. So it is natural to ask whether the Dubrovin–Saito construction of the Frobenius manifolds corresponding to simple singularities admits a generalization, that produces solutions of the open WDVV equations. In our paper we present such a generalization for the singularities of types A and D. For the A-singularity our construction gives a polynomial solution that coincides with the one from [BCT18, BCT19]. For the D-singularity our solution has a simple pole along the variable s.

Additionally, in both A- and D-cases our solution of the open WDVV equations has the following remarkable feature. The Saito Frobenius manifold of a simple singularity has two natural coordinate systems. The first one is given by the parameters of a universal unfolding of a simple singularity. The second coordinate system is given by the flat coordinates of the metric. We show that for the singularities A and D the transition functions between these two coordinate systems coincide with the coefficients of powers of the variable s in the expansion of our solution of the open WDVV equations.

The Saito Frobenius manifolds of simple singularities together with their certain submanifolds form a class of Frobenius manifolds, that is, via a construction of B. Dubrovin [Dub98], in a natural bijection with the class of finite irreducible Coxeter groups (see also [Zub94]). This class of Frobenius manifolds plays a fundamental role in the theory of Frobenius manifolds, because of the following result of C. Hertling, conjectured by B. Dubrovin [Dub98]. Recall that a Frobenius manifold potential F is called homogeneous, if there exists a vector field E of the form

$$E=\sum_{\alpha=1}^{N}\underbrace{(q_\alpha t^\alpha+r^\alpha)}_{=:E^\alpha}\frac{\partial}{\partial t^\alpha},\quad q_\alpha,r^\alpha\in\mathbb{C},\quad q_1=1, \tag{1.6}$$

such that

$$E(F)=E^\alpha\frac{\partial F}{\partial t^\alpha}=(3-\delta)F+\frac{1}{2}A_{\alpha\beta}t^\alpha t^\beta+B_\alpha t^\alpha+C,\quad \text{for some } \delta,A_{\alpha\beta},B_\alpha,C\in\mathbb{C}.$$

The number δ is called the *conformal dimension* and the vector field E is called the *Euler vector field*. C. Hertling proved that any generically semisimple Frobenius manifold (see Section 2.1 for definition), whose potential is polynomial, $F\in\mathbb{C}[t^1,\dots,t^N]$, and homogeneous with the Euler vector field of the form $E=q_\alpha t^\alpha\frac{\partial}{\partial t^\alpha}$, where $q_\alpha>0$, can be expressed as the product of the Frobenius manifolds corresponding to finite irreducible Coxeter groups [Hert02, Theorem 5.25].

In the second part of the paper we study the space of polynomial solutions of the open WDVV equations associated to the Frobenius manifolds of finite irreducible Coxeter groups. Note that all solutions of the open WDVV equations, considered in the works [HS12, PST14, BCT18, BCT19], are associated to a homogeneous Frobenius potential F and, moreover, the function F^o satisfies the homogeneity condition

$$E^\alpha\frac{\partial F^o}{\partial t^\alpha}+\frac{1-\delta}{2}s\frac{\partial F^o}{\partial s}=\frac{3-\delta}{2}F^o+D_\alpha t^\alpha+\widetilde{D}s+E,\quad \text{for some } D_\alpha,\widetilde{D},E\in\mathbb{C}. \tag{1.7}$$

We see that the degree of the variable s is determined by the conformal dimension of the Frobenius manifold. We will call a solution of the open WDVV equations homogeneous, if it satisfies condition (1.7).

In our paper we describe the space of homogeneous *polynomial* solutions of the open WDVV equations associated to the Frobenius manifolds of *all* finite irreducible Coxeter groups. In particular, we show that this space is non-empty only for the Coxeter groups A_N, B_N and $I_2(k)$.

Remark 1.1. *Note that the Frobenius manifolds of finite irreducible Coxeter groups are generically semisimple and in* [BB19, Section 6.2] *we proved that for a homogeneous Frobenius manifold potential a homogeneous solution of the open WDVV equations always exists in a neighbourhood of a semisimple point.*

Our approach to study solutions of the open WDVV equations is based on the following crucial observation of P. Rossi. Let $F=F(t^1,\dots,t^N)$ be a Frobenius manifold potential and $F^o=F^o(t^1,\dots,t^N,s)$ be a solution of the open WDVV equations satisfying (1.5). Then the $(N+1)$-tuple of functions $\big(\eta^{1\mu}\frac{\partial F}{\partial t^\mu},\dots,\eta^{N\mu}\frac{\partial F}{\partial t^\mu},F^o\big)$ forms a *vector potential* of a *flat F-manifold*. This allows us to use the theory of flat F-manifolds to study solutions of the open WDVV equations.

Remark 1.2. *As it was noticed by the anonymous referee of our paper, there is a similarity between the open WDVV equations and the Monge–Ampére equations. The referee also suggested to view the system of equations* (1.3), (1.4) *for the function F^o as a "nonlinear Lax pair" for the system of equations* (1.1) *for the function F. We think that these interesting observations can become a starting point for future research.*

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2. Flat F-manifolds and Frobenius manifolds

In this section we recall the definitions and the main properties of flat F-manifolds and Frobenius manifolds. We also explain how solutions of the open WDVV equations correspond to flat F-manifolds of special type.

2.1. Flat F-manifolds.

The notion of a flat F-manifold was introduced in [Man05].

Definition 2.1. *A flat F-manifold $(M, \nabla, \circ)$ is the datum of a complex analytic manifold M, an analytic connection ∇ in the tangent bundle TM, an algebra structure $(T_pM, \circ)$ with unit e on each tangent space analytically depending on the point $p \in M$ such that the one-parameter family of connections $\nabla + z\circ$ is flat and torsionless for any $z \in \mathbb{C}$, and $\nabla e = 0$.*

For a flat F-manifold $(M, \nabla, \circ)$ consider flat coordinates t^α, $1 \le \alpha \le N$, $N = \dim M$, for the connection ∇ such that $e = \frac{\partial}{\partial t^1}$. Then locally there exist analytic functions $F^\alpha(t^1, \dots, t^N)$, $1 \le \alpha \le N$, such that the second derivatives

$$c^\alpha_{\beta\gamma} = \frac{\partial^2 F^\alpha}{\partial t^\beta \partial t^\gamma} \tag{2.1}$$

give the structure constants for the multiplication $\circ$,

$$\frac{\partial}{\partial t^\beta} \circ \frac{\partial}{\partial t^\gamma} = c^\alpha_{\beta\gamma} \frac{\partial}{\partial t^\alpha}.$$

From the associativity of the multiplication and the fact that the vector field $\frac{\partial}{\partial t^1}$ is the unit it follows that

$$\frac{\partial^2 F^\alpha}{\partial t^1 \partial t^\beta} = \delta^\alpha_\beta, \qquad 1 \le \alpha, \beta \le N, \tag{2.2}$$

$$\frac{\partial^2 F^\alpha}{\partial t^\beta \partial t^\mu} \frac{\partial^2 F^\mu}{\partial t^\gamma \partial t^\delta} = \frac{\partial^2 F^\alpha}{\partial t^\gamma \partial t^\mu} \frac{\partial^2 F^\mu}{\partial t^\beta \partial t^\delta}, \qquad 1 \le \alpha, \beta, \gamma, \delta \le N. \tag{2.3}$$

The N-tuple of functions $(F^1, \dots, F^N)$ is called the *vector potential* of our flat F-manifold.

Conversely, if M is an open subset of $\mathbb{C}^N$ and $F^1, \dots, F^N \in \mathcal{O}(M)$ are functions satisfying equations (2.2) and (2.3), then these functions define a flat F-manifold $(M, \nabla, \circ)$ with the connection ∇, given by $\nabla_{\frac{\partial}{\partial t^\alpha}} \frac{\partial}{\partial t^\beta} = 0$, and the multiplication $\circ$, given by the structure constants (2.1).

Remark 2.2. *Let $M \subset \mathbb{C}^N$ be an open subset in the Zariski topology. The tangent spaces T_pM can be endowed with an algebra structure, algebraically depending on the point $p \in M$, using the following construction. Denote by $\mathcal{O}^{\mathrm{alg}}$ the sheaf of algebraic functions on M. Let R be an $\mathcal{O}^{\mathrm{alg}}(M)$-algebra, which is free as an $\mathcal{O}^{\mathrm{alg}}(M)$-module with a basis $\phi_1, \dots, \phi_N \in R$. Denote by $v_1, \dots, v_N$ the standard coordinates on $\mathbb{C}^N$ and by $\mathcal{T}^{\mathrm{alg}}_M$ the algebraic tangent sheaf of M. Define an isomorphism of $\mathcal{O}^{\mathrm{alg}}(M)$-modules $\Psi\colon \mathcal{T}^{\mathrm{alg}}_M(M) \to R$ by $\Psi(\frac{\partial}{\partial v_i}) := \phi_i$. Thus, the sheaf $\mathcal{T}^{\mathrm{alg}}_M$ becomes a sheaf of $\mathcal{O}^{\mathrm{alg}}$-algebras that endows the tangent spaces T_pM with an algebra structure algebraically depending on the point $p \in M$.*

Remark 2.3. *Consider an analytic manifold M with an algebra structure $(T_pM, \circ)$ on each tangent space analytically depending on the point $p \in M$. We see that a connection ∇, endowing our manifold M with a flat F-manifold structure, can be completely described by a choice of*

coordinates $t^1, \dots, t^N$ on M such that the structure constants $c^\alpha_{\beta\gamma}$ of multiplication in these coordinates satisfy the integrability condition

$$\frac{\partial c^\alpha_{\beta\gamma}}{\partial t^\sigma} = \frac{\partial c^\alpha_{\beta\sigma}}{\partial t^\gamma}$$

together with the condition $c^\alpha_{1,\beta} = \delta^\alpha_\beta$. In this paper we will construct flat F-manifolds exactly by presenting flat coordinates as above.

A flat F-manifold $(M, \nabla, \circ)$ is called *conformal*, if it is equipped with a vector field E, called the *Euler vector field*, such that $\nabla\nabla E = 0$, $[e, E] = e$ and $\mathcal{L}_E(\circ) = \circ$. This means that in the flat coordinates the Euler vector field E has the form

$$E = \underbrace{(q^\alpha_\beta t^\beta + r^\alpha)}_{=:E^\alpha} \frac{\partial}{\partial t^\alpha}, \quad q^\alpha_\beta, r^\alpha \in \mathbb{C}, \quad q^\alpha_1 = \delta^\alpha_1,$$

and the vector potential $(F^1, \dots, F^N)$ satisfies the condition

$$E^\mu \frac{\partial F^\alpha}{\partial t^\mu} = q^\alpha_\beta F^\beta + F^\alpha + A^\alpha_\beta t^\beta + B^\alpha, \quad \text{for some } A^\alpha_\beta, B^\alpha \in \mathbb{C}. \tag{2.4}$$

A point $p \in M$ of an N-dimensional flat F-manifold $(M, \nabla, \circ)$ is called *semisimple* if T_pM has a basis of idempotents $\pi_1, \dots, \pi_N$ satisfying $\pi_k \circ \pi_l = \delta_{k,l}\pi_k$. Moreover, locally around such a point one can choose coordinates u^i such that $\frac{\partial}{\partial u^k} \circ \frac{\partial}{\partial u^l} = \delta_{k,l}\frac{\partial}{\partial u^k}$. These coordinates are called the *canonical coordinates*. In particular, this means that the semisimplicity is an open property. The flat F-manifold $(M, \nabla, \circ)$ is called semisimple, if a generic point of M is semisimple.

2.2. **Frobenius manifolds.** For a complex analytic manifold M we denote by $\mathcal{T}_M$ the analytic tangent sheaf of M.

Definition 2.4. *A flat F-manifold $(M, \nabla, \circ)$ is called a Frobenius manifold if the tangent spaces T_pM are equipped with a symmetric non-degenerate bilinear form η analytically depending on the point $p \in M$ such that $\nabla\eta = 0$ and for any $X, Y, Z \in \mathcal{T}_M$ the following condition is satisfied:*

$$\eta(X \circ Y, Z) = \eta(X, Y \circ Z).$$

The connection ∇ is then the Levi-Civita connection associated to the form η. A Frobenius manifold will be denoted by the triple $(M, \eta, \circ)$. The form η is traditionally called a metric.

Let $(M, \eta, \circ)$ be a Frobenius manifold and consider the flat coordinates $t^1, \dots, t^N$ of the metric η and the vector potential $(F^1, \dots, F^N)$. Then locally there exists an analytic function F such that $F^\alpha = \eta^{\alpha\beta}\frac{\partial F}{\partial t^\beta}$ and $\frac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta}$, where $(\eta_{\alpha\beta})$ is the matrix of the form η in the coordinates $t^1, \dots, t^N$. The function F satisfies the WDVV equations (1.1) and is called the Frobenius manifold potential.

A Frobenius manifold $(M, \eta, \circ)$ is called conformal if the corresponding flat F-manifold is conformal and the metric η satisfies the condition

$$\mathcal{L}_E\eta = (2 - \delta)\eta, \quad \text{for some } \delta \in \mathbb{C},$$

where $\mathcal{L}_E$ denotes the Lie derivative. The number δ is then called the conformal dimension of the Frobenius manifold. The Frobenius manifold potential F satisfies then the condition

$$E(F) = (3 - \delta)F + \frac{1}{2}A_{\alpha\beta}t^\alpha t^\beta + B_\alpha t^\alpha + C, \quad \text{for some } A_{\alpha\beta}, B_\alpha, C \in \mathbb{C}.$$

In the theory of Frobenius manifolds it is typically assumed that one can choose flat coordinates such that the matrix $\left(\frac{\partial E^\alpha}{\partial t^\beta}\right)$ is diagonal and so the Euler vector field has form (1.6).

The papers [Dub96, Dub99] contain a systematic study of the theory of Frobenius manifolds.

2.3. **Extensions of flat F-manifolds and the open WDVV equations.** Consider a flat F-manifold structure, given by a vector potential $(F^1, \dots, F^{N+1})$ on an open subset $M \times U \in \mathbb{C}^{N+1}$, where M and U are open subsets of $\mathbb{C}^N$ and $\mathbb{C}$, respectively. Suppose that the functions $F^1, \dots, F^N$ don't depend on the variable t^{N+1}, varying in U. Then the functions $F^1, \dots, F^N$ satisfy equations (2.3) and, thus, define a flat F-manifold structure on M. In this case we call the flat F-manifold structure on $M \times U$ an *extension* of the flat F-manifold structure on M.

Consider the flat F-manifold associated to a Frobenius manifold, given by a potential $F(t^1, \dots, t^N) \in \mathcal{O}(M)$ and a metric η, $F^\alpha = \eta^{\alpha\mu}\frac{\partial F}{\partial t^\mu}$, $1 \le \alpha \le N$. It is easy to check that a function $F^o(t^1, \dots, t^N, s) \in \mathcal{O}(M \times U)$ satisfies equations (1.3), (1.4) and (1.5) if and only if the $(N+1)$-tuple $(F^1, \dots, F^N, F^o)$ is a vector potential of a flat F-manifold. Here we identify $s = t^{N+1}$. This defines a correspondence between solutions of the open WDVV equations, satisfying property (1.5), and flat F-manifolds, extending the Frobenius manifold given. This observation belongs to Paolo Rossi.

3. Saito Frobenius manifolds

In this section we recall the Dubrovin–Saito construction of a Frobenius manifold structure on the parameter space of a universal unfolding of a simple singularity.

Let us first recall the list of polynomials defining simple singularities:

$$\begin{aligned}
f_{A_N}(x,y) =& \frac{x^{N+1}}{N+1} + y^2, && N \ge 1,\\
f_{D_N}(x,y) =& \frac{x^{N-1}}{N-1} + xy^2, && N \ge 4,\\
f_{E_6}(x,y) =& x^4 + y^3,\\
f_{E_7}(x,y) =& x^3y + y^3,\\
f_{E_8}(x,y) =& x^5 + y^3.
\end{aligned}$$

The associated *local algebra* is defined by

$$\mathcal{A}_W := \mathbb{C}[x,y] \Big/ \left(\frac{\partial f_W}{\partial x}, \frac{\partial f_W}{\partial y} \right),$$

where W is one of the types A_N, D_N or E_N. Because $x = y = 0$ is an isolated critical point of f_W, the local algebra $\mathcal{A}_W$ turns out to be a finite-dimensional vector space. Denote $\dim \mathcal{A}_W =: N$. A *universal unfolding* of the singularity of f_W is a function $\Lambda_W \colon \mathbb{C}^2 \times \mathbb{C}^N \to \mathbb{C}$ of the form

$$\Lambda_W(x, y, v_1, \dots, v_N) = f_W + \sum_{i=1}^{N} v_i \phi_i, \quad \phi_i \in \mathbb{C}[x,y],$$

such that the classes of polynomials $\phi_1, \dots, \phi_N$ form a basis of the local algebra $\mathcal{A}_W$. Explicitly, universal unfoldings of the ADE singularities are given by

$$\begin{aligned}
\Lambda_{A_N} &= \frac{x^{N+1}}{N+1} + y^2 + \sum_{k=1}^{N} v_k x^{k-1},\\
\Lambda_{D_N} &= \frac{x^{N-1}}{N-1} + xy^2 + \sum_{k=1}^{N-1} v_k x^{k-1} + v_N y,\\
\Lambda_{E_6} &= x^4 + y^3 + v_1 + v_2 x + v_3 y + v_4 x^2 + v_5 xy + v_6 x^2 y,\\
\Lambda_{E_7} &= x^3 y + y^3 + v_1 + v_2 x + v_3 y + v_4 x^2 + v_5 xy + v_6 x^3 + v_7 x^4,\\
\Lambda_{E_8} &= x^5 + y^3 + v_1 + v_2 x + v_3 y + v_4 x^2 + v_5 xy + v_6 x^3 + v_7 x^2 y + v_8 x^3 y.
\end{aligned}$$

Consider the quotient ring

$$\widehat{\mathcal{A}}_W := \mathbb{C}[x, y, v_1, \dots, v_N] / \left(\partial_x \Lambda_W, \partial_y \Lambda_W\right).$$

As a $\mathbb{C}[v_1, \dots, v_N]$-module, the space $\widehat{\mathcal{A}}_W$ has dimension N with a basis given by the classes $[\phi_1], \dots, [\phi_N] \in \widehat{\mathcal{A}}_W$ of the polynomials $\phi_1, \dots, \phi_N$. Identifying the $\mathbb{C}[v_1, \dots, v_N]$-modules $\mathcal{T}^{\mathrm{alg}}_{\mathbb{C}^N}(\mathbb{C}^N)$ and $\widehat{\mathcal{A}}_W$ via the isomorphism Ψ_W defined by

$$\Psi_W\left(\frac{\partial}{\partial v_k}\right) := [\phi_k], \quad 1 \le k \le N,$$

by Remark 2.2, we endow the tangent spaces $T_p\mathbb{C}^N$ with a multiplication. The structure constants of it are polynomials in the coordinates $v_1, \dots, v_N$.

A flat metric can be introduced as follows. It is easy to see that there exist unique positive rational numbers $q_x, q_y, q_1, \dots, q_N$ such that

$$q_x x \frac{\partial \Lambda_W}{\partial x} + q_y y \frac{\partial \Lambda_W}{\partial y} + \sum_{k=1}^{N} q_k v_k \frac{\partial \Lambda_W}{\partial v_k} = \Lambda_W.$$

There is a unique index $1 \le l \le N$ such that the number q_l is minimal. For the singularities A_N and E_N we have $l = N$ and in the D_N-case we have $l = N - 1$. Denote by $(c_v)^k_{i,j}$ the structure constants of multiplication on $\mathbb{C}^N$ in the coordinates $v_1, \dots, v_N$. Define a bilinear form η_W on $\mathbb{C}^N$ by

$$\eta_W\left(\frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_j}\right) := (c_v)^l_{i,j}.$$

This bilinear form together with the multiplication, introduced above, define a Frobenius manifold structure on $\mathbb{C}^N$, often called the Saito Frobenius manifold. It is semisimple and conformal with an Euler vector field E_W given by

$$E_W = \sum_{k=1}^{N} q_k v_k \frac{\partial}{\partial v_k}.$$

The conformal dimension is $\delta = 1 - q_l$.

Remark 3.1. *The coordinates $v_1, \dots, v_N$ are not flat whenever $N \ge 3$.*

There exist unique *global* flat coordinates $t^i(v_1, \dots, v_N)$ on $\mathbb{C}^N$ such that

$$t^i(v_1, \dots, v_N) = v_i + O(v_*^2).$$

They satisfy the quasi-homogeneity condition

$$E_W(t^i(v_1, \dots, v_N)) = q_i t^i(v_1, \dots, v_N), \tag{3.1}$$

and, hence, the Euler vector field in the flat coordinates t^i is given by

$$E_W = \sum_{i=1}^{N} q_i t^i \frac{\partial}{\partial t^i}.$$

The Frobenius manifold structure in the flat coordinates t^i is described by a polynomial potential $F_W(t^1, \dots, t^N) \in \mathbb{C}[t^1, \dots, t^N]$, which we fix by requiring that it doesn't contain monomials of degree less than 3. Then the polynomial F_W satisfies the condition

$$E_W(F_W) = (3 - \delta) F_W. \tag{3.2}$$

Explicit formulas for the flat coordinates of the Saito Frobenius manifolds of simple singularities are given in [NY98]. For the A_N-case the formula is

$$t^\gamma(v_1, \dots, v_N) = \sum_{\substack{\alpha_1, \dots, \alpha_N \ge 0 \\ \sum (N+2-i)\alpha_i = N+2-\gamma}} \frac{1}{N+1-\gamma} \prod_{k=0}^{\sum \alpha_i - 1} (N + 1 - \gamma - k(N+1)) \frac{\prod v_i^{\alpha_i}}{\prod \alpha_i!}, \quad 1 \le \gamma \le N.$$

For the D_N-case the formula is

(3.3)

$$t^\gamma(v_1,\dots,v_N)=\sum_{\substack{\alpha_1,\dots,\alpha_{N-1}\geq 0\\ \sum(N-i)\alpha_i=N-\gamma}}\left(-\frac{1}{2}\right)^{\sum\alpha_i-1}\prod_{k=0}^{\sum\alpha_i-2}(2\gamma-1+2k(N-1))\,\frac{\prod v_i^{\alpha_i}}{\prod\alpha_i!},\quad 1\leq\gamma\leq N-1,$$

$$t^N(v_1,\dots,v_N)=v_N.$$

Note that in this case the coordinates $t^1,\dots,t^{N-1}$ depend only on $v_1,\dots,v_{N-1}$.

4. Extended r-spin theory and the open WDVV equations for the A-singularity

Here we explain how the Saito Frobenius manifold of the A-singularity together with a certain solution of the open WDVV equations appear in the intersection theory on the moduli spaces of algebraic curves.

Let $r=N+1$. For integers $0\leq\alpha_1,\dots,\alpha_n\leq r-1$, using the geometry of algebraic curves with an r-spin structure, one can construct a cohomology class

$$W^r_{0,n}(\alpha_1,\dots,\alpha_n)\in H^{2d}(\overline{\mathcal{M}}_{0,n},\mathbb{Q}),\quad d=\frac{\sum\alpha_i-(r-2)}{r},$$

called *Witten's class*, on the moduli space $\overline{\mathcal{M}}_{0,n}$ of stable curves of genus 0 with n marked points (see e.g. [PPZ19]). Here we assume that the class $W^r_{0,n}(\alpha_1,\dots,\alpha_n)$ is equal to zero, if d is not an integer or $d<0$. The class $W^r_{0,n}(\alpha_1,\dots,\alpha_n)$ vanishes, if one of the α_i's is equal to $r-1$. Consider the generating series

$$F_{r\text{-spin}}(t^1,\dots,t^{r-1}):=\sum_{n\geq 3}\frac{1}{n!}\sum_{0\leq\alpha_1,\dots,\alpha_n\leq r-2}\left(\int_{\overline{\mathcal{M}}_{0,n}}W^r_{0,n}(\alpha_1,\dots,\alpha_n)\right)\prod_{i=1}^n t^{\alpha_i+1}.$$

The functions F_{A_N} and $F_{r\text{-spin}}$ are related by [JKV01a]

$$F_{A_N}(t^1,\dots,t^N)=(-r)^{-3}F_{r\text{-spin}}((-r)t^1,\dots,(-r)t^N).$$

This is one of the simplest cases of mirror symmetry. Denote

$$\langle\tau_{\alpha_1}\dots\tau_{\alpha_n}\rangle_{A_N}:=\left.\frac{\partial^n F_{A_N}}{\partial t^{\alpha_1}\dots\partial t^{\alpha_n}}\right|_{t^*=0},\quad 1\leq\alpha_1,\dots,\alpha_n\leq N.$$

We have (see e.g. [LVX17, page 4])

$$\langle\tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_3}\rangle_{A_N}=\delta_{\alpha_1+\alpha_2+\alpha_3,N+2},\qquad \langle\tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_3}\tau_{\alpha_4}\rangle_{A_N}=-\min(\alpha_i-1,N+1-\alpha_i).\tag{4.1}$$

These formulas will be used later.

In [JKV01b] the authors noticed that the construction of Witten's class $W^r_{0,n}(\alpha_1,\dots,\alpha_n)$ can be extended to the case when $\alpha_1=-1$ and $0\leq\alpha_2,\dots,\alpha_n\leq r-1$. In [BCT19] the authors considered the generating series

$$F^{\text{ext}}_{r\text{-spin}}(t^1,\dots,t^r):=\sum_{n\geq 2}\frac{1}{n!}\sum_{0\leq\alpha_1,\dots,\alpha_n\leq r-1}\left(\int_{\overline{\mathcal{M}}_{0,n+1}}W^r_{0,n+1}(-1,\alpha_1,\dots,\alpha_n)\right)\prod_{i=1}^n t^{\alpha_i+1}$$

and proved that it satisfies the open WDVV equations, associated to the potential $F_{r\text{-spin}}$, together with property (1.5). Here one should identify $t^r=s$. It occurs that after a simple transformation the function $F^{\text{ext}}_{r\text{-spin}}$ also gives the generating series of intersection numbers on the moduli space of r-spin disks, introduced in [BCT18, Theorem 1.3].

Introduce a function $F^o_{A_N}(t^1,\dots,t^N,s)$ by

$$F^o_{A_N}(t^1,\dots,t^N,s):=(-r)^{-2}F^{\text{ext}}_{r\text{-spin}}((-r)t^1,\dots,(-r)t^N,(-r)s).$$

Clearly, it satisfies the open WDVV equations, associated to F_{A_N}, together with condition (1.5). In [BCT18, Proposition 5.1] the authors found an explicit formula for the coefficients of $F^{\text{ext}}_{r\text{-spin}}$ that gives

$$\left\langle \tau_{\alpha_1} \dots \tau_{\alpha_n} \sigma^k \right\rangle^o_{A_N} = \begin{cases} (n+k-2)!, & \text{if } \sum_{i=1}^n (N+2-\alpha_i) + k = N+2, \\ 0, & \text{otherwise,} \end{cases}$$

where we use the notation

$$\left\langle \tau_{\alpha_1} \dots \tau_{\alpha_n} \sigma^k \right\rangle^o_{A_N} := \left. \frac{\partial^{n+k} F^o_{A_N}}{\partial t^{\alpha_1} \dots \partial t^{\alpha_n} \partial s^k} \right|_{t^*=s=0}.$$

The first examples read[1]:

$$\begin{aligned}
F_{A_2} &= \frac{t_1^2 t_2}{2} - \frac{t_2^4}{24}, \\
F^o_{A_2} &= \frac{s^4}{12} + \frac{t_2 s^2}{2} + t_1 s + \frac{t_2^2}{2}; \\
F_{A_3} &= \frac{t_1^2 t_3}{2} + \frac{t_1 t_2^2}{2} - \frac{t_2^2 t_3^2}{4} + \frac{t_3^5}{60}, \\
F^o_{A_3} &= \frac{s^5}{20} + \frac{t_3 s^3}{3} + \frac{t_2 s^2}{2} + \left(t_1 + \frac{t_3^2}{2} \right) s + t_2 t_3; \\
F_{A_4} &= \frac{t_1^2 t_4}{2} + t_1 t_2 t_3 + \frac{t_2^3}{6} - \frac{t_2^2 t_4^2}{4} - \frac{t_2 t_3^2 t_4}{2} - \frac{t_3^4}{12} + \frac{t_3^2 t_4^3}{6} - \frac{t_4^6}{120}, \\
F^o_{A_4} &= \frac{s^6}{30} + \frac{t_4 s^4}{4} + \frac{t_3 s^3}{3} + \left(\frac{t_4^2}{2} + \frac{t_2}{2} \right) s^2 + (t_1 + t_3 t_4) s + \frac{t_4^3}{6} + \frac{t_3^2}{2} + t_2 t_4.
\end{aligned}$$

In [Bur18] the author proved that the coefficients of the function $F^o_{A_N}$ are related to the expression of the coordinates v_k in the terms of the flat coordinates $t^1, \dots, t^N$ on the Saito Frobenius manifold of the singularity A_N by

$$\frac{\partial F^o_{A_N}}{\partial s} = \frac{s^{N+1}}{N+1} + \sum_{k=1}^N s^{k-1} v_k(t^1, \dots, t^N). \tag{4.2}$$

5. Generalized Dubrovin–Saito construction for the singularities A and D

In this section, for the singularites of types A and D we present a generalization of the Dubrovin–Saito construction that produces a flat F-manifold that extends the Saito Frobenius manifold and, therefore, gives a solution of the open WDVV equations. In the A_N-case this solution coincides with the function $F^o_{A_N}$. In both A- and D-cases, the coefficients of powers of the variable s in this solution coincide with the transition functions between two coordinate systems on the Saito Frobenius manifold.

5.1. **A_N-case.** Consider the space $M^{\text{ext}}_{A_N} := \mathbb{C}^{N+1}$ with coordinates $v_1, \dots, v_{N+1}$. Consider the quotient ring

$$\widehat{\mathcal{A}}^{\text{ext}}_{A_N} := \mathbb{C}[x, y, w, v_1, \dots, v_{N+1}] \Big/ \left(w - \partial_x \Lambda_{A_N}, \partial_y \Lambda_{A_N}, wx - v_{N+1} w \right).$$

As a $\mathbb{C}[v_1, \dots, v_{N+1}]$-module, the space $\widehat{\mathcal{A}}^{\text{ext}}_{A_N}$ is free of dimension $N+1$, and the elements $[1], [x], \dots, [x^{N-1}], [w]$ form a basis. To show that any other element $[x^i y^j w^k]$ can be expressed in terms of them, first note that, obviously, $[y] = 0$ and $[wx] = v_{N+1}[w]$. We also see that

$$[w^2] = \left[w \left(x^N + \sum_{k=2}^N (k-1) v_k x^{k-2} \right) \right] = \left(v_{N+1}^N + \sum_{k=2}^N (k-1) v_k v_{N+1}^{k-2} \right) [w]. \tag{5.1}$$

[1]We follow the convention of B. Dubrovin and use variables with lower indices for the flat coordinates in particular examples.

Using the relation $[x^N] = [w] - \sum_{k=2}^{N}(k-1)v_k[x^{k-2}]$, we can express any element $[x^p]$ with $p \geq N$ in terms of the elements $[1], [x], \ldots, [x^{N-1}], [w]$.

Identifying the $\mathbb{C}[v_1, \ldots, v_{N+1}]$-modules $\mathcal{T}^{\mathrm{alg}}_{M^{\mathrm{ext}}_{A_N}}(M^{\mathrm{ext}}_{A_N})$ and $\widehat{\mathcal{A}}^{\mathrm{ext}}_{A_N}$ via the isomorphism $\Psi^{\mathrm{ext}}_{A_N}$ defined by

$$\Psi^{\mathrm{ext}}_{W}\left(\frac{\partial}{\partial v_k}\right) := [x^{k-1}], \quad 1 \leq k \leq N, \qquad \Psi^{\mathrm{ext}}_{W}\left(\frac{\partial}{\partial v_{N+1}}\right) := [w],$$

by Remark 2.2, we endow the tangent spaces $T_pM^{\mathrm{ext}}_{A_N}$ with a multiplication and, clearly, the structure constants of it are polynomials in the coordinates $v_1, \ldots, v_{N+1}$.

Consider the flat coordinates $t^\alpha = t^\alpha(v_1, \ldots, v_N)$, $1 \leq \alpha \leq N$, the potential $F_{A_N}(t^1, \ldots, t^N)$ of the Frobenius manifold of the singularity A_N and the function $F^o_{A_N}$, described in Section 4.

Theorem 5.1.
1. The coordinates $t^1(v_1, \ldots, v_N), \ldots, t^N(v_1, \ldots, v_N)$ and $t^{N+1} := v_{N+1}$ together with the multiplicative structure on $M^{\mathrm{ext}}_{A_N}$, constructed above, define a flat F-manifold structure on $M^{\mathrm{ext}}_{A_N}$.
2. The vector potential of this flat F-manifold is given by $\left(\eta^{1\alpha}_{A_N}\frac{\partial F_{A_N}}{\partial t^\alpha}, \ldots, \eta^{N\alpha}_{A_N}\frac{\partial F_{A_N}}{\partial t^\alpha}, F^o_{A_N}\right)$, where we identify $s = t^{N+1}$.

Proof. We denote by $(c^{\mathrm{ext}})^\alpha_{\beta\gamma}$ the structure constants of multiplication in the coordinates $t^1, \ldots, t^{N+1}$ and by $(c^{\mathrm{ext}}_v)^\alpha_{\beta\gamma}$ the structure constants of multiplication in the coordinates $v_1, \ldots, v_{N+1}$.

In order to prove the theorem, we have to check the following equations:

$$(c^{\mathrm{ext}})^\alpha_{\beta\gamma} = \sum_{\mu=1}^{N}\eta^{\alpha\mu}_{A_N}\frac{\partial^3 F_{A_N}}{\partial t^\mu \partial t^\beta \partial t^\gamma}, \qquad 1 \leq \alpha \leq N, \qquad 1 \leq \beta, \gamma \leq N+1, \tag{5.2}$$

$$(c^{\mathrm{ext}})^{N+1}_{\alpha\beta} = \frac{\partial^2 F^o_{A_N}}{\partial t^\alpha \partial t^\beta}, \qquad 1 \leq \alpha, \beta \leq N+1. \tag{5.3}$$

Since the subspace $\mathbb{C}[v_1, \ldots, v_{N+1}]\langle[w]\rangle$ is an ideal in the ring $\widehat{\mathcal{A}}^{\mathrm{ext}}_{A_N}$ and the quotient by this ideal coincides with the ring $\widehat{\mathcal{A}}_{A_N}$, we have

$$(c^{\mathrm{ext}}_v)^a_{b,c} = \begin{cases} (c_v)^a_{b,c}, & \text{if } 1 \leq a, b, c \leq N, \\ 0, & \text{if } 1 \leq a \leq N \text{ and one of the indices } b, c \text{ is equal to } N+1. \end{cases}$$

This implies equation (5.2) and it remains to prove (5.3).

Suppose $1 \leq \alpha \leq N$ and $\beta = N+1$. Since $[xw] = v_{N+1}[w]$, we have $(c^{\mathrm{ext}}_v)^{N+1}_{k,N+1} = v^{k-1}_{N+1}$ for $1 \leq k \leq N$, and, therefore,

$$(c^{\mathrm{ext}})^{N+1}_{\alpha,N+1} = \sum_{k=1}^{N}\frac{\partial v_k}{\partial t^\alpha}(c^{\mathrm{ext}}_v)^{N+1}_{k,N+1} = \sum_{k=1}^{N}\frac{\partial v_k}{\partial t^\alpha}v^{k-1}_{N+1} \overset{\text{eq.(4.2)}}{=} \frac{\partial^2 F^o_{A_N}}{\partial t^\alpha \partial t^{N+1}}.$$

Suppose $\alpha = \beta = N+1$. Then we compute

$$(c^{\mathrm{ext}})^{N+1}_{N+1,N+1} = (c^{\mathrm{ext}}_v)^{N+1}_{N+1,N+1} \overset{\text{eq.(5.1)}}{=} v^N_{N+1} + \sum_{k=2}^{N}(k-1)v_k v^{k-2}_{N+1} \overset{\text{eq.(4.2)}}{=} \frac{\partial^2 F^o_{A_N}}{\partial (t^{N+1})^2}.$$

Finally, if $1 \leq \alpha, \beta \leq N$, then from the associativity of the algebra $\widehat{\mathcal{A}}^{\mathrm{ext}}_{A_N}$ we get

$$(c^{\mathrm{ext}})^{N+1}_{\alpha\beta}(c^{\mathrm{ext}})^{N+1}_{N+1,N+1} = (c^{\mathrm{ext}})^{N+1}_{\alpha,N+1}(c^{\mathrm{ext}})^{N+1}_{\beta,N+1} - \sum_{\mu=1}^{N}c^\mu_{\alpha\beta}(c^{\mathrm{ext}})^{N+1}_{\mu,N+1}.$$

Since the function $F^o_{A_N}$ satisfies (1.4) and, as we have just proved, $(c^{\mathrm{ext}})^{N+1}_{\gamma,N+1} = \frac{\partial^2 F^o_{A_N}}{\partial t^\gamma \partial t^{N+1}}$ for $1 \leq \gamma \leq N+1$, we obtain $(c^{\mathrm{ext}})^{N+1}_{\alpha\beta} = \frac{\partial^2 F^o_{A_N}}{\partial t^\alpha \partial t^\beta}$. □

5.2. **D_N-case.** Consider the space $M^{\rm ext}_{D_N} := \mathbb{C}^N \times \mathbb{C}^*$ with coordinates $v_1, \dots, v_{N+1}$. Consider the quotient ring

$$\widehat{\mathcal{A}}^{\rm ext}_{D_N} := \mathbb{C}[x, y, w, v_1, \dots, v_N, v_{N+1}, v_{N+1}^{-1}] \Big/ \left(w - v_{N+1}\partial_x \Lambda_{D_N}, \partial_y \Lambda_{D_N}, 2wx - v_{N+1}^2 w\right).$$

As a $\mathbb{C}[v_1, \dots, v_N, v_{N+1}, v_{N+1}^{-1}]$-module, the space $\widehat{\mathcal{A}}^{\rm ext}_{D_N}$ is free of dimension $N+1$ with a basis $[1], [x], \dots, [x^{N-2}], [y], [w]$. To show that any other element $[x^i y^j w^k]$ can be expressed in terms of them, first note that

$$[xy] = -\frac{v_N}{2}[1], \qquad [wx] = \frac{v_{N+1}^2}{2}[w], \qquad [wy] = -\frac{v_N}{v_{N+1}^2}[w], \tag{5.4}$$

where the last equation follows from the first two. Similarly to the A_N-case, we have

$$[w^2] = \left(\frac{v_{N+1}^{2N-3}}{2^{N-2}} + \frac{v_N^2}{v_{N+1}^3} + \sum_{k=2}^{N-1}(k-1)v_k \frac{v_{N+1}^{2k-3}}{2^{k-2}}\right)[w]. \tag{5.5}$$

Using that $[w - v_{N+1}\partial_x \Lambda_{D_N}] = 0$, we obtain

$$[y^2] = \frac{1}{v_{N+1}}[w] - [x^{N-2}] - \sum_{k=2}^{N-1}(k-1)v_k[x^{k-2}]. \tag{5.6}$$

Multiplying this equation by $[x]$, we get the relation

$$[x^{N-1}] = \frac{v_{N+1}}{2}[w] + \frac{v_N}{2}[y] - \sum_{k=1}^{N-1}(k-1)v_k[x^{k-1}], \tag{5.7}$$

that allows to express any element $[x^p]$ with $p \geq N-1$ in terms of the elements $[1], [x], \dots, [x^{N-2}]$, $[y]$, $[w]$.

Identifying the $\mathbb{C}[v_1, \dots, v_N, v_{N+1}, v_{N+1}^{-1}]$-modules $\mathcal{T}^{\rm alg}_{M^{\rm ext}_{D_N}}(M^{\rm ext}_{D_N})$ and $\widehat{\mathcal{A}}^{\rm ext}_{D_N}$ via the isomorphism $\Psi^{\rm ext}_{D_N}$ defined by

$$\Psi^{\rm ext}_W\left(\frac{\partial}{\partial v_k}\right) := [x^{k-1}], \quad 1 \leq k \leq N-1, \qquad \Psi^{\rm ext}_W\left(\frac{\partial}{\partial v_N}\right) := [y], \qquad \Psi^{\rm ext}_W\left(\frac{\partial}{\partial v_{N+1}}\right) := [w],$$

we endow the tangent spaces $T_p M^{\rm ext}_{D_N}$ with a multiplication and, clearly, the structure constants of it belong to the ring $\mathbb{C}[v_1, \dots, v_N, v_{N+1}, v_{N+1}^{-1}]$.

Consider the flat coordinates $t^\alpha = t^\alpha(v_1, \dots, v_N)$, $1 \leq \alpha \leq N$, and the potential $F_{D_N}(t^1, \dots, t^N)$ of the Frobenius manifold of the singularity D_N. Let $t^{N+1} := v_{N+1}$ and define a function $F^o_{D_N}(t^1, \dots, t^{N+1})$ by

$$F^o_{D_N} := \left(\sum_{k=1}^{N-1} \frac{v_k v_{N+1}^{2k-1}}{2^{k-1}(2k-1)} + \frac{v_{N+1}^{2N-1}}{2^{N-2}(2N-1)(2N-2)} + \frac{v_N^2}{2v_{N+1}}\right)\Bigg|_{v_i = v_i(t^*)}.$$

Theorem 5.2.
1. The coordinates $t^1(v_1, \dots, v_N), \dots, t^N(v_1, \dots, v_N)$ and $t^{N+1} = v_{N+1}$ together with the multiplicative structure on $M^{\rm ext}_{D_N}$, constructed above, define a flat F-manifold structure on $M^{\rm ext}_{D_N}$.
2. The vector potential of this flat F-manifold is given by $\left(\eta^{1\alpha}_{D_N}\frac{\partial F_{D_N}}{\partial t^\alpha}, \dots, \eta^{N\alpha}_{D_N}\frac{\partial F_{D_N}}{\partial t^\alpha}, F^o_{D_N}\right)$.

Proof. We denote by $(c^{\rm ext})^\alpha_{\beta\gamma}$ the structure constants of multiplication in the coordinates $t^1, \dots, t^{N+1}$ and by $(c^{\rm ext}_v)^\alpha_{\beta\gamma}$ the structure constants of multiplication in the coordinates $v_1, \dots, v_{N+1}$.

In order to prove the theorem, we have to check the following equations:

$$(c^{\rm ext})^\alpha_{\beta\gamma} = \sum_{\mu=1}^{N} \eta^{\alpha\mu}_{D_N} \frac{\partial^3 F_{D_N}}{\partial t^\mu \partial t^\beta \partial t^\gamma}, \qquad 1 \leq \alpha \leq N, \qquad 1 \leq \beta, \gamma \leq N+1, \tag{5.8}$$

$$(c^{\rm ext})^{N+1}_{\alpha\beta} = \frac{\partial^2 F^o_{D_N}}{\partial t^\alpha \partial t^\beta}, \qquad 1 \leq \alpha, \beta \leq N+1. \tag{5.9}$$

Since the subspace $\mathbb{C}[v_1,\dots,v_N,v_{N+1},v_{N+1}^{-1}]\langle[w]\rangle$ is an ideal in the ring $\widehat{\mathcal{A}}^{\mathrm{ext}}_{D_N}$ and the quotient by this ideal coincides with the ring $\widehat{\mathcal{A}}_{D_N}$, we have

$$(c_v^{\mathrm{ext}})^a_{b,c}=\begin{cases}(c_v)^a_{b,c}, & \text{if } 1\le a,b,c\le N,\\ 0, & \text{if } 1\le a\le N \text{ and one of the indices } b,c \text{ is equal to } N+1.\end{cases}$$

This implies equation (5.8) and it remains to prove (5.9).

We have two substantially different cases: the case $\alpha\in\{N,N+1\}$ or $\beta\in\{N,N+1\}$ and the case $\alpha,\beta\in\{1,2,\dots,N-1\}$.

Case $\alpha\in\{N,N+1\}$ or $\beta\in\{N,N+1\}$. If $\alpha,\beta\in\{N,N+1\}$, then $(c^{\mathrm{ext}})^{N+1}_{\alpha,\beta}=(c_v^{\mathrm{ext}})^{N+1}_{\alpha,\beta}$ and $\frac{\partial^2 F^o_{D_N}}{\partial t^\alpha\partial t^\beta}=\frac{\partial^2 F^o_{D_N}}{\partial v_\alpha\partial v_\beta}$. The equation $(c_v^{\mathrm{ext}})^{N+1}_{\alpha,\beta}=\frac{\partial^2 F^o_{D_N}}{\partial v_\alpha\partial v_\beta}$ immediately follows from formulas (5.4), (5.5) and (5.6).

If $1\le\alpha\le N-1$ and $\beta=N+1$, then

$$(c^{\mathrm{ext}})^{N+1}_{\alpha,N+1}=\sum_{k=1}^{N-1}\frac{\partial v_k}{\partial t^\alpha}(c_v^{\mathrm{ext}})^{N+1}_{k,N+1}\overset{\text{eq.(5.4)}}{=}\sum_{k=1}^{N-1}\frac{\partial v_k}{\partial t^\alpha}\frac{v_{N+1}^{2k-2}}{2^{k-1}}=\frac{\partial}{\partial t^\alpha}\frac{\partial F^o_{D_N}}{\partial v_{N+1}}=\frac{\partial^2 F^o_{D_N}}{\partial t^\alpha\partial t^{N+1}}.$$

If $1\le\alpha\le N-1$ and $\beta=N$, then

$$(c^{\mathrm{ext}})^{N+1}_{\alpha,N}=\sum_{k=1}^{N-1}\frac{\partial v_k}{\partial t^\alpha}(c_v^{\mathrm{ext}})^{N+1}_{k,N}\overset{\text{eq.(5.4)}}{=}0=\frac{\partial}{\partial t^\alpha}\frac{\partial F^o_{D_N}}{\partial v_N}=\frac{\partial^2 F^o_{D_N}}{\partial t^\alpha\partial t^N}.$$

Case $\alpha,\beta\in\{1,2,\dots,N-1\}$. We have to check that

$$(c^{\mathrm{ext}})^{N+1}_{\alpha\beta}=\frac{\partial^2 F^o_{D_N}}{\partial t^\alpha\partial t^\beta}\Leftrightarrow\sum_{1\le a,b\le N-1}\frac{\partial v_a}{\partial t^\alpha}\frac{\partial v_b}{\partial t^\beta}(c_v^{\mathrm{ext}})^{N+1}_{a,b}=\sum_{k=1}^{N-1}\frac{\partial^2 v_k}{\partial t^\alpha\partial t^\beta}\frac{v_{N+1}^{2k-1}}{2^{k-1}(2k-1)},$$

that is equivalent to the equation

$$(c_v^{\mathrm{ext}})^{N+1}_{a,b}=-\sum_{k=1}^{N-1}\frac{\partial v_k}{\partial t^\gamma}\frac{\partial^2 t^\gamma}{\partial v_a\partial v_b}\frac{v_{N+1}^{2k-1}}{2^{k-1}(2k-1)},\quad 1\le a,b\le N-1. \tag{5.10}$$

Let us compute the structure constants $(c_v^{\mathrm{ext}})^{N+1}_{a,b}$. Introduce polynomials $\omega_k\in\mathbb{Q}[v_1,\dots,v_{N-1}]$, $k\ge 0$, by

$$\omega_k:=\sum_{\substack{\alpha_1,\dots,\alpha_{N-1}\ge 0\\ \sum(N-i)\alpha_i=k}}s_1^{\alpha_1}\cdots s_{N-1}^{\alpha_{N-1}}\frac{(\sum\alpha_i)!}{\prod\alpha_i!},\quad \text{where } s_i:=(1-i)v_i \text{ for } 1\le i\le N-1.$$

The first few functions ω_k are

$$\omega_0=1,\quad \omega_1=s_{N-1},\quad \omega_2=s_{N-2}+s_{N-1}^2,\quad \omega_3=s_{N-3}+2s_{N-2}s_{N-1}+s_{N-1}^3.$$

The functions ω_k satisfy the recursion relation

$$\omega_{k+1}=\sum_{i=1}^{N-1}s_{N-i}\omega_{k+1-i},\quad k\ge 0, \tag{5.11}$$

where we adopt the convention $\omega_j:=0$ for $j<0$.

Lemma 5.3. *For $1\le a,b\le N-1$ we have*

$$c^{N+1}_{a,b}=\sum_{k=0}^{a+b-N-1}\omega_k\frac{v_{N+1}^{2(a+b)-2N-1-2k}}{2^{a+b-N-k}}. \tag{5.12}$$

Proof. Equation (5.12) is equivalent to the following formula:

$$\mathrm{Coef}_{[w]}[x^{p-2}] = \sum_{k=0}^{p-N-1} \omega_k \frac{v_{N+1}^{2p-2N-1-2k}}{2^{p-N-k}}, \quad 2 \le p \le 2N-2, \tag{5.13}$$

where $\mathrm{Coef}_{[w]}[x^{p-2}]$ denotes the coefficient of $[w]$ in the expression for $[x^{p-2}]$ in terms of the basis elements $[1], [x], \dots, [x^{N-2}], [y], [w]$. For $p \le N$ formula (5.13) is obvious because both sides of it are equal to zero.

Suppose $p \ge N+1$. Multiplying both sides of equation (5.7) by $[x^{p-N-1}]$, we get the relation

$$\mathrm{Coef}_{[w]}[x^{p-2}] = \frac{v_{N+1}^{2p-2N-1}}{2^{p-N}} + \sum_{k=1}^{N-1} s_k \mathrm{Coef}_{[w]}[x^{p+k-N-2}],$$

that allows to compute the coefficients $\mathrm{Coef}_{[w]}[x^{p-2}]$ recursively. Then, using also relation (5.11), formula (5.13) can be easily proved by induction. □

Using the lemma, we see that equation (5.10) can be equivalently written as

$$\sum_{k=0}^{a+b-N-1} \omega_k \frac{v_{N+1}^{2(a+b)-2N-1-2k}}{2^{a+b-N-k}} = -\sum_{k=1}^{N-1} \frac{\partial v_k}{\partial t^\gamma} \frac{\partial^2 t^\gamma}{\partial v_a \partial v_b} \frac{v_{N+1}^{2k-1}}{2^{k-1}(2k-1)} \Leftrightarrow$$
$$\Leftrightarrow \sum_{k=1}^{N-1} \omega_{a+b-N-k} \frac{v_{N+1}^{2k-1}}{2^k} = -\sum_{k=1}^{N-1} \frac{\partial v_k}{\partial t^\gamma} \frac{\partial^2 t^\gamma}{\partial v_a \partial v_b} \frac{v_{N+1}^{2k-1}}{2^{k-1}(2k-1)}.$$

So we have to prove that

$$\frac{\partial v_k}{\partial t^\gamma} \frac{\partial^2 t^\gamma}{\partial v_a \partial v_b} = -\frac{2k-1}{2} \omega_{a+b-N-k} \Leftrightarrow \frac{\partial^2 t^\gamma}{\partial v_a \partial v_b} = -\sum_{k=1}^{N-1} \frac{2k-1}{2} \omega_{a+b-N-k} \frac{\partial t^\gamma}{\partial v_k}. \tag{5.14}$$

Recall that $t^\gamma(v_1, \dots, v_{N-1})$ is a quasi-homogeneous polynomial of degree $N-\gamma$, if we put $\deg v_a = N-a$. This implies that both sides of the last equation in (5.14) are zero if $a+b \le N$. Let us assume now that $a+b \ge N+1$. The last equation in (5.14) is equivalent to

$$\frac{\partial^2 t^\gamma}{\partial v_a \partial v_b} - \sum_{i=1}^{a-1} s_{N-i} \frac{\partial^2 t^\gamma}{\partial v_{a-i} \partial v_b} = -\sum_{k=1}^{N-1} \frac{2k-1}{2} \frac{\partial t^\gamma}{\partial v_k} \left(\omega_{a+b-N-k} - \sum_{i=1}^{a-1} s_{N-i} \omega_{a-i+b-N-k} \right).$$

Note that for $i \ge a$ we have $a-i+b-N-k < 0$ and, therefore, by (5.11), the expression in the brackets is equal to zero unless $k = a+b-N$. So we come to the following equivalent identity:

$$\frac{\partial^2 t^\gamma}{\partial v_a \partial v_b} - \sum_{i=1}^{a-1} s_{N-i} \frac{\partial^2 t^\gamma}{\partial v_{a-i} \partial v_b} = -\frac{2(a+b-N)-1}{2} \frac{\partial t^\gamma}{\partial v_{a+b-N}}, \quad \begin{array}{c} 1 \le a, b, \gamma \le N-1, \\ a+b \ge N+1. \end{array} \tag{5.15}$$

Note that both sides of (5.15) are quasi-homogeneous polynomials of degree $a+b-\gamma-N$. Differentiating both sides by $\frac{\partial^{\sum \alpha_i}}{\partial v_1^{\alpha_1} \dots \partial v_{N-1}^{\alpha_{N-1}}}$, putting $v_j = 0$ and using formula (3.3), wee see that equation (5.15) is equivalent to the following family of identities:

$$-\frac{1}{2} A \left(2\gamma - 1 + 2(N-1) \sum \alpha_i \right) + A \sum_{i=1}^{a-1} (N-i-1)\alpha_{N-i} = -A \frac{2(a+b-N)-1}{2} \Leftrightarrow$$
$$\Leftrightarrow A \left(-\gamma - (N-1) \sum \alpha_i + \sum_{i=1}^{a-1} (N-i-1)\alpha_{N-i} + a + b - N \right) = 0, \tag{5.16}$$

that should be true for any tuple $\alpha_1, \dots, \alpha_{N-1} \geq 0$ such that

$$\sum_{i=1}^{N-1}(N-i)\alpha_i = a+b-\gamma-N, \tag{5.17}$$

and where $A = \left(-\frac{1}{2}\right)^{\sum \alpha_i} \prod_{k=0}^{\sum \alpha_i - 1}(2\gamma - 1 + 2k(N-1))$. Condition (5.17) implies that $\alpha_i = 0$, if $i \leq N-a$. Therefore, the summation $\sum_{i=1}^{a-1}$ in (5.16) can be replaced by the summation $\sum_{i=1}^{N-1}$ and, using (5.17), we immediately see that the expression in the brackets in (5.16) vanishes. This completes the proof of the theorem. □

Taking into account the discussion about the relation between solutions of the open WDVV equations and flat F-manifolds from Section 2.3, we get the following result.

Corollary 5.4.
1. The function $F^o_{D_N}$ satisfies the open WDVV equations together with condition (1.5) *and the quasi-homogeneity property*

$$\sum_{\alpha=1}^{N} q_\alpha t^\alpha \frac{\partial F^o_{D_N}}{\partial t^\alpha} + \frac{1-\delta}{2} s \frac{\partial F^o_{D_N}}{\partial s} = \frac{3-\delta}{2} F^o_{D_N}.$$

2. We have

$$v_k(t^1, \dots, t^N) = \begin{cases} 2^{k-1}(2k-1)\mathrm{Coef}_{s^{2k-1}} F^o_{D_N}, & \text{if } 1 \leq k \leq N-1, \\ \sqrt{2\mathrm{Coef}_{s^{-1}} F^o_{D_N}}, & \text{if } k = N. \end{cases}$$

Example 5.5. *Here are the Frobenius manifold potentials for the singularities D_4 and D_5 together with the constructed solutions of the open WDVV equations:*

$$F_{D_4} = \frac{t_1^2 t_3}{2} + \frac{t_1 t_2^2}{2} - \frac{t_1 t_4^2}{2} - \frac{t_2 t_3 t_4^2}{4} - \frac{1}{12} t_2^3 t_3 + \frac{t_2^2 t_3^3}{24} - \frac{t_3^3 t_4^2}{24} + \frac{t_3^7}{3360},$$

$$F^o_{D_4} = \frac{s^7}{168} + \frac{t_3 s^5}{20} + \left(\frac{t_3^2}{8} + \frac{t_2}{6}\right) s^3 + \left(\frac{t_3^3}{12} + \frac{t_2 t_3}{2} + t_1\right) s + \frac{t_4^2}{2s},$$

$$F_{D_5} = \frac{t_1^2 t_4}{2} + t_1 t_2 t_3 - \frac{t_1 t_5^2}{2} + \frac{t_2^3}{6} - \frac{t_2 t_3^3}{6} - \frac{t_2^2 t_3 t_4}{4} - \frac{t_2 t_4 t_5^2}{4} - \frac{t_3^2 t_5^2}{8} + \frac{t_2^2 t_4^3}{24} + \frac{t_3^4 t_4}{16} + \frac{t_2 t_3^2 t_4^2}{8} - \frac{t_3 t_4^2 t_5^2}{8}$$
$$- \frac{t_3^3 t_4^3}{48} - \frac{t_4^4 t_5^2}{64} + \frac{t_3^2 t_4^5}{160} + \frac{t_4^9}{32256},$$

$$F^o_{D_5} = \frac{s^9}{576} + \frac{t_4 s^7}{56} + \left(\frac{t_4^2}{16} + \frac{t_3}{20}\right) s^5 + \left(\frac{t_4^3}{12} + \frac{t_3 t_4}{4} + \frac{t_2}{6}\right) s^3 + \left(\frac{t_4^4}{32} + \frac{t_3 t_4^2}{4} + \frac{t_2 t_4}{2} + \frac{t_3^2}{4} + t_1\right) s + \frac{t_5^2}{2s}.$$

Note that $F^o_{D_4}$ and $F^o_{D_5}$ have simple poles along $s = 0$. In particular, the dependence on s in both cases is not polynomial.

6. Polynomial solutions of the open WDVV equations for finite irreducible Coxeter groups

In this section we first recall a description of the Frobenius manifolds corresponding to finite irreducible Coxeter groups, and then describe the space of homogeneous polynomial solutions of the associated open WDVV equations.

6.1. **Frobenius manifolds of finite irreducible Coxeter groups.** *Finite Coxeter groups* are finite groups of linear transformations of a real N-dimensional vector space V, generated by reflections. The complete list of finite irreducible Coxeter groups is given by (the dimension of the space V equals the subscript in the name of the group)

$$A_N, N \geq 1 \qquad D_N, N \geq 4, \qquad E_6, \qquad E_7, \qquad E_8, \tag{6.1}$$

$$B_N, N \geq 2, \qquad F_4, \qquad H_3, \qquad H_4, \qquad I_2(k), k \geq 3, \tag{6.2}$$

with the exceptional isomorphisms $A_2 \cong I_2(3)$ and $B_2 \cong I_2(4)$. By a construction of B. Dubrovin [Dub98], for such a group W the complexified space of orbits $M_W := (V \otimes \mathbb{C})/W \cong \mathbb{C}^N$ carries a Frobenius manifold structure. For the Coxeter groups A_N, D_N and E_N the corresponding Frobenius manifolds coincide with the Saito Frobenius manifolds of simple singularities. By a result of J.-B. Zuber [Zub94], the Frobenius manifold potentials corresponding to the remaining irreducible Coxeter groups can be explicitly described by

$$
\begin{aligned}
F_{B_N}(t^1,\dots,t^N) =& F_{A_{2N-1}}(t^1,0,t^2,0,\dots,t^{N-1},0,t^N),\\
F_{I_2(k)}(t^1,t^2) =& F_{A_{k-1}}(t^1,0,\dots,0,t^2),\\
F_{F_4}(t^1,t^2,t^3,t^4) =& F_{E_6}(t^1,0,t^2,t^3,0,t^4),\\
F_{H_4}(t^1,t^2,t^3,t^4) =& F_{E_8}(t^1,0,t^2,0,0,t^3,0,t^4),\\
F_{H_3}(t^1,t^2,t^3) =& F_{D_6}(t^1,0,t^2,0,t^3,\sqrt{-1}t^2).
\end{aligned}
\tag{6.3}
$$

All the Frobenius manifolds corresponding to finite irreducible Coxeter groups are semisimple.

6.2. **Euler vector field.** We see that for any finite irreducible Coxeter group W, acting on an N-dimensional real vector space V, the associated Frobenius manifold is described by the polynomial potential $F_W(t^1,\dots,t^N)$ satisfying the quasi-homogeneity condition

$$\sum_{\alpha=1}^{N} q_\alpha t^\alpha \frac{\partial F_W}{\partial t^\alpha} = (3-\delta)F_W, \quad q_\alpha > 0.$$

The numbers $q_1,\dots,q_N$ have the following interpretation. Consider the symmetric algebra $S(V\otimes\mathbb{C})$. The subring $S(V\otimes\mathbb{C})^W$ of W-invariant polynomials is generated by N algebraically independent homogeneous polynomials, whose degrees $d_1,\dots,d_N \geq 2$ are uniquely determined by the Coxeter group. The maximal degree h is called the *Coxeter number* of W. Then we have

$$q_\alpha = \frac{d_\alpha}{h}, \qquad \delta = 1 - \frac{2}{h}.$$

Note that then in the homogeneity condition (1.7) for solutions of the open WDVV equations the degree of the extra variable s becomes

$$\frac{1-\delta}{2} = \frac{1}{h}.$$

6.3. **Homogeneous polynomial solutions of the open WDVV equations.** In this section we describe the space of homogeneous polynomial solutions of the open WDVV equations associated to the Frobenius manifolds of finite irreducible Coxeter groups. It occurs that for the Coxeter groups different from A_N, B_N and $I_2(k)$ there are no such solutions. We prove it in Section 6.3.1. For the groups A_N, B_N and $I_2(k)$ all solutions can be obtained from the function $F^o_{A_N}$, as is explained in Section 6.3.2.

Consider an irreducible Coxeter group W, the potential F_W and a homogeneous polynomial solution F^o of the open WDVV equations, satisfying (1.5). Note that equations (1.3)-(1.5) involve only the second partial derivatives of F^o and that adding constant and linear terms in the variables $t^1,\dots,t^N$ and s to F^o just changes the constants D_α, $\widetilde{D}$ and E in condition (1.7). If we remove constant and linear terms in the variables $t^1,\dots,t^N$ and s from the function F^o, then it will satisfy the condition

$$\sum_{\alpha=1}^{N} q_\alpha t^\alpha \frac{\partial F^o}{\partial t^\alpha} + \frac{1-\delta}{2} s \frac{\partial F^o}{\partial s} = \frac{3-\delta}{2} F^o. \tag{6.4}$$

6.3.1. *Irreducible Coxeter groups different from A_N, B_N and $I_2(k)$.*

Theorem 6.1. *Let W be a finite irreducible Coxeter group different from A_N, B_N and $I_2(k)$. Consider the corresponding Frobenius manifold potential F_W. Then there are no homogeneous polynomial solutions F^o of the associated open WDVV equations satisfying property* (1.5).

Proof. As we already explained above, we can assume that F^o doesn't contain constant and linear terms in the variables $t^1, \dots, t^N$ and s and satisfies condition (6.4).

Let W be one of the groups D_N, E_6, E_7 or E_8. Let us rewrite equations (1.4) in the coordinates $v_1, \dots, v_N$ and s:

$$(c_v)^\mu_{\alpha\beta}\frac{\partial^2 F^o}{\partial v_\mu \partial s} + \frac{\partial^2 F^o}{\partial v_\alpha \partial v_\beta}\frac{\partial^2 F^o}{\partial s^2} + \frac{\partial t^{\widetilde{\alpha}}}{\partial v_\alpha}\frac{\partial t^{\widetilde{\beta}}}{\partial v_\beta}\frac{\partial^2 v_\mu}{\partial t^{\widetilde{\alpha}}\partial t^{\widetilde{\beta}}}\frac{\partial F^o}{\partial v_\mu}\frac{\partial^2 F^o}{\partial s^2} = \frac{\partial^2 F^o}{\partial v_\alpha \partial s}\frac{\partial^2 F^o}{\partial v_\beta \partial s}, \quad 1 \le \alpha, \beta \le N,$$

where $(c_v)^\mu_{\alpha\beta}$ denotes the structure constants of multiplication in the coordinates v_μ. Clearly, $\frac{\partial F^o}{\partial v_\mu}\Big|_{v_*=s=0} = 0$. Since $\delta \ge 0$, we have $\frac{3-\delta}{2} > 2 \cdot \frac{1-\delta}{2}$. This implies that $\frac{\partial^2 F^o}{\partial s^2}\Big|_{v_*=s=0} = 0$. Therefore,

$$\frac{\partial}{\partial v_\gamma}\left((c_v)^\mu_{\alpha\beta}\frac{\partial^2 F^o}{\partial v_\mu \partial s}\right)\Big|_{v_*=s=0} + \frac{\partial}{\partial v_\gamma}\left(\frac{\partial^2 F^o}{\partial v_\alpha \partial v_\beta}\frac{\partial^2 F^o}{\partial s^2}\right)\Big|_{v_*=s=0} = \frac{\partial}{\partial v_\gamma}\left(\frac{\partial^2 F^o}{\partial v_\alpha \partial s}\frac{\partial^2 F^o}{\partial v_\beta \partial s}\right)\Big|_{v_*=s=0},$$

for any indices $1 \le \alpha, \beta, \gamma \le N$. We will prove that this equation can't be true by finding indices $2 \le \alpha, \beta, \gamma \le N$ such that

$$(c_v)^\mu_{\alpha\beta}\big|_{v_*=0} = 0, \qquad \frac{\partial (c_v)^\mu_{\alpha\beta}}{\partial v_\gamma} = A\delta^{\mu,1}, \quad A \in \mathbb{C}^*, \tag{6.5}$$

$$\frac{\partial^2 F^o}{\partial v_\alpha \partial v_\beta} = 0, \qquad \frac{\partial}{\partial v_\gamma}\left(\frac{\partial^2 F^o}{\partial v_\alpha \partial s}\frac{\partial^2 F^o}{\partial v_\beta \partial s}\right) = 0. \tag{6.6}$$

Case $W = D_N$, $N \ge 4$. We have $\delta = \frac{N-2}{N-1}$, $\frac{1-\delta}{2} = \frac{1}{2(N-1)}$ and $q_k = \begin{cases} \frac{N-k}{N-1}, & \text{if } 1 \le k \le N-1, \\ \frac{N}{2(N-1)}, & \text{if } k = N. \end{cases}$

Let us choose $\alpha = 2$ and $\beta = \gamma = N$. From $\frac{\partial \Lambda_{D_N}}{\partial y} = 2xy + v_N$ we see that $\frac{\partial}{\partial v_2} \circ \frac{\partial}{\partial v_N} = -\frac{1}{2}v_N\frac{\partial}{\partial v_1}$, that implies the properties in line (6.5). We have

$$q_2 + q_N = \frac{N-2}{N-1} + \frac{N}{2(N-1)} = \frac{3N-4}{2(N-1)} > \frac{2N-1}{2(N-1)} = \frac{3-\delta}{2} \Rightarrow \frac{\partial^2 F^o}{\partial v_2 \partial v_N} = \frac{\partial^3 F^o}{\partial v_2 \partial v_N \partial s} = 0,$$

$$2q_N + \frac{1-\delta}{2} = \frac{2N}{2(N-1)} + \frac{1}{2(N-1)} = \frac{2N+1}{2(N-1)} > \frac{3-\delta}{2} \qquad \Rightarrow \frac{\partial^3 F^o}{\partial v_N^2 \partial s} = 0,$$

that gives the properties in line (6.6). So the theorem is proved for the case $W = D_N$.

Case $W = E_6$. We have $\delta = \frac{5}{6}$, $\frac{1-\delta}{2} = \frac{1}{12}$ and $(q_1, \dots, q_6) = \left(1, \frac{3}{4}, \frac{2}{3}, \frac{1}{2}, \frac{5}{12}, \frac{1}{6}\right)$. Let us choose $\alpha = \beta = \gamma = 3$. From $\frac{\partial \Lambda_{E_6}}{\partial y} = 3y^2 + v_3 + v_5 x + v_6 x^2$ we see that $\frac{\partial}{\partial v_3} \circ \frac{\partial}{\partial v_3} = -\frac{1}{3}v_3\frac{\partial}{\partial v_1} - \frac{1}{3}v_5\frac{\partial}{\partial v_2} - \frac{1}{3}v_6\frac{\partial}{\partial v_4}$, that implies the properties in line (6.5). We have $2q_3 = \frac{4}{3} > \frac{13}{12} = \frac{3-\delta}{2}$, implying $\frac{\partial^2 F^o}{\partial v_3^2} = \frac{\partial^3 F^o}{\partial v_3^2 \partial s} = 0$, that gives the properties in line (6.6) and proves the theorem for $W = E_6$.

Case $W = E_7$. We have $\delta = \frac{8}{9}$, $\frac{1-\delta}{2} = \frac{1}{18}$ and $(q_1, \dots, q_7) = \left(1, \frac{7}{9}, \frac{2}{3}, \frac{5}{9}, \frac{4}{9}, \frac{1}{3}, \frac{1}{9}\right)$. Choose $\alpha = 3$, $\beta = 4$ and $\gamma = 2$. From $\frac{\partial \Lambda_{E_7}}{\partial x} = 3x^2 y + v_2 + 2v_4 x + v_5 y + 3v_6 x^2 + 4v_7 x^3$ we see that $\frac{\partial}{\partial v_3} \circ \frac{\partial}{\partial v_4} = -\frac{1}{3}v_2\frac{\partial}{\partial v_1} - \frac{2}{3}v_4\frac{\partial}{\partial v_2} - \frac{1}{3}v_5\frac{\partial}{\partial v_3} - v_6\frac{\partial}{\partial v_4} - \frac{4}{3}v_7\frac{\partial}{\partial v_6}$, that implies the properties in line (6.5).

We have

$$\begin{aligned} q_3+q_4=\frac{11}{9}>\frac{19}{18}=\frac{3-\delta}{2} \qquad &\Rightarrow \qquad \frac{\partial^2 F^o}{\partial v_3\partial v_4}=0,\\ q_2+q_3+\frac{1-\delta}{2}=\frac{3}{2}>\frac{3-\delta}{2} \qquad &\Rightarrow \qquad \frac{\partial^3 F^o}{\partial v_2\partial v_3\partial s}=0,\\ q_2+q_4+\frac{1-\delta}{2}=\frac{25}{18}>\frac{3-\delta}{2} \qquad &\Rightarrow \qquad \frac{\partial^3 F^o}{\partial v_2\partial v_4\partial s}=0, \end{aligned}$$

that implies the properties in line (6.6). This proves the theorem for $W=E_7$.

Case $W=E_8$. We have $\delta=\frac{14}{15}$, $\frac{1-\delta}{2}=\frac{1}{30}$ and $(q_1,\dots,q_8)=\left(1,\frac{4}{5},\frac{2}{3},\frac{3}{5},\frac{7}{15},\frac{2}{5},\frac{4}{15},\frac{1}{15}\right)$. Choose $\alpha=\beta=\gamma=3$. From $\frac{\partial\Lambda_{E_8}}{\partial y}=3y^2+v_3+v_5x+v_7x^2+v_8x^3$ we see that $\frac{\partial}{\partial v_3}\circ\frac{\partial}{\partial v_3}=-\frac{1}{3}v_3\frac{\partial}{\partial v_1}-\frac{1}{3}v_5\frac{\partial}{\partial v_2}-\frac{1}{3}v_7\frac{\partial}{\partial v_4}-\frac{1}{3}v_8\frac{\partial}{\partial v_6}$, that implies the properties in line (6.5). We have $2q_3=\frac{4}{3}>\frac{31}{30}=\frac{3-\delta}{2}$, implying $\frac{\partial^2 F^o}{\partial v_3^2}=\frac{\partial^3 F^o}{\partial v_3^2\partial s}=0$, that completes the proof of the theorem for $W=E_8$.

For the groups H_3, H_4 and F_4 we are going to use the explicit formulas for the corresponding Frobenius potentials from the paper [Zub94]. Note that these potentials are related to the ones, given by (6.3), by certain rescallings $F_W(t^1,\dots,t^N)\mapsto F_W(\lambda_1t^1,\dots,\lambda_Nt^N)$, $\lambda_i\in\mathbb{C}^*$, but this doesn't affect our proof.

For the groups F_4 and H_4 the corresponding potentials, computed in [Zub94], are

$$\begin{aligned} F_{F_4}=&\frac{t_4^{13}}{185328}+\frac{t_3^2t_4^7}{252}+\frac{t_2^2t_4^5}{60}+\frac{t_2t_3^2t_4^3}{6}+\frac{t_3^4t_4}{12}+\frac{t_2^3t_4}{6}+\frac{t_1^2t_4}{2}+t_1t_2t_3,\\ F_{H_4}=&\frac{t_4^{31}}{245764125000}+\frac{t_3^2t_4^{19}}{1539000}+\frac{t_3^3t_4^{13}}{10800}+\frac{t_2^2t_4^{11}}{4950}+\frac{t_2t_3^2t_4^9}{360}+\frac{t_3^4t_4^7}{120}+\frac{t_2^2t_3t_4^5}{20}+\frac{t_2t_3^3t_4^3}{6}+\frac{t_3^5t_4}{20}\\ &+\frac{t_2^3t_4}{6}+\frac{t_1^2t_4}{2}+t_1t_2t_3. \end{aligned}$$

Note that the equation

$$\frac{\partial}{\partial t_2}\left(c^\mu_{2,2}\frac{\partial^2F^o}{\partial t_\mu\partial s}\right)\Big|_{t_*=s=0}+\frac{\partial}{\partial t_2}\left(\frac{\partial^2F^o}{\partial t_2\partial t_2}\frac{\partial^2F^o}{\partial s^2}\right)\Big|_{t_*=s=0}=\frac{\partial}{\partial t_2}\left(\frac{\partial^2F^o}{\partial t_2\partial s}\frac{\partial^2F^o}{\partial t_2\partial s}\right)\Big|_{t_*=s=0},$$

where $c^\gamma_{\alpha\beta}$ are the structure constants of multiplication in the coordinates t_μ, can't be true, because $c^\mu_{2,2}\big|_{t_*=0}=0$, $\frac{\partial c^\mu_{2,2}}{\partial t_2}=\delta^{\mu,1}$ and $\frac{\partial^2F^o}{\partial t_2\partial t_2}=0$, that follows from (6.4).

For the group H_3 the Frobenius manifold, computed in [Zub94], is

$$F_{H_3}=\frac{1}{2}t_1^2t_3+\frac{1}{2}t_1t_2^2+\frac{1}{20}t_2^2t_3^5+\frac{1}{6}t_2^3t_3^2+\frac{t_3^{11}}{3960}.$$

The general form of a polynomial function $F^o_{H_3}(t_1,t_2,t_3,s)$ satisfying (1.5) and (6.4) is

$$F^o_{H_3}=st_1+c_9st_2t_3^2+c_8s^3t_2t_3+c_7s^5t_2+c_6st_3^5+c_5s^3t_3^4+c_4s^5t_3^3+c_3s^7t_3^2+c_2s^9t_3+c_1s^{11},\quad c_k\in\mathbb{C}.$$

Suppose that it satisfies equation (1.4) with $\alpha=3$, $\beta=2$. A direct computation shows that, applying the derivative $\frac{\partial^2}{\partial t_2^2}$ to both sides of it, we get 2 on the left-hand side and 0 on the right-hand side. This contradition proves the theorem for the case of the group H_3. □

6.3.2. *Coxeter groups* A_N, B_N *and* $I_2(k)$. Define

$$\begin{aligned} F^o_{B_N}(t^1,\dots,t^N,s):=&F^o_{A_{2N-1}}(t^1,0,t^2,0,\dots,t^{N-1},0,t^N,s), &&N\geq 2,\\ F^o_{I_2(k)}(t^1,t^2,s):=&F^o_{A_{k-1}}(t^1,0,\dots,0,t^2,s), &&k\geq 3. \end{aligned}$$

Let $F^{o,-}_{I_2(k)}:=2t^1s-F^o_{I_2(k)}$ and denote $F^{o,+}_{I_2(k)}:=F^o_{I_2(k)}$.

Note that if a function $F^o(t^1,\dots,t^N,s)$ satisfies the open WDVV equations, then the function $\lambda^{-1}F^o(t^1,\dots,t^N,\lambda s)$ also satisfies them for any $\lambda\neq 0$. Moreover, if $F^o|_{s=0}=0$, then the

substitution $\left(\lambda^{-1}F^o(t^1,\dots,t^N,\lambda s)\right)\big|_{\lambda=0}$ is well defined and is a solution of the open WDVV equations.

Theorem 6.2. *Let W be one of the groups A_N, B_N or $I_2(k)$. Then all polynomial solutions F^o of the open WDVV equations satisfying* (1.5) *and* (6.4) *are given by the family*

$$F^o = \begin{cases} \lambda^{-1}F^o_{A_N}(t^1,\dots,t^N,\lambda s), & \lambda\in\mathbb{C}^*, \quad \text{if } W=A_N,\ N\geq 2,\\ \lambda^{-1}F^o_{A_1}(t^1,\lambda s), & \lambda\in\mathbb{C}, \quad \text{if } W=A_1,\\ \lambda^{-1}F^o_{B_N}(t^1,\dots,t^N,\lambda s), & \lambda\in\mathbb{C}, \quad \text{if } W=B_N,\ N\geq 2,\\ \lambda^{-1}F^o_{I_2(k)}(t^1,t^2,\lambda s), & \lambda\in\mathbb{C}^*, \quad \text{if } W=I_2(k),\ k \text{ is odd},\\ \lambda^{-1}F^{o,\pm}_{I_2(k)}(t^1,t^2,\lambda s), & \lambda\in\mathbb{C}, \quad \text{if } W=I_2(k),\ k \text{ is even}. \end{cases}$$

Proof. Case $W=A_N$. We have $q_\alpha=\frac{N+2-\alpha}{N+1}$ and $\delta=\frac{N-1}{N+1}$. The case $N=1$ is obvious. Suppose that $N\geq 2$ and F^o is a solution of the open WDVV equations, satisfying (1.5) and (6.4). For an n-tuple $\overline{\alpha}=(\alpha_1,\dots,\alpha_n)$, $1\leq\alpha_i\leq N$, denote

$$\left\langle\tau_{\overline{\alpha}}\sigma^k\right\rangle^o=\left\langle\tau_{\alpha_1}\dots\tau_{\alpha_n}\sigma^k\right\rangle^o:=\left.\frac{\partial^{n+k}F^o}{\partial t_1^\alpha\dots\partial t^{\alpha_n}\partial s^k}\right|_{t^*=s=0}.$$

This number is non-zero only if $k=k(\overline{\alpha}):=N+2-\sum_{i=1}^n(N+2-\alpha_i)$.

Note that

$$c^\gamma_{2,\beta}=\begin{cases}0, & \text{if } \gamma>\beta+1,\\ 1, & \text{if } \gamma=\beta+1,\\ O(t^*), & \text{if } \gamma\leq\beta,\end{cases} \tag{6.7}$$

that follows from (4.1). Setting $t^*=0$ in equation (1.4) with $\alpha=2$ and $2\leq\beta\leq N$, we get

$$\left\langle\tau_\alpha\sigma^\alpha\right\rangle^o=(\alpha-1)!\left(\left\langle\tau_2\sigma^2\right\rangle^o\right)^{\alpha-1},\quad 2\leq\alpha\leq N,\qquad \left\langle\tau_2\tau_N\right\rangle^o\left\langle\sigma^{N+2}\right\rangle^o=N!\left(\left\langle\tau_2\sigma^2\right\rangle^o\right)^N.$$

Differentiating equation (1.4) with $\alpha=2$ and $\beta=N$ by $\frac{\partial}{\partial t^2}$ and setting $t^*=s=0$, we get $-1+\left\langle\tau_2\tau_N\right\rangle^o\left\langle\tau_2\sigma^2\right\rangle^o=0$, where we use formula (4.1) for the numbers $\left\langle\tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_3}\tau_{\alpha_4}\right\rangle_{A_N}$. We see that $\left\langle\tau_2\sigma^2\right\rangle^o\neq 0$ and

$$\left\langle\sigma^{N+2}\right\rangle^o=N!\left(\left\langle\tau_2\sigma^2\right\rangle^o\right)^{N+1}\neq 0.$$

After the rescaling $F^o(t^1,\dots,t^N,s)\mapsto\lambda^{-1}F^o(t^1,\dots,t^N,\lambda s)$ with an appropriate constant $\lambda\neq 0$ we get $\left\langle\tau_2\sigma^2\right\rangle^o=1$ and, therefore,

$$\left\langle\tau_\alpha\sigma^\alpha\right\rangle^o=(\alpha-1)!=\left\langle\tau_\alpha\sigma^\alpha\right\rangle^o_{A_N},\quad 1\leq\alpha\leq N,\qquad \left\langle\sigma^{N+2}\right\rangle^o=N!=\left\langle\sigma^{N+2}\right\rangle^o_{A_N}.$$

Consider now an n-tuple $\overline{\alpha}=(\alpha_1,\dots,\alpha_n)$, $1\leq\alpha_i\leq N$, with $n\geq 2$ and $k(\overline{\alpha})\geq 0$. Differentiating equation (1.4) with $\alpha=\alpha_1$ and $\beta=\alpha_2$ by $\frac{\partial^{n-2}}{\partial t^{\alpha_3}\dots\partial t^{\alpha_n}}$ and setting $t^*=0$, we get the recursion

$$\frac{\left\langle\tau_{\overline{\alpha}}\sigma^{k(\overline{\alpha})}\right\rangle^o}{k(\overline{\alpha})!}=\sum_{\substack{I\sqcup J=\{1,\dots,n\}\\ 1\in I,\,2\in J}}\frac{\left\langle\tau_{\overline{\alpha}_I}\sigma^{k(\overline{\alpha}_I)}\right\rangle^o\left\langle\tau_{\overline{\alpha}_J}\sigma^{k(\overline{\alpha}_J)}\right\rangle^o}{(k(\overline{\alpha}_I)-1)!(k(\overline{\alpha}_J)-1)!}-\sum_{\substack{I\sqcup J=\{1,\dots,n\}\\ 1,2\in I,\,J\neq\emptyset}}\frac{\left\langle\tau_{\overline{\alpha}_I}\sigma^{k(\overline{\alpha}_I)}\right\rangle^o\left\langle\tau_{\overline{\alpha}_J}\sigma^{k(\overline{\alpha}_J)}\right\rangle^o}{k(\overline{\alpha}_I)!(k(\overline{\alpha}_J)-2)!}, \tag{6.8}$$

where for a subset $I=\{i_1,\dots,i_{|I|}\}\subset\{1,\dots,n\}$, $i_1<\dots<i_{|I|}$, we denote $\overline{\alpha}_I:=(\alpha_{i_1},\dots,\alpha_{i_{|I|}})$. The correlators $\langle\cdot\rangle_{A_N}$ don't appear in this recursion because for any subset $I\subset\{1,\dots,n\}$ and an index $1\leq\mu\leq N$ we have

$$\sum_{i\in I}(N+2-\alpha_i)+(N+2-\mu)\leq\sum_{i=1}^n(N+2-\alpha_i)+(N+2-\mu)=$$
$$=2N+4-k(\overline{\alpha})-\mu<2N+4$$

and, therefore, $\langle \tau_{\overline{\alpha}_I} \tau_\mu \rangle_{A_N} = 0$. The recursion (6.8) determines all the numbers $\left\langle \tau_{\overline{\alpha}} \sigma^{k(\overline{\alpha})} \right\rangle^o$ starting from the numbers $\left\langle \sigma^{N+2} \right\rangle^o$ and $\langle \tau_\alpha \sigma^\alpha \rangle^o$. So we conclude that $F^o = F^o_{A_N}$.

Case $W = B_N$. We have $q_\alpha = \frac{N+1-\alpha}{N}$ and $\delta = \frac{N-1}{N}$. The function $F^o_{B_N}$ satisfies the open WDVV equations together with equations (1.5) and (6.4), because, as one can easily check using the quasi-homogeneity of the function $F^o_{A_{2N-1}}$, the correlator $\langle \tau_{\alpha_1} \dots \tau_{\alpha_n} \tau_\mu \rangle_{A_{2N-1}}$ vanishes, if all the α_i's are odd and μ is even.

Suppose that F^o is a solution of the open WDVV equations, satisfying (1.5) and (6.4). Since $F_{B_2} = F_{I_2(4)}$, we will consider the B_2-case together with the cases $W = I_2(k)$ later. So we assume that $N \geq 3$. Note that a correlator $\left\langle \tau_{\alpha_1} \dots \tau_{\alpha_n} \sigma^k \right\rangle^o$ vanishes unless $\sum_{i=1}^N (N+1-\alpha_i) + \frac{k}{2} = N + \frac{1}{2}$. Setting $t^* = 0$ in equation (1.4) with $\alpha = 2$, we get the relations

$$\begin{aligned} \left\langle \tau_\alpha \sigma^{2\alpha-1} \right\rangle^o &= \frac{(2\alpha-2)!}{2^{\alpha-1}} \left(\left\langle \tau_2 \sigma^3 \right\rangle^o \right)^{\alpha-1}, \quad 2 \leq \alpha \leq N, \\ \langle \tau_2 \tau_N \sigma \rangle^o \left\langle \sigma^{2N+1} \right\rangle^o &= \frac{(2N-1)!}{2^N} \left(\left\langle \tau_2 \sigma^3 \right\rangle^o \right)^N. \end{aligned}$$

Differentiating equation (1.4) with $\alpha = 2$ and $\beta = N-1$ by $\frac{\partial}{\partial t^2}$ and setting $t^* = s = 0$, we get $\langle \tau_2^2 \tau_{N-1} \tau_N \rangle_{B_N} + \langle \tau_2 \tau_N \sigma \rangle^o = 0$. Since, by (4.1), $\langle \tau_2^2 \tau_{N-1} \tau_N \rangle_{B_N} = -1$, we conclude that $\langle \tau_2 \tau_N \sigma \rangle^o = 1$ and

$$\left\langle \sigma^{2N+1} \right\rangle^o = \frac{(2N-1)!}{2^N} \left(\left\langle \tau_2 \sigma^3 \right\rangle^o \right)^N.$$

Suppose that $\left\langle \tau_2 \sigma^3 \right\rangle^o \neq 0$, then $\left\langle \sigma^{2N+1} \right\rangle^o \neq 0$. After the rescaling $F^o(t^1, \dots, t^N, s) \mapsto \lambda^{-1} F^o(t^1, \dots, t^N, \lambda s)$ with an appropriate constant $\lambda \neq 0$ we get $\left\langle \tau_\alpha \sigma^{2\alpha-1} \right\rangle^o = \left\langle \tau_\alpha \sigma^{2\alpha-1} \right\rangle^o_{B_N}$ and $\left\langle \sigma^{2N+1} \right\rangle^o = \left\langle \sigma^{2N+1} \right\rangle^o_{B_N}$. In the same way, as in the A_N-case, there is a recursion similar to (6.8), that reconstructs all the correlators $\left\langle \tau_{\alpha_1} \dots \tau_{\alpha_n} \sigma^k \right\rangle^o$ with $n \geq 2$. Therefore, $F^o = F^o_{B_N}$.

Suppose that $\left\langle \tau_2 \sigma^3 \right\rangle^o = 0$, then $\left\langle \sigma^{2N+1} \right\rangle^o = 0$. Consider the decomposition

$$F^o = \sum_{k=0}^N P_k(t^1, \dots, t^N) s^{2k+1}, \quad P_k \in \mathbb{C}[t^1, \dots, t^N].$$

Consider an index l such that $P_l \neq 0$ and $P_{>l} = 0$. Since $l < N$, the polynomial P_l can't be a constant. Suppose $l > 0$, then equation (1.4) implies that

$$\frac{\partial P_l}{\partial t^\alpha} \frac{\partial P_l}{\partial t^\beta} = \frac{2l}{2l+1} P_l \frac{\partial^2 P_l}{\partial t^\alpha \partial t^\beta}, \quad 1 \leq \alpha, \beta \leq N.$$

The space of solutions of the differential equation $(f')^2 = \frac{2l}{2l+1} f f''$ for a function $f = f(x)$ is formed by the family $f = C_1(x + C_2)^{-2l}$, $C_1, C_2 \in \mathbb{C}^*$, together with the constant solution $f = C$, $C \in \mathbb{C}$. Since P_l is a non-constant polynomial, we come to a contradition. Therefore, $l = 0$.

In this case system (1.4) is equivalent to the system

$$c^\gamma_{\alpha\beta} \frac{\partial P_0}{\partial t^\gamma} = \frac{\partial P_0}{\partial t^\alpha} \frac{\partial P_0}{\partial t^\beta}, \quad 1 \leq \alpha, \beta \leq N.$$

For $\alpha = 2$ we get the relations

$$\frac{\partial P_0}{\partial t^{\beta+1}} + \sum_{1 \leq \gamma \leq \beta} c^\gamma_{2,\beta} \frac{\partial P_0}{\partial t^\gamma} = \frac{\partial P_0}{\partial t^2} \frac{\partial P_0}{\partial t^\beta}, \quad 2 \leq \beta \leq N-1,$$

that recursively determine all the derivatives $\frac{\partial P_0}{\partial t^\beta}$ starting from the derivatives $\frac{\partial P_0}{\partial t^2} = t^N$ and $\frac{\partial P_0}{\partial t^1} = 1$. This completely determines the polynomial P_0. We conclude that $F^o = \left(\lambda^{-1} F^o_{B_N}(t^1, \dots, t^N, \lambda s) \right)\big|_{\lambda=0}$.

Case $W = I_2(k)$. We have $q_1 = 1$, $q_2 = \frac{2}{k}$, $\delta = \frac{k-2}{k}$ and $F_{I_2(k)} = \frac{(t^1)^2 t^2}{2} + \alpha_k \frac{(t^2)^{k+1}}{(k+1)!}$, $\alpha_k \neq 0$. The function $F^o_{I_2(k)}$ satisfies the open WDVV equations together with equations (1.5) and (6.4), because $\langle \tau_{\alpha_1} \dots \tau_{\alpha_n} \tau_\mu \rangle_{A_{k-1}} = 0$, if $\alpha_i \in \{1, k-1\}$ and $\mu \notin \{1, k-1\}$ [Zub94, Section 1].

Note that if a function F^o satisfies property (1.5), then all the open WDVV equations are automatically satisfied except equation (1.4) with $\alpha = \beta = 2$.

Suppose $k = 2l+1$, $l \geq 1$. A polynomial $F^o(t^1, t^2, s)$, satisfying (1.5) and (6.4), has the form

$$F^o = t^1 s + \sum_{i=0}^{l+1} \beta_i \frac{s^{2l+2-2i}}{(2l+2-2i)!} \frac{(t^2)^i}{i!}, \quad \beta_i \in \mathbb{C}.$$

Suppose that the open WDVV equations are satisfied. Equation (1.4) with $\alpha = \beta = 2$ is equivalent to

$$\begin{aligned} &\frac{\partial^2 F_{I_2(2l+1)}}{\partial (t^2)^3} + \frac{\partial^2 F^o}{\partial (t^2)^2} \frac{\partial^2 F^o}{\partial s^2} - \left(\frac{\partial^2 F^o}{\partial t^2 \partial s} \right)^2 = 0 \Leftrightarrow \\ \Leftrightarrow &\alpha_{2l+1} \frac{(t^2)^{2l-1}}{(2l-1)!} + \left(\sum_{i=2}^{l+1} \beta_i \frac{s^{2l+2-2i}}{(2l+2-2i)!} \frac{(t^2)^{i-2}}{(i-2)!} \right) \left(\sum_{i=0}^{l} \beta_i \frac{s^{2l-2i}}{(2l-2i)!} \frac{(t^2)^i}{i!} \right) \\ &- \left(\sum_{i=1}^{l} \beta_i \frac{s^{2l+1-2i}}{(2l+1-2i)!} \frac{(t^2)^{i-1}}{(i-1)!} \right)^2 = 0. \end{aligned}$$

The expression on the left-hand side of the last equation has the form $\sum_{i=0}^{2l-1} (t^2)^{2l-1-i} s^{2i} P_i(\beta_0, \dots, \beta_{l+1})$, where

$$P_0 = \frac{\alpha_{2l+1}}{(2l-1)!} + \frac{\beta_{l+1}\beta_l}{(l-1)!l!}, \qquad P_i = \frac{\beta_{l+1}\beta_{l-i}}{(l-1)!(l-i)!(2i)!} + Q_i(\beta_{l-i+1}, \dots, \beta_{l+1}), \quad 1 \leq i \leq l,$$

and Q_i are polynomials in $\beta_{l-i+1}, \dots, \beta_{l+1}$. We see that $\beta_{l+1} \neq 0$ and the equations $P_i = 0$, $0 \leq i \leq l$, determine the coefficients $\beta_0, \dots, \beta_l$ in terms of the coefficient β_{l+1}. Thus, $F^o = \lambda^{-1} F^o_{I_2(2l+1)}(t^1, t^2, \lambda s)$ for some $\lambda \neq 0$.

Suppose $k = 2l$, $l \geq 2$, and a polynomial F^o satisfies the open WDVV equations together with equations (1.5) and (6.4). Then F^o has the form

$$F^o = t^1 s + \sum_{i=0}^{l} \beta_i \frac{s^{2l+1-2i}}{(2l+1-2i)!} \frac{(t^2)^i}{i!}, \quad \beta_i \in \mathbb{C},$$

and equation (1.4) with $\alpha = \beta = 2$ is equivalent to

$$\begin{aligned} &\alpha_{2l} \frac{(t^2)^{2l-2}}{(2l-2)!} + \left(\sum_{i=2}^{l} \beta_i \frac{s^{2l+1-2i}}{(2l+1-2i)!} \frac{(t^2)^{i-2}}{(i-2)!} \right) \left(\sum_{i=0}^{l-1} \beta_i \frac{s^{2l-1-2i}}{(2l-1-2i)!} \frac{(t^2)^i}{i!} \right) \\ &- \left(\sum_{i=1}^{l} \beta_i \frac{s^{2l-2i}}{(2l-2i)!} \frac{(t^2)^{i-1}}{(i-1)!} \right)^2 = 0. \end{aligned}$$

The expression on the left-hand side has the form $\sum_{i=0}^{2l-2} (t^2)^{2l-2-i} s^{2i} P_i(\beta_0, \dots, \beta_l)$, where

$$\begin{aligned} P_0 =& \frac{\alpha_{2l}}{(2l-2)!} - \frac{\beta_l^2}{((l-1)!)^2}, \\ P_i =& \gamma_i \beta_l \beta_{l-i} + Q_i(\beta_{l-i+1}, \dots, \beta_l), \quad \gamma_i = \frac{2l(i-1)}{(l-1)!(2i)!(l-i)!}, \quad 1 \leq i \leq l, \end{aligned}$$

and Q_i are polynomials in $\beta_{l-i+1}, \dots, \beta_l$. We see that the equation $P_0 = 0$ determines β_l up to a sign and then the equations $P_i = 0$, $2 \leq i \leq l$, determine the coefficients $\beta_0, \dots, \beta_{l-2}$ in terms of β_{l-1}. Thus, $F^o = \lambda^{-1} F^{o,\pm}_{I_2(2l)}(t^1, t^2, \lambda s)$ for some λ. □

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