Yandex × PME

ВЫСШАЯ ШКОЛА ЭКОНОМИКИ
НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ УНИВЕРСИТЕТ

# PROCEEDINGS OF THE PME AND YANDEX RUSSIAN CONFERENCE: **TECHNOLOGY AND PSYCHOLOGY FOR MATHEMATICS EDUCATION**

Moscow, 2019

Yandex × PME

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# PROCEEDINGS OF THE PME AND YANDEX RUSSIAN CONFERENCE: **TECHNOLOGY AND PSYCHOLOGY FOR MATHEMATICS EDUCATION**

Plenary Lectures, Plenary Discussions,
Research Reports, Oral Communications, Poster Presentations

Editor: Anna Shvarts

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# PREFACE

The second conference in the row of PME regional conferences was organized in Moscow, Russia on 18–21st of March 2019. This conference aimed to facilitate a dialogue between mathematics education community in Russia and international researchers in mathematics education. The conference theme was Technology and Psychology for Mathematics education, and the participants primarily focused on the psychological aspects of mathematical abilities and mathematical processing, as well as on the efficiency and particularities of the technologically supported learning. However, there was a range of presentations about teacher education, teaching methods for diverse students, educational issues from a neuroscience perspective, and other aspects of mathematics education.

Fifty-five Russian participants from 16 Russian regions and 32 international and former-Russian participants from 15 countries, including India, China, and Nepal took part in the conference. The conference spanned over four days and for most of the time it was run in 2 parallel sessions. Altogether, there were 61 individual contribution, 5 plenary lectures and 2 plenary discussion. The poster session had 21 presentations. The conference was open to the broad audience and about 200 listeners attended the conference sessions.

## WELCOMING NOTES

The International Group for the Psychology of Mathematics Education (PME) exists to promote lively and productive interaction between mathematics education researchers across the globe. Before the Moscow regional conference, we have organized 42 international conferences, the PME conference in 2018 attracting 689 participants to Umeå, Sweden. Yet, we see that some countries have been underrepresented in PME conferences. To make it easier for researchers from these areas to join PME, we initiated regional conferences. I am very excited that this second regional PME conference is organized in Russia. The Russian mathematics education has a strong tradition unlike any other, and it is really a loss for PME that so few Russian mathematics educators and education researchers have participated in PME. This conference has brought together Russian mathematics educators and active PME members. We hope that you make new friends and build new research collaboration networks. We hope to see more Russian researchers and educators in the future PME conferences. We welcome you to the PME family.

*Markku S. Hannula,*
President of The International Group for the Psychology of Mathematics Education

Dear Conference Participants, I'm happy to welcome you in Moscow, at Yandex. Yandex technologies would have never developed the way they have without Russia's strong traditions in mathematics and fundamental sciences. We believe it is crucial to support and stimulate young generations' interest in mathematics and offer them opportunities in this field. For this reason, we developed a range of educational programs, from courses in programming for high school to advanced courses in data analysis to a digital learning platform for primary school. Personally, I have drawn great inspiration from teaching programming and algebra, which I have been doing at Moscow State University since 2001, and I am now grateful for this opportunity that we have to explore the advances of technology and psychology and their application in teaching mathematics.

*Dr Elena Bunina,*
Professor of Higher Algebra at Moscow State University,
CEO of Yandex in Russia, Human Resources Director at Yandex

Russian mathematical education for gifted students is a unique phenomenon in the international landscape. Our school of thought in pure mathematics is well known, as well as the outstanding results of USSR and Russian school children at the international competitions in mathematics. Russian scholars also contributed greatly to the field of educational and development psychology, and such names as Vygotsky, Leontiev, Davydov, and Krutetsky still inspire many specialists all over the world. The dialogue between Russia and other countries in the field of research in mathematics education evanesced after 1917. I reckon it should be re-established and maintained at the new level, as a joint effort of specialists from the variety of disciplines. I believe there are many researchers in Russia, who are curious and excited to investigate what it means to understand mathematics. How subtle communication between a teacher and a student can support the latter, or which aspects of the teaching process can be efficiently outsourced to AI, are just two examples of studies that can prove useful to improve learning results across the country. Our conference, themed "Technology and Psychology for Mathematics Education", acted as an invitation for the Russian researchers to join the international community and to share their experience, results, doubts and inspiration. I am happy that conference days were full of discussions, and I am looking forward to collaboration projects originating from this conference.

*Anna Shvarts,*
The Conference Chair

# THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

The International Group for The Psychology in Mathematics Education has been was founded at the Third International Congress on Mathematics Education (ICME-3) in 1976 and is an official subgroup of the International Commission for Mathematical Instruction (ICMI).

All information concerning PME and its constitution can be found at the PME website: http://www.igpme.org/

## THE GOALS OF PME

The major goals of the group are:

- to promote international contact and exchange of scientific information in the field of mathematical education;
- to promote and stimulate interdisciplinary research in the aforesaid area; and to further a deeper and more correct understanding of the psychological and
- other aspects of teaching and learning mathematics and the implications thereof.

## THE FORMER PME PRESIDENTS

Efraim Fischbein, Israel

Richard R. Skemp, UK

Gerard Vergnaud, France

Kevin F. Collis, Australia

Pearla Nesher, Israel

Nicolas Balacheff, France

Kathleen Hart, UK

Carolyn Kieran, Canada

Stephen Lerman, UK

Gilah Leder, Australia

Rina Hershkowitz, Israel

Chris Breen, South Africa

Fou-Lai Lin, Taiwan

João Filipe Matos, Portugal

Barbara Jaworski, UK

Peter Liljedahl, Canada

Markku S. Hannula, Finland (present)

### PME CONFERENCES

The PME conferences are the main annual meetings for the researchers from all around the world who are interested in psychology and other research issues in mathematics education. During 5 days, the members of PME have the opportunity to communicate personally with each other during working groups, poster sessions and many other activities. Every year the conference is held in a different country.

### REGIONAL PME CONFERENCES

PME provides annual funding for a regional conference to support researchers in regions currently underrepresented at PME. This initiative aims to supportthe development of a regional research community that pursues the goals of PME, and by doing so, encourage researchers from that region to actively participate in future PME conferences and help them in preparing top quality PME contributions. The first regional PME conference: South America was conducted in Chili in 2018. PME and Yandex Russian conference is the second conference that is funded by this initiative.

### PME MEMBERSHIP AND TRAVELLING SUPPORT INFORMATION

Membership is open to people involved in active research consistent with the aims of PME, or professionally interested in the results of such research. Membership is on an annual basis and depends on payment of the membership fees. PME has between 700 and 800 members from about 60 countries all over the world.

There is limited financial assistance for attending PME conferences by the researcher from underrepresented countries available through the Richard Skemp Memorial Support Fund.

## YANDEX

Yandex is a technological company that makes intelligent products and services that help people solve their day-to-day problems both online and offline. Complex technologies behind Yandex products and services are unique. That is what enables us to do things that once would have sounded like magic.

Our team of specialists represents many scientific disciplines, including mathematics, data analysis, programming, linguistics, and many others. Besides working on products and technologies at Yandex, some of our experts teach, lecture and train students and young specialists.

We use the leading methodological and technological expertise to develop cutting-edge solutions for digital education. Our education platform facilitates personalized education in the Russian language and mathematics for primary school. It has been tested by 4,200 students in 73 schools in 15 regions in Russia and has received overwhelmingly positive feedback. Analytical tools that we provide allow teachers to follow the progress of each of their students, pinpoint factors that boost students' performance, and choose the best tactics to achieve top results.

Read more about Yandex at www.yandex.com/company.

# CONFERENCE ORGANIZERS

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Keith Jones, University in Southampton

Roza Leikin, University of Haifa

Elena Kardanova, Nat_ional Research University Higher School of Economics

Sergey Polikarpov, Moscow Pedagogical State University

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Lyubov Galitskaya

Alexandra Ledneva

Ekaterina Lomachenkova

Anna Smulyanskaya

Anna Shirokova-Koens

Anna Shvarts (PME representative)

## SUPPORTERS OF THE CONFERENCE

International Commission on Mathematical Instruction (ICMI)

Institute of Education, National Research University Higher School of Economics

Institute for Mathematics and Computer Science, Moscow Pedagogical State University

Psychological Institute, Russian Academy of Education

Russian Federal Institute of Education Development

National Institute of Quality Education

Russian Psychological Society

Freudenthal Institute for Science and Mathematics education, Utrecht University

## FORMAT OF THE PRESENTATIONS AND REVIEWING PROCESS

The participants were invited to submit contributions in one of the following formats.

### RESEARCH REPORTS (RR)

Research Report is 20-minute talk presenting original piece of research in mathematics education, followed by a 20-minute discussion. Proposal for Research Report can have the maximum length of eight pages. It should provide the theoretical framework, preliminary results, and a discussion of these results. It is important to state what is new in this research, how the study builds on past findings, and/or how it has developed new directions and pathways.

We have received 48 proposals for a Research Report; each of them was reviewed by 3 experts (2 international and 1 Russian). Four research Reports were accepted after the first stage of review and are 15 RR were accepted after revision. 11 RR were invited to resubmit as Oral Communication and 12 were invited to resubmit as Poster Presentation.

### ORAL COMMUNICATIONS (OC)

Oral Communication is a 10-minute talk presenting a study in a context of broader research in mathematics education, followed by a panel discussion in a group of authors on the same or related subject. Proposal for Oral Communication is one page long. The number of submitted OC proposals was 37 and 7 RR proposals were resubmitted as Oral Communication. In the end, 25 Oral Communications were accepted for the presentation at the conference.

### POSTER PRESENTATIONS (PP)

Poster Presentation demonstrates a piece of research in a visual format with a flexibly structured discussion during poster session. This format most benefit the research that is best communicated visually. The number of submitted PP proposals was 32, and 1 RR was resubmitted as a Poster Presentation. In the end, 21 Poster Presentations were accepted.

Altogether, we received 123 submissions and 65 of them were accepted for the presentation.

### LIST OF THE REVIEWERS

We are grateful to all reviewers who guaranteed the high-level of the scientific program:

Hatice Akkoç, Turkey
Mette Susanne Andresen, Norway
Samuele Antonini, Italy
Michal Ayalon, Israel
Marita Barabash, Israel
Patrick Barmby, South Africa
Angelika Bikner-Ahsbahs, Germany
Alexey Vladislavovich Borovskikh, Russia
Ludmila Ivanovna Bozhenkova, Russia
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Andonis Zagorianakos, Greece

# PLENARY LECTURES

# THE EVOLUTION OF MATHEMATICS EDUCATION RESEARCH: RUSSIA'S PLACE IN THIS GLOBAL MOVEMENT

**Norma Presmeg**

Illinois State University, Normal, USA

*After outlining differences between research in pure mathematics, and mathematics education research, I highlight aspects of the founding and evolution of the International Group for the Psychology of Mathematics Education (PME), from its inception in 1976. This evolution entailed changes in theoretical paradigms, and in methodologies for research in mathematics education that were considered legitimate, ranging from an early paradigm in which only rigorous statistical research had scientific standing, through several decades of increasing acceptance of the value of qualitative research, to a more recent perception that quantitative and qualitative research methodologies have different purposes, and that each has its place—resulting in increasing use of conceptual lenses that make use of mixed methods of various types. In Russia, the Soviet psychologist V.A. Krutetskii was ahead of his time in recognizing the potential for depth in research that involved clinical interviewing of 'capable' mathematics students with various individual differences, in their approach to types of mathematical problems that he collected and categorized. I highlight Krutetskii's book, first published in Russian in 1968, which was translated into English in 1976, and which formed a strong theoretical core for my own initial research on visual thinking in teaching and learning mathematics, starting in 1982 and continuing for several decades.*

## 1. MATHEMATICS RESEARCH AND MATHEMATICS EDUCATION RESEARCH. WHAT ARE WE TALKING ABOUT?

Mathematics is central to the endeavors of mathematicians, mathematics educators, and mathematics education researchers alike. However, it appears at times as if we are talking past each other, because there are distinct fields in question. The debate initiated by Ted Eisenberg and elaborated in the book edited by Fried and Dreyfus [2014] — *Mathematics and Mathematics Education: Searching for Common Ground* — is still relevant. Thus I constructed a diagram to portray the distinctness of mathematics itself, mathematics education, and research in pure mathematics and in mathematics education respectively [Presmeg, 2014]. Each of the subsets in the ellipses (Fig. 1) may be regarded as the *topic* of the wider set in which it rests. *Mathematics*, mathematics *education*, and mathematics education *research* may be conceptualized as being nested, like Russian dolls, with mathematics at the center, nested in mathematics education, which is nested in turn in mathematics education research. In contrast, the topic addressed in pure *mathematics* research is mathematics. The topic of mathematics *education* is also mathematics; and the topic of mathematics education *research* in not mathematics *per se*, but mathematics education.

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Technology and Psychology for Mathematics Education. P. 19–31. Moscow, Russia: HSE Publishing House.

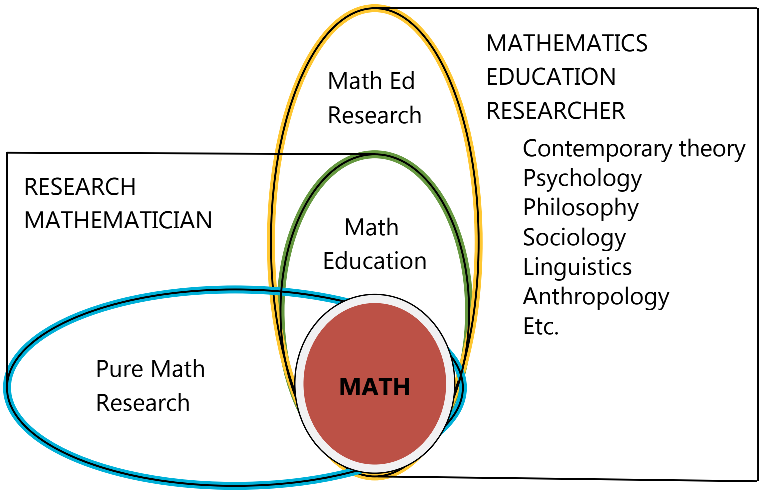



***Fig. 1.*** Mathematics at the center
of mathematics research and mathematics education research

On the one hand, mathematicians are engaged in research in various fields of mathematics, and in mathematics education insofar as they teach mathematics. On the other hand, mathematics education researchers are not engaged primarily with research in pure mathematics, except insofar as they may be also mathematicians. Their research embraces the nested model of various aspects of the "complex human worlds" ([Presmeg, 1998], with hints of Bruner's *Actual Minds, Possible Worlds*, 1986) involved in the teaching and learning of mathematics at all levels. It is clear from this conceptualization that mathematics education research cannot simply be a branch of *applied mathematics*. Unlike, for instance, *business calculus* — which could be regarded as a branch of applied mathematics — in mathematics education research we are not applying the principles of mathematics to the topic of the research, namely mathematics education.

As I have pointed out many times (e.g., [Presmeg 1998]), mathematics as a field is thousands of years old, and mathematicians have taught mathematics for thousands of years, but mathematics education *research* as a field in its own right is less than a century old, notwithstanding the important 1908 meeting in Rome at which the International Commission on Mathematics Instruction (ICMI) was founded. The International Mathematics Union (IMU) was founded in 1920 as an international community of mathematicians [Furinghetti, Giacardi, 2010] and ICMI is an affiliate of the IMU. I attended the bright celebration of the ICMI centenary meeting in Rome in 2008, and I appreciated the founding of ICMI, and the founders who were all eminent mathematicians. But mathematics education research has matured as a field in the last half-century, as evidenced by a proliferation of journals, conferences, and kinds of research that it embraces. Despite the title of Sierpinska and Kilpatrick's [1998] edited book that followed a high-level conference on mathematics education research's *search for identity*, we are still trying to find out who we are! But it is clear that mathematics education research is now an established field, and many universities internationally are acknowledging this fact, e.g., by establishing mathematics education professorships.

The experience of Sweden in this regard is a case in point [Presmeg, 2014]. I elaborate on this evolution as it appears in the history of PME, in the next section.

## 2. THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS DUCATION (PME)

At the meeting of the Third International Congress on Mathematical Education (ICME-3) in 1976, several eminent scholars, including Hans Freudenthal from The Netherlands, decided to establish an international association devoted to mathematics learning and teaching (PME) which would meet annually in various countries. The PME-1 meeting was held at Utrecht in 1977, with 86 participants. PME-2 was in Osnabrück, Germany, and PME-3 in Warwick, England. I was privileged to attend PME-4, in Berkeley, California, in 1980. With its high-level presentations and valuable international interaction among participants, it quickly became my favorite conference: I have now attended 29 PME meetings, 27 of them consecutively, from PME-12 in Veszprem, Hungary, in 1988, to PME-38 in Vancouver, Canada, in 2014, where there were 865 participants. This Regional PME conference in Moscow includes an invitation to all Russian researchers concerned with mathematics education to take their place in this fruitful international association.

As suggested by the name of the PME organization, *psychology* was a dominant theoretical field in its early years. However, as perceptions of teaching and learning mathematics evolved, it was recognized that *social* and *cultural* aspects of this field were just as important as individual ones, and for several years in the 1990s there was lively debate about changing the name of PME to be more inclusive in its focus. For historical reasons the name did not change, but in its inclusiveness PME has evolved to reflect the paradigms of the field, both theoretically and empirically. This evolution of the field included changes in conceptual lenses and methodologies, which I address in the next section.

## 3. QUANTITATIVE AND QUALITATIVE METHODOLOGIES

In an age when only rigorous quantitative studies were deemed *scientific* in Western educational research, Krutetskii [1976] wrote as follows:

> It is hard to understand how theory or practice can be enriched by, for instance, the research of Kennedy, who computed, for 130 mathematically gifted adolescents, their scores on different kinds of tests and studied the correlation between them, finding that in some cases it was significant and in others not. The process of solution did not interest the investigator. But what rich material could be provided by a study of the process of mathematical thinking in 130 mathematically able adolescents! [p. 14]

According to this statement, in the wider field he was ahead of his time. During the decades of the 1980s and 1990s, the fine grain of qualitative research to which Krutetskii was referring overtook the quantitative paradigm and became dominant. My own early research, starting in 1982 and inspired by Krutetskii's work, involved both mild statistical analyses to determine trends, and the deeper *think-aloud* clinical interviews that he promulgated [Presmeg, 1985]. It was the qualitative methodology, rather than the quantitative, that allowed for depth of understanding of the mathematical thought processes of the 52 visualizers in my study, and their teachers. Krutetskii's contribution is outlined further in the next section. Currently there is greater appreciation for the potential and place of both quantitative and qualitative research in our field, because their strengths and disadvantages complement each other — they have different purposes. The generalizability of statistical methods is counterbalanced by the finer insights of qualitative interviews and observations, resulting in increasing use of *mixed methods* of various types in mathematics education research [Bikner-Ahsbahs, Knipping, Presmeg, 2015].

In Soviet Russia, no mental testing was done for psychological purposes after 1936, although achievement tests were still used in schools. Krutetskii [1976, Ch. 2] criticized the excessive use of psychological testing using factor analysis in the West, although he did use factor analysis himself, for the purpose of supporting his position [p. xv]. The significance of his work is that he demonstrated powerfully the advantages of clinical interview techniques using think-aloud procedures, for depth of understanding of individual differences in children's mathematical processing.

## 4. KRUTETSKII'S WORK ON THE PSYCHOLOGY OF MATHEMATICAL ABILITIES IN SCHOOLCHILDREN

Vadim Andreyevich Krutetskii (1917–1991) graduated in 1941 with a degree in economic geography from Moscow State University, and received his PhD in 1950 from the USSR Academy of Pedagogical Sciences in Moscow, where he remained for nearly 30 years, becoming deputy director of the Research Institute of General and Educational Psychology (Wikipedia). His work on individual differences became known to English-speaking scholars in 1963, when he presented a brief paper on mathematical abilities at the 17th International Congress on Psychology in Washington DC. Four of his papers were translated into English in 1969 for the publication of Vol. 2, *The Structure of Mathematical Abilities,* in the series Soviet Studies in the Psychology of Learning and Teaching Mathematics. However, it was the translation of his 1968 book (Soviet monograph, 431 pages, 25,000 copies) into English in 1976 with the title *The Psychology of Mathematical Abilities in Schoolchildren,* that caused the editors Jeremy Kilpatrick and Izaak Wirszup to consider the potential impact of his work to be comparable to that of Piaget.

Although Krutetskii [1976] on the basis of his research worked out a typology of the processes of thinking of "mathematically able adolescents", his research extended to what he called both "capable" and "incapable" high school mathematics students [Krutetskii, 1969], and to the individual differences amongst them. He defined *ability* as "a personal trait that enables one to perform a given task rapidly and well, in contrast to a habit or skill, which is a characteristic of one's activity" [Krutetskii, 1976, p. xiii]. The types of *abilities* (rather than 'ability') that he identified based on the extant literature at the time, and on his research results, were as follows:

1. *Analytic*, with a predominant verbal-logical component; visual-pictorial components and spatial concepts are weakly developed, but are not required by capable pupils in this category.

2. *Geometric*, with a very strong visual-pictorial component dominating an above-average verbal-logical component. These pupils feel a need for visual thinking.

3. Harmonic, with the verbal-logical and visual-pictorial components in equilibrium. One subset of pupils in this category — *abstract-harmonic* — can use visual supports but they do not help; a second subset — *pictorial-harmonic* — can use visual supports and they are helpful.

Krutetskii's [1976, p. 350] analysis of the structure of mathematical abilities — distinguishing what he called "very capable and capable" mathematics schoolchildren from those called "incapable" — included the following components:

1. *Obtaining mathematical information.* The ability for formalized perception of mathematical material, for grasping the formal structure of a problem.

2. *Processing mathematical information.* Abilities for logical thought, generalization, curtailment, flexibility, clarity and logical economy, and reversibility.

3. *Retaining mathematical information.* A generalized memory for mathematical relationships.

4. *General synthetic component.* A mathematical cast of mind — seeing the world through mathematical eyes.

It is noteworthy that swiftness of processing, computational abilities, memory for symbols, numbers and formulas, and even abilities for spatial concepts and visualizing abstract mathematical relationships, were not included in this structure — they were *optional* for high performance! These aspects determined the *type* of mathematical processing, not its efficacy.

In the remainder of this paper, I shall concentrate on the strengths and difficulties associated with visualization in teaching and learning high school mathematics. As a brief introduction, consider a problem from Series XXIII, Vol. 2 of Krutetskii's [1976] problem bank:

*How much does a brick weigh, if it weighs 1 kg plus half a brick?* [p. 158]

How many of you solved it using an algebraic equation (mentally or written down)?

How many of you used a visual image (either imagined or drawn in a picture)?

Some people imagine a scale with a brick on one pan, and a 1 kg weight with half a brick in the other. Then they discard half a brick from both sides: the scale is still in equilibrium. Thus half a brick weighs 1 kg, and the whole brick weighs 2 kg.

## 5. VISUALIZATION RESEARCH BASED ON KRUTETSKII'S FORMULATION

Based on Krutetskii's work, I used a model (e.g., [Presmeg, 2014, p. 211]) that challenged the analytical-visual dichotomy that had been used in early research studies in

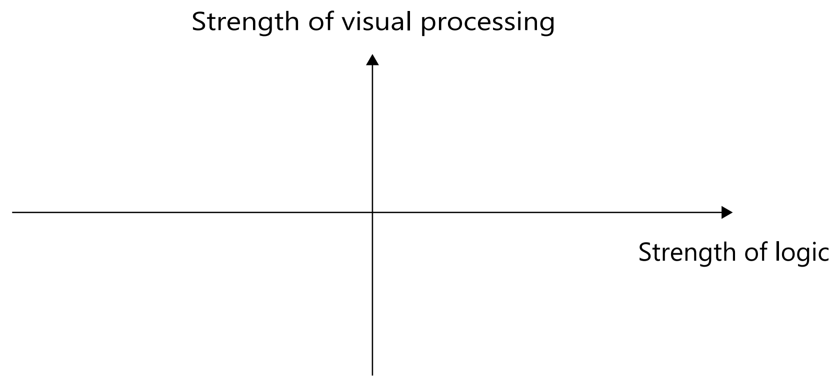



*Fig. 2.* Orthogonal axes for logic and visualization in mathematics

this field (e.g., [Lean, Clements, 1981]). Following Krutetskii's [1976] formulation, the strength of *logic* (and analysis) determines the effectiveness of mathematical problem solving, whereas the presence or absence of *visualization* determines its type. That is, *all* mathematical thinking involves logic (which could be depicted on the $X$ axis, Fig. 2), but mathematical visualization is orthogonal to it (on the $Y$ axis) and could be present or absent. My characterization of visualization from the 1980s went beyond Bishop's [1980] distinction between Interpreting Figural Information (IFI) and Visual Processing (VP), although these provided a starting point. Krutetskii's [1976] theoretical formulation was central in my research. For him, strength of logic determines the effectiveness of mathematical thinking, whereas visualization is optional. There is no duality between logical analysis and visualization in an either-or sense.

My research identified individuals in all four quadrants of this model [Presmeg, 1985] according to their mathematical logic and preference. In my work, visualization could be of the form of mental visual imagery (internal representations) — but it could also be of the form of inscriptions of various kinds (external representations). In keeping with the Peircean semiotic framework I used in my later research [2006], my working definition is that a visual image is a mental sign vehicle involving visual or spatial information, whereas inscriptions are external sign vehicles.

## PREFERENCE FOR VISUALIZATION IN MATHEMATICS

In mathematics, sign vehicles are often of a visual nature; even algebraic symbolism has a structure and needs to be seen, either mentally or in written form. However, one might talk more broadly about individual *preferences* for visualization in mathematics, and guided by the work of Suwarsono [1982] who worked with seventh graders in Australia, I constructed an instrument to measure the mathematical visuality of high school students (grades 11 and 12) and their mathematics teachers. Although most of the problems of Krutetskii's Series implicitly imparted information about the preferences for visual thinking of his interviewees, the instrument I constructed was designed to measure such preferences explicitly. I collected a 'problem bank' of several hundred problems, some of which were drawn from other sources (e.g., [Kordemsky, 1981]) and some from Krutetskii's [1976] Series XXIII–XXVI, although no diagrams were included in my instrument because they might induce visual thinking. These were all 'word problems' without any figural content, which were rigorously pilot-tested for solution with and without visual means. Parts A (6 items) and B (12 items) of the preference instrument were designed for students in the last years of high school; Parts B (the same 12 items) and C (6 more difficult items) were intended for their mathematics teachers. After standardization and checks for validity and reliability, the instrument was used to select teachers of a range of styles, and visualizers in their grade 12 mathematics classes. A *visualizer* is a person who prefers to use visual methods (including visual imagery) to solve problems that are capable of solution by visual and nonvisual means, as in my instrument. The frequency distribution graphs of visuality scores indicated that for most populations this frequency follows a normal, Gaussian, distribution. But there are people at both ends of the scale: some who seldom, if ever, feel the need to visualize, and others for whom it is not an option, they always feel the need. Those whose visuality scores were above the median were taken to be the visualizers.

Briefly, the results of this initial research [Presmeg, 1995] were surprising in several ways. After a year of intensive observation of lessons in the mathematics classes of 13 teachers, and clinical interviews with 54 visualizers in their classes, data analysis yielded the following results:

- For the teachers, a Teaching Visuality (TV) score (based on triangulation of observations, interview data with teachers and students, analyzing 12 elements of their teaching) was only weakly correlated with the Mathematical Visuality scores they obtained on the preference instrument (Spearman's rho = 0.404). According to the TV scores, the teachers fell neatly into three groups (Table 1): visual, nonvisual, and a middle group that used visual means in their teaching, but also stressed abstraction and generalization. It made sense that some teachers who had little preference for visual methods in their solving of mathematical problems, would nevertheless use visualization in their teaching because they believed it was beneficial for their pupils.
- For the visual students, five different kinds of imagery were identified as they solved mathematics problems of various kinds from the school-leaving national examinations of previous years: concrete pictorial imagery; kinaesthetic (involving

Table 1

**Three groups of teachers, according to their Teaching Visuality scores**

| Nonvisual Group | | Middle group | | Visual Group | |
|---|---|---|---|---|---|
| **Teacher** | **Score** | **Teacher** | **Score** | **Teacher** | **Score** |
| Mrs Crimson | 2 | Mr Blue | 7 | Mr Red | 9 |
| Mr Black | 3 | Mrs Turquoise | 7 | Mrs Gold | 9 |
| Mr Brown | 4 | Mrs Green | 7 | Mrs Silver | 10 |
| Mr White | 3 | Mr Grey | 6 | Mrs Pink | 9 |
| | | | | Miss Mauve | 10 |

physical movement); dynamic (the image itself is moved or transformed), memory images of formulas (often involving spatial components); and pattern imagery (pure relationships stripped of concrete details).

- All the difficulties experienced by these visualizers involved generalization in some way. Prototypical images could curtail flexibility and limit processing. Concrete imagery often had mnemonic advantages, but was not helpful unless the limitations could be overcome. There were two ways in which generalization could be achieved, namely, through concrete imagery that was metaphoric, capturing a mathematical principle or relationship; and through pattern imagery, that was in itself of a generalized nature, depicting pure mathematical relationships.
- The visualizers in classes of nonvisual teachers struggled, attempting to memorize blindly. But the visualizers with visual teachers *also* struggled! Their teachers were enthusiastic in their visual teaching, but did not understand the generalization difficulties with which their visual students were struggling — which were not problems for the teachers themselves. It was the teaching by teachers in the middle group that was most beneficial for the visualizers, enabling them to use their preferred visual mode without its limitations.

When this early research was conducted, there were few studies in this field. Subsequently, research on visualization became a mainstream category, as reflected in the increasing number of research reports in this area presented in PME conferences (see the historical account by Presmeg in 2006, which documents such research in PME 1976–2006). Many of these later studies addressed how visualization could best be incorporated in the mathematics curricula at various levels. There is still scope especially for research that incorporates the changing nature of visualization caused by the explosion of technology and its availability in the classroom. In my chapter in the first PME Handbook [Presmeg, 2006], I put forward 13 "big research questions", many of which still offer opportunities for scientific studies that forward our knowledge in this field, started by Krutetskii so many decades ago. These questions were used in a closing commentary to analyze the papers in an issue of *ZDM — The International Journal on Mathematics Education*, which brought the field of research on visualization up to date at that point [Rivera, Steinbring, Arcavi, 2014]. These questions were as follows:

1. What aspects of pedagogy are significant in promoting the strengths and obviating the difficulties of use of visualization in learning mathematics?
2. What aspects of classroom cultures promote the active use of effective visual thinking in mathematics?
3. What aspects of the use of different types of imagery and visualization are effective in mathematical problem solving at various levels?
4. What are the roles of gestures in mathematical visualization?
5. What conversion processes are involved in moving flexibly amongst various mathematical registers, including those of a visual nature, thus combating the phenomenon of compartmentalization?

6. What is the role of metaphors in connecting different registers of mathematical inscriptions, including those of a visual nature?
7. How can teachers help learners to make connections between visual and symbolic inscriptions of the same mathematical notions?
8. How can teachers help learners to make connections between idiosyncratic visual imagery and inscriptions, and conventional mathematical processes and notations?
9. How may the use of imagery and visual inscriptions facilitate or hinder the reification of processes as mathematical objects?
10. How may visualization be harnessed to promote mathematical abstraction and generalization?
11. How may the affect generated by personal imagery be harnessed by teachers to increase the enjoyment of learning and doing mathematics?
12. How do visual aspects of computer technology change the dynamics of the learning of mathematics?
13. What is the structure and what are the components of an overarching theory of visualization for mathematics education? [Presmeg, 2006, p. 227].

## 6. AFTERWORD: KRUTETSKII'S LEGACY IN THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

Krutetskii's Series of problems in 26 categories [1976, Ch. 8, p. 98ff.], which he used for qualitative and quantitative research, are still a treasure trove for researchers interested in school mathematics teaching and learning, not only in the area of preference for visualization in mathematics, but concerning all aspects of the effective fostering of the components of mathematical abilities through teaching. His careful categorization and gradation of the problems in the Series provides a groundwork that has been scarcely touched by researchers. The categories of problems spanned all of the elements of the *Structure of Mathematical Abilities* that his analyses confirmed:

| | | |
|---|---|---|
| *Obtaining mathematical information* | (interpretation of a problem): | Series I–IV |
| *Information processing* | (generalization): | Series V–XII |
| | (flexibility): | Series XIII–XVI |
| | (reversibility): | Series XVII |
| | (reasoning and logic): | Series XVIII–XXI |
| *Information retention* | (mathematical memory): | Series XXII |
| *Typology* | (types of mathematical ability): | Series XXIII–XXVI |

In closing, I illustrate the advantages of visual thinking, especially with the affordances of today's technology, using some of Krutetskii problems from his *Typology* series, which were also part of Section C of my preference for visualization instrument (called the Mathematical Processing Instrument: MPI). The following problem [Ibid., p. 159], which he took from Kordemsky, was the final problem in the section of the MPI intended for high school mathematics teachers:

C-6: A train passes a telegraph pole in a quarter of a minute, and in three quarters of a minute it passes completely through a tunnel 540 meters long. What is the train's speed in meters per minute, and its length in meters?

The cognitive complexity of this problem seems to reside in synchronizing the pole and the tunnel. Many people resort to algebraic solutions. However, a creative visual solution to the problem is as follows (Fig. 3).

The first segment of the journey is where the train is before it enters the tunnel. Each segment takes the train a quarter minute. It takes the front of the train half a minute to go the length of the tunnel. Thus, the speed of the train is 540 × 2 = = 1080 metres/minute, and the length of the train is half the length of the tunnel, i.e., 270 metres.

In my original research [Presmeg, 1985] some of the participants drew a picture of the train, complete with engine and coaches and smoke emerging from the stack, in a diagram that elaborated on the abstract one in Fig. 3. The concrete details seemed unnecessary. However, Susana Carreira ((Personal email communication, 2014); [Presmeg, 2018]) pointed out that with today's available technology, it is easy to add concrete details to the picture, as she demonstrated (Fig. 4). Such details could be importance resources in students' sense making, and could enhance their positive affect and incentive to solve the problem. Unlike the algebraic process, the visual method yields an instant solution.

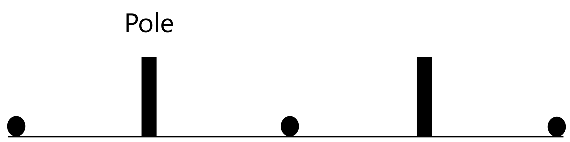



*Fig. 3.* A creative visual solution to the Train in Tunnel problem

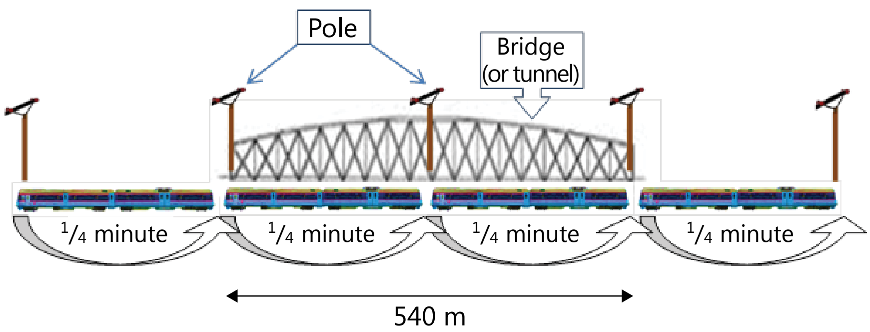



*Fig. 4.* Pattern imagery and concrete imagery: Susana's creative concrete solution

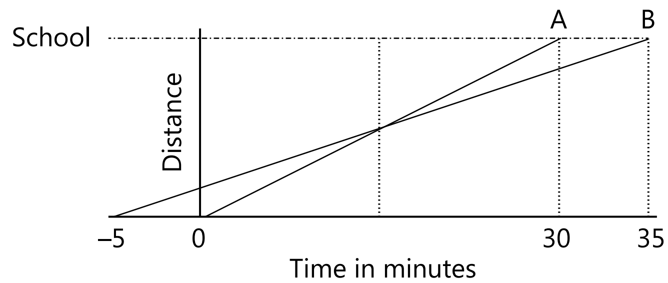



*Fig. 5.* A visual solution to the boys walking to school problem

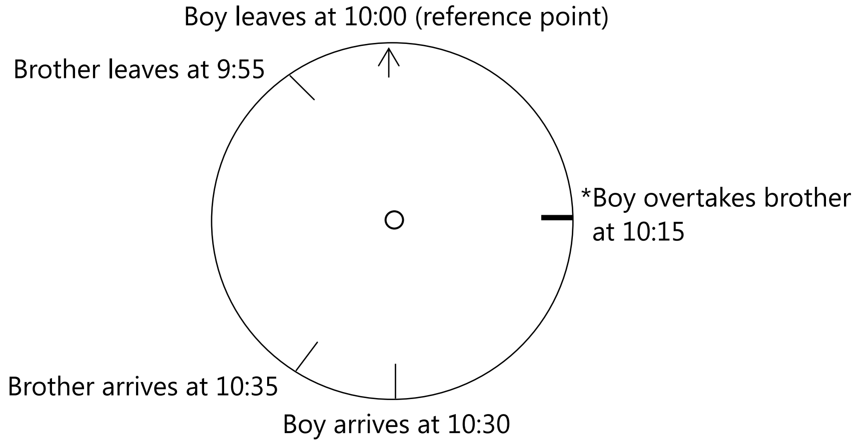



*Fig. 6.* A solution using the metaphor of a clockface

Two visual solutions provided by interviewees to the third problem in Section C of the MPI, are shown in Fig. 5 and 6.

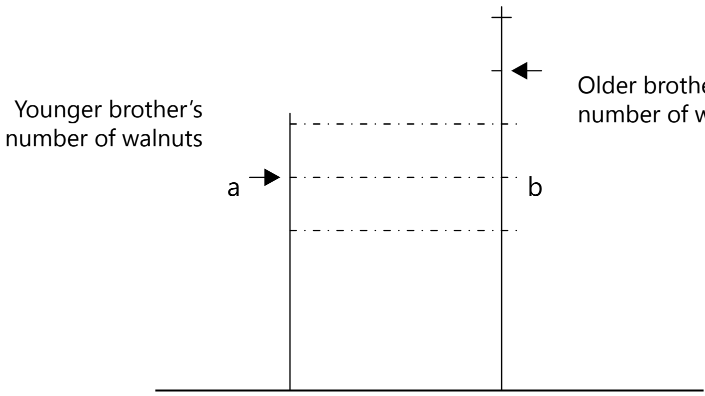
C-3: A boy walks from home to school in 30 minutes, and his brother takes 40 minutes. His brother left 5 minutes before he did. In how many minutes will he overtake his brother?

Finally, some problems are less visual in the sense that it is more difficult to use a visual solution than an algebraic one for most people. Nevertheless, visualizers naturally and creatively solve them visually (e.g., Fig. 7)!

C-4: An older brother said to a younger, "Give me eight walnuts, then I will have twice as many as you do." But the younger brother said to the older one, "You give me eight walnuts, then we will have an equal number." How many walnuts did each have?

I have given an account *only* of the work of Krutetskii in the psychology of mathematics education (PME) in Russia. Taking into account the rich legacy, which will be elaborated by others in this conference, of other Russian researchers who have contributed to mathematics education both theoretically and empirically, I look forward to Russian scholars' further contributions to the international field of research in mathematics education.

C-4. Solution 2: I drew a diagram to represent the number of walnuts.




From the conditions of the problem, the top half of line b is divided into 4 equal parts, each representing 8 walnuts. Thus 7 of these parts represents the older brother's number of walnuts and 5 parts represents the younger brother's. Thus the younger brother had 40 and the older brother 56 walnuts.

***Fig. 7.*** A creative visual solution to the walnuts problem

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# RESEARCHING VYGOTSKY, AND RESEARCHING WITH VYGOTSKY IN MATHEMATICS EDUCATION

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*Vygotsky's approach to child development has had, and continues to have, a major impact on teaching and learning and on research in education. In this talk I will show some of that work and its significance in the field of mathematics education, whilst pointing to misunderstandings circulating in the community. Vygotsky died young, aged just 37. It is evident, given how he developed and changed his ideas as he approached death, that he had so much more to say but did not have time to write enough nor elaborate fully his theory. Thus we continue to research him and his ideas, as well as work to apply his insights in our studies of teaching and learning. I propose future directions for research in mathematics education building on Vygotsky's thought.*

## INTRODUCTION

The non-Russian speaking world came late to Vygotsky's work; the education field even later. Theories of teaching and learning go back many millennia, in Greece back to Socrates through Plato, for example, and many Chinese scholars believe the history of education in China can be traced back at least as far as the 16th century BC, with Confucius being a major influence from the fifth century BC. For thinkers such as these, ideas and concepts concerning how people learn and how people should learn, as well as related theories of how teaching should be framed, were derived from their philosophies of humanity and culture. In the last 100 years or so educationalists have turned, in large part, to theories in psychology and in particular to what psychologists have to say about how the human mind develops.

We have been very fortunate to have two great thinkers and researchers in child development in the twentieth century, both born in the year 1896: Jean Piaget and Lev Semyenovich Vygotsky. We know that Vygotsky read Piaget's writings. Indeed he wrote a preface for the Russian edition of Piaget's first two books, published in 1932 by Gosizdat, the State Publishing House founded in Russia in 1919. Towards the end of his life, when writing about verbal communication, he wrote the following:

> Piaget: the emergence of dispute = the emergence of verbal thinking. All forms of verbal communication between adult and child later become psychological functions. A general law: Every function appears on the scene twice in the child's cultural development, i.e., on two levels, first the social, and then the psychological, first between people as an interpsychological category, and then within the child. Cf.: La loi du dkcalage [The law of "blocking" or shifting-French] in Piaget [Vygotsky, 1989, p. 58].

Thus he expressed appreciation of Piaget but also pointed out, there and in many other places, where he disagreed with Piaget. It seems that Piaget did not encounter


In *Shvarts A.* (Ed.). Proceedings of the PME and Yandex Russian conference: Technology and Psychology for Mathematics Education. P. 32–47. Moscow, Russia: HSE Publishing House.

Vygotsky's work until some decades after Vygotsky's death. Much has been written about the differences between their thinking (see e.g., [Lerman, 1996; Steffe, Thompson, 2000; Lerman, 2000*a*]) and whether their theories can be somehow merged (see e.g., [Confrey, 1994]). Perhaps Bruner's explanation of their differences is the clearest:

> So should we try to combine Piaget and Vygotsky into a common system in the hope of explaining both extremes of this astonishing human variability? I think that would be naïve. The justifiable pedagogical optimism of cultural revolutionaries is not just the sunny side of the equally justified stoicism of principled pedagogical "realism". The two perspectives grow from different world views that generate different pedagogical strategies, different research paradigms, perhaps even different epistemologies, at least for a while. Better each go their own way. Let the Dionysian partisan activists specialize in finding leverages of change — e.g. how collaborative learning environments empower learners, what scaffolding helps learners over what seemed before to be "innate" constraints. But also let the Apollonian realists explore "natural" constraints and seek out the regularities they impose on development, wherever found in whatever culture [Bruner, 1996, p. 14].

In my work I have taken the same line in relation to the differences between their ideas, as I indicated in my 1996 paper. Before discussing research in our sub-field of mathematics education I want to provide some evidence that, as I mentioned above, we in the non-Russian speaking world came late to Vygotsky's whole view of the development of mind as a cultural-historical process. In the former USSR well known researchers and theoreticians such as Davydov, Zinchenko, Talyzina, [1982] and Ilyenkov [Davydov, 1994] were developing the work of Vygotsky, Luria, Leontiev and others but few outside had access to literature before this time or even that body of developing work.

In a search I carried out some twenty years ago I wrote that the earliest references in mathematics education to Vygotsky were as follows:

> in PME proceedings, Crawford [1988];
>
> in *Educational Studies in Mathematics* in a review of Wertsch [1981] by Crawford [1985], but the first mention in an article, Bishop [1988];
>
> in the journal *For the Learning of Mathematics*, Cobb [1989];
>
> in the *Journal for Research in Mathematics Education*, English [1993];
>
> in the *Journal of Mathematical Behavior*, Schmittau [1993]
>
> [Lerman, 2000*b*, p. 25].

Vygotsky died more than 50 years before our first acknowledgement to his ideas, as revealed in my search. This is not the place to try to identify the reasons for this time gap: these reasons include political and language/translation issues. Suffice to say that many of us have been working hard to understand Vygotsky's thought — not easy given how young he was when his life ended and how little he was able to develop his ideas and carry out research in that short period of time. Indeed it is evident that he continued revising quite radically how he understood child development from a cultural-historical perspective even on his deathbed.

> During the first decade of the 21st century, Vygotsky's descendants have given Ekaterina Zavershneva access to the family archive. In the process, she uncovered a wealth of unpublished and unheard-of private notes. In these notes that were written near the end of his life, Vygotsky expresses discontent with his own theory, the one most people who read Vygotsky think they are familiar with; and he deems them insufficient, requiring a complete overhaul and revision. In particular, although he had spent much of his scholarly life critiquing and attempting to overcome the Cartesian dualism that is characteristic of psychology then as now he had failed. In the notes, he acknowledges the remnants of Cartesian dualism in his work, including an over-emphasis of the intellectual over affect and the practical. The Cartesianism also characterizes current theoretical approaches, especially in (radical, social) constructivism; but, as philosophers have shown, the spectres of Cartesianism exist even in embodiment and enactivist theory. To overcome these remnants in his own work, Vygotsky turned to the philosopher Baruch Spinoza… [Roth, 2017, p. vii].

What made the task of understanding and researching Vygotsky's ideas so difficult for us was that our sub-field of mathematics education, and the field of education more widely, was profoundly Piagetian. Jean Piaget, born in the same year as Vygotsky, lived until he was 84, publishing hundreds of books and articles, and there are any number of books and studies by scholars based on his work. This alone is in huge contrast to publications by Vygotsky.

In this talk I will begin by elaborating on the originality of Vygotsky's thought in this Piagetian context. I will then review some of the research directions that have been taken by mathematics education researchers working with Vygotsky's thought. I cannot even attempt to be comprehensive, given such a diversity of rich work going on around the world but I will aim to point to some of the major directions, including those that have, in my view and those of some others, misunderstood Vygotsky. I will end with looking at future directions for research drawing, in particular, on the fragments of his revised thinking at the very end of his life.

## VYGOTSKY'S WORK IN A PIAGETIAN WORLD

Bruner [1996, p. 9] expresses well the central question Vygotsky was grappling with all his working life in child development: "…the most central question for Vygotsky is how a culture's symbolic tools manage through social interaction to get from 'outside' into our 'inside' repertory of thought." Of course how that question is answered in terms of the role of others informs our role as teachers. The inverted commas in Bruner's statement imply what we now know was more firmly expressed in Vygotsky's re-think in his last days, that the distinction between outside and inside is misleading, even false.

The history of developing Vygotskian research in education outside of Russia is of a community struggling to operationalise what was a body of work slowly emerging from decades in which it was just not available, one which ran against the dominant ideas at the time, and which was philosophically quite other.

In 1976 Wood, Bruner and Ross coined the term 'scaffolding' as a metaphor for the task of the teacher, according to Vygotsky's theory as they interpreted it, in particular the zone of proximal development (ZPD). They wrote:

> …it involves a <…> process that enables a child or novice to solve a problem, carry out a task or achieve a goal which would be beyond his unassisted efforts. This scaffolding consists essentially of the adult "controlling" those elements of the task that are initially beyond the learner's capacity, thus permitting him to concentrate upon and complete only those elements that are within his range of competence [Wood, Bruner, Ross, 1976, p. 90].

The notion of scaffolding has become widespread, almost ubiquitous, in education, particularly that of young children, and it was certainly a worthy attempt to introduce a key element of Vygotsky's ideas at that time. Educational thought then, and still today in most Western countries, was dominated by 'ages and stages', the recontextualisation of Piaget's stages of conceptual development to a world in which the teacher controlled what and when children should learn based on an understanding of conceptual development as a biological process. The elements of Piaget's constructivist theory that are more difficult to recontextualise to the classroom, namely assimilation, accommodation and reflective abstraction, are nevertheless central to his theory. In our sub-field of mathematics education these elements were taken up and addressed seriously, during the 1980s and 1990s, in what became known as 'radical' constructivism, radical in the argument that these processes can only be carried out by the individual. The teacher's interventions, including in particular tasks to be set, are chosen based on the teacher's 'knowledge' of the individual child's understanding, and are limited to the hope that the task might lead the child towards reflective abstraction. I have put 'knowledge' in inverted commas because in constructivism, especially the radical version, the teacher can never know the child's constructions, the child's knowledge, she can only conjecture what that might be. The separation of the child from the teacher and indeed the world are at the heart of Piaget's Kantian inspired theory of learning and hence teaching. Scaffolding, therefore, is just a different way of framing what is seen as the teacher's control of the child's learning.

I want to point out here that, in my view, Wittgenstein presents the clearest argument against the whole idea of private languages, as is implied by constructivism, in its radical version at least. In the book *Philosophical Investigations* [Wittgenstein, 1958] paragraph 243 and from 256 onwards he shows that what words and things mean, or signify, is how they are used; there is nothing more. For example:

> 264. "Once you know what the word stands for, you understand it, you know its whole use."

Whilst Vygotsky insisted on the recognition that the teacher has to be seen as central in the process of the notion of the ZPD, it is much more radical in that Vygotsky intended a quite different role of the teacher. She, together with parents, more knowledgeable others, artefacts, and tools, carry history and culture. The interaction of the child with these mediators are central; the child's consciousness forms as one with the interaction. What things mean in a cultural-historical setting constitute the child's mind.

> We know the general law: first a means of acting on others, then on oneself. In this sense, all cultural development has three stages: development in itself, for others, and for oneself (e.g., a demonstrative gesture — at first it is simply a failed grasping movement aimed at an object and

> designating an action; then the mother understands it as an instruction; and, finally, the child begins to point) [Vygotsky, 1989, p. 56].

It is therefore clear and not at all surprising how quite fundamental concepts, such as those of space, shape, and number can and do vary hugely across cultures, as anthropologists have demonstrated. In mathematics education the extensive work of Saxe [2012] (see also [Morris, 2014] on probability) in investigating different conceptions of mathematics in different cultures is further illustration of the fundamental importance of Vygotsky's work. Such concepts are not universal. Universality is to be found in the process of the formation of mind, which takes place always in a specific cultural-historical setting. This is what Wittgenstein intends, in the quote above, when he says: "once you know what the word stands for", which can only be acquired from others, since it precedes the child in that specific culture and history.

What must be kept in mind is that Vygotsky's theory is a Marxist theory, and is therefore driven by dialectical processes, working from the intersubjective to the intrasubjective.

> It is not the consciousness of men that determines their being but, on the contrary, their social being that determines their consciousness [Marx, 1859, p. 328–329].

The ZPD functions as a symbolic space [Meira, Lerman, 2009] in which dialectical forces are at play. The child's actual development is present as the ZPD emerges and confronts the 'scientific knowledge' brought to the mediation by the teacher, parent, more knowledgeable others, artefacts, and/or tools. Vygotsky has set out the process by which concepts develop, from heaps, through complexes, to potential concepts and 'full' concepts, in a dialectical manner. In this way one can see the ZPD as pulling the child into her/his tomorrow.

Bruner is right when he suggests that Vygotsky comes from a different theoretical and philosophical orientation to Piaget. One should work with Piaget, or Vygotsky, and not try to take these two world views together. Further, this requires working with Vygotsky (the same applies to Piaget) in his cultural-historical setting, understanding his world view, seeing how it has been applied, how it developed, during his short lifetime. And it is Vygotskian to see that researchers and writers today must take his work into our cultural-historical setting and develop it as we see necessary. In the next section I will look at some of that current work, including my own, and re-examine it from within a Vygotskian world view.

## CURRENT RESEARCH INSPIRED BY VYGOTSKY

Activity theory (AT) and teaching-learning in the ZPD have been major areas of research in mathematics education in the last two decades. Research on collaborative work in mathematics classrooms has looked to Vygotsky for the application of notions of learning as a social or sociocultural process. In the last decade there has also been a substantial new area of work of which he has been a major influence, that of commognition. I will discuss and illustrate briefly each of these areas of research.

### Activity theory

Historically, developments can be seen as beginning with Vygotsky's meditational triangle, called the first generation AT, through the work of Leontiev, and the concepts of activity, action and operation, called second generation, to Engeström's developed meditational triangles, called third generation AT, or actually CHAT, cultural-historical activity theory. Each of these remains a source of inspiration for research in mathematics education; it is not the case that each new generation has replaced its predecessor. For example, in a study of undergraduates' choices and uses of tools in their study of mathematics, including lecturers' notes, attendance at lectures, online resources, other students, etc., Anastasakis [2018], reviewing literature in the field, found support in Kapitelinin and Nardi [2006] in particular for the view that tool use is not adequately elaborated in general in recent work in AT and turned to Leontiev's second generation AT for his analysis. He writes:

> The main ideas used from Leontiev's perspective, include the principles of object-orientedness (all human activities are directed towards their objects), activities' hierarchical structure (activities consist of layers), mediation (our relationship with the "objective" world is mediated by tools) and development (activities develop over time). Two modifications of Leontiev's version AT will be also incorporated: the separation of motive and object [Kaptelinin, Nardi, 2006] and the addition of ensembles/purposes as a fourth layer in an activity's hierarchical structure [González et al., 2009; Anastasakis, 2018, p. 20–21].

A range of studies on AT in mathematics education can be found, in the special issue volume 20 of *International journal for technology in mathematics education* (2013), though there have been several other such collections. These special issues indicate the significance of AT in research in our field.

### Zones of proximal development

Substantial research has been carried out by mathematics education researchers working with Vygotsky's zone of proximal development. Whilst at first it seems to have been taken up by most as a way of structuring teaching, of supporting learning, and direction in the choice of tasks by teachers, Vygotsky clearly meant much more, as it emerged in his thinking. It is a metaphor for the whole process of learning, from the young baby through all adult life. Given his fundamental view of knowledge being first on the social plane and then on the internal plane, the ZPD captured how that process takes place, by focusing on problem solving. In fact it needs to be seen as the mechanism by which the child is pulled into the whole of culture and knowledge at that time, in that place, and through which the child is pulled into her/his tomorrow.

I can only give a few snapshots here of work in the ZPD. The three examples I provide deal, respectively, with the emergence of a ZPD, the non-emergence of a ZPD and the mutual ZPD in a teacher-student interaction.

The first is that of Meira and Lerman [2009], in which a study of a very young, almost pre-verbal child's interactions with his nursery teacher are analysed for how concepts, language, and ways to interact with a teacher/adult develop in the interaction.

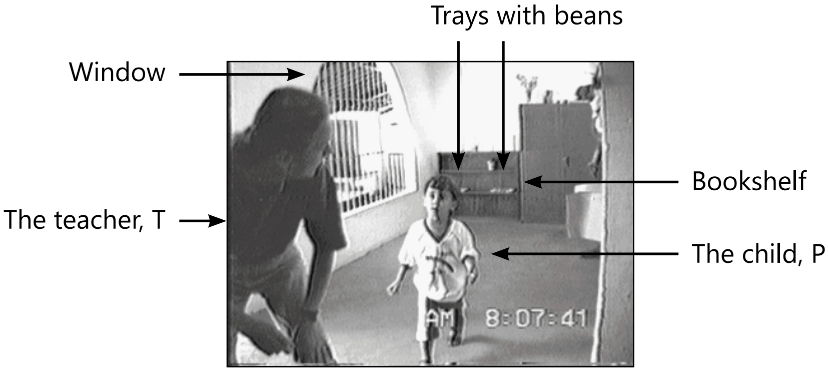


***Photo 1***
Fragment 1 — 8:07:41

1. P: (Walks to teacher) Tia. (A tender nickname for teachers in Brazilian K-4 schools which literally means "aunt".)
2. T: (Turns back and directs her gaze to child) Yeah?

A few minutes past eight o'clock (before class begins), five people were in the room: Pedro, 2yrs 6mths (the oldest pupil), leaning against the window on the left (see Photo 1) and looking alternately outside and inside the room; the teacher, collecting class materials at one end of the room (and not noticing Pedro's behavior behind herself); the researcher (who did not enter the scene or speak at any time during the filming), controlling the video equipment and standing near the camera; and an assistant teacher holding a second child in her arms… neither of whom had any direct participation in the episode analyzed here. On the bookshelf by the back wall, two plastic trays held an experimental horticulture where beans were placed to grow on cotton wool. During the time Pedro was at the window (about two minutes), he might have spotted the trays but he paid no obvious particular attention to that location, as indicated by the time he spent looking in the direction of the bookshelf where the trays stood. Then, Pedro left the window and walked towards the teacher calling for "Tia" (see Photo 1). Up to this point, the teacher had not paid any attention to Pedro and indeed was gazing in an opposite direction. As Pedro walked towards her and called for "Tia", however, she broke her previous flow of action, began a movement of standing up to face the child and replied with "Yeah?", in a way that expressed her openness to the child's request for attention.


***Photo 2***
Fragment 2 — 8:07:42

3. T: (Walks towards bookshelf, closely followed by the child) It's growing up, isn't it Pedro?

Fragment 2 shows a critical passage: the moment when the teacher said "It's growing up, isn't it Pedro?", as a clear reference (at least as we take into account what follows) to the state of the beans planted in trays located 5 meters away on the bookshelf. Notice that Pedro had not yet said anything (other than "Tia" in fragment 1) that explicitly revealed the goal of his request for attention. On the other hand, he made use of a pointing gesture that seemed to direct the teacher's attention towards the bookshelf (as seen in Photo 2). But the child was pointing backwards, facing the teacher, and not looking in any other direction up to the point in which his pointing gesture begins to fade down. Pedro's pointing did not have a well marked target and, at 5 meters from the bookshelf, could not indicate the trays to the exclusion of other things on the bookshelf, or even inside the room or through the window. In order to visualize this, we have drawn shapes on Photo 2 indicating a reasonable area of targets for Pedro's pointing (the ellipse) and the area where the trays were located (the rectangle), showing that the first do not totally overlap the second and do include other possible targets. Notice further that the teacher may have had a different visual perspective on Pedro's pointing, since she stands higher and to the left of the camera. Despite the absence of more determined, less ambiguous clues for initiating conversation on any specific topic, the teacher replied very promptly with a responsive question which sets the content of the interactions that followed: "It's growing up, isn't it Pedro?"

What were the circumstances which could account for the teacher replying in such a way? In the morning this video was taken, the researcher arrived in the classroom with the teacher (before any of the pupils) and witnessed no conversation about the beans up to the moment showed in fragment 2. On the other hand, the teacher had declared that the plantation on cotton wool seemed to be highly valued by the pupils in this classroom, in particular due to her previous indication that she would move the beans to the backyard horticulture. Yet, is it possible to determine how did she "know" about Pedro's interest at that specific moment? We suggest that the teacher did not know! She could only guess on the basis of underdetermined clues such as previous conversations with this pupil and others, and the child's pointing to the general location where the trays could be found. Whatever set of clues the teacher used to specify the content of Pedro's motives in calling her (assuming he did have one), he reacted as an attentive audience by not rejecting the general conversational theme suggested by the teacher (the growing beans, so it seems) and following her as she walked towards the bookshelf at the opposite end of the room (as seen in the next photo). We see in these moves by Pedro and the teacher the emergence of a ZPD, as we shall argue [Meira, Lerman, 2009, p. 209–211].

The analysis and discussion go on, but for the purposes of this paper and talk I want to emphasise a number of key points:

- the emergence of the ZPD in the moment, it does not pre-exist the interaction. Too often the ZPD is seen to be a kind of physical bubble the child brings with her/him into the classroom, which the teacher has to 'see' or estimate in order to choose what task to give the child;

- the love and care of the teacher as a necessary element for what can be called the 'catching of each other's attention'. She could have continued focusing on her job of setting up the room for the day;
- the unity of the child's development with cultural-historical knowledge of the scientific and horticultural communities. There is no separation of knowledge and consciousness, culture and person, external reality and consciousness; they are a unity. I will return to this latter point in the final section of this paper.

In this second example, taken from Lerman [2001], a class of 13/14 year olds is working on an exercise in simplifying ratios, sometimes presented as fractions, first arithmetic and then algebraic. For the algebraic questions the teacher had given instructions to use either substitution of numbers for the letters so that students could see how to simplify by dividing both terms by common factors, or remain within algebra and divide both terms by common letters. The students work in pairs and the following analysis was carried out on a transcript of two students called here D and M. In a video-stimulated recall interview the teacher said that M was the much more able student who was helping D, who would not manage the tasks on his own, and it appears from an earlier event that the students were aware of how the teacher perceived their respective abilities.

> The… extract below is work which D and M undertook… when they began work on part (f), [*ab* : *ab* = …]
>
> 1. M: What? Equals ab? (Pause, D looks on M's page) Equals *ab*?
> 2. D: Yeah.
> 3. M: No, it equals one.
> 4. D: Wait a second…
> 5. M: 'Cause one, (punching calculator buttons) twelve times tw… no. One, look, look, look. One times two, divide one times two… it shouldn't equal four (M appears to be substituting the values one and two for *a* and *b*).
> 6. D: (laughs)
> 7. M: Um, yeah, it's, 'cause I'm doing (punching buttons) one times two, divide one times two, equals one.
> 8. D: So that's cancelled. The two *b*'s are cancelled out.
> 9. M: Equals one.
> 10. D: Right? The two *b*'s are cancelled out.
> 11. M: Hey, where'd my pen go? No come on, look, look, look, look. You've got to do BODMAS. Watch, watch, watch, watch (punching buttons). One times two, divide one… come on, one times two. That's stuffed up (with emphasis). One.
> 12. D: …I'm going to… this is… better…
> 13. M: Look, look, look, look at this one, look at this one.
> 14. D: …Hang on…
> 15. M: Divide.

16. D: …I'm going to do these, this one first.
17. M: Equals 1, it does equal 1. I've got to do this first [Lerman, 2001, p. 63–64].

In this extract I suggest that a ZPD does not emerge, they do not catch each other's attention, in part because they each chose to use one of the strategies in the teacher's instruction but not the same (lines 7 and 8 show M using substitution and D using cancelling common terms), in part because the calculator came between the two students, both literally and metaphorically, and in part because their relationship was set up by the teacher so that D would be led by M. M took on the 'teaching role', could not 'hear' D, and the chance to work together in a ZPD was missed. We might note D's use of the more generalizable strategy at line 8 in relation to the teacher's judgement of relative ability of the two boys.

In the final example, taken from Roth and Radford [2010] we will see the teacher also learning.

> In this lesson, the children are in the process of classifying "mystery objects" that they are pulling from a black plastic bag. The 22 children sit in a circle, the center of which develops into space for a classification of the objects [Fig. 1]. Each child gets a turn pulling an object and then either placing it on a colored paper with other "like" objects or create a new group. They are asked not to use color or size as a way of distinguishing objects, though most of the children continue to do so. The two teachers teaching the unit — Mrs. Turner, who is the regular classroom teacher, here in the lead, and a university professor — have stated previously (in their planning meeting preceding the lesson) their intent to allow the children to arrive at a classification system in which all objects are grouped according to their geometric properties, that is, as cubes, spheres, rectangular prisms, and so on. To achieve this end, Mrs. Turner interacts with each child so that at the end of its turn, the object has found its place according to what we recognize in the practice of Euclidean Tridimensional Geometry, its geometrical properties.
>
> …For instance, it is Connor's turn, a child sitting in the circle. He has pulled what he eventually comes to know to be a cube, but he has classified it on its own rather than with the other cubes on the floor. Following the interactions with Mrs. Turner, he ends up giving his mystery object its appropriate place. At this point, Mrs. Turner utters, her intonation falling toward the end as if she were making a statement, "em an what did we say that group was about," while pointing from afar toward Connor [Fig. 1]. There is a long pause developing, much longer than research has shown to exist in teacher–student interactions. Connor then takes a turn and utters with rising intonation characteristic of questions, "What do you mean like?" all the while touching his mystery object.
>
> T: Em and what did we say that group was about?*
>
> C: What do you mean like?
>
> T: What was the… What did we put for the name of that group? What's written on the card?
>
> C: Squares.
>
> T: Square and…
>
> J: Cubes.
>
> T: Cube. Does it meet the criteria of having the square or the cube?
>
> …

> Yet when we take an approach to the analysis in which each word uttered in the transcript is a thing in the consciousness of both, then the analytic situation changes. In fact, we may say that not only does Mrs. Turner guide Connor to the point of naming what his group was about, but Connor also guides Mrs Turner toward what she needs to do to assist him. Connor, in fact, exhibits considerable cultural competence, which allows the conversation to unfold. Even though the intonation of Mrs. Turner has descended as is common in statements, Connor, in responding, indicates that he understands her to want something from him. By responding, Connor comes to inhabit the public space of interaction and opens up possibilities for intersubjectivity to appear. Surely, in doing so, he shows to be ready to engage in actions that are not premeditated. He exposes himself. The *question* of what she wants is problematic, rather than the fact that she wants something from him. He allows her to know more than that he has simply not understood. His lack of understanding may have arisen from not listening or not hearing what she has said. But in this situation he might have asked, “What did you say?” thereby indicating that the problem is a failure to hear rather than a failure to comprehend. In asking Mrs. Turner what she means, Connor not only responds by stating a failure to understand what she wants, but in fact guides Mrs. Turner through what to do next: state what she really means to say by uttering “what did we say this group was about” [Roth, Radford, 2010, p. 300–302].

As in the first example of the very young child in the nursery, the teacher here learns what the student needs by catching his attention, his meaning, in order to make a decision how to respond. Both are pulled into their ZPDs in this interaction and learn from it; the consciousness of both develops.

### *Group collaborative work in classrooms*

As I indicated in the introduction, there are some misunderstandings in research in our sub-field. Perhaps the most common is the assumption that group work and the negotiation of mathematical meaning is inherently Vygotskian. First, I would argue that a ZPD can emerge in the interaction between a child (or adult) and an artefact [Graven, Lerman, 2014; Abtahi, Graven, Lerman, 2017]; between a teacher and a whole class; a teacher and one student (as in Meira and Lerman above); two students, or other situations. Second, how groups work, how they are managed by the teacher, is an absolutely key element in the learning, or otherwise, of students (see e.g., [Boaler, Staples, 2008]). Group work that takes place without drawing on elements of the ZPD largely replicates the usual organisation of learning in a classroom, in which some students succeed and many others do not. Third, knowledge precedes everyone. It can be ‘negotiated’ in the sense that the ideas of each person in the group should be acknowledged, but the teacher (or a knowledgeable other) is responsible for the shift from the everyday to the scientific; put another way, for the ascent from the abstract to the concrete. The notion of ‘negotiation’ comes, I suggest, from the social constructivist attempts to incorporate Vygotsky’s radical insights into constructivism [Lerman, 1996]. Indeed we can see the assumption that group work is inherently Vygotskian as a misunderstanding. Bringing everyday notions into the ZPD is essential but it is the teacher who must pull the students into their tomorrow through instruction towards scientific concepts.

### *Commognition*

A substantial area of work, informing a greatly increasing body of researchers around the world, is that of the so-called ‘commognition’, a term drawing together ‘communication' and 'cognition'. The programme of Anna Sfard, the originator of the term, is an attempt to overcome the separation of mind and society by conceiving of developing consciousness using the metaphor of *participation* in the surrounding discourses, in distinction to an *acquisition* metaphor more commonly held by educational researchers, to the extent that even the internal plane is an engagement in communication with internalised others. Of course the fundamental notion, as recognised by Sfard, is Vygotsky's. For example, he wrote, in a paper published in 1989 but written by him in his final years:

> …thinking is speech (conversation with oneself) [Vygotsky, 1989, p. 57].

Some of the key ideas of commognition are summarised in Thoma and Nardi [2016], the references being to Sfard [2008]:

> Mathematics is seen as a discourse and doing mathematics is seen as engaging with mathematical discourse. The rules followed by the participants of the discourse are distinguished in *object-level* rules ("narratives about regularities in the behavior of objects of the discourse" [P. 201]) and *metarules* which "define patterns in the activity of the discursants trying to produce and substantiate object-level narratives" [Ibid.]. Discourses are described in terms of four characteristics: *word use, visual mediators, endorsed narratives* and *routines*. More specifically, *word use* refers to the use of words specific to the discourse or everyday words (colloquial discourse) which may have different meaning when used in this discourse. In the mathematical discourse *word use* includes mathematical terminology (i.e. integers) and some words with special meaning in mathematics (i.e. disjoint sets). The *visual mediators* are objects and artifacts used to describe objects of the discourse. Some examples of visual mediators in the mathematical discourse are symbols and diagrams (i.e. Venn diagrams). *Endorsed narratives* are "sequence(s) of utterances, spoken or written, framed as a description of objects of relations between objects, or of activities with or by objects" [P. 223]. In the mathematical discourse an example of an endorsed narrative is a definition or a theorem. Finally, *routines* are a set of metarules describing patterns in the activity of the discursants. Some examples of routines in the mathematical discourse are the routines of proving and defining.

## FUTURE DIRECTIONS

It is to be expected that future research will continue to develop ideas of work in the ZPD, in the ascent from abstract to concrete, in concept development, in the participation metaphor and other areas of Vygotsky's work, including commognition. At the same time, as shown by Roth [2017] we are still learning about Vygotsky's theories, in particular those of his final years.

Vygotsky was always concerned with ethics, the relation of the individual to others, and in particular the responsibility of society, through parents, teachers and others, for bringing children into their futures. How this process takes place was the whole theme of his psychological studies. Radford has developed a research programme built around the ethics of teaching, learning, relationships in the classroom and identities, couched in terms of a theory of objectification [Radford, 2016]. In this respect the ethical ideas of Levinas are central (e.g., [1998]).

The theme of Roth's book is Vygotsky's debt to Spinoza. Indeed Roth refers to 'thinking with the Spinozist-Marxian Vygotsky' to illustrate the significance of ethics to his final ideas. It may well be that it was only in those last months that he clarified his own thinking and brought Spinozan ethics more to the fore. Had he lived longer he would certainly have revised his ideas in relation to the ZPD and other aspects of his pedagogic work.

Vygotsky's interest in Spinoza was in his monist ethics:

> Mind and Body are one and the same individual thing, conceived now under the attribute of Thought and now under the attribute of Extension [Spinoza, p. 259].

Roth writes:

> One entry in his personal notebooks, dated to some time between 1931 and 1933, reads like a programmatic instruction to himself: 'Bring Spinozism to life in Marxist psychology'. Indications of where the thoughts occurring to him were leading are apparent from his personal notes and the final pieces of writing that were published only posthumously [2017, p. 1–2].

Vygotsky's work is essentially Spinozan, since every function of the child is first social and then psychological, and hence there is a unity between mind and body. We must try, therefore, to perceive where Vygotsky's theories may have turned if he intended to take a deeper Spinozan position.

Roth has brought to the fore Marx's emphasis on the term 'societal'. The distinction between social and societal is about what is universal within society. When students do mathematics together it is social. The interventions of the teacher are societal because the universal, that is cultural, or 'scientific', is brought by the teacher.

I would take this towards Marxist sociology of education, as I have written elsewhere (e.g., [Lerman, 2017]). In drawing on Marx sociologists such as Bernstein take the same approach to teaching-learning as Vygotsky [Bernstein, 1993; 2000]. Bernstein shows how education reproduces social advantage and disadvantage; different social backgrounds provide children with different linguistic and therefore intellectual resources. In general children from more advantaged homes acquire the language of schooling, that is the scientific or the societal, prior to coming to school. They are therefore ready for learning in a way that children from less advantaged backgrounds are not (see e.g., [Cooper, Dunne, 2000]). This is not a deficit model but a marking out of the Spinozist-Marxian Vygotsky's identification between societal and individual, the methods of differential distribution of symbolic power. Bringing sociological insights and in particular methodologies (see e.g., [Morgan, Tsatsaroni, Lerman, 2002]) to other ways of researching Vygotsky and researching with Vygotsky in mathematics teaching-learning will enrich and extend our knowledge and our practice as researchers and as teachers.

### *Notes*

* The transcript's format has been simplified here, as there is no space to include information about the transcription conventions used by the authors.

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# A NEW WORLD: EDUCATIONAL RESEARCH ON THE SENSORIMOTOR ROOTS OF MATHEMATICAL REASONING

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*Recent developments in the theory and methods of cognitive science are enabling educational researchers to evaluate empirically the historical thesis that mathematical concepts are grounded in sensorimotor activity. My presentation will survey results from several recent design-research studies that have used eye-tracking techniques to capture the moment at which a student first sees the world in a new way. For the student, this spontaneous perceptual construction serves as a handy solution for coordinating the control of an interactive system. In turn, through cultural mediation this construction evolves into a new way of reasoning that becomes a mathematical concept. I will speculate on implications for educational technology.*

I would like for you to know what I mean, when I talk about sensorimotor perception. And I would like for you to realize how new sensorimotor perceptual structures emerge through goal-oriented interaction with the environment. I further hope to convince you that learning new mathematical concepts begins with the formation of new sensorimotor perceptual structures. And I will show you how we can now track this process. The better we understand all this, the better we can design for learning.

## A NEW WORLD: PERCEPTION REVISITED


*a* *b* *c*

**Fig. 1.** *a* — robotics surgery; *b* — the Red Sea at Sinai; *c* — a coral reef

Figure 1*a* is a recent photograph of my father, Dr. Jack Abrahamson, Professor of Surgery. Jack will turn 90 later this year. Don't worry, he's not operating any more. The picture was taken last year, when he was touring the robotics operating theatre at the Medical School, University of California San Francisco. But I want to tell you a story about a non-medical experience Jack had. It was 40 year ago. So this is back in the 70s. The era of Leonid Brezhnev, I guess. I grew up in Israel. One summer we trav-


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elled south to Sinai, to spend a vacation in a Bedouin village on the coast of the Red Sea (Fig. 1*b*). My brothers and I were underwater divers, so we had seen the coral reef before (Fig. 1*c*). But my father, although he grew up by the sea, in South Africa's Cape Province, and was an able seaman, had never used flippers or snorkel to go down under and see what lies below the water surface. Eventually, we talked him into trying. Jack put on a snorkel, a mask, and a pair of flippers and swam out to the reef. From far away, we watched his snorkel bob over and under the water. Finally, he came out and walked back up the beach. We'd never seen him before quite that way. Back then, Dad was a very serious person. Now his face was glowing, his jaw dropped, his eyes popped out. He uttered only three words: "A new world."

Now, to be sure, the reef had always been there, just below the water. The point was to see it — for *him* to see it, and, for that, one often needs the right conditions, including perhaps some special gear, and possibly some encouragement from those who create the conditions and facilitate the encounter. And yet those who create this encounter may, too, experience a new world — of witnessing another person learning.

So it's a new world in two senses. For the individual person — let's call them a student — the activity enables a new perceptual relation with the environment. You apprehend details and structures which were always there, even if *you*'re experiencing them for the very first time. You might reflect on what you see and express it to yourself and other people. This new world is about a new phenomenal experience in the world as you found it. Yet for the observer — let's call them a researcher — the activity might enable seeing what the student is seeing in a way that has not been available before. Now, unfortunately, we could not see what Jack saw underwater or how he came to see it and make sense of it — we didn't have access to all that. To achieve that sight, one needs to "go underwater," that is, to use special instruments, like eye-tracking devices. This new world is about scientific breakthrough in modeling how new sensorimotor perception is formed when a person engages in goal-oriented situated activities, such as solving motor-control problems.

## A NEW WORLD OF THEORY, TECHNOLOGY, AND METHODS

This paper is about the new world in both of these senses: a world of students seeing new structures, and a world of researchers who see this process as it is happening.

Upfront, here's the paper's take-home message. Three recent developments in the field of mathematics education research — in theory, technology, and methods — have been converging in a way that's enabling us to re-think early stages of conceptual learning, to create conditions that foster these stages effectively, according to some instructional objectives, and to understand how these early stages play out and how they might be monitored and steered. We're talking about theory of *embodied cognition*, technology for *embodied interaction*, and methods of *learning analytics*.

These are exciting days to be a cognitive scientist with an interest in the teaching and learning of mathematical concepts, because these three developments in the field have now matured and have come to a confluence.

### *Theory: embodied cognition*

One development is in the philosophy of knowledge, epistemology, which is challenging the theories that researchers have been putting forth and evaluating with respect to how people learn. If once it was taken for granted that the brain functions as a computer-like central processing unit that is cut off from the sensorimotor modalities, now there is increasing interest in alternative proposals by which all cognitive activity is intrinsically perceptuomotor. These ideas, which have loosely been called embodiment theory, and include arguments that the mind is embodied, embedded, extended, and enactive, are renewing the field's interest in pre-AI models of the mind and of learning processes [Newen, Bruin, Gallagher, 2018; Shapiro, 2014]. Cognition, embodiment theorists argue, is not the manipulation of abstract symbols but, rather, cognition is perceptuomotor activity. Cognition is *modal*, in the sense that it is constituted in the various modalities, that is, by the very same type of neural activity that enables us to function in the world through sensation, perception, imagination, motor action, and by engaging cultural forms, artifacts, language, and practices that have evolved to support our survival and thriving. And so, how we think depends on how we know to move in the world, how we know to operate on various objects. Educators can therefore affect students' cognitive development by shaping their experiences with objects.

Some embodiment theorists claim there is no such thing as abstract entities. Rather, there are imaginary concrete forms. These can be complex and counterfactual, in the sense that they cannot be materialized. But mathematicians, like fantasy novelists and readers, are glad to suspend their disbelief and treat these objects as real or pseudo-real. Indeed, these objects may not be directly accessible to our senses as percepts in the world, but still they are phenomenologically real. Just as dreams are experienced as real, no matter how fanciful they are (cf. [Hutto, Myin, 2017]).

Embodiment theory ranges in its philosophical commitments and radicalism. Yet, however articulated, it is emboldening us to investigate and leverage the sensorimotor foundations of mathematical concepts [Abrahamson, Lindgren, 2014; Lindgren, Johnson–Glenberg, 2013].

### *Technology: human–computer interfaces for naturalistic embodied interaction*

Another development is in technology, and in particular in Human–Computer Interaction (HCI). HCI is a major branch of engineering concerned with designing and evaluating technological systems for humans to interact efficiently with computational platforms. Recent advances in HCI include new platforms with human–computer natural-user interfaces that allow for intuitive, discovery-based interaction with information structures encoded in software [Antle, 2013; Dourish, 2001]. These advances in HCI are creating opportunities for educational designers to build learning environments in computational platforms that enable naturalistic inquiry through embodied interaction. Of particular interest are environments that implement activity genres for mathematics teaching and learning, in which students first build pre-symbolic, qualitative understandings of new notions through manual interaction with material or virtual objects and only later adopt mathematical formulations of these physical movements [Abrahamson, 2014]. For instance, we can cast a mathematical concept in the form of an embodied-interaction regimen that privileges particular movement patterns, that is, particular enactment of situated skill. *This form of computational technology thus offers an epistemic interface between a mathematical concept and human sensorimotor perception, where both are cast in the movement modality, and the objective is for the human, through concerted inquiry, to figure out and match the computer's hidden movement form* [Howison et al., 2011].

### *Methods: eye tracking and multimodal learning analytics*

And, finally, as educational researchers, how might we monitor this process? In particular, how do we capture a student's sensorimotor perception and any changes in this perception as the student engages in embodied-interaction learning activities? What might it even mean to know how a child is seeing the world? As I will explain, the instruments of eye tracking can help us gain some purchase on determining where a child is gazing and, through that, and in triangulation with the child's actions, speech, and gesture, to make educated guesses about how the child is seeing the world she is interacting with. Thus, a third development is in methods, in particular, instruments for measuring sensorimotor activity in a variety of modalities, then integrating these data and presenting them for analysis to search for behavioral patterns and trends in these patterns. This means that when students interact with the technological platforms we design for them to learn mathematical concepts, we can monitor our data, even in real time, for the emergence, regulation, and refinement of sensorimotor routines [Worsley et al., 2016].

So those are three lines of development in our field that are relevant to the work my lab does with our international collaborators — theory of embodiment, technology for embodied interaction, and multimodal learning analytics. Combined, these developments in theory, technology, and methods have created opportunities for the kind of research that led me to state that these are exciting days to be a cognitive scientist with interest in mathematics teaching and learning.

I would like to tell you about one line of research that our lab began exploring about ten years ago, which is building on this synergy of theory, technology, and methods. I will start by setting for you the context of this work, and then I will focus on a particular theoretical construct we call an "attentional anchor." I will explain this construct, and I will present empirical work suggesting that this construct might play a role in future research more widely, on both the theory and the practice of mathematics teaching and learning.

The crux of the innovation is that we can use eye-tracking devices in order to monitor for changes in the way that mathematics students perceive the visual displays they are studying and manipulating. These changes in perception mark the formation of new sensorimotor schemes — new ways of perceiving the world so as to act upon it — and, in turn, these new perceptual structures constitute things we can measure, model, symbolize, and discuss, so that they become mathematical objects. Having real-time access to how students are perceiving the environment, I will suggest, may change the way we teach, both in person and through artificially intelligent interfaces.

These are still early days in this line of research, and so some of the empirical data I will discuss are yet preliminary, and the applications are still being engineered. However, I would like to use the opportunity of this paper to share with you these developments and hopefully the excitement.

## THE WORLD ANEW: LEARNING BY SEEING THINGS IN NEW WAYS

**Fig. 2.** Friedrich Fröbel "gifts": a curricular regimen of pedagogical manipulatives

Many of us here are in the business of creating stuff for kids to learn math by. The idea goes back at least two centuries, to Friedrich Fröbel (see Fig. 2), who invented kindergarten, and it has been popular through the work of Maria Montessori, Caleb Gattegno, Vasily Davydov, Daniil Elkonin, Zoltan Diénès, and many others. Seymour Papert called these things "objects to think with." And many of us have built their academic careers around investigating how people learn mathematical concepts through interacting with objects. I'm in this business, too, of creating, evaluating, and theorizing pedagogical regimens, including media and activities, for students to learn mathematical concepts.

Much of the literature in the research field of mathematics education, certainly in the collected proceedings from annual meetings of the International Group for the Psychology of Mathematics Education (PME) and its regional subsidiaries, such as this inaugural PME Yandex Russia meeting, is on how children learn through engaging with pedagogical materials and, therefore, how this learning should be facilitated and assessed. And a central idea in this body of research is that through engaging these materials, usually in an attempt to accomplish some particular assigned task, the students develop new perceptions of the environment that are vital to learning the concept in question. These new ways of orienting toward the environment are designed by mathematics educators so as to align with our civilization's cultural heritage comprising productive ways of organizing our collective behaviors. And so these new ways

***Fig. 3.*** The author and Neomi Sarah, at 4 years old, using hand formations to discuss the commutative property of multiplication. Ten fingers are 2 × 5 or 5 × 2

of *perceiving* the world emerge through, are mediated by, and are integrated in the use of new forms of *operating* on the world; forms that enable students to participate in the social enactment of cultural practices. Most parents do this intuitively, when they teach their children to count. Educators seek to emulate this naturalistic pedagogical acumen by formulating and theorizing effective principles for cultivating mathematical knowledge beyond counting (e.g., see Fig. 3).

This paper, too, is about children coming to perceive the world in new ways through operating on it. Where I am hoping to push the conversation forward is in suggesting that mathematics educational research is now at a point where we might revisit what we mean when we talk about students learning to perceive the world in new ways. In particular, I wish to suggest that we could pay more attention to the physical movements that students enact as they learn. I will argue that *by paying more attention to how children move, when they learn mathematical concepts, we could do a better job in theorizing the cultivation of perception*. For researchers, the new world is that, using eye-tracking instruments, we can see the moment a child comes to see the world in a new way. This insight could lead us to rethink the design of educational artifacts.

## TOGGLING THE WORLDS: PERCEPTION OF AMBIGUOUS FIGURES

Movement is difficult to talk about, because — well…, it keeps moving! So in order to say something about the phenomenology of movement, let us step back and begin by speaking about the phenomenology of something much simpler — static images.

Ambiguous figures are popular, because, similar to optical illusions more generally, they offer an intriguing perceptual experience (see Fig. 4). As we shift our foveal visual orientation onto different regions of these images, our perceptual construction of the image toggles between two alternative and often mutually exclusive potential meanings of the image [Tsal, Kolbert, 1985]. In turn, *reflecting* on this experience, we may realize that visual sensory perception is active (not passive), constructed (not inherent),

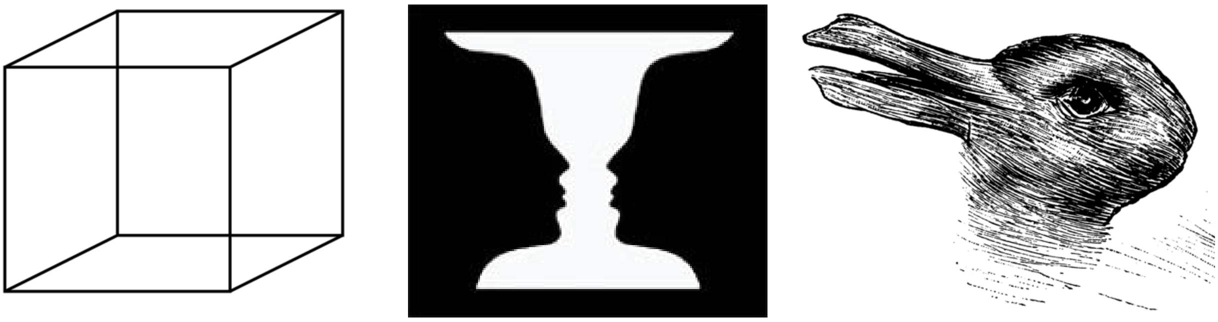


*Fig. 4.* Three famous illustrations bearing ambiguous figures.
From left: Louis Albert Necker's cube; Edgar Rubin's vase; and Joseph Jastrow's duck/rabbit

relational (not monistic), subjective (not objective), and mostly tacit (not conscious). Consequently, given these important insights onto sensory perception, these images are often used in introductory courses on sensation and perception.

Yet I would like to point out that with all these famous images, we are not asked to do anything physical — just to observe and interpret. Can we change how a person sees an object by asking them to do different things with that object? To accomplish this, it may help to select an object that we commonly use actively as a tool. As in the case of the classical ambiguous images, above, I am about to create experiential circumstances that could affect how you frame your perception of a sensory display. But, unlike these images, I will attempt to manipulate your sensory perception of the object through changing your motor orientation toward the object. Ready?

## AFFORDANCES, THE PHENOMENOLOGICAL QUALITY OF THINGS

Consider a pencil (see Fig. 5).

*Fig. 5.* A pencil — what it *is* is what you do with it

Now imagine that you are about to use it in each of the following ways: to

- write
- erase
- sharpen
- pop a balloon
- drum
- scratch your back

- hand it to a child
- twirl it around your finger
- balance it in the middle
- balance it on the tip
- save a ladybird from a puddle
- break it in two

As you considered putting the pencil to these various proposed uses, you may have noticed your body orienting in different ways, each appropriate to the object's specific *ad hoc* utility relative to its proposed function. Your sensory organs, such as your eyes, may have shifted to specific regions of the object, such as its middle (to balance it there), even as your motor organs, such as your hands, arms, and upper torso, tensed and shifted ever so slightly, in preparation to enact a particular form of contact, such as a grasp, each type of contact attuned to particular properties of the object. These types of sensorimotor impressions, which are both nuanced and ephemeral, bind us to objects in the environment by way of proto-action perception, which Gibson [1977] called *affordances*. Note that an affordance is not "in" the object irrespective of the observing organism, nor is it "in" the organism irrespective of the object. Rather, an affordance is inherently relational [Heft, 1989]. When you consider writing with a pencil, you both see the pencil in a way that is specific to its writing function and you organize your motor capacity to enact writing movements.

You have just participated voluntarily in an experimental activity designed to achieve two technical objectives: (a) to sever your physical orientation toward an object from your sensory perception of the object; and (b) to keep changing the purpose and consequent morphology of your unconsummated imaginary engagement with the object. I invited to you engage in this humble introspective exercise, because I could thus occasion for you an opportunity to experience the manifold of constituent somatic, kinaesthetic, and proprioceptive micro-sensations of your preparatory motor disposition toward a perceptual construction of the environment. Normally, in the stream of doing things with objects, these feeble constituent qualia of sensorimotor activity — how you are seeing an object, and how your body is preparing to engage it — are tacitly and irreducibly enmeshed below the radar of consciousness. As Mechsner writes,

> affordances are not only perceived as properties of the affording object. By way of an educated phenomenological sensitivity we may also experience the specific way our body is related to the objects of our interest [2003, p. 240].

In this sense, what an object *is*, at least in our ongoing unreflective phenomenology — which, arguably, characterizes the vast majority of our humdrum hominid experience — is how we are *using* it. In fact, the object will not be salient or accessible to our consciousness as a thing, unless our flow of immersive being-in-the-world breaks down [Koschmann, Kuuti, Hickman, 1998]. But I needed for you to unpack your natural being-in-the-world into its respective sensory and motor factors, because I wish to discuss the constitutive role of sensory perception in forming our capacity to engage in the motor enactment of movement forms by which we accomplish the control of task-oriented manipulation. As such, I needed for you to temporarily experience perception and action as disjoint.

Now, of course, we all know that the image in Fig. 5 is a pencil. That is, we might, on the one hand, take the categorial stance that a pencil is a pencil is a pencil; regardless of what we intend to do with this pencil, the simple fact stands that this still is no more yet no less than just that — a pencil. As such, there would be little to any room for ecological psychology in the scholarly work of sorting and defining the phenomenal manifold. By this oppositional view, there is much epistemic utility in acknowledging the objective identity of objects, which transcends all contextual and intentional circumstances, impervious to the hazards of the observer's knowledge, skill, objective, sentiment, dispositions, or wherewithal. Notwithstanding, as mathematics-education researchers, who care to understand how students learn through manipulating objects they encounter in instructional activities, it is, on the other hand, important for us to query the source and constitution of a child's contextual orientation toward these objects.

Intellectual concern for the implicit meanings students bear for artifacts they are manipulating as well as for the emergence of explicit mathematical meanings from these activities is typical to scholarship on individuals' guided mathematical sense-making. This concern is discussed from a variety of perspectives in our field's literature on the epistemology, ontology, and ontogenesis of mathematical entities, such as through the framework of instrumental genesis [Vérillon, Rabardel, 1995], radical constructivism [Steffe, Kieren, 1994] sociocultural theory [Saxe, Gearhart, Seltzer, 1999; Sfard, 2002; Stetsenko, 2002], or various semiotic approaches [Bartolini Bussi, Mariotti, 2008; Font, Godino, Gallardo, 2013; Radford, 2014].

I draw much inspiration from these contributions to the literature, and yet my interest is on what I believe is a giant gap, in most of this work, with respect to modelling how humans engage objects and wherefrom concepts therefore emerge. What is missing is movement [Sheets–Johnstone, 2015]. In theorizing students' learning through manipulating objects, researchers for the most focus on the outcomes of manipulation — *what* the students do with the things, such as sorting, joining, or counting them. What is less theorized is *how* the students manipulate the objects, that is, the movement forms that students enact in performing the assigned tasks. I wish to add to the field's conversation on learning-through-manipulating a focus on students' experience of moving — the moving itself that gets things done in the learning space (e.g., see [Sinclair, 2018]). Our earlier exercise with the pencil (Fig. 5) was designed to sensitize the reader to the tacit phenomenological stuff that, I maintain, much of learning is made of. As I now explain, the passage from unreflective phenomenology of movement to reflective consciousness of objects hinges on the construction of perceptual structures. Thus, the very perceptual structures that enable the enactment of movement are cognitive grounds for what will become a mathematical notion.

## ENACTIVISM BY DESIGN — FROM PHENOMENOLOGY TO CONCEPTS

If humans' pervasive phenomenology is unreflective immersive doing in the world, where do mathematical concepts come from? And then, once we are satisfied with some working hypothesis about the origins of mathematical concepts, how might we put this theory into practice, in the form of activities for students to learn mathematical concepts? As I now explain, my laboratory's design-based research efforts have been inspired by *enactivism* [Varela, Thompson, Rosch, 1991], a theoretically informed and empirically validated epistemological perspective from the philosophy of cognitive science. Still, applying a high-level theory to educational practice often requires a mid-level pragmatic framework [Ruthven et al., 2009]. We have been applying enactivism to our pedagogical agenda by formulating *embodied design*, a pedagogical framework for mathematics education. Embodied design articulates heuristic principles for building and implementing activity genres that draw on students' naturalistic perceptual and motor capacities [Abrahamson, 2009; 2014; 2015; 2017]. In particular, the *action-based genre* of embodied design delineates steps for engineering learning environments that foster conceptual learning at the sensorimotor–sociocultural interface.

In their seminal book, The Embodied Mind, Varela et al. [1991, p. 173] explain their philosophy of cognitive science, enactivism, as follows:

> In a nutshell, the enactive approach consists of two points: (1) perception consists in perceptually guided action and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided.

At the Embodied Design Research Laboratory in the Graduate School of Education at the University of California Berkeley, we have been evaluating the potential relevance of the enactivist position as a guiding framework for building mathematics learning activities and examining students' experiences using these resources. As designers, we use the enactivist credo to reverse-engineer our learning environment.

We begin by asking ourselves, what are the <u>cognitive structures</u> we would like our students to develop? Answering this question depends on our learning objective. If, for example, we would like for the students to develop the concept of proportionality, then we consider what might be a dynamic instantiation of this concept; that is, we are looking for some movement composition that mathematicians would recognize as a clear schematic exemplar of the target concept. Pratt and Noss [2010] use the term *phenomenalization* to capture this creative process of making "concrete" realizations of "abstract" concepts. For example, we might determine a movement composition for proportionality, in which two objects rise side by side at different speeds (see Fig. 6). Once we have designed this movement form, which we call a *conceptual choreography*, we then ask how a student might enact this movement form. This particular movement form of two rising objects could be enacted bimanually, with each hand raising one object. As such, we have determined a <u>sensorimotor pattern</u> that students should develop through participating in the activity we are about to create. From here, we deduce what would be the <u>actions</u> composing this sensorimotor pattern, here it would

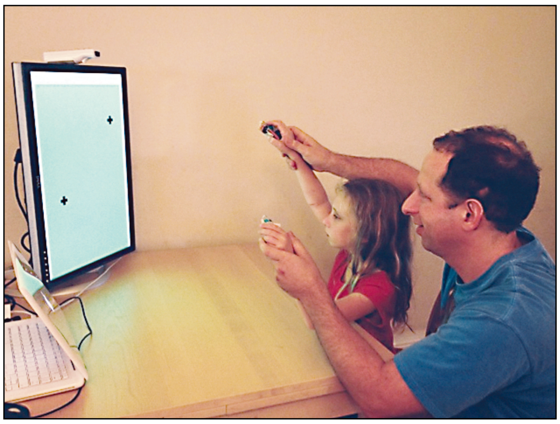


*Fig. 6.* The author and Neomi Sarah, at 8 years old, using remotes to co-enact a conceptual choreography for proportionality, in which two cursors rise at a 1 : 2 ratio

be a left-hand action and a right-hand action, where each hand is rising. Finally, we ask how these actions might be perceptually guided so as to perform the conceptual choreography. That is, what could be a particular perceptual orientation toward the sensory display of the two rising hands that could facilitate the coordination of this bimanual motor action in proportionate form? As we will explain, one way of controlling two hands moving at different speeds, which can otherwise be a challenging feat, is to focus on the spatial interval between the hands. This gap has to increase as the hands rise and decrease as they come back down.

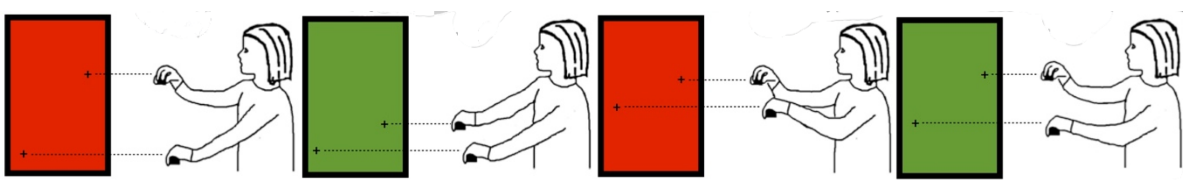
## BUILDING A NEW WORLD: EVALUATING EMBODIED DESIGN

I have demonstrated how a philosophy of cognitive science (enactivism) can be implemented in the form of a pedagogical framework (embodied design). In particular, I have exemplified how *we cast developmental stages of enactivist ontogeny as structural and procedural elements in the action-based genre of embodied design for mathematics learning*. As such, *our engineered learning environments, which are carefully designed HCI systems, are crafted to simulate the ecological conditions that would elicit from students the development of new sensorimotor perceptual structures*. We had determined these particular structures as pivotal for students to experience the phenomenalization of a mathematical concept that is targeted by our design. In turn, *students engaged in our activities develop these perceptual structures spontaneously, as their pragmatic means of solving an emergent motor-control problem they encounter in the course of attempting to perform an assigned task involving the manipulation of objects*. For example, they construct a new Gestalt — the spatial interval between two virtual objects on a screen — as their "steering wheel" for coordinating the bimanual work of moving two objects in parallel at different speeds. We use the phrase *Mathematics Imagery Trainer* to name this type of learning environment that we build for students' sensorimotor perceptual construction of proto-conceptual structures. These sensorimotor perceptual structures evolve from proto-conceptual to conceptual once students adopt mathematical frames of reference, as we explain below. *By token of eliciting naturalistic behaviour* *within a highly crafted environment and nurturing these behaviors into normative disciplinary expression, the Mathematics Imagery Trainer fosters conceptual learning at the sensorimotor–sociocultural interface.*

The Trainer featured in Fig. 6 was the first of our attempts to leverage, evaluate, and investigate the enactivist–constructivist thesis — viz. that cognitive structures emerge from recurring action-oriented sensorimotor patterns — as a *modus operandi* for crafting educational design. Results from clinical testing of the Trainer were first presented at PME-NA 32 [Reinholz et al., 2010]. Students who participated individually or in pairs in our task-based semi-structured clinical interviews were able to enact movement forms that satisfied the task requirement of keeping the screen green while raising or lowering the virtual objects. Their multimodal explanations suggested that they were developing a succession of increasingly sophisticated strategies for solving the bimanual motor-control problem that they encountered in the course of attempting to perform the task. Through iterated attempts, the study participants became conscious of new dimensions of operating the objects, and they explored for optimal values along these dimensions.

Figure 7 illustrates paradigmatically the sequence of interaction events commonly observed across students, as they figure out that their hands should move not at the same speed but at different speeds.

When we then introduced symbolic artefacts onto the screen — first a grid, and later numerals along the vertices (see Fig. 8) — students endorsed these features into their sensorimotor scheme, and yet in so doing the scheme changed. We concluded that



*a* *b* *c* *d*

***Fig. 7.*** Solving a movement riddle in the Mathematics Imagery Trainer, a child learns to move physically in a new way; she then articulates her movement formally as governed by a proportional function

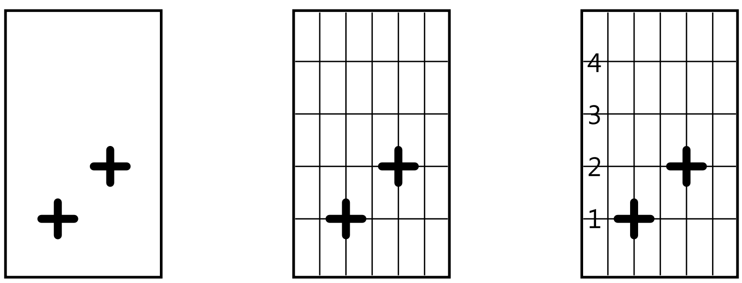



***Fig. 8.*** Three configurations of the Mathematics Imagery Trainer's computer interface. From left: cursors only; with a grid; and with numerals

*students had identified in the figural features of these didactically interpolated symbolic artefacts certain relevant utilities for enhancing either the enactment, evaluation, or explanation of their extant control strategy; in so doing, they implicitly assimilated the features as frames of reference, thus shifting from naïve to cultural forms of organizing and understanding their situated actions*. We noted in particular that introducing the grid caused students to change their bimanual strategy from moving their hands simultaneously through the continuous space and using qualitative language to describe their strategy (e.g., "The higher I go, the bigger the gap") to moving their hands sequentially through the discretized space and using quantitative language (e.g., "For every 1 I go up on the left, I go up 2 on the right"). As such, the students developed the activity's intended psychological–discursive forms of acting and reasoning without the researchers offering any direct instruction, demonstration, or formatting [Abrahamson et al., 2011].

Participants were also able to coordinate among polysemous visualizations of the environment, for example explaining why raising their right hand 2 units every time they raise their left hand 1 unit (Fig. 9iii) means that the spatial interval between their hands should steadily increase (Fig. 9ii). As such, we noted the pedagogical potential of the activity design and, specifically, of the Trainer environment, to foster important conceptual reasoning using non-inscriptional media, that is, even before pen is set to paper [Abrahamson et al., 2014].

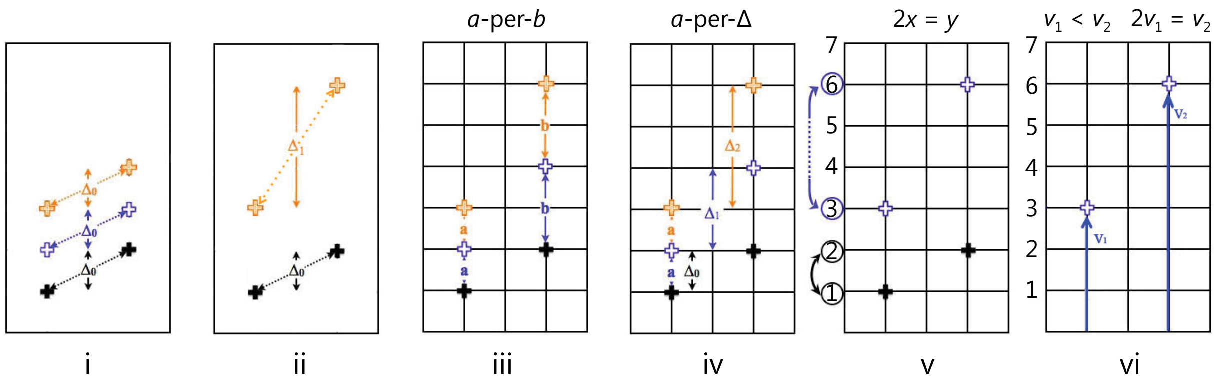



***Fig. 9.*** Student generated solution strategies for the make-the-screen-green problem (the case of a 1:2 ratio)

Research efforts are still underway to evaluate how sensorimotor competence, which students develop through participating in activities using the Trainer, could possibly be cast as constituting forms of knowing that the community of mathematics-education researchers and practitioners would recognize as relevant to normative disciplinary practice in educational settings. A logical and anticipated way forward here is to demonstrate how these ways of knowing play out, when study participants, who have engaged with the activities, then set to engage in solving problems that the field generally appreciates as constituting measures of domain-specific subject-matter content knowledge. This approach could potentially translate to methods of assessing what students gain conceptually through developing new movement forms for manipulating objects according to specified task requirements.

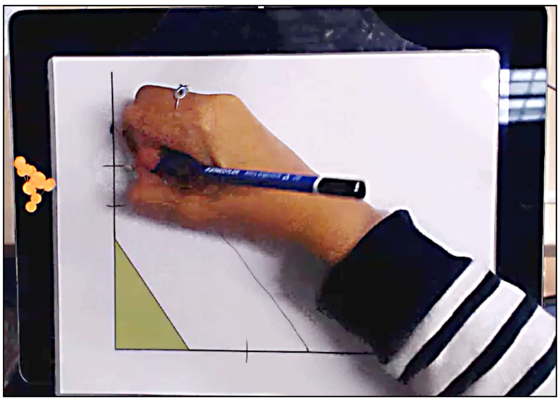


***Fig. 10.*** A student who had imagined a diagonal line on a tablet, to solve a 3 : 2 orthogonal ratio problem, now recreates this line on paper, using measurement

Some studies are showing promise. A multi-classroom-based doctoral dissertation [Petrick, 2012] tentatively concluded that students who engaged with the Trainer and other analogous activities advanced conceptually more than a comparative group (see also in [Abrahamson, 2012]). Related studies resulted in similar empirical findings, suggesting the pedagogical potential of Trainer activities in the domains of proportionality [Bongers, Alberto, Bakker, 2018], the Cartesian coordinate system [Duijzer et al., 2017], geometrical area [Shvarts, 2017], and parabolas [Shvarts, Abrahamson, 2019]. For example, Bongers et al. [2018] demonstrated effective semiotic transitions from description to inscription (see Fig. 10): Students who had invented and manipulated imaginary sensorimotor perceptual structures on a tablet, as their pragmatic solutions to the problem of coordinating the enactment of solution movements, then recreated and thus materialized these percepts using pencil and paper; what more, they spontaneously used multiplication to measure proportional segments of these constructions, such as building a set of dilated similar right triangles. For further review of empirical findings from the project, see Abrahamson and Bakker [2016].

## ATTENTIONAL ANCHORS INTO THE STUDENT'S NEW WORLD

As educational *designers*, we were thus learning more about the action-based activity genre of embodied design [Bakker, Shvarts, Abrahamson, 2019]. Yet as design-based *researchers*, we hoped to get a tighter theoretical grip on the evolution of sensorimotor perceptual structures: we wanted to witness and monitor the micro-process of a new structure emerging into students' interaction routines to become an object of reflection and mathematical modeling. This is where eye-tracking technology offered opportunities. Our study participants were seeing a new world by constructing sensorimotor perceptual structures, and now we, too, were about to see a new world by capturing the students' embodied learning process.

The emergence of a new world means that objects that had been latent to the environment have become salient to the human subject. Yet how might we theorize a phenomenal object that comes forth from the background? Enactivism suggests that perceptual structures emerge from the background when they repeatedly facilitate our engagement in tasks requiring the performance of new movement forms. Yet how precisely is this happening? How should we operationalize the micro-process of perceptual emergence? To answer this question, we sought more radical stances.

Hutto and Sánchez–García [2015] propose a radical-enactivist interpretation of skilled athletic performance. They interpret skilled performance as utilizing specialized action-oriented relations with the environment. These relations are perceptual “anchors” into the environment that determine attentional routines guiding effective motor action. These *attentional anchors* are thus action-oriented sensorimotor perceptual constructions—the perceptual components of affordances. Abrahamson and Sánchez–García [2016] borrow the construct of attentional anchors to refer to the structures that study participants purportedly constructed to solve Trainer motor-control problems, such as using the spatial interval between the cursors to facilitate bimanual coordination.

Attentional anchors (hence AA) might originate in a gaze that is strategically cast between two or more manipulated objects, such as the cursors, so as to maintain them in peripheral vision, similar to a juggler who gazes not at the balls themselves but at an empty spot above her [Hutto, Sánchez–García, 2015]. That is, *students discover and use an AA, because it enables them to perform movements that conserve a select dynamic stability of an emergent system they are thus building and transforming.*

The function of perception in organizing motor action is possibly more critical than the literature has surmised. Empirical findings from studies of perception and action suggest that AAs can be generated independent of sensory access to one’s actuating limbs. Thus, perception takes the lead. When participants cannot see their hands, still,

> voluntary movements are organized by way of a representation of the perceptual goals, whereas the corresponding motor activity, of sometimes high complexity, is spontaneously and flexibly tuned in [Mechsner et al., 2001, p. 69].

As such, generating AA may be a natural inclination of biological organisms’ embodied cognitive architecture. Perception is sentient enactment [Noë, 2006].

Led by collaborating researchers at Utrecht University, the next study applied eye-tracking methods to monitor the sensory behavior of students engaged in the solution of Trainer motor-control problems. Corroborating and expanding on our earlier clinical findings, we now had a new form of empirical data that we could put forth as evidence supporting our hypothesis that students’ task-effective bimanual coordination is associated with changes in the composition of their perceptual orientation toward the sensory display (see Fig. 11). We concluded that attentional anchors serve a vital function in the accomplishment of coordinated bimanual action. Moreover, our study participants’ mathematical discourse *about* these perceptual structures suggested that they constitute important cognitive pivots from unreflective engagement to disciplinary reasoning. These findings recur across a set of variants on the original Trainer task [Abrahamson et al., 2016].

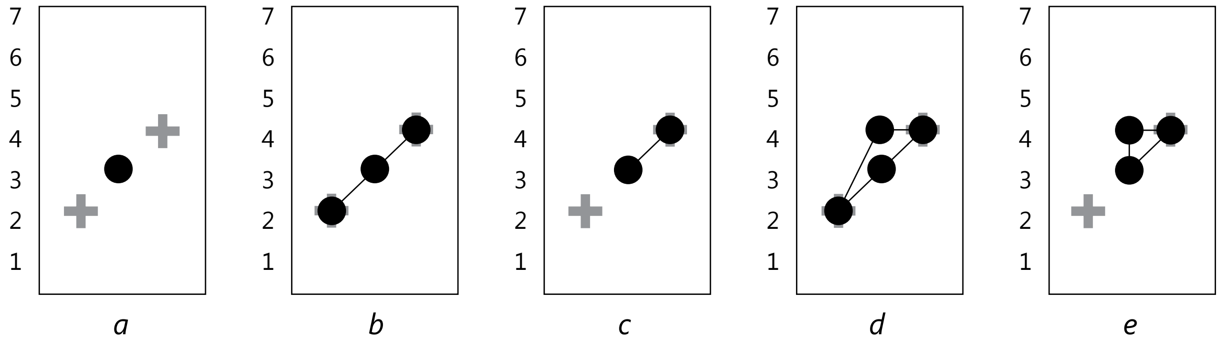



*fig. 11.* Dynamical gaze patterns reveal a variety of attentional anchors. The focal gaze point between the cursors is at an un-manipulated location. Pattern *b* was the most prevalent among participants

## NEW WORLD EXPANDING: THE TRAINER BEYOND PROPORTION

Now equipped with these new empirical data of students' combined multimodal problem-solving behaviors — both clinical and eye-tracking data — we felt more confident in claiming that: (a) the Mathematics Imagery Trainer environment realizes Abrahamson's action-based genre of embodied design; and (b) this genre achieves its objectives of fostering students' development of sensorimotor perceptions bearing semiotic potential as grounding new mathematical notions. Next we turned to apply these theoretical, pedagogical, and methodological ideas to the design of additional Trainer activities. Here I will briefly mention two more designs for grounding mathematical concepts in sensorimotor perception.

### *Parabolas*

Figure 12 features two configurations of a Trainer for parabolas. Here, the triangle is green only when $BC = AC$. $A$ is fixed at the parabola's focus, $B$ runs along the horizontal dashed line immediately below $C$, and the student manipulates only Vertex $C$. By keeping the triangle green while moving Vertex $C$, the student effectively inscribes a parabola curve. (Note that Labels $A$, $B$, and $C$ as well as the dashed lines in this figure are used only here to illustrate the design for readers of this text: these lines are never

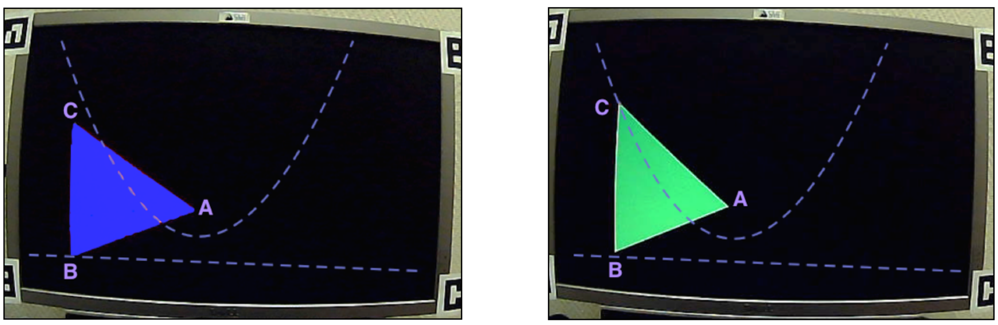



*Fig. 12.* An action-based embodied design for parabolas. By manipulating Vertex $C$, in an attempt to keep the triangle green, students find themselves inscribing a parabola

shown to the students, as they engage in the activity.) Participant college students learned to move in green, and then they were guided to derive a definition of the parabola from geometrical properties of the isosceles triangle and auxiliary constructions [Shvarts, Abrahamson, 2019]. The key cognitive event, along this solution process, was perceiving the isosceles triangle. Once they saw it, participants immediately became more fluent in operating the device according to task specifications.

### *Trigonometry*

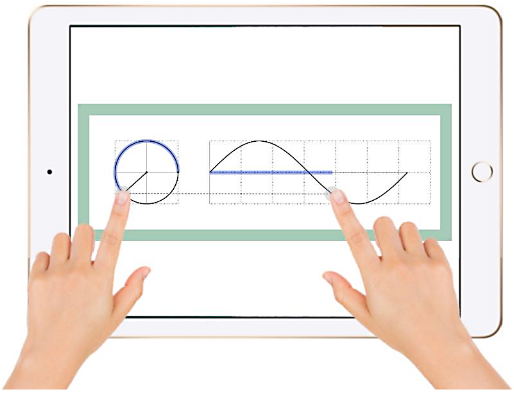


***Fig. 13.*** An action-based embodied design for trigonometry. The frame is green, if the unit-circle's radian (on left) corresponds to the sine-graph's *x*-value (on right)

Figure 13 features a Trainer activity for trigonometry. Here, the student slides their left-hand fingertip on the perimeter of a unit circle, while sliding the right-hand fingertip on a sine graph. Whenever the radian value on the circle corresponds to the *x*-value in the sine graph, the rectangular frame around the interactive zone becomes green. The student needs to keep the frame green while moving both hands. Data analysis of a pilot study with participant college students suggests that they imagined a horizontal line segment connecting the two fingertips (not shown in Fig. 13).

The horizontal-line attentional anchor seemed to help participants keep the two fingers at the same height. Mathematized, this imaginary line then came to mean that the left- and right fingertip positions are equally high or low on the grid, thus sharing the same *y*-value, which is sin(*x*). This awareness appeared further to support the enactment of green-keeping movement [Alberto et al., 2019].

## CONCLUSION: THE WORLD TODAY, THE WAY I SEE IT

Mathematical concepts are grounded in sensorimotor perceptions that emerge as practical solutions for the efficient enactment of goal-oriented ecologically coupled movements. Sociocultural reframing of these sensorimotor perceptions occurs through the timely mediation of symbolic artifacts, when learners participate in facilitated cultural practice. Students adopt these artifacts spontaneously as readily available means of enhancing their goal-oriented actions, yet in so doing they surreptitiously appropriate heritage conceptual systems that enable them to participate in the discourse and social enactment of cultural–historical mathematical practice.

I believe it is important to help students maintain their original sensorimotor perceptions as their means of grounding mathematical concepts, even as we support them in appropriating and exercising the complementary powerful cultural devices of mathematical practice. Just how teachers should do this remains an open question. However, I further believe that the multimodal affordance of natural communication, particularly the combination of speech and gesture, bears promise [Abrahamson et al., 2012; Flood, 2018; Fuson, Abrahamson, 2005]. As such, one way forward is to study how teachers engage with students' grounded sensorimotor perceptions to sustain ecological meaning in mathematical concepts.

## MOVING FORWARD: EXPLORING OUR NEW WORLD TOGETHER

It has been thrilling to discover the new world of sensorimotor perception that facilitates our engagement with the environment, orienting our every mundane operation, and to conjecture as to the horizons this discovery opens up for educational theory and practice. There are ethical issues at stake. For example, the action-based genre of embodied design is reshaping our approach to the mathematics learning of sensorily diverse students, such as those who are blind or visually impaired [Abrahamson et al., in press].

Like Columbus, however, it has been sobering to realize that the novelty of this new world is truly subjective. This new world as I found it had always been there, eons before my collaborators and I came along to first cast our eyes on it and claim it. Yet unlike Columbus, this new world that our research is now colonizing has truly always been our own — ours, yours, everyone's. This beautiful coral reef has been there forever below the surface of our own sensorimotor waters, waiting.

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# BRAIN AND MATHEMATICS: IMPLICATIONS FOR EDUCATION

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*Numbers and calculations elicit brain activity in a set of brain areas. Number tasks refer to numerical processes that do not involve formal mathematical operations, whereas calculations tasks involve operations such as addition, subtraction and multiplication. Functional magnetic resonance imaging (fMRI) studies with adults show that the parietal, cingulate and insular cortices are critical for processing number tasks; calculation tasks also implicate the prefrontal cortex extensively. Children engage similar areas in posterior parts of the brain; however, prefrontal parts of the brain do not show consistent involvement, suggesting a reorganization of function across development. Instead the insular cortex, a brain area associated with awareness and emotion is highlighted in children's problem solving. Potential implications for education in terms of teacher's professional development and children's learning experiences are discussed.*

## SOME BASICS ABOUT BRAIN ANATOMY

The fact that the brain is made up out of billions of neurons is a relatively new idea. In the late ninetieth century Santiago Ramon y Cajal spent a lot of his time trying to convince scientists at the time that the brain was made up of individual cells. His discovery was made using tissue-staining methods and his claims were confirmed later with the invention of the electron microscope in the 1930s. In 1949 Donald Hebb explained that neurons that fire together wire together. Indeed, neurons communicate through synapses and electrical signals travel across them to take messages to close and distant parts of the body [Kolb, Wishaw, 2003]. Recent ground-breaking discoveries suggest that neurons not only fire and wire together, they can also change with experience (i.e., neuroplasticity) and new neurons can be created (i.e., neurogenesis) even in adult brains (e.g., [Drapeau, Abrous, 2008]). These neuroanatomical principles are fundamental on how we view the brain and provide a positive message for education as a science of training neurons in young minds.

The brain is anatomically organized in two hemispheres and five lobes (e.g., occipital, parietal, temporal, frontal and limbic), connected with a dense bundle of fibbers called the corpus callosum [Kolb, Wishaw, 2003]. Serious injuries (i.e., lesions) to any part of the brain typically correspond to changes in behavioural functions. Although this purported that the brain is organized in terms of specific areas of function (e.g., [Broca, 1861; Wernicke, 1874]), Alexander Luria was likely the first to claim that regions in the brain do not act in isolation [Luria, 1970]. The latter relates to identification of several

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functional networks in the brain that work in consonance to create thoughts, actions and emotions in our everyday life.

## MATHEMATICS AND THE BRAIN

Mathematics played a key role human history and is fundamental on how we are going to evolve as a society. Mathematical cognition is a quintessential ability in humans. Young children can quickly decipher which pile has the most candy or who is getting the biggest piece of cake. Most children are exposed to formal mathematics during school age years or earlier. Our society is driven by knowledge, innovation and discovery, and mathematics serves a fundamental role in this process. According to the world economic forum an estimated 65% of children who are now entering elementary school will graduate to work on completely new occupations that do not exist today (Chapter 1: The Future of Jobs and Skills, 2016). Although, our world is changing rabidly and future labor market requirements are not fully understood, researchers agree that future workforce will have a higher demand for Science, Technology, Engineering and Mathematics (STEM) majors [Fayer, Lacey, Watson, 2017]. Mathematics is essential for STEM majors and occupations that are progressively more on demand. Traditionally, mathematics has been a core subject in school curricula and research shows that success in math relies in part on the child's age and their readiness to receive instruction [Agostino Johnson, Pascual-Leone, 2010]. Advances in mathematical performance coincide with the development of fundamental cognitive processes such as mental competence and the protracted development of brain structures such as the pre-frontal cortex (Pascual-Leone et al., 2010). Neuroscientists have investigated the brain areas related to numerical and mathematical processes. About twenty years ago, a neurofunctional model was proposed to explain the brain areas that support mental arithmetic, mainly focusing on the functions of the parietal cortex, in posterior parts of the brain [Dehaene, Cohen, 1997; Dehaene et al., 1993]. Although this model was based mainly on lesion patient studies, it has stimulated a substantial body of neuroimaging research with healthy individuals.

### *Magnetic Resonance Imaging* (*MRI*)

MRI is a non-invasive method used to take high-resolution images of the inside of the human body. It utilizes a strong magnet and radio waves to provide detailed images of the living brain; it does not use *x*-rays. The same machine can be used to take anatomical scans or functional scans. Functional MRI measures changes in blood flow that take place in an active part of the brain, an indirect measures of neuronal activity. fMRI signal is used to understand how different mental functions are represented in the brain.

MRI is safe for children and adults as long as proper safety guidelines are followed. Because the machine uses strong magnets we must ensure that anyone going in the MRI room removes any metallic objects that he or she may be wearing (jewelry, hair clips, watches etc.). Importantly, all participants are carefully screen to confirm that they do not have any metal implants (e.g., pacemakers), because the magnetic field will interfere with the device. Another important consideration for individuals entering the

MRI machine is claustrophobia, a fear of small places. Although an MRI can feel narrow for adults, who are bigger, it feels more spacious and confortable for the children, who are smaller. Our challenge with children is that we have to make our games (i.e., tasks they play while in the MRI) interesting for them so that we keep them engaged while we are imaging their brain in action. From experience we figured out that several short games (4 × 5 minutes) are better for children than one longer one (20 minutes). Moreover, when we are scanning anatomical images that do not require a cognitive activity, we play a movie for children so they can stay entertained during that time.

### *Math games with neuroimaging*

Because the MRI scanner is the way it is, cognitive games we show participants need to be carefully designed. These games are carefully designed so that researchers know exactly what a participant is looking at what exact time. Very few studies have examined complex mathematical problems with fMRI (e.g., [Krueger et al., 2010]). This is in part because of increased individual differences in performance levels on more complex mathematical problems, as increased individual differences would present themselves in different problem solving strategies and in turn variable time to generate a solution. The majority of fMRI studies examine simple judgements with numbers and mathematical operations. I call number tasks the math games that have no mathematical operations. For example, participants are presented with a group of dots on the left of the screen and a group of dots on the right of the screen and asked to indicate with a button press which group of has more dots; this is non-symbolic number processing. Symbolic number tasks involve symbols (e.g., Arabic numeral) and participants are asked to indicate with a button press which number is bigger. Calculation tasks require a mathematical operation, addition, subtraction, multiplication and division. The majority of fMRI studies have examined brain responses to addition problems and the least number of studies have examine brain responses to on division problems.

I started investigating brain responses to numbers and calculations after some work on cognitive competence, as this was my main research focus. Specifically, I am interested to understand how many things the human brain can hold and manipulate in mind before we start going into overload. For my graduate work I had developed some tasks that use visual-spatial stimuli (i.e., colours; Colour Matching Tasks; [Arsalidou, Pascual-Leone, Johnson, 2010]) that manipulated difficulty across six levels. Behavioural data showed that children and adolescents progressively passed more levels of the tasks based on their mental-attentional capacity, which corresponded to their age [Ibid.]. These results were consistent with past theoretical predictions [Pascual-Leone, 1970] and empirical findings [Pascual-Leone, Baillargeon, 1994]. Importantly, when we administered this task in the fMRI we obtained graded increases in cortical activity in several brain regions that included the parietal, prefrontal and cingulate cortices. We replicated findings obtained with the visual-spatial using verbal tasks (i.e., with letters as stimuli) that follow the same parametric design with six levels of difficulty [Powell et al., 2014]. Numbers however can be considered as visual-spatial and verbal stimuli. The neuroimaging literature on numerical processes focused mainly on the parietal cortex, which is critical in processing visual-spatial aspects of numbers [Dehaene et al., 2003 for review]. The contribution the prefrontal cortex and other brain regions in the healthy human brain is not very well substantiated. Since no single study is definitive, we had to look at reported findings using a meta-analyses approach [Arsalidou, Taylor, 2011]. fMRI meta-analyses compile data reported across many studies to identify the activation likelihood of any location in the brain to identify concordance across studies. So we compiled adult data from fMRI studies that examined number tasks and data from fMRI studies that examined calculations task.

Number tasks showed significant activation likelihood estimation (ALE; [Eickhoff et al., 2009]) values in several regions that included the parietal cortex bilaterally and the cingulate cortex [Arsalidou, Taylor, 2011]. No significant concordance was observed in dorsolateral prefrontal areas, known for higher-order cognition (e.g., [Christoff et al., 2009; Arsalidou et al., 2013]). Thus, deciphering the cost of items on a menu for example is more likely to engage parietal cortices. Calculation tasks showed significant ALE values in the same areas and in addition showed extensive activation in prefrontal cortices bilaterally. This area included dorsolateral prefrontal cortices associated with coordination of mental actions [Christoff et al., 2009]. For individuals who solve math problems regularly and have to plan, strategize and coordinate mental action in the service of reaching a solution would activate their prefrontal cortex.

Which hemisphere is implicated in different math problems (i.e., addition, subtraction, multiplication) remains a question in mathematical cognition. Laterality indices from adult fMRI meta-analyses suggest a differential implication [Arsalidou, Taylor, 2011]. Regions of interest were selected anatomically, using Brodmann areas (BA). Korbinian Brodmann was a German neurologist who spends many hours looking at the histology of the cells in different parts of the brain. He defined 52 areas as speculated that if these cells have different structures they should perform different function. His work was published in 1909. Neuroscientists to this day use these cytoarchitechtonic mappings to identify with more specificity different locations in the brain. Laterality indices in our results show that addition is left hemisphere dominant in parietal (BA 7, 40) and prefrontal (BA 9, 46) cortices [Ibid.]. Subtraction is more mixed showing both bilateral and left dominant areas in parietal and prefrontal cortices. Multiplication is mainly right dominant or right lateralized (BA 7, 40 and 46) with the exception of BA9 in the prefrontal cortex, which was left dominant.

### *Right-left-right hypothesis*

Critically, all these mathematical tasks use numbers whether this is addition, subtraction or multiplication, thus it is not clear why different hemispheres are implicated in different math problems. The classic material-specific hypothesis that the left hemisphere processes verbal information and the right hemisphere processes visual information cannot account for these data. Instead we need to look into other perspectives. Specifically, in the Juan Pascual-Leone proposed a hypothesis that stems from the developmental literature and strategy use that supports a process-specific approach [Pascual-Leone, 1987; Pascual-Leone et al., 1995]. In our recent paper we described it as the right-left-right hypothesis for hemispheric dominance [Arsalidou et al., 2018]. It explains hemispheric dominance not in terms of the material in the task but in terms of the task familiarity and a trade off between the mental-attentional capacity of the individual and the mental-demand of the task. Particularly, when the task is very easy such as when we solve simple number tasks, activity should favour the right hemisphere. When tasks are effortful but still possible for the individual then the left hemisphere should be favoured. When the task is too difficult, often when the participant is failing the task, then the right hemisphere should be favoured. We can consider the right hemisphere as coming to the rescue of the situation. The brain is in a state for looking at well-known ways of doing things in an effort to identify a way to solve a very difficulty or novel problem, and this happens usually through trial and error. When a method is established the left hemisphere should be able to take over. This is a novel hypothesis and we need to investigate it further. A critical point is that it is difficult to find brain areas associated with the mental-demand of the task being much higher than the mental attentional capacity of the individual. This is because at this state, participants start to fail the task. Reasonably, it is customary in fMRI studies to report results that are related to correct responses rather that incorrect responses. Therefore to test this hypothesis we need original studies that examine this directly and encourage future studies to report findings related to incorrect responses so that we can use meta-analyses to understand the literature.

The trade off between mental-attentional capacity of the individual and mental-demand of the task changes as a function of age. According to developmental theory, mental-attentional capacity increases by one unit every other year after the age of 3 years, such that it reaches 7 units in 15–16 years olds and adults [Pascual-Leone, 1970; Pascual-Leone, Johnson, 2005]. As mental attentional capacity increase what used to be a difficult tasks for a child becomes an easy task. Thus, according to the right-left-right hypothesis, something that is very easy for older children would engage the right hemisphere, whereas the same task may be effortful (but possible) for younger children and would engage the left hemisphere. This hypothesis has indirect support from fMRI studies with children that examined number and calculation tasks.

### *fMRI studies with children and math*

Many studies have investigated brain responses associated with number and calculation tasks in children. Our meta-analyses have compiled data from 344 children that performed number tasks and 501 children that examined calculation tasks [Arsalidou et al., 2018]. For both groups the majority of children were between seven to thirteen years old. Within this age range mental-attentional capacity ranges from 3 to 5 units, according to theoretical predictions, reflecting a different cognitive competence. ALE results show that children engage right hemisphere areas in parietal and insular cortices for number tasks, whereas calculation tasks engage left parietal cortex and insular cortices bilaterally. Notably, no significant concordance was observed in dorsolateral prefrontal cortices for number or calculation tasks. This finding may be due to variability in problem solving strategies that may be reflected in different hemispheres, as suggested by the right-left-right hypothesis. In other words children at different ages may recruit the prefrontal cortex differentially, thus agreement across studies is not being observed. Importantly, children implicate extensively the insular cortex when solving calculation tasks. The insular cortex has not been previously emphasized in its role in mathematical cognition. Instead the insula has been involved in introspective processes, awareness and consciousness [Barrett, Simmons, 2015]. It has been implicated in all sorts of qualitatively different mental functions such as emotion, cognition and sensorimotor processes [Duerden et al., 2013; Uddin et al., 2014; Gasquoine, 2014; Kurth et al., 2010]. Some have suggested that it is involved in a salience network, together with the cingulate cortex, that directs attention to salient information in the environment [Uddin, 2015; Menon, 2015]. We suggest that the insular cortex may play a critical role in motivated goal directed action [Arsalidou, Pascual-Leone, 2016] that is related to task engagement.

In summary, prefrontal cortices associated with higher order cognition appears to undergo a functional reorganization during childhood and that factors associated with motivation and perhaps task engagement is critical for children's performance in mathematical processes.

## IMPLICATIONS FOR EDUCATION

Neuroanatomical findings on neuroplasticity and neurogenesis send a positive, empowering message to education. Awareness that a change takes place in the brain every time a child engages in learning activities can be enormously empowering to educators. In other words knowledge that educators can help morph the anatomy of an organ in a child is very powerful. These conclusions are not limited to childhood, as adult brains can change from learning. Neurofunctional findings suggest that dynamic functional changes occur in childhood and adolescence, such as in the prefrontal cortex, structures associated with higher-order cognitive processes. Interestingly, in mathematical cognition and other core cognitive processes such as working memory [Yaple, Arsalidou, 2018], the insular cortex plays a critical role. As we suggest that this is key region related to motivation and task engagement this highlights for education the need for appropriate school engagement practices. Teachers are the experts in the classroom, and research shows that teaching educators about the brain improved teacher knowledge and confidence [Dubinsky et al., 2013]. Perhaps teaching children about the brain would also improve their knowledge and confidence, although this remains to be substantiated with research.

Specific findings of neuroscience can highlight mechanisms of learning and lead to more powerful evidence-based education. I finish with a positive message of collaboration among neuroscientists, educators that is supported by technology to improve methods of teaching and education in the future.

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# COMPUTERS IN THE PRODUCTIVE LEARNING OF MATHEMATICS

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*The main ideas of productive learning in mathematics appeared much earlier than the development of computers. In the works of Wertheimer, Hadamard, Polya, and others, the basic principles of a productive approach to teaching mathematics were formulated. At the same time, the practice of mathematics teaching did not rely on those results because of the contradiction of the principles to the current state of learning technologies. The emergence of the computers in everyday life has opened up opportunities for technological support of productive learning methods that were previously technologically inefficient. Papert demonstrates in his work the possibilities of using a computer as a tool for a student's development, as it was developed in Vygotsky's ideas about the role of a tool in child development. However, the transition to a new educational system structure requires a large period of time, during which a new educational culture will be formed on the basis of modern technological capabilities and a new generation of teachers who perceive this culture as their own will emerge. This report will present an analysis of the possibilities, both realized and unrealized, of using a computer to support productive learning.*

## PRODUCTIVE THINKING IN MATHEMATICS

Let us consider a teacher's simple question to students: "Do you know that medians intersect at a single point?"

The answer: "Yes!"

Next question: "Do you know why medians intersect at a single point?" In response, first a long pause, then uncertainly, "We once somehow proved it, we tried to remember."

What do we want to hear in response to the question "Why"? We are waiting for a justification of the answer, the proof of this fact. But what do we mean by proof? Do we want to hear the memorized text from a textbook, as in the examples that Wertheimer gives in his book "Productive Thinking"? [Wertheimer, 1945]

Or do we want to hear an explanation based on understanding, which is manifested by one's ability to provide an explanation in layman's terms?

If a student does not know how to answer in a reasonable timeframe, it likely means that he/she did not "catch" or "seize" this idea (i.e., did not grasp its meaning). In other words, he/she could not translate it into his/her internal language, after which he/

In *Shvarts A.* (Ed.). Proceedings of the PME and Yandex Russian conference:
Technology and Psychology for Mathematics Education. P. 77–90. Moscow, Russia: HSE Publishing House.

she could explain it to him/herself and others without going into details. Let us call it understanding.

When speaking about the technological support of teaching mathematics, it is reasonable to highlight the influence of a computer on the formation of internal intellectual mechanisms. Thus, the problem of the computer's influence on a student's understanding of the material being studied appears interesting.

This topic has many different aspects and titles. For example, Khinchin discussed the phenomenon of "formal knowledge" as opposed to thoughtful knowledge [Khinchin, 1963].

What do children want to know when they ask the question "why" or other so-called smart questions?

We understand that they are not interested in the true cause of this or that phenomenon, but they do want to connect something new they have discovered at that moment to their already existing ideas about the world.

Genetic mechanisms encourage a person to build a reliable intellectual system in which all new information is "comprehended" by binding it to more basic knowledge and ideas.

This basis consists of intellectual elements which are connected with human senses (such as vision) and of basic ideas received in childhood. Minsky calls this phenomenon the principle of investment [Minsky, 1986].

Let us return to the question about medians. How many students do you think would answer this question?

Among ordinary schoolchildren who do not study in special math schools, and who do not plan to become mathematicians upon graduation, we would get very few answers.

This is what Wertheimer wrote about in his book "Productive Thinking", and what Khinchin called formal knowledge. A new idea which is not yet accepted by a student remains isolated knowledge, knowledge that does not develop the student's thinking and will not be applied in other situations.

What could be the answers to this question? In other words, what answers would show understanding? The first possible answer?

The first answer is based on spatial representations. Medians intersect at a single point because three planes intersect at a single point. The answer is unusual, but it can be explained with that clear picture (see Fig. 1).

The second answer is based on a concept of physics (see Fig. 2). The medians intersect at one point because it is the center of gravity of the triangle (or of its vertices). Surprisingly, within the Russian-language Internet, there is only one picture (left) out of several hundred connected with medians. It is the same situation within the English-language Internet (middle). Note that the pictures to the right do not explain anything — this is just an illustration of the concept of the center of gravity of a figure.

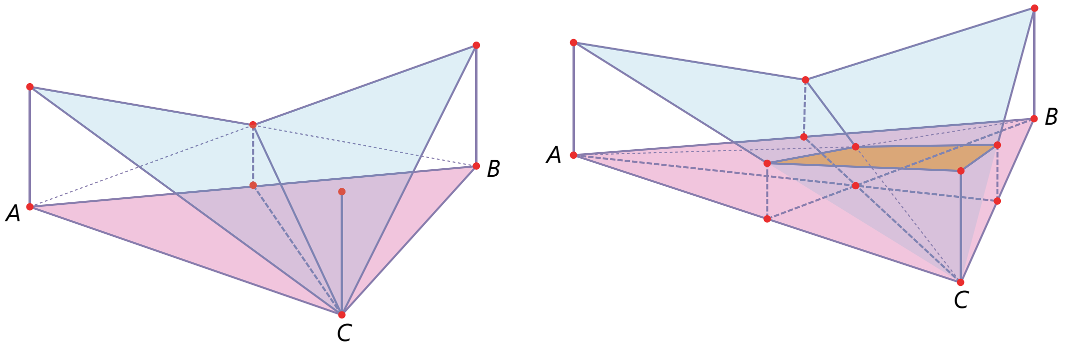


*Fig. 1.* Medians intersect at a single point because three planes intersect at a single point

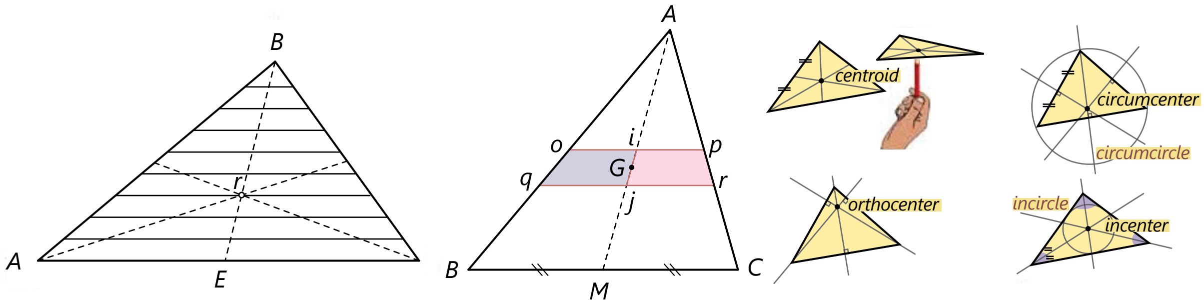


*Fig. 2.* The medians intersect at one point because it is the center of gravity of the triangle

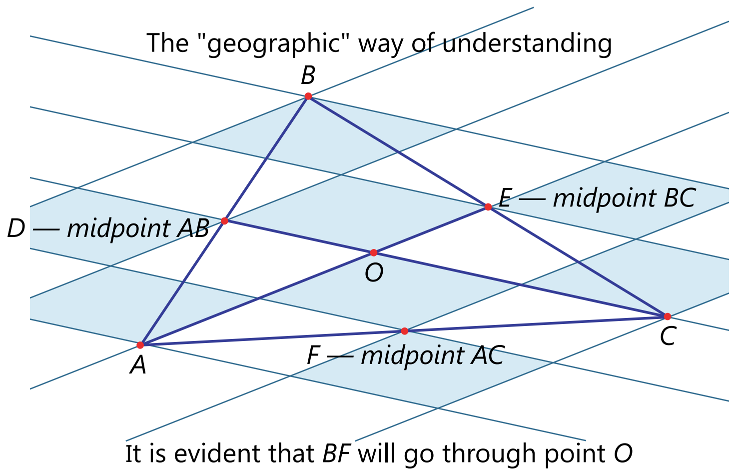


*Fig. 3.* The medians intersect at a single point because it is evident within an appropriate "geographical" grid

The explanation is that we cut the triangle into strips and find out from obvious considerations the centers of their gravity. Then we combine the results, and the centers of gravity of the strips lie on the median. And then we "intersect" the results, saying that the same considerations can be applied to another median, and therefore the center of gravity of the whole triangle will be at their intersection.

The third answer is based on a "geographical" representation (see Fig. 3). The medians intersect at a single point because if we draw a grid of the straight lines parallel to two medians and passing through the vertices and the sides' midpoints, the third median will pass through the diagonals of the cells of the grid and therefore necessarily through the intersection point of the two given medians. From this interpretation, it is obvious that the ratio of median's segments at the intersection point is 2 : 1.

A mathematician can make a claim to each of the previous examples saying that such evidence is not a proof, and he/she will be right.

What is the role of proofs? In the book "Society of Mind", Minsky presents a perspective on the proof as building a chain of related judgments so that the destruction of only one of its links leads to the destruction of the whole structure [Minsky, 1986].

Thus, the logical conclusion exists not as a way of thinking, but as a way to verify knowledge, to ensure the reliability of the building of science. This refers to knowledge in its social, not personal, sense. That is, a person convinces him/herself not by a rigorous proof, but by convincing others through proofs which are accepted by the community as a format for the preservation and transmission of objective knowledge.

## COMPUTER TOOLS: EXPERIMENTS, EVIDENCE AND PROOF

Among all approaches to technological support in learning mathematics, Dynamic Geometry is the most successful. This fact is confirmed by the large number of implementations of this idea and the large number of teachers and students taking advantage of its possibilities.

Using dynamic geometry, teachers and authors of educational materials solve various problems, from fast drawings of attractive figures to the preparation of tasks for mathematical Olympiads.

We will discuss several fundamental possibilities provided by dynamic geometry to support productive learning, that is, to support independent search activity. Let us consider the problem of constructing an inscribed circle. There are three stages of solving the problem of constructing an inscribed circle in dynamic geometry:

1) a "manual" attempt to touch the circle by the triangle — but the movement of the vertices of the triangle destroys the solution;
2) construction of the center of the inscribed circle by the intersection of bisectors, and constructing a circle centered at this point — the movement of the vertices destroys this solution too;
3) construction of an additional point of the circle which is the base of the perpendicular from the center to the side — the movement of the vertices does not destroy this last solution.

There is a serious psychological difference in solving this problem with and without a computer (see Table 1).

Table 1

**Solving this problem with and without a computer**

| Without computer: | With computer: |
|---|---|
| A student draws a picture with a circle inscribed in a triangle, looking at which the teacher can understand that the student understands the meaning of the word "inscribed". | A student constructs the solution and checks it him/herself by moving points (which can be done by a teacher too). |
| A student gives explanations to the drawing: about finding the center of an inscribed circle and the point on it (constructing of bisectors and perpendicular). The teacher should read the explanations and either accept them or indicate incompleteness or incorrectness. | The place of explanations is occupied by the algorithm of construction, which is indirectly expressed by the correct dynamic drawing, but can be considered in terms of algorithmic operations (script). |
| If a student has proposed an original solution, the teacher will hardly have enough time to check it promptly, without stopping to work with other students | Novel solutions are checked as well as the standard one, and a student can convince him/herself and the teacher of its correctness simply by moving vertices |

Now back to the example of the intersection point of the medians of a triangle. Will it be enough for the next experiment to convince students that medians of a triangle intersect at a single point? (Fig. 4).

After the experiment, students will agree that the medians intersect at a single point, but will not answer the question "Why?" The reason is that this experiment does not provide any link between the ideas the student has and the results. Knowledge will be isolated, and therefore formal.

Consider the differences in the two given examples of computer experiments. In the task of constructing a circle inscribed in a triangle, students answer the question, "Why is a circle built like this?" with a response something like: "Because the center of the circle lies at the intersection of bisectors, and at the tangency point the radius is perpendicular to the tangent."

At the same time, students may not answer the question, "Why do these bisectors intersect at one point?" But for this task it is not important. More important is that in this experiment three facts turned out to be connected together (the inscribed circle, the intersection of the bisectors, the perpendicularity of the radius to the tangent). It is the connection, not the knowledge of each fact separately, that is the goal of the constructive activity.

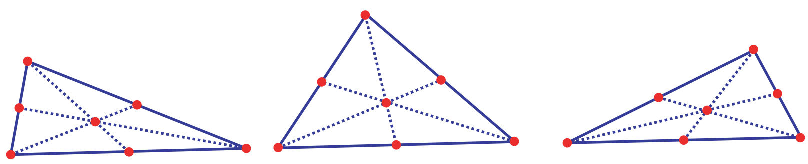

*Fig. 4.* Attempt to convince students that medians of a triangle intersect at a single point

Let us summarize this discussion. Why does mathematics teaching need tools?

From a psychological point of view, Vygotsky showed the role of tools in the development of a child: external objects (tools) are the main way to assist a person (child) with their intellectual operations [Vygotsky, 1934/1994]. Vygotsky and Leontyev studied a mechanism of internalization. If a student does not have some intellectual operation, it is necessary to “carry it outward”. After performing actions with this external “reflection” “by hand”, external operations will be automatically transferred to internal ones, and after the internalization a new psychological tool will be formed [Leontiev, 1977/2009].

A question arises: what kind of activity with an external instrument will be adequate to the internal psychological instrument? There is an example about the development of spatial thinking [Yakimanskaya, 1971]: if a student “makes a short stool (taburetka)” manually, most of the problems with the formation of spatial thinking will be solved.

From a mathematics point of view, Poincaré noted that mathematics is based on operating with external tools: “What is geometry for a philosopher? This is the study of a certain group. Which one? Groups of rigid body motions. How do you define this group without forcing some rigid bodies?” [Poincaré, 1921].

From a teaching technology point of view (mathematics and computer science), Papert studied the creation of “smart things” such as computer instruments (models) as a means to carry outward manipulations with mathematical concepts. An example is teaching students in differential equations through programming “dynamic turtles”.

Let us return to the example of the experimental verification that the medians intersect at a single point. What intellectual structure will correspond to the process of moving a triangle?

According to Minsky, this will be a trans-frame, which is defined by three parameters: the beginning of movement, the end, and the description of the transformation process.

The fact that it is difficult for us to “experience it yourself” confirms the phenomenon of magic tricks, when we see a magician’s actions and know their beginning and end but the result nonetheless contradicts our expectation.

That is, a person “loses mental control”, transmits it to a computer, and his conviction by this experiment will not be greater than by constructing a triangle with medians on paper using a compass and a ruler.

Another perspective is that of kinesthetic sensations and understanding. Papert has an observation of how a little girl draws a circle with the LOGO-turtle, making experiments with her body: step forward, turn right at a small angle, step forward, turn right at the same angle, and so on, until we return to the starting point [Papert, 1980].

In this example, the dynamic representation of the circle is replaced by the elementary and familiar actions of the local movement of one’s body, familiar to every child.

Is it possible to reduce the movement of the triangle with medians to such local actions, ensuring that the intersection of the medians is invariant when the position of the vertices changes?

This approach is used in some mathematical problems. For example, the classical proof of the Pythagorean theorem uses the idea of dissection, based on the visual idea that the area of a figure (in fact, the figure itself) does not change when it moves. This psychological effect is also reflected in the teaching of mathematics: some mathematicians cannot agree whether congruent (that is, received by movements) figures are called equal. Another application in math is the "small movement" (stirring) method.

The use of external objects for internalization in the pre-computer era can be considered as well. When discussing the issue of launching the internalization mechanism, it should be noted that it is constantly used in the teaching of mathematics. Following Vygotsky's idea about replacing the tools of the external world with signs, you can look at the language means of mathematics from this point of view. The emergence of a geometric view of algebraic concepts, therefore, is a way to use the mechanisms of visual thinking to translate into an internal plan of new algebraic ideas. Dieudonne proposed a radical idea: not to use drawings in the teaching of geometry [Dieudonne, 1969]. His motivation can be understood, but it is difficult to agree with such a dismissive attitude to the psychology of learning.

Another example is graph theory. It turned out that the visual presentation helps students to interpret many problems of the theory of binary relations. This example is remarkable as graphs represent artificial objects, a specially created language. The effectiveness of graph theory (the language of which no one disputes) shows the important role of the use of intermediate artificial objects between the mathematical concept and its object implementation.

## INTRODUCTION OF MANIPULATORS IN AN INFORMATION LEARNING ENVIRONMENT

To launch a mechanism of internalization for the formation of mathematical concepts, it is not enough to use only the basic tools and domain models.

Special tools are needed, such that working with them triggers the translation process.

These special tools — let us call them manipulators — should reflect the properties of the mathematics objects as well as the features of operating with them mentally. One can consider the graph theory as an analogy.

In his preface to the book "Mindstorms: Children, Computers, and Powerful Ideas" Seymour Papert writes that he learned how to multiply numbers with the help of gears [Papert, 1980]. This implies that operating with gears can lead to the internalization of ideas associated with the multiplication of numbers.

In the framework of the "Construct, Test, Explore" contest (in which participants are offered dynamic models and constructive tasks), elementary school pupils were asked to make clocks from gears.

Consider some features of using the domain model and a manipulator for this task and results obtained in this experiment.

The upper part of Fig. 5 (below) depicts a dynamic model from the "Clock" task of the "Construct, Test, Explore" contest [Akimushkin, Majtarattanakon, Pozdniakov, 2014].

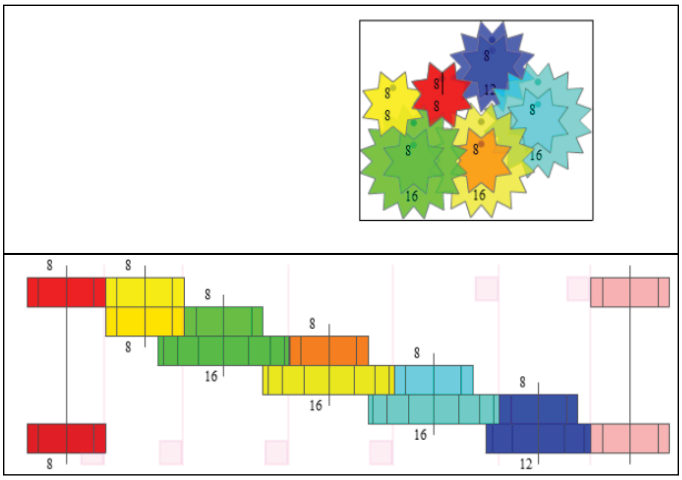

***Fig. 5.*** Interface of the "Clock" task of the "Construct, Test, Explore" contest

Participants construct a clock by adding or removing double gears, changing their location and number of teeth. For participants to associate a clock device with factorization, the manipulator used two forms of gearing system presentation: one presentation based on the physical arrangement of objects on the plane, and another based on conditional arrangement in a line with sequential links — a rotation transmission scheme. Such a representation of the clock structure allows pupils to operate with gears in two "languages" simultaneously: the technical language of gears and the language of mathematics by which the multiplication of rational numbers is implicitly represented. The presence of such a correspondence allows us to consider the manipulator as a means for launching the mechanism of internalization for the formation of ideas about the fractions multiplication: $8/8 \times 8/16 \times 8/16 \times 8/16 \times 8/12 = 1/12$.

## DISCUSSION OF THE EXPERIMENT RESULTS

The presence of a computer tool led to a large number of original solutions based on different factorizations and different locations on the plane (more than 80 different principal designs were proposed and there were no two completely identical solutions among 647 participants).

Among 647 participants, about 30% found the exact solution. This coefficient shows that these participants had the formation of ideas on the relationship of numbers and the mechanical interpretation of multiplication; it can also be said that they have correct ideas about rational number multiplication.

Effect 1/12 = 0.0833. Some participants from the lower grades (about 3–5%) found an approximate solution with great accuracy, but did not understand that the large clock hand rotates 12 times faster than the small one.

One of the important conclusions is that operating with carriers of meanings does not always lead to elicitation and acceptance of meanings.

Thus, it can be said that only 30% of participants worked with the manipulator, allowing for the construction of a proportion with any rational coefficient and leading to elicitation of the concept of the coefficient of proportionality.

This percent value indicates the effectiveness of the manipulator used and the fact that operating with carriers of meanings does not always lead to the elicitation of these meanings.

## DESIGN AND VERIFICATION: THE ROLE OF EXAMPLES AND COUNTEREXAMPLES IN TEACHING

Consider the possibility of efficient communication with "artificial intelligence": instead of assigning a computer the role of the "teacher" and a student the role of a performer, we will change the roles.

The student will design solutions and explain them to a computer program using some appropriate language. The computer will verify the solutions based on the given conditions and examples and will produce reasonable comments.

Note that the success of dynamic geometry can be explained by the fact that dynamic geometries verify solutions to constructive geometric tasks on an almost infinite set of examples.

Let us try to simulate the work of a computer program which does not only check the student's answer, but actively "resists" — it tries to refute the answer, looking for counterexamples.

Consider the "Verifier" program and its application in teaching of calculus. "Verifier" is a computer program for dialogue with a student through responses of a complex structure (predicates). The program is based on parametric classes of functions and solutions in the form of predicates. "Verifier" compares the predicate entered by the student with the reference answer on a set of examples determined by variation of these parameters [Mantserov, 2006].

The result of such a comparison can be one of three types (see Fig. 6):

1. On all generated examples, the entered response and reference answer take the same values.
2. On a particular generated example, the entered predicate takes the value "true", but the reference one is "false" (which indicates an extraneous solution).
3. On a particular generated example, the entered predicate takes the value "false", and the reference one is "true" (which means a missing solution).

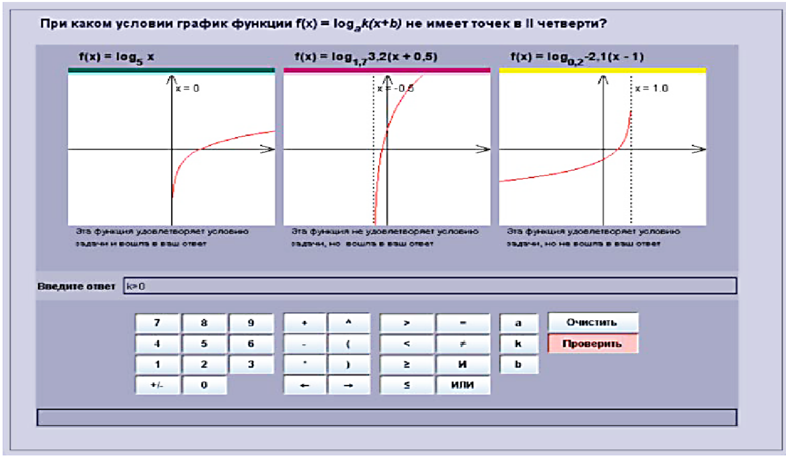


***Fig. 6.*** Task example in the "Verifier" environment:
For what values of parameters *a*, *k*, and *b* does the function graph $y = \log_a k(x + b)$ have no points in the second quadrant?

In the first case, the system reports that it was not possible to refute the student's solution; in the other two, a generated example is demonstrated within the corresponding environment (for example, a graph with marked signs of inconsistency and verbal comments).

<u>Role of examples in teaching.</u> Working with examples and counterexamples has a deeper purpose than it seems at first glance.

An analysis of the process of reasoning with general concepts shows that the proof is based on the actualization of basic examples by interpreting common actions with abstract objects.

It is enough to compare the effect of pronouncing the words "ring", "field", and "group" with the effect of pronouncing the phrases "ring of polynomials", "field of residues of dividing into a prime number", and "group of rotations of the tetrahedron" in the first courses of technical universities to notice the greater influence of the second set on the students' readiness to perceive the teacher's further speech.

The use of examples has roots in the instrumental basis of mathematical concepts, on the one hand, and ways of "carrying intellectual activity outward" on the other hand.

<u>Role of counterexamples in teaching.</u> The role of counterexamples is to limit the scope of interpretations on which a generalization is built.

Note that generalization does not arise due to the actions of the teacher, but appears automatically as one of the essential properties of the internalization mechanism.

For example, students easily transfer the properties of operations with real (or rational) numbers to modular arithmetic, but they do not immediately distinguish the features of arithmetic with a composite module. Therefore, counterexamples are required to show uncertainty in performing the division operation and, due to it, the difference between Z/5Z and Z/6Z.

## USING DIFFERENT INTERPRETATIONS OF MATHEMATICAL CONCEPTS TO SUPPORT PRODUCTIVE MENTAL ACTIVITY

One of the serious mistakes in the development of computer support is the use of only a single methodological or psychological theory. Each theory emphasizes only a small part of all aspects of learning from dozens or even hundreds of such aspects in real teaching. Dynamic geometry became popular because it is not tied to a single pedagogical theory. This is just a subject model of the study area.

Thus, the availability of a subject model should be considered an essential aspect of high quality technological support for learning.

Similarly, when creating didactic superstructures above the subject model, one should not try to solve the problem of supporting all aspects of learning, but select only the most essential ones which contribute to the formation of the student's mental tools.

Let us consider how you can connect dialogue support through examples and counterexamples with an arbitrary object environment (not only with classes of parametric tasks).

Let us consider one solution to this problem using an example of the DM&TI Olympiad (Discrete Mathematics and Theoretical Informatics Olympiad), in which traditional "Olympiad" tasks combined with constructive activity are supported by a computer [Pozdniakov, 2018].

As a part of computer support, manipulators are used that simulate such concepts as: graphs, logic schemes, state machines, regular expressions, Turing machines, etc.

Constructive tasks allow for experiments with models, checking solutions with examples, the set of which a student can change by him/herself. These examples play the role of feedback when a student searches for a solution.

However, complete correctness (not only of examples) is verified after the conclusion of the Olympiad.

An example of a task on regular sets and expressions is: "Find such a regular expression that defines words in the alphabet $\{a, b, c\}$ which does not contain one of the three given letters".

Another interpretation of a similar problem is in terms of the state machine.

Why do we need different interpretations? Different interpretations allow us to create meaningful internal links in the material under study.

Feynman called it "the Babylonian tradition": "I know this, I know that and it is as if I know this; from here I conclude everything else. Maybe tomorrow I will forget something, but I will remember something, and the remnants will be able to restore everything anew. I don't know very well where to start or how to finish, but I always have enough information in my head, so if I forget some of them, I can still recover it" [Feynman, 2000]. The presence of several representations (for example, algebraic and geometric) allows one to "jump over" from one interpretation to another and thereby to overcome challenging places that can have different subjective difficulties in multiple interpretations.

Examples of mathematically equivalent but psychologically different representations of mathematical ideas. Three different interpretations of the same idea have been described above. Let us add two more:

1. Finite state machine.
2. Regular expressions.
3. Regular grammars.
4. Turing machine.
5. Markov algorithm.

The first three concepts are mathematically equivalent, but they connect with different types of basic ideas:

- A finite state machine is represented by graphs, which rely on visual thinking.
- Regular expressions are represented by algebraic formulas, demonstrating operational knowledge.
- Regular grammars are a linguistic way of representing.

The other two concepts are equivalent definitions of the algorithm and they psychologically represent its different aspects:

- Turing machine — computing, operational aspect.
- Markov algorithm — linguistic aspect.

Finally, the finite state machine is a special case of the algorithm. All five interpretations can be used in tasks based on the first three concepts.

When discussing dynamic geometry, we did not pay attention to the following aspect: an internal representation of all the geometric objects in dynamic geometry is analytic. That is, we are dealing with the environment of analytical geometry. However, the language in which the user communicates with dynamic geometry is the language of synthetic geometry, the language of constructions, transformations, etc.

This suggests that the efficiency of using a computer to support a student's productive mental activity can be based on the difference in the ways people and computers perform tasks.

Consider this possibility on the example of so-called "smart tasks".

## CAN A COMPUTER PROGRAM CHECK A TASK SOLUTION WITHOUT KNOWING THE CORRECT ANSWER? CONCLUSIONS

When using tasks with automatic verification, it is assumed that at first someone will invent a task, solve it and enter it into a computer. Then the student will solve the problem and enter the answer into the computer. This answer will be compared with the reference one and the student will receive a response to the entered answer.

There are two obvious drawbacks in this "division of labor". First, if the author makes a mistake in the decision, the student will receive an inadequate response. Second, when solving tasks which are specially prepared in advance and if the task is difficult, the student will not be able to get a response to the subtasks set by him/herself.

Consider such a computer support that allows students to describe task conditions themselves and to then check input solutions without knowing a reference answer. Let us consider media for support of combinatorics tasks, which are set by the student during the solution process. To do this, it is necessary to have a way to adequately describe the tasks, which will allow the computer to solve this problem. "Wise Tasks" have two possibilities: 1) to use special "combinatorics language" or 2) a simplified interface to choose constraints [Bogdanov, 2006].

After the inputting condition, "Wise Tasks" enumerates all combinations and finds the answer (it will be a large integer and cannot give any useful information to the student).

The student inputs an answer in traditional form using combinatorial operations: arithmetic operations with natural numbers, factorial, *k*-combinations, etc.

The program calculates the result using a "large" integers format and compares it with the one calculated by the computer at the previous step.

Thus, this project did not require any serious machine intelligence in order to support activity corresponding to the principles described in the works of Polya [1954; 1981; 2004].

## CONCLUSIONS

Learning is a social process and the expediency of using those or other computer capabilities is not obvious. It is necessary to continue searching for new forms of human–computer interactions aimed at developing students and their deep understanding of mathematics.

The introduction of a computer into the educational process should not be related to the suppression of mental activity in favor of a computer unless the student operates with more general conceptual categories.

There exist successful examples of using a computer to support productive thinking and it would be worthwhile to examine the roots of such success and to develop technologies for effective uses of computer support in actual teaching practice.

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# PLENARY DISCUSSIONS

# OPENING DISCUSSION. SCIENTIFIC RESEARCH IN EDUCATION AS A FACTOR OF DIGITAL ECONOMY AND KNOWLEDGE ECONOMY GROWTH

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The conference started by a plenary discussion between the Russian officials, namely, Isak Froumin, Yury Zinchenko, Igor Remorenko, and Ivan Yaschenko, Yandex representative, Natalia Chebotar and the PME president elected, Markku S. Hannula. They discussed the role of mathematics education research in the current societal and economic situation.

# COMPUTER-BASED TECHNOLOGY IN MATHEMATICS EDUCATION

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The second plenary discussion was organized as a debate between two extreme positions concerning the future transformation of mathematics education under the pressure of technology: a radical change versus an absence of specific transformation. The first position "Computer-based technology will change and already is changing mathematics education dramatically" was defended by Aleksey Semenov and Roza Leikin and the second position "Computer-based technology is just one more tool in teaching and learning mathematics, just like a blackboard or a calculator" was defended by Vladimir Dubrovsky and Angelika Bikner-Ahsbahs.

The following questions were answered by the experts, as they were defending each position:

- How does technology influence the goals, scope and expected results of mathematics education?
- How does technology influence teachers' responsibility and role in a classroom?
- How does technology influence the understanding of mathematical concepts?
- How does technology transform the individualization of mathematics education?

# TECHNOLOGY AND PSYCHOLOGY FOR MATHEMATICS EDUCATION: COLLABORATION OF RUSSIAN AND INTERNATIONAL COMMUNITIES

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**Norma Presmeg**

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**Yakov Abramson**

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The focus of this discussion was on educational research as an interdisciplinary collaboration. Marie Arsalidou explained the role of neuroscience and the need for tight collaboration between neuroscientists and educators for achievement of educationally meaningful research results. Anastasia Sidneva clarified the peculiarity of a methodological position that psychological research takes as researchers enter the field of education: the tradition of Russian psychology concerns with the investigations of educational phenomena from the transformative perspective rather than from the that studies processes perspective of independent from researcher. Markku S. Hannula explicated the role of PME conferences in international collaboration as an opportunity to discuss a concrete educational problem that waits for clarification and solution.

Further, the experts turned to the issue of collaboration between researchers and practitioners. Norma Presmeg highlighted that orientation towards the better teaching is not merely a methodological choice, but the goal of mathematics education research as a field. Yakov Abramson stressed the importance of choosing a psychological guideline for a teacher.

All the conference participants had a chance to contribute their vision of the next steps in the development of Russian mathematics education research community. As the following steps, there were such initiatives as spreading information through an e-mail list between participants, publishing a special issue in one of the specialized

Russian journals, organizing the next conference, PME-Russia in Saint Petersburg in 2021. The discussion was followed by the introduction of the main journals and key conferences for the researchers in mathematics education by Anna Shvarts. She specifically focused on PME conferences and on an opportunity to receive traveling support through the Skemp Fund.

# RESEARCH REPORTS

# ROLE OF THE TOOL FOR TEACHING TOWARDS A MODELLING PERSPECTIVE ON DIFFERENTIAL EQUATION SYSTEMS

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*This paper reports on a study in progress of the role played by various ICT tools for students' formation of mathematical concepts. In the study experienced teachers (14 in all) documented their modelling by differential equations of a system. Their reports were analysed with regard to signs of concept formation and tool use. The aim, data and method of the inquiry is presented together with an excerpt from the analysis and reflections upon the study so far.*

## INTRODUCTION

The aim of this paper is to discuss the role played by computer-based ICT tools for teaching towards a modelling perspective on differential equations (DE) from a mathematics education point of view in terms of emergent modelling. Basis for the paper is a study in progress of teachers' written reports from a mathematics education course on DE and modelling. The study resumed results and experiences from comparable research on the interplay between students' modelling for concept formation in DE, and the use of ICT [Rasmussen, Whitehead, 2018; Ju, Kwon, 2007; Yackel et al., 2003; Andresen 2006]. This study will inquire teachers doing modelling projects in a masters' course. The teachers used a variety of tools of their own choice during the course. The idea of the study was to analyse a more complex situation in the light of results from simpler ones and throw light on questions like: Is it possible to identify common trends regarding tool use in the upper secondary classroom and in the groups of teachers? How should the influence of different tools on the modelling processes be identified and interpreted? And what does this mean for the learning about DE?

## BACKGROUND

Earlier research had focused on upper secondary school students' modelling of DE with CAS software, based on Blanchard et al. [2002] that take a dynamical systems' view on DE. The main outcome of [Andresen, 2006] was that DERIVE (from Texas Instruments) had become a supportive tool for students' concept formation. The use of DERIVE combined with the qualitative and dynamical approach showed potentials to support the students' model recognition and capability to understand and criticize the use of ready-made models in DE. However [Ibid.] reported that the use of CAS in general, depending on the teacher, tended to change the focus of attention into technical and practical aspects. The research revealed a potential danger of teaching 'models and modelling' in a design where mathematics served as a bare tool for prompting

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solutions and correct answers. Such teaching designs could evolve at the expense of 'modelling for concept formation', which focuses on emergent models and gives the students a chance to build up insight into mathematics theory and knowledge, based on experiences from their own mathematical activities. The present study was initiated with the aim to gain knowledge and experiences to avoid such danger.

*Research question:*

*What roles do computer-based ICT tools play for the concept formation, when DE are taught as a means for modelling dynamic systems?*

## THEORETICAL FRAMEWORK

### *Modelling for concept formation: 'Emergent Models'*

The studies in this paper were based on the constructionists' view that learning is a process of constructing and modifying conceptions [Cobb, Yackel, Wood, 1992]. They were also based on Hans Freudenthal's view on mathematics as a human activity [Freudenthal, 1991]. According to this view, students should have the opportunity to reinvent mathematics by horizontal and vertical mathematizing. This view constitutes the basic principles of Realistic Mathematics Education (RME). Horizontal and vertical mathematizing may be modelled [Gravemeijer, Stephan, 2002] by the passing of four levels of activity (situational, referential, general and formal), where a new mathematical reality is created at each level and raising from one level to the next is driven by reflections, which substantiate the progressive mathematizing. Gravemeijer's four-level-model of mathematising formed the basis for study of potentials for learning mathematics from mathematical modelling, that is, basis for development of the design heuristics of *emergent models* described in [Gravemeijer, Stephan, 2002]. In the case of DE models the essential concepts encompass solution, slope field, equilibrium solution etc.

### *Explorative and Expressive Work*

At functional level and for concept formation, explorative work is distinct from expressive work. The modelling process at functional level by Morten Blomhøj and Tomas Højgaard Jensen [2003] is an example of expressive work: modelling encompasses six sub-processes, each of them requiring creative non-routine activities. At functional level, explorative work aims at inquiry of existing constructions or artefacts like mathematical models or computer standard routines. Expressive work at a functional level aims at creation, for example of a solution or description of a mathematical problem. Following the RME conceptualising of modelling in [Gravemeijer, Stephan, 2002], concept formation should take both the explorative and the expressive approach into account [Andresen, 2006].

### *Interplay between tool and modelling*

According to the theory of Instrumental genesis, an artefact like a laptop with CAS software does not in itself serve as a tool for anybody. It becomes useful, and then denoted an instrument, only after the user's formation of mental utilisation scheme(s). [Drijvers, Gravemeijer, 2005]. The instrumental genesis proceed through activities in "The two-sided relationship between tool and learner as a process in which the tool in a manner of speaking shapes the thinking of the learner, but also is shaped by his thinking" [Ibid.]. In the case of modelling, the dynamical systems' view on DE will generate a learning environment where ICT tools are needed for both expressive and explorative work. The tools serve to supply the students' mathematical activity with a technical dimension and to support the students in developing flexibility of the mathematical concepts [Andresen, 2006], for example in the form of fluent changes between different representations in line with the 'rule of four' in mentioned by Rasmussen and Whitehead [2018].

## METHODOLOGY

### *Materials for analysis*

The study in progress inquires the work of a group of teachers, who followed the course 'Modelling in and for mathematics teaching and learning' in our masters' programme in mathematics education. The programme prerequisites two years of professional experience as teacher in lower or upper secondary school and at least 60 European Credit Transfer System (ECTS) points in mathematics, meaning one year of full-time study at university level. The modelling course was a 15 ECTS points course, compulsory part of our programme. It was based on the textbook [Blanchard, Devaney, Hall, 2002], and encompassed lectures, tasks and two projects. This paper focus on students working in pairs on one of the two projects, after their first 36 lessons on first order linear and non-linear DE systems, qualitative, quantitative and numerical methods, modelling and problem solving, and Laplace transformations. The aim of the project was, according to the guidelines: *To formulate, complete and present a project that encompasses the building and/or revision of a simple differential equation model using appropriate digital tools*. The guidelines proposed a structure of the report encompassing *i) Introduction and research question, ii) Building or description of the model and discussion of it, iii) Qualitative and quantitative evaluation of the model, iv) Conclusion and discussion, v) Perspectives*. Purposes of the project were: 1) to learn about differential equations by doing a modelling project, and 2) to get personal experiences with learning mathematics from doing a modelling project. The study analyses reports, prepared 2014–2016 from this project, from all our programme's students' modelling with differential equations, 14 cases in all. The aim is, in each of the 14 cases to study the interplay between the students' modelling process, and the ICT tool they had used.

### *Clusters of reports*

After the first reading through, the cases were coded (most of the codes are omitted in this paper) according to: *i) Mathematical model:* Predator — prey models (PP), Epidemic models (SIR), models of a harmonic oscillator (HO), Exponential growth (Ex), Logistic growth (Lo), and Others (Ot). All these models were elaborated and illustrated in the textbook too; *ii) ICT tool:* The students' ICT competencies spanned from almost novice with calculator and Excel, to experienced users of GeoGebra, Mathlab and POLYMATH. There are several ready-made ICT tools available on the web for exploration and inquiry of differential equations models; *iii) Data source:* The students found data on the web, with a few exceptions who on their own got, or from the beginning had, access to suitable data in an area of interest. A few tables in from the textbook were also used; and *iv) Modelling method:* The cases represented a variety of methods. In some cases, the students took one model and a seemingly corresponding data set as their starting point (Start Mod) and made efforts to estimate the model's parameters to fit to the data. Others described a step- by-step procedure with more and more complex models (Step-by-step) aiming at fitting the model to the data. Use of Graphs was common for checking; some found an analysis solution or numerical solution. Many used analytical methods like Jacobi determinant and eigenvalues etc. to check the model.

The qualitative analysis of the 14 cases will be carried out in a series of sub-analyses. For each sub-analysis, the cases will be grouped in clusters according to one or more of the criteria. The conclusion to the sub-analysis will be reached as synthesis of the results from inquiry of each cluster.

### *Analysis of concept formation:*

The analysis will be based on Gravemeijer's four — level — model and follow the format described above. It aims to find signs of emergent models that could indicate progress in the students' concept formation. Such signs are found by identification of mathematical activities and reflections, in the form of deductions and statements in the reports. They are interpreted as possible single steps in the entire developmental progression in the students' thinking. Meta-statements and reflections about the process written in the single reports support the interpretations.

Example 1: The 14 cases may be grouped in clusters following the codes PP, SIR, etc. for sub-analysis of formation of concepts specific for the individual model. Inquiry of the clusters might reveal similarities and differences between emergences of model-specific concepts involved in PP, SIR etc.

### *Analysis of tool use*

The students' actual use of one or more tool(s) in each report will be characterised regarding being explorative or expressive. The single statements and deductions related to the tool will be interpreted in terms of explorative and expressive, relating to the two sides in the relationship between tool and learner according to the theory of instrumental genesis. The single step of the students' activity will be characterised as mainly imposed by the tool or as mainly the students' own enterprise. Further, the analysis aims to study to what degree the students had generated an instrument of the ICT tool.

Example 2: For sub-analysis of tool use, to which the mathematical model was particularly close related in the cases where the tool was a ready-made test box, the cases may be grouped in clusters according to criteria i) and ii). It might be the case that ready-made tools are used mainly for exploration of the model it was tailored for.

## DATA AND ANALYSIS

Table 1

| Group/Code | Reports | Characteristics |
|---|---|---|
| *A* | 1, 4 | (SIR), (Start Mod) |
| *B* | 2, 13 | (PP), (Start Mod) |
| *C* | 5, 6, 9,10, 11, 14 | (PP), (Step-by-step) |
| *D* | 7, 12 | (Lo), (SIR), (Step-by-step) |

Table 1 shows an example of sub-analysis from the study in progress. It shows the 14 cases grouped in cluster $A, B, C, D$ (and none: case 3 and 8) after the criteria i) model and iv) method.

### *Excerpt from the analysis. Case: Report 12*

'Development of HIV infected persons in the world'.

Method: Started with model for logistic growth and found analytical solution. Filled in data in a GeoGebra spreadsheet with regression. Estimated the parameters and graphed the solution which did not fit. Continued with results of the regression, not with the solution. Tested other models by trial and error; cancelled polynomial regression because the graph fitted badly. Turned into quantitative and qualitative inquiries of the SIR model. Summary: Analytical solution of simple equations, regression in GeoGebra, Qualitative inquiry based on ready-made test-box.

Concept formation: Horizontal mathematization (data to parameters and variables), several shifts between 'model of', and 'model for' in both directions, shifts between representations. Shift from general to formal level (change to SIR) and back (testing on data). Summary: Flexible understanding of the SIR model and its content, including reflections and critical reflections.

Tool use: The first parts using GeoGebra dominated by own enterprise; expressive use of well-known instruments. Second part with the SIR model: explorations, apparently influenced by the tool (equilibrium points, null-clines), but also alternating with expressive modelling (estimation and interpretation of parameters). Summary: Well-known tools used as a forerunner for generation of an instrument from the new, ready-made test-box.

## DISCUSSION

The experienced mathematics teachers in this case documented their mathematical activities and reflections in their report in a convincing way meaning that they seemed to be familiar with the mathematics involved, and with the tools used in the first part of the report. For example, they used a first-person perspective in their writings in the report (see [Ju, Kwon, 2007]). The teachers, though, were not familiar with modelling; neither did they know the differential equations model in advance. Therefore, the case could be useful for inquiry of learning by modelling in a favour-able context regarding learning potentials in mathematics. There seemed to be a successful generation of instrument from the ready-made test-box in the case. It will be interesting to see whether this is a common trend in cluster *D* and whether such a trend can be identified in a cluster of cases using ready-made test-boxes. Genesis of an instrument might, further, be found as a common trend in a cluster of cases using a step-by-step method. Regarding concept formation as well as tool use, it will be interesting to relate the results of inquiries of cluster *D* and cluster *A* in a search for potential similarities and discrepancies between the work with SIR in groups *D* and *A*. Besides, there may for example be interesting similarities and discrepancies between the step-by-step work in groups *D* and *C*, with SIR and the predator — prey models.

## CONCLUSIONS AND PERSPECTIVES

Andresen [2006] showed that the students became quite familiar with DERIVE and used it as their instrument, for example for initial inquiry of tasks or problems. There were no apparent examples of tool use that directly supported mathematical concept formation in the area of differential equations. The students' learning outcome was mostly within modelling and model recognition. This new study has potentials to go deeper into the formation of mathematical concepts during the modelling process since the teachers were asked to document this process: The variety of tools used combined with the quantity of models gives room for inquiry of interplay between tools and modelling. The relatively small number of reports was considered by inquiry of each single report as a case and then grouping them into clusters depending of the theme of the sub-analysis. Therefore, this new study in progress will, hopefully, serve to support development of teaching designs where dynamic and modelling perspectives go hand in hand with advanced technology, towards students' robust understanding of differential equations.

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# THINKING INSIDE THE POST: INVESTIGATING THE DIDACTICAL USE OF MATHEMATICAL INTERNET MEMES

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*We venture in the almost unexplored field of mathematical Internet memes, with the aims of investigating their didactical features in a teaching and learning setting. The work is framed within the research field studying the links between emotions and mathematical thinking and takes off with a schematization of the meanings carried by a meme, formulated through an a-priori analysis of spontaneous web productions and results of an exploratory experiment. The analysis is then compared to the data collected in a teaching experiment conducted at high school level. Results sustain the conjectured meanings structure and elicit evidence of students' emotions and of their role in the learning process initiated by the interaction with memes.*

## INTRODUCTION: MEMES AS NEW LEARNING OBJECTS

We argue that mathematical Internet memes can morph into effective learning objects if paired with adequate teaching practices, pointed at harnessing memes' social, emotional and communicative potentials and funnel them into teaching and learning assets. The present study aims at verifying the robustness of an a-priori schematization of the meanings carried by memes and test their didactical use in a learning setting.

## WHAT IS A MATHEMATICAL INTERNET MEME

Mathematical Internet memes are a special kind of Internet memes, which in turn are a subset of memes, "unit(s) of cultural transmission" that propagate themselves by imitation [Dawkins, 1976, p. 249]. The common feature that characterizes Internet memes is that they are pieces of digital media that spread virally through social channels, reaching a large audience in a very short time. They are built of "verbal and pictorial parts, which unfold their meaning through collective semiosis" [Osterroth, 2018, p. 6] they can be in the form of viral images, videos or files created by users following collectively established rules that govern the so-called *memesphere* and they are widely shared through social platforms with a satirical or humorous intent. According to Shifman [2014, p. 15] "while seemingly trivial and mundane artefacts", memes "reflect deep social and cultural structures" and "epitomize the very essence of the so-called Web 2.0 era". Although they score an ever-increasing number of appearances on social platforms (the hashtag #memes hit 67 million of occurrences on Instagram in January 2019), they can be called *famous strangers:* well known to net citizens worldwide, but


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***Fig. 1.*** Original template (1*a*) and mathematical variations (1*b* and 1*c*)

totally foreign for those who are not familiar with social media culture. As a matter of fact, up to now, they are understudied by academic research.

We start with a web found example: in Fig. 1*a*, we see the original viral image of the *Who Killed Hannibal?* meme, Fig. 1*b* and 1c show two mathematical variations.

Which information is necessary to understand them? First, we need to recognize them as memes. Second, we have to connect some evidence to the background image: we should know the original image to identify the remixing in the mathematical variations. In Fig. 1*b* Hannibal is unaffected by the shooter and in 1c he doubles himself as a consequence of being shot at. Third, we have to understand the mathematical meanings represented symbolically: in 1*b* the notion that the exponential function $y = e^x$ remains untouched by the differential operator and in 1*c* the fact that $y = e^{2x}$ doubles when the first derivative is taken. In our preliminary web survey on social media, we have encountered dozens of mathematically-themed groups, with hundreds of users reacting, commenting and questioning about the image or the mathematical part of memes like the one analysed here. This suggests that only those who succeed in grasping all levels fully understand the meme, laugh and feel part of this mathematically skilled community that emerged spontaneously in the digital world.

## THE MEANINGS OF A MEME

In a previous study [Bini, Robutti, 2019], through the a-priori analysis of web productions and the results of an exploratory experiment, we identified three partial meanings that build up the full meaning of an Internet meme:

- The first partial meaning is *structural* and lies in its being a meme, namely to have a specific and shared structure and graphics (font, colour, text position).
- The second partial meaning is *social* and lies in the shared conventions of viral images, compositional setups and syntaxes (Fig. 1*a*).
- The third partial meaning is *specialised* and lies in images, symbols or text referring to a specific topic (mathematical, or other) (Fig. 1*b* and 1*c*).

The first two meanings ground in the popular culture rules that govern the *memesphere*, while the third calls some mathematical knowledge and skills into action. The interplay of all three *partial meanings* is needed to unlock what we call the *full meaning* of the meme, which triggers a sense of surprise and fun. Here we intend meaning within a "sphere of practice", adhering to a common set of rules, where "mathematical meanings are constructed" [Kilpatrick et al., 2005, p. 10].

For students, who are fully fledged net citizens and access the first two meanings easily, the obstacle in grasping the full meaning usually lays in understanding its specialised meaning (the mathematical content). In an educational setting, we hypothesise that this final hurdle, that makes the act of cracking the meme even more rewarding, could turn out as one of the meme's significant didactical feature. In fact, the introduction of some attuned *desirable difficulties* in the learning process can improve long-term retention, since "in responding to the difficulties and challenges, the learner is forced into more elaborate encoding processes and more substantial and elaborate retrieval processes" [Bjork, 1994, p. 192]. On the other hand, teachers — who are usually not familiar with social media trends — may be shut out of the first two meanings, and therefore of the full meaning of the meme. This can be called an *undesirable difficulty* that creates a barrier between teachers and students, grounding on the digital culture vs. school culture cliché. This paper aims at opening a breach in this barrier, introducing the idea of *didactical meme*: a mathematical Internet meme used in the classroom for teaching and learning purposes.

## THEORETICAL FRAMEWORK

Memes are a totally new phenomenon in mathematical education research and there is no history in literature of suitable theories to frame them. Our first exploratory approach [Bini, Robutti, 2019] was based on the results of an a-priori analysis of memes and aimed at describing their role in education from a cognitive point of view. The Boundary Object perspective, as introduced by Star, Griesemer [1989], and Sfard's [2008] theory about discourse and communicational approach to cognition seemed appropriate to ground our analysis on.

Further investigations, involving new data on the memes design process (described in the *Data and analysis* paragraph), steered our focus from the cognitive to the emotional aspects of students' interplay with memes. We were faced with evidence that "emotions also affect cognitive processing in several ways: they bias attention and memory and activate action tendencies" [Zan et al., 2006, p. 118]. The cultural-historical approach introduced by Radford [2015], a "cultural conception of emotions and their role in thinking in general and mathematical thinking in particular" [p. 26], seems fit for our case. In fact, according to Radford, emotion and thinking are strictly connected: "from a cultural-historical perspective, emotions are both subjective and cultural phenomena simultaneously; they are entrenched in physiological processes and conceptual and ethical categories through which individuals perceive, understand, reflect, and act in the world" [p. 35]. In particular, his idea that "contemporary cultural ideas of learning and learners are conveyed by schools and other social institutions, family, and mass culture" [Radford, 2015, p. 46] valorize the social value of memes, which, through shares and likes, act as social currency in the memesphere.

To sum up, using the words of an anonymous Reddit user, memes are "like inside jokes between millions of people": they find their reason for being in reactions and root deeply into emotions. We argue that a didactical meme can be a conveyor of cognitive and emotional elements, taking advantage of a fact neural scientists agree on, i.e. that "emotional arousal often leads to stronger memories" [LeDoux, 2007], as memories about emotional situations are normally stored both in explicit and implicit memory systems. The research questions of the study are: RQ0) What is the students' familiarity with social platforms and memes? RQ1) Are the meanings of didactical memes recognised by students the same as we described in the a-priori analysis? RQ2) What is the role of emotions in a learning process involving didactical memes?

## METHODOLOGY

This work presents the pilot study of a PhD thesis (one of the authors'), involving a class group of 22 $10^{th}$ grade students, who created their own didactical memes on the topic of linear systems and recorded videos with the explanations of the specialised meaning. Due to page restriction, didactical memes involved in the study will be now on referred to simply as memes. Data collected are: individual entry forms and worksheets, memes and videos created at school by students working in pairs (3h), screencast and video recordings of the memes and videos production processes by two selected pairs, individual feedback forms and reflective worksheets, video recordings of the collective discussion guided by the teacher (2h). Observed pairs were picked out coupling students with mixed mathematical and linguistic abilities, to facilitate the emergence of the expected meanings and their interaction. Students' creations have been gathered in collective spaces (Padlet walls shared via Google Classroom) that mimicked the social media environment, allowing the coveted reactions.

## DATA AND ANALYSIS

The entry online form answered RQ0, assessing that 100% of the students were familiar with social platforms (83% declared to visit them "several times a day"), with memes ("an image with funny text") and had some interaction with general purpose (i.e. non-didactical) memes (100% declared to like and/or share them on social platforms, one student identified himself as a meme creator). The entry worksheet, administered by the teacher, used the meme in Fig. 2 to check if our a-priori identified partial meanings matched those recognised by students.

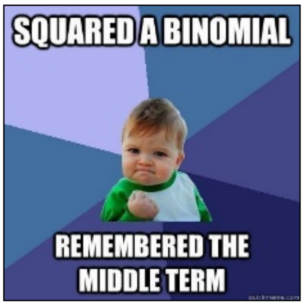


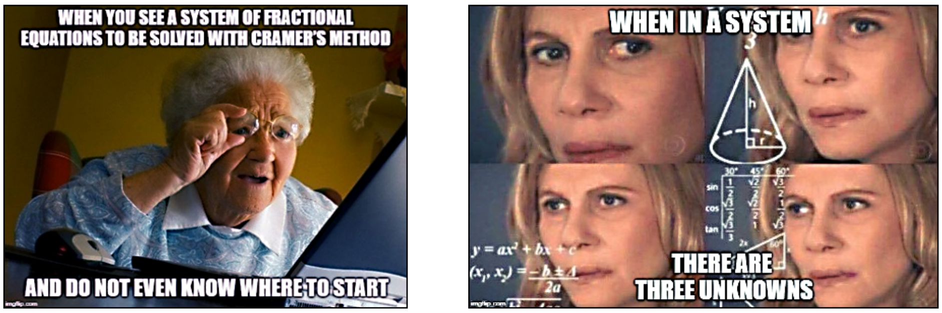

*Fig. 2.* The entry worksheet mathematical meme

All answers support our a-priori analysis of the partial meanings of a meme (RQ1). In Table 1 we show expected partial meanings (not revealed in the questionnaire), questions, and samples of students' answers (selected because particularly effective).

Table 1

**Assessment of the meanings of a meme**

| Meaning | Questions | Answers |
|---|---|---|
| Structural | In your opinion, what leads you to say that this is a meme and not a cartoon? | S1: From the classic meme font, that is in capital letters, and the fact that it is divided into two parts, a sentence above and one below that do not cover the image |
| Social | What do you think is the purpose of the image chosen as the background of the meme? | S2: The child's gesture means "hurray I've done it" and implies that, as finding the middle term is difficult, when you do you are happy |
| Specialised | Which mathematical topic is referred to in this meme? | 95% identified and explained the rule applied to square a binomial and 81% agreed on the fact that the middle term is usually forgotten |




*a* *b*

***Fig. 3.*** Pilot study focus couples' memes (captions translated by authors)

Subsequently, we analysed the products (Fig. 3*a* and 3*b*) and the production processes of the two focus pairs, to identify the interaction between partial meanings.

All memes were created through a Meme Generator website that automatically imposes the compositional rules, so we shall take the structural meaning for granted and focus on the interplay of the social and specialised ones. Hereafter we summarize the key moments of the selected pairs' memes production processes.

Figure 3*a*: pair 1 (two girls) started with the idea of creating a meme on what they identified as the most difficult aspect of the assigned topic (specialised meaning: Cramer's rule and fractional equations). In the Meme Generator website, they looked for a template whose social meaning matched the emotion that these mathematical difficulties stirred. In the explanatory video, they connected the two meanings, clarifying that "the expression of the old woman in the meme represents our faces when we see [simultaneous] fractional equation to be solved with Cramer's rule".

Figure 3*b*: pair 2 (two boys) browsed through the various templates in the website, laughing and quoting a variety of possible captions related to their feelings and experienced difficulties in mathematics (fractions, binomial expansions, systems), to create something funny and original ("in my opinion everybody will use the first [images], we could differentiate ourselves…"), because if the template and/or subject were already used by someone else, there would be less chance of gaining likes.

In both cases, grounding on the shared structural level, we witness something that we did not consider in our a-priori analysis: a dynamic interplay between social and specialised meanings in the design activity, deeply rooted on emotions (Fig. 4).

Pair 1 follows the left-pointing red arrow and pair 2 the right-pointing black arrow, but in both cases students' facial expressions, choice of words and physical reactions showed that emotions — aroused by a dynamic coaction of mathematical content and cultural constructs [Radford, 2015, p. 29] — enabled the connection between meanings and acted as origin and ultimate goal of the interaction with the meme (RQ2).

A similar double path illustrates the class group dealing with memes in Fig. 5, created by the researcher and presented in a worksheet the following lesson — asking students to describe their specialised meanings — as a start off for the discussion.

The meme in Fig. 5*a* was greatly appreciated ("this is beautiful"): its specialised meaning was immediately understood and connected to the social meaning through an emotional interlacing ("it looks like Viola — the smartest student in the class group — when we solve problems together"). Here the specialised meaning, i.e. the ability to recognize that the given problem is best solved using the elimination method, is processed first and then connected to the social meaning resulting in a successful didactical meme that prompted a deep collective discussion: we are moving along the left-pointing red arrow in Fig. 4 scheme.

On the other hand, the meme in Fig. 5*b* puzzled students: in the worksheet a significant share (64%) correctly described the mathematical meaning ("for the comparison method we need to apply the transitive property", "the comparison method is justified by the transitive property" (in the Italian curriculum, the comparison method refers

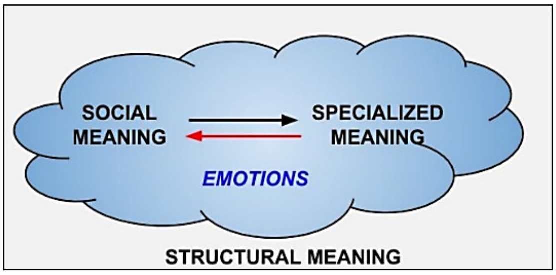
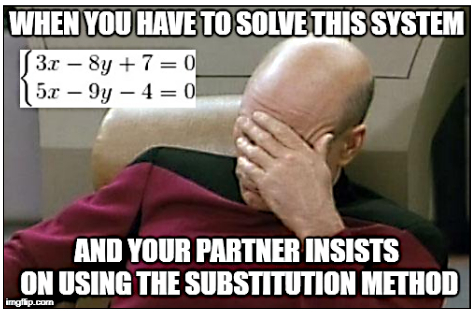
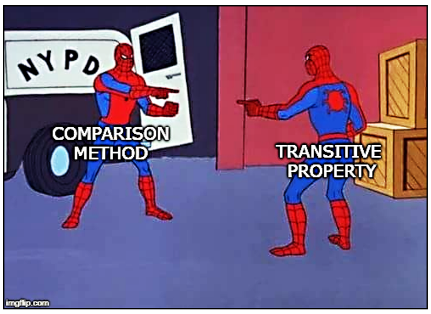



***Fig. 4.*** The meanings interplay and the role of emotions




*a* *b*

***Fig. 5.*** The reflective worksheet memes

to a 2 × 2 linear system solving technique where the same unknown is obtained from both equations and then the right-end sides expressions are joined to get a single variable equation)). Discussing the meme later, they showed mixed feelings ("I did not understand it so well because of this transitive property"). The following excerpt clarifies the unfolding of the interplay between the different meanings:

> Student: No, I did not remember very well what the transitive property was.
>
> Teacher: But looking at the meme image, what would you say?
>
> Student: That they are two similar things, two equal things (the social meaning of the image is to describe situations in which two very similar elements meet).

Upon further inspection, the majority of the students admitted they did not remember what the transitive property was and a recap of the property was given by the teacher.

> Teacher (after the recap, addressing the whole class group): The things you wrote [in the worksheet], those of you who wrote them correctly, did you write them because you remembered the transitive property or just to make sense of this meme and say that they are almost the same thing because the image tells us this?
>
> Students yelling in chorus: Because of the meme!
>
> Teacher: I do not know whether to be happy or not, though…
>
> Researcher: Now that we have used this meme to recall that they are the same thing, do you think you will remember when you use the comparison method, do you think you will associate it with the transitive property more consciously?
>
> Students in chorus: Yes, definitely.

In this case, knowledge is built moving along the black arrow of Fig. 4, from the social to the specialised meaning and emotions are deeply tangled with the whole path. Finally, in the feedback questionnaire, 81% of the students answered positively to the question whether they had learnt or understood something better ("yes, also checking at the other memes created"), 86% scored more than 7 in a 1–10 rating scale question about "having created the meme will help you remember this topic better?". In the following days, the teacher reported that "we are working on the transitive property of equality: I was amazed by my pupils' attention to names… when I explain I do not give much importance to names. But I think it's for the idea of *making a good impression*".

## DISCUSSION: A NEW COGNITIVE OPPORTUNITY

To sum up, we started with an a priori analysis that led to the three meanings structure of a meme and was confirmed by students' entry worksheets (RQ1). When we observed the processes of memes' creation and interaction with them, we saw that the three partial meanings were not handled in a fixed order, but that the development is more complex and dynamic, with different access points. Students' physical reactions and utterances showed that emotions drive the relation with memes (both as creators and users) and allow shifting from the specialised to the social meaning and vice versa (RQ2). This emotional involvement turns out as the meme's most powerful affordance: from the learning point of view it guides and motivates students to understand the meme's specialised meaning. From the teaching point of view, it can be exploited by teachers who can use memes to reply to students' memes, resulting in a *memetic* adaptation of the *semiotic game* [Arzarello et al., 2009], focused on the memes' mathematical specialised meanings. This use of memes can foster language awareness and further mathematical reflections, as shown by the teacher's testimony.

Although these results seem encouraging, our work is far from complete, more investigation is needed to dig deeper into the affordances of memes: for example, a mid and long-term assessment to evaluate the connection between the emotional situations aroused by memes and students' retention of the associated knowledge. Anyway, we think that this almost uncharted territory is worth exploring, because, even if digital culture can be labelled by someone as a non-culture, facing the evidence that our students are emotionally embedded in it, we think it would be educationally valuable to embrace it and turn it into a cognitive opportunity.

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# COGNITIVE MOTIVATION AS A CRUCIAL FACTOR IN MATHEMATICS EDUCATION

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*The article relates the leading determinants of giftedness to the specific example of mathematical giftedness. Students and post-graduates of mathematical specialities at prestigious Moscow universities (n = 100) are the participants of the current research. Giftedness is diagnosed with a technique designed in the frame of the Creative Field Method using mathematical material. The findings are compared with the results of techniques that examine intellect and personality. It has been demonstrated that the general intellect correlates to successfully mastering mathematical activity, but it cannot definitely predict mathematical giftedness. The manifestation of the latter depends on cognitive motivation and the so-called "worldview activeness".*

## INTRODUCTION

Identifying and supporting children with general and special giftedness is a topical issue at the state level in many countries. For the most part, the education of gifted children is carried out according to formal criteria, such as school progress and winning various contests. Those circumstances call for a scientifically justified definition of giftedness.

## THEORETICAL BACKGROUND

When diagnosing giftedness, the established tradition is to reduce it to a high level of ability as demonstrated in the work of Binet, Eysenck, Terman, Spearman, Wechsler, Stern. However, the results of modern research show the impossibility of explaining the achievements of gifted children and adults by the specifics of their intellect. Numerous works cover various traits of a gifted person and his/her motivation [Bogoyavlenskaya, 2011; Melik-Pashaev, 2018; Renzulli, 1984; Heller, Perleth, Lim, 2005; Csikszentmihalyi, 1997]. Nevertheless, according to Panov, a peculiar "diagnostics paradox" can be observed: it is stated in psychology that giftedness should not be reduced to the level of ability, and yet it is diagnosed, as a rule, by evaluation of various abilities [Panov, 2014].

This contradiction can be seen with utmost clarity in the example of mathematical giftedness: the scientific literature shows that it is defined in most cases based on individual abilities in mathematical processes [Krutetzkii, 1969; Yakimanskaya, 2004; Presmeg, 2006].


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Meanwhile, some studies of mathematical giftedness cover the role of personality traits in its structure. Goldin supposes that the student's affective system occupies a central place in his/her cognition and its influence can either raise or lower cognitive activity [Goldin, 2002]. Jensen mentions that schoolchildren who are oriented at a problem continue to solve it even when difficulties arise. Those who do not have a considerable interest in the problem make an effort only as necessary not to fail [Jensen, 1973]. Bargdill and Starko discuss the role of the internal motivation for creativity development as well: the higher the level of a child's internal motivation, the more probable are creative solutions and discoveries [Bargdill, Starko, 2006]. McLeod finds a positive correlation between the attitude toward a problem and achievements in various classes [McLeod, 1992]. Plucker and Renzulli suppose that a positive attitude to the subject can be considered as an index of creative potential [Plucker, Renzulli, 1999].

According to our theory of giftedness and creativity [Bogoyavlenskaya, 1971; 2011], at the stage of mastering an activity a corresponding level of the general intellect is necessary, but the future performance of the activity is determined by the person's system of motives and values. One person might simply solve the problem. Another one, infatuated by the process during problem solving becomes absorbed, considers the activity more widely, and beyond reaching the initial goal discovers new regularities. It is the development of the activity through one's initiative that is considered as a unit of creativity and characterizes a personality, in whose structure cognitive motivation dominates. Yet giftedness is defined as the ability to demonstrate creativity — developing the activity by one's initiative. This approach has not been applied previously to the topic of mathematical giftedness.

International studies of the specifics of mathematical giftedness in the context of problem correlation of general, special and creative abilities is a relatively recent phenomenon [Hong, Aqui, 2004; Kattou et al., 2013; Kontoyianni et al., 2013; Leikin, Pitta-Pantazi, 2013; Pitta-Pantazi et al., 2011; Sriraman, Haavold, Lee, 2013], whereas in the Soviet/Russian psychology field the correlation between various kinds of abilities has been studied since the second half of the 20th century [Rubinstein, 1960; Teplov, 1961; Krutetskii, 1969]. In Shadrikov's theory [Shadrikov, 2010], abilities are understood as characteristics of functional systems: special abilities are considered to be general ones which have acquired the characteristic of responsiveness under the influence of the demands of the activity. Thus, the contradiction dissolves and the issue of the nature of special abilities gets a definite answer. Abilities are considered at three levels: the level of the individual (natural abilities); the level of the activity subject (special abilities) and the level of the personality (including his/her moral sphere). In Shadrikov's opinion, abilities at the level of the personality represent giftedness, which can transfer to creativity.

## EMPIRICAL RESEARCH

The research aim is the analysis of the psychological structure of mathematical giftedness and defining its cognitive and personal components.

*Research hypotheses*

1. Successfully mastering mathematical activity should be connected with the level of the general intellect, but a high intellectual level without considering personality traits does not definitely predict mathematical giftedness.

2. The manifestation of mathematical giftedness should depend on the dominating cognitive motivation of the personality (which can be seen in constructive motivation and infatuation with the subject) and the so-called “worldview activeness”.

*Research method*

The Creative Field Method allows diagnosing such personality trait as the Intellectual Initiative (II) — the development of an activity by one’s initiative (extending beyond the limits of the given task). The method defines three levels of work: (1) successfully mastering the given activity (stimulus-productive level); (2) the ability to develop the activity by one’s initiative, which allows discovering new regularities (heuristic level); and (3) proving the discovered regularities (creative level). In the frame of this method, the Coordinate System technique, using mathematical material, has been designed and applied by researchers [Bogoyavlenskaya, 1971; 2011; Petukhova, 1976].

The advanced Raven’s Progressive Matrices [Raven, 2002] is a non-verbal intellect test designed for a finer differentiation when the participant’s abilities exceed the medium level. The test includes two series, the first containing twelve tasks and the second comprised of thirty-six tasks.

The diagnostics of the motivational structure of personality [Milman, 2005] describes the following motivational scales reflecting the main personality orientations:
consumer tendency (motivation of life support, motives of comfort and safety, status and prestige motivation) and constructive tendency (motivation of general activeness, motivation of creative activeness, motivation of the public benefit). Each scale is divided into two subscales: the ideal state of the motive and its real state.

The technique of evaluating worldview activeness [Leontiev, Ilchenko, 2007] includes 13 pairs of statements referring to various aspects of human life. Each pair has the same beginning and two variants of an ending (*A* and *B*). The participant is asked to evaluate the degree of his/her concordance with the variant of the statement from 0 to 100% (0 — totally disagree, 100% — completely agree). The instruction states that the sum does need to be 100%. A participant can also introduce his/her own variants of the answer if not satisfied with those provided. There are 4 types of answers: 1) definite answers ($A$ = 100%, $B$ = 0% or vice versa), 2) combinations of two variants ($A + B \leqslant 100\%$), 3) crossing responses ($A + B > 100\%$) and 4) self-contained answers. The first two types of answers indicate the worldview passivity of a person, expressed in the uncritical acceptance of another's opinion. The answers of the third type indicate the worldview multidimensionality, such person has a broader view of the world, assumes that there are hidden dimensions in the world. The answers of the fourth type are attributed to the highest type of worldview activeness, such people clearly have a desire for change, tolerance for uncertainty and a sense of meaningfulness of their lives.

### *Research participants*

Between 2011 and 2017, 100 participants (68 males and 32 females; ages 18 to 34 years, $M = 23{,}55$, $\sigma = 2{,}76$) took part in the research. Forty-two of them were students of mathematical specialities at Moscow universities (MSU, MIPT, MEPHI, BMSTU), 15 were PhD students and 43 were mathematicians and programmers, eleven having PhDs in physical-mathematical sciences.

## RESEARCH RESULTS AND DISCUSSION

As the result of the Coordinate System Technique, the following distribution according to the II levels has been obtained: 72 people at the stimulus-productive level; 24 people at the heuristic level; and 4 at the creative level. For the purposes of statistical processing, the participants at the creative and heuristic levels were united in the same group because of the small number of the former.

The special feature of the Creative Field Method is that it allows diagnosing both the intellectual initiative (extending beyond the limits of the given task) and the general level of mental capabilities (in the frame of mastering the given layout of the activity). When comparing the results of the Raven's Test and the indices of educability in the Coordinate System Technique, correlations with the time to solve problem are obtained ($r = -0.35$, $p < 0.001$). There are no correlations between the results of the Raven's Test and the indices of the intellectual initiative in the Coordinate System Technique. The analysis of the results of the two groups of participants shows that those at the stimulus-productive level do not differ in the Raven's Test results from the participants at the heuristic level ($t = -0.65$, $p = 0.52$). Thus, based on the obtained results, it is possible to conclude that it would not be correct to rely on high scores in intellectual tests as the only sufficient criterion for the identification of giftedness.

The application of Milman's technique shows that, in the ideal plan, participants at heuristic and stimulus-productive levels have similar results. The only difference is in the motivation of comfort and safety: in the ideal plan, the motivation of comfort for the heuristic level is less important than for the participants at the stimulus-productive level ($t = 4.36$, $p < 0.001$). The main differences can be seen in the participants' real motivation. Thus, those at heuristic and stimulus-productive levels differ in the motivation of general activeness ($t = -4.04$, $p < 0.001$), motivation of creative activeness ($t = -5.90$, $p < 0.001$) and motivation of the public benefit ($t = -3.82$, $p < 0.001$). According to Milman, those three kinds of motivation form a constructive tendency of the personality. In the open questions of the questionnaire, the participants at the heuristic level describe real situations from their lives confirming their answer choices. Thus, one participant at the heuristic level became the youngest member of the jury of the All-Russian Olympiad of Schoolchildren in Mathematics. Another participant at the heuristic level is now a Chair of the Organizing Committee of the Tournament of the Cities in Moscow and one of three members of the Central Organizing Committee with a casting vote.

According to Leontiev, creative synthesis is defined as finding a new solution by revision of the question's statement, from which comes an incompatibility of the experience elements. In this case, the maximum degree of "worldview activeness" is demanded from the participant [Leontiev, Ilchenko, 2007]. It can be rightfully brought into correlation with intellectual initiative (see [Bogoyavlenskaya, 2011]) as it relates to the going beyond the limits of the given activity as well.

The data obtained in the Technique of Worldview Activeness were recoded from two possible variants into four. There were 19.5% definite answers ($A$ = 100%, $B$ = 0% or vice versa), 29.1% combinations of two variants ($A + B \leqslant 100\%$), 16.4% crossing responses ($A + B > 100\%$) and 35% self-contained answers. Meanwhile, the participants of the heuristic and stimulus-productive levels differed significantly in the distribution of the frequencies of various answer types ($\chi^2 = 59.579$, df = 3, $p < 0.001$). According to the results of the correlation analysis of the frequencies of various answer types in the Technique of Worldview Activeness and main indices of the Coordinate System Technique, a correlation between the frequency of the self-contained answers and manifestations of the intellectual initiative ($r = 0.545$, $p < 0.001$) has been found. There is also an inverse correlation of intellectual initiative and the frequency of definite answers ($r = -0.453$, $p < 0.001$) and the combination of two variants ($r = -0.219$, $p = 0.029$).

To define the interconnection between cognitive and personality components, we carried out a logistic regression analysis. Belonging to the stimulus-productive or heuristic level categories of the II, manifestation of mathematical giftedness is considered as the dependent variable. For the predictors at the first stage, we introduced personality components: worldview activeness and constructive motivation. At the second stage, a cognitive component (Raven's general intellect index) was added. The prognosis of the regression Model 1, including only personality components, proved correct for 88% of the participants. When adding the cognitive component (Raven's general intellect index) into Model 2, there were no sufficient changes. Based on the obtained results, we accepted Model 1, which shows that worldview activeness ($\beta = 1.15$, $p < 0.001$) and constructive motivation ($\beta = 0.455$, $p = 0.001$) are the significant predictors of mathematical giftedness. General intellect according to the obtained data does not predict the manifestation of mathematical giftedness.

Without knowing it, the research participants gave very precise definitions of how various qualities relate to each other in the structure of mathematical giftedness.

Participant #16 (stimulus-productive level): "Well, kindness or something else… are not the qualities to define or not to define a mathematician. The matter is, that what can help in the work is needed: diligence, persistence, intuition. If you are talented and diligent, that is enough."

Participant #69 (heuristic level): "In my opinion, interest, motivation and love for mathematics — that is all that needed. Interest is quite enough. If a person is really interested in something certain, then he can achieve everything no matter what he had in the beginning."

Participant #98 (heuristic level): "In any case there should be a passion for solving the problem, this eagerness to disclose it, to find the unknown. And if there is such eagerness, then to master a set of tools or scientific materials, maybe a language to read other articles, to communicate with colleagues from other countries will be no problem. So, the main thing is passion."

Those fragments illustrate the differences between participants of the stimulus-productive and heuristic levels. If the former talk about the things which are "important and needed for work", then the latter prioritize interest and "passion", and the other qualities by this attitude become just "instruments".

## CONCLUSION

In the current work, the theory of giftedness and creativity [Bogoyavlenskaya, 1971; 2011] has been applied to the topic of special giftedness using the example of mathematical giftedness. According to this theory, mathematical giftedness is understood as the ability to develop an activity by one's initiative in the sphere of mathematics and is discovered as a system quality, integrating cognitive and personality components when cognitive motivation dominates in the personality structure. The participants of the current research were not random, it was students and post-graduates of mathematical specialities at prestigious Moscow universities, each one achieved a success in mathematics. The study is shown that the general intellect (on the advanced Raven's Progressive Matrices) is connected with mastering mathematical activity (on the Coordinate System technique), but it cannot definitely predict mathematical giftedness. Its manifestation depends on the cognitive orientation of the personality (which appears as constructive motivation for and infatuation with the subject matter) and the so-called "worldview activeness". Thus, based on the obtained results, it is possible to conclude that it would not be correct to rely on high scores in intellectual tests as the only sufficient criterion for the identification of giftedness The teaching mathematics should consist of not only disseminating certain knowledge, but of forming by students cognitive motivation (as internal process) in the first place.

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# FROM PROTOTYPICAL PHENOMENON TO DYNAMIC FUNCTIONAL SYSTEM: EYE-TRACKING DATA ON THE IDENTIFICATION OF SPECIAL QUADRILATERALS*

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*In the paper we consider the characteristics of the perceptive system as a possible basis for the well-known prototypical phenomenon in geometry. Analysis of eye-tracking data revealed intense involvement of extrafoveal processes in the categorical identification of squares and rectangles in canonical orientation, while rotated orientation required stronger foveal processing. Also, pronounced individual differences were found in perceptual strategies. Identification tasks that relied on an inclusion relation between squares and rectangles required more time but not fixations, pointing to logical rather then perceptual difficulty. The results are interpreted in light of the culture-historical approach to categorical identification as a functional system of extrafoveal, foveal and logical processes.*

## INTRODUCTION

Investigations of geometrical concepts and their relations with visual representations has a long history in mathematics education research. One of the key theoretical challenges and correspondent learning difficulties is related to the prototypical phenomenon [Hershkowitz, 1989]: Why is a right triangle better recognised when it has vertical and horizontal legs? Why, when being asked about a rectangle, do students first think of a shape that is vertical or horizontal? Traditionally, geometrical concepts have been considered under the light of the prototypical theory, however this theory states rather than explains the mixed visual and conceptual nature of the concepts. Up to now, "the difficulty to interpret the sensory/cognitive dichotomy" remains one of the major theoretical issues [Sinclair et al., 2016]. In this paper we question more deeply this relation between sensory and conceptual processes in their application to the categorisation of geometrical shapes.

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In *Shvarts A.* (Ed.). Proceedings of the PME and Yandex Russian conference: Technology and Psychology for Mathematics Education. P. 122–129. Moscow, Russia: HSE Publishing House.

In line with the systemic structure of higher mental functions in culture-historical theory [Vygotsky, 1934/1965], we claim that the prototypical structure of geometrical concepts might be at least partially explained as a result of the dynamical and systemic organisation of a variety of cognitive processes in an identification task. Our systemic approach highlights the tight and yet dynamical interrelation between conceptual and perceptual processes.

The evidence is based on the analysis of a categorical visual search in which a participant is required to identify the shape that corresponds to the target geometrical concept as quickly as possible. We use eye-tracking data to explore the relative roles of extrafoveal perception (the most automatized and usually preattentive process), foveal perception and language processes in geometrical shape recognition.

Our **research questions** concern the existence of preferred shapes, which are easier to recognise at the periphery. Confirmation of this hypothesis would support the theoretical assumption that conceptual processes are deeply intertwined with perceptual abilities, including at the level of mostly preattentive extrafoveal vision. We also question the factors that influence the involvement of peripheral processes to describe the dynamical constitution of possible perceptual strategies.

## THEORETICAL FRAMEWORK

### *Prototype phenomenon*

The theory of prototypes [Rosch, 1978] has been involved in the explanation of the phenomenon that students better identify and more often produce images of particular examples of geometrical shapes [Hershkowitz, 1989]. While in cognitive science this theory was developed as an elaboration of the classical concept definition, as a conjunction of attributes by adding a weighted role of the attributes, in mathematics education the connection with visual experience is widely accepted. One of the classical attitudes is based on Heile's theory that considers visual experience as the first level of familiarisation with geometrical concepts, and this level needs to be overcome. However, the question of whether the levels are consequential and whether the visual images behind the concepts need to be overcome or synthetised with the logical constraints of verbal definitions is still under consideration [Sinclair et al., 2016]. A special and most difficult identification task concerns the inclusion relations between geometrical concepts.

### *Categorical perception as a higher mental function*

According to the cultural-historical approach, higher mental functions emerge as a systemic unity, a "meaningful *functional system*" with "plastic, changeable interfunctional relations" that constitute "complex dynamic systems, which have to be considered as the result of integration of elementary functions" [Vygotsky, 1934/1965]. This systemic perspective of Vygotsky's heritage contributes to the complex dynamic system approaches that attract more and more attention in the educational literature (see more in [Jörg, 2016]). When applied to perceptual-categorical processes, this approach stresses the qualitative transformation of cultural perception, that is an "immediate fusion of the processes of concrete thinking and perception" [Vygotsky, 1987, p. 269] in which "we can no longer separate the perception of the object as such from its meaning or sense" [p. 299].

On the one hand, the perceptual processes per se are transformed by mathematics education and expertise [Krichevets, Shvarts, Chumachenko, 2014; Radford, 2010]; on the other hand, categorisation processes cannot departure from the sensory base. The traces of sensory experience and organisation in the functional system of geometrical concepts identification are in the focus of this study.

### *Foveal and extrafoveal processes in perception*

Let us focus on human visual abilities. The construction of an eye itself determine that visual abilities strongly depend on viewing eccentricity: spatial resolution is high for objects in the foveal region and becomes progressively impaired in the periphery [Rovamo, Virsu, 1979]. Despite the obvious drawbacks of peripheral (or extrafoveal) vision, it plays a significant role in our life. In situations when an object is rather large and separated from others, it can be easily detected using only extrafoveal vision [Levi, 2008]. The literature on visual search shows that such characteristics as shape, color, orientation, depth and motion can be identified extrafoveally and preattentively (for a review, see [Wolfe, Horowitz, 2017]). Some papers provide evidence that such a higher-level feature as direction of illumination can be processed using extrafoveal vision as well [Ramachandran, 1988]. Different factors may influence the extrafoveal processing speed and performance, including the similarity of distractors to a target [Reingold, Glaholt, 2014].

### *Canonical spatial directions: horizontal and vertical*

Horizontal and vertical, or canonical, orientations are the best-exposed directions in human surroundings, especially if we analyse cultural environments that include buildings, furniture and other artificial elements (e.g., [Coppola et al., 1998]). It has been shown in many behavioral and neurophysiological studies that a stimulus with vertical and horizontal orientations is processed quicker and better in many perceptual tasks [Appelle, 1972]. Apparently, the better recognition of vertical and horizontal directions in comparison to others has influenced the historical development of mathematical notations, such as Cartesian coordinates [Krichevets et al., 2014]. In this paper we investigate how this natural sensitivity to horizontal and vertical directions might contribute to the structure of geometrical concepts and presumably lead to learning difficulties.

## METHOD

**Equipment and materials.** Four geometrical shapes with an approximate size 4–6° of visual angle were presented in each stimulus; their centres were placed 12° from the screen centre (see Fig. 1).

Stimuli were displayed using SMI Experiment Center 3.3 on a 21" monitor at a 1280 × × 1024 pixel resolution and a 60-Hz refresh rate. Participants were seated approximately 60 cm distance from the monitor. Eye movements of both eyes were recorded

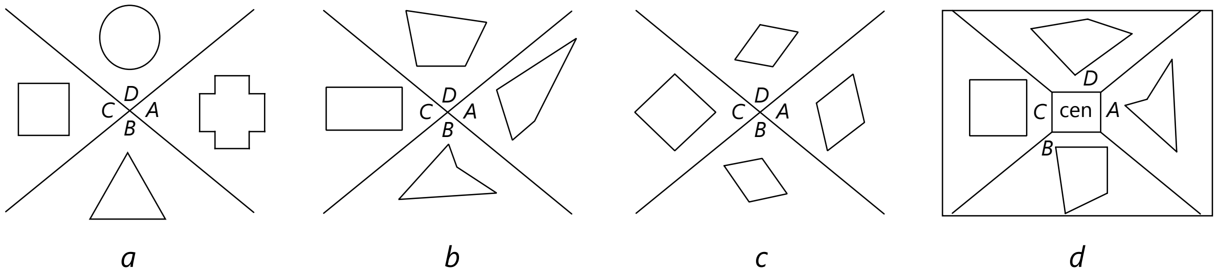


*Fig. 1.* Examples of stimuli in (*a*) Experiment 1 and (*b*, *c*) Experiment 2. (*d*) Areas of interest for the data analysis

with a sampling rate of 120 Hz by SMI RED and iViewX software. Nine-point calibration was considered valid if the average error in a validation test was less than 0.5°.

**Participants and procedure**. 33 undergraduate and graduate students with normal or corrected to normal vision took part in the study: 20 in Experiment 1 and 13 in Experiment 2. Participants were required to search for a shape that corresponded to the target concept that was named before each trial. The trials were initiated by a 500 ms gaze on the fixation cross in the centre of the screen. The instruction was to press the space button and name the target area (*A*, *B*, *C* or *D*, see Fig. 1) as quickly and as accurately as possible. The researcher tracked the correctness of the responses and only correct trials were further analysed.

A circle, square, triangle and cross (see Fig. 1*a*) were the target shapes in Experiment 1, which consisted of 24 trials. In Experiment 2 we had three experimentally varied factors: (1) the target concept: rectangle or square; (2) the distractors that could be either similar to the target (square among rhombuses or rectangle among parallelograms) or dissimilar to the target (square or rectangle among irregular quadrilaterals); and (3) the shape's position: prototypical (on their base) or rotated. So we had a 2 × 2 × 2 within-subjects design with two levels of each factor. The target area was quasi-randomized. Figures 1*b* and 1*c* show some samples of the stimuli. Altogether there were 96 trials.

**Data analysis.** The screen was divided into five areas of interest (*A*, *B*, *C*, *D* and center) as shown in Fig. 1*d*. We merged all sequential fixations in the area to one visit and then a sequence of visits were analysed to calculate the order number of the first visit to the target area (FirstT parameter), and reaction times were also measured. If we assume that a participant does not use extrafoveal vision while searching, the target area could be visited first, second, third or fourth in the stochastic sequence of visited areas, and thus the mean FirstT parameter would be (1 + 2 + 3 + 4) / 4 = 2.5. We compared the parameter FirstT with 2.5 to judge the involvement of extrafoveal analysis. SPSS v.22 and Matlab 2008b were used.

## RESULTS

### *Experiment 1*

The results showed clear evidence of an extrafoveal solution of this task: 372 of 480 trails (77.5%) were solved without any fixations. There were also strong individual differences in the amount of fixations needed (Crosstab 20*5, Chi-Square = 258, df = 76, $p < 0.000001$). In 15% of trials we observed the first fixation to be immediately at the target area; one participant solved 14 out of 24 trails with one fixation per trial. This implies that in these cases extrafoveal vision is strongly involved, but a person performs an additional saccade for a foveal check. Individual differences revealed that participants can choose a more “risky” strategy, or a “safer” one with an additional foveal check. Thus the involvement of extrafoveal vision depends not only on physiological mechanisms but also on personal strategies and attitude to the task.

*Experiment 2*

One participant gave 12 wrong answers, so we excluded her results from the group statistics. The average FirstT (order number of the first visit to the target area) was 1.09 (SE = 0.031). Both group ($t = -45.3$) and individual ($t$-statistic varied from −5.04 ($p = 0.000002$) to 24.9) results demonstrated strong difference from a random sequence of the visited areas (hypothetical value 2.5) and evidenced involvement of extrafoveal vision.

484 out of 1241 trails (39%) were solved simultaneously, without any saccades. Crosstab analysis demonstrated strong individual differences (Chi-Square is about 500, df = 48, $p < 0.000001$). Table 1 shows the results of some participants with different strategies, however there were several participants with each strategy. Participant *A* found the target shapes almost without any fixations, using only extrafoveal vision. In contrast, participant *D* mostly did not use extrafoveal vision and stared at a few areas before she found the target one. Interestingly, there is a variety of patterns apart from these two: participant *B* could answer without any fixations, relying on extrafoveal vision in some trials, but needed a few fixations and foveal analysis in others. The results of participant *C* are particularly interesting: in most of the trials she did one fixation to the target shape, and thus her extrafoveal vision mostly provided the correct answer but she performed an additional confirmatory saccade.

We conducted ANOVA with three factors: target concept and shape, rotation, and similar versus dissimilar distractors. All factors highly significantly influenced the number of visited areas (FirstT) (factor rotation: $F(1, 10) = 31.2$, $p < 0.0003$ and factor distractors: $F(1, 10) = 37.46$, $p < 0.0002$) and reaction time (factor rotation: $F(1, 11) = 134$, $p < 0.000001$ and factor distractors: $F(1, 11) = 219$, $p < 0.000001$) (see Fig. 2). So, the noncanoni-

Table 1

**Individual differences in perceptual strategies of some participants**

| Number of areas visited before target shape was found in each trial | | | | | | |
|---|---|---|---|---|---|---|
| | **0** | **1** | **2** | **3** | **4** | **Sum** |
| *A* | 82 | 7 | 5 | 1 | 1 | 96 |
| *B* | 36 | 30 | 23 | 5 | 2 | 96 |
| *C* | 11 | 63 | 13 | 9 | 0 | 96 |
| *D* | 5 | 34 | 38 | 16 | 3 | 96 |

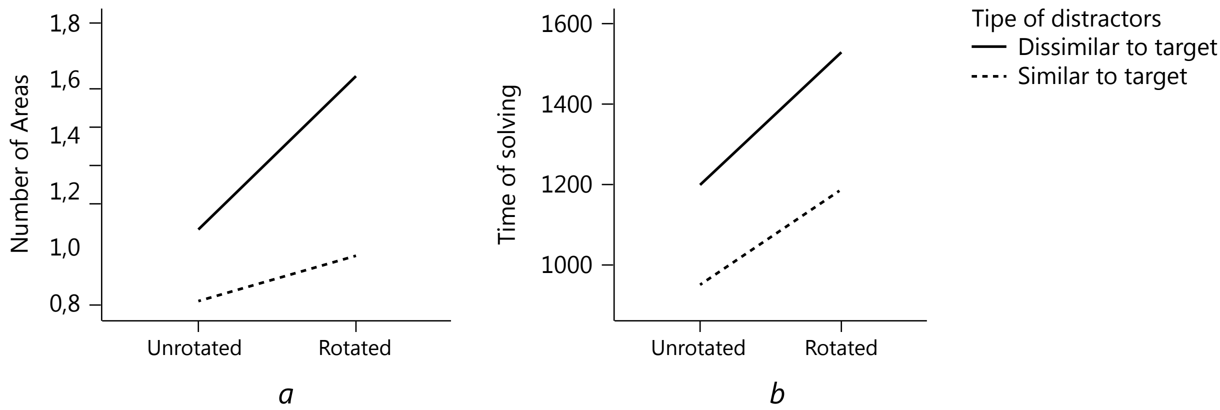


***Fig. 2.*** The influence of the factors of rotation and similarity with the distractor on (*a*) the number of areas visited before the target and (*b*) the reaction time

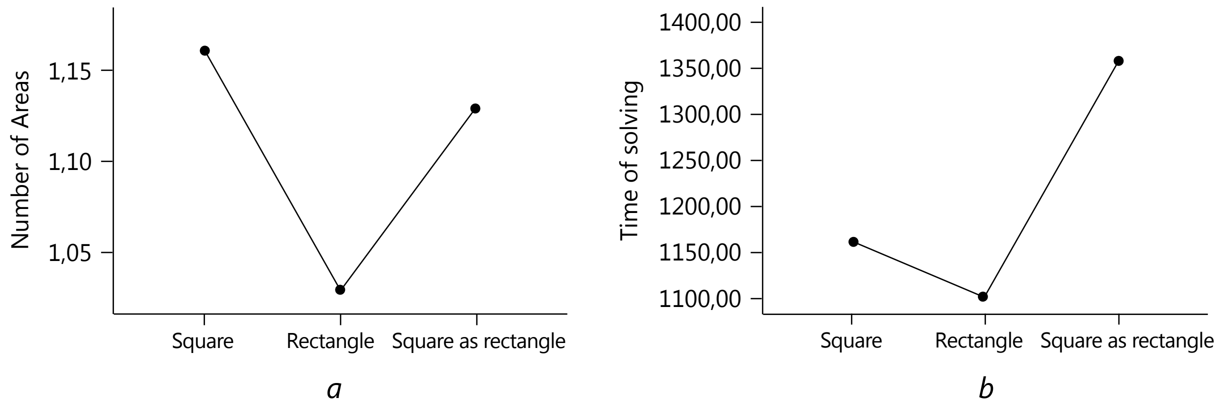


***Fig. 3.*** The influence of the target concept on (*a*) the number of areas visited before the target and (*b*) the reaction time

cal rotation of the target shape and similarity with the distractors made identification task more difficult and the involvement of extrafoveal processes decreased. There was also an interaction between factors found for both variables (for FirstT, $F(1, 10) = 8.07$, $p = 0.018$ and reaction time $F(1, 11) = 9.96$, $p = 0.008$): The combination of rotation and similar distractors provided more than a cumulative increase of difficulty. The same influences of the factors were observed for each of the target concepts.

The differences between the number of areas visited and time of solving dependent on target shape are also very interesting (see Fig. 3). Identification of a square as belonging to rectangles required more time ($F(1, 11) = 14.9$, $P = 0.003$), but did not require additional fixations ($F(1, 11) < 1$). We assume that the task of identifying a square as a rectangle posed a non-perceptual but logical difficulty for participants.

## DISCUSSION AND PRACTICAL APPLICATIONS

As we have demonstrated in the first experiment, extrafoveal analysis is sufficient for distinguishing simple geometrical shapes: all participants were able to detect the target shape extrafoveally. However, some participants used it to confirm their extrafoveal hypothesis by an additional saccade. This result clearly demonstrates the complexity of perceptual processes, and the variety of possible strategies in the categorisation task in which extrafoveal and foveal processes may play different roles. The same principle of systemic organisation of foveal and extrafoveal processes under the task constraints was observed in the second experiment: individual differences clearly showed a variety of possible strategies. Extrafoveal vision might be involved more or less and may be sufficient for the decision or require additional confirmation.

While our participants almost never made identification mistakes, the group tendencies in the involvement of extrafoveal processes demonstrate the possible ground for an orientation effect behind the prototypical phenomenon [Herskowitz, 1989]. The shapes in canonical orientation were mostly identified by extrafoveal vision, while in case of rotated exposure some foveal analysis was often required, especially in the cases of similarity between the target shape and distractors.

The congruence of the results in geometrical categorisation with traditional findings in visual search for perceptually given stimuli demonstrates that perceptual processes deeply intervene with conceptual processes: we observed the influence of distractor similarity [Reingold, Glaholt, 2014] and an orientation effect [Appelle, 1972] in geometry. The level of visual experience is not overcome by later logical processes, as Hiele's theory would suppose [Sinclair et al., 2016], and the role of visual perception is not limited to a congruency with the frame of reference [Herskowitz, 1989]. Visual perception, including foveal and extrafoveal processes, appears to take an essential part in the complex functional system of categorical identification, dynamically constituted under constraints of each task and individual differences.

The third component of this functional system is logical reasoning: the identification tasks that relied on inclusion relations required more time but not fixations, thus evidencing logical rather than perceptual difficulty.

Possible educational applications include practice of the functional system as a whole in a rich perceptual environment rather than merely discussion of the shapes' attributes. A variety of identification tasks in the sets of shapes with different distractors and shape orientations might serve as a good ground for the development of stable identification and problem solving. In case of inclusion relations, an additional logical articulation would be required.

## CONCLUSIONS

We have demonstrated the relevance of low-level perceptual qualities of the stimuli for categorical identification of geometrical shapes. These low-level qualities, such as canonical orientation of the figures, might partially determine the prototypical phenomenon. According to our results, the prototypical phenomenon is not limited to distinguishing a weighted list of attributes, but can be seen even in the involvement of extrafoveal perception in different identification tasks. These findings show that perceptual and conceptual processes tightly intertwine in the solution of an identification task. Identification that involves inclusion relations requires additional preparations independent from the foveal or extrafoveal analysis that presumably process logical relations between categories.

Strong individual differences in the distribution between foveal and extrafoveal processes reveal a complex functional system rather than a particular stable mechanism serving the task of categorical identification in geometry. This functional system is dynamically constituted as an ensemble of extrafoveal, foveal and logical processes in the response to particular task constraints and individual preferences.

## ACKNOWLEDGEMENTS

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# SCHOOL PUPILS' INTELLECTUAL DEVELOPMENT DURING MATHEMATICAL TEACHING: THE ROLE OF EDUCATIONAL TEXTS

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*The main purpose of the research is to show the role of psychodidactic approach in modern school mathematical education and to present the psychodidactic typology of development-focused educational texts for students of grades 5 to 9. Each type of text creates conditions for the enrichment of the main components of the mental experience of students (cognitive, conceptual, metacognitive, and intentional) as a factor of their intellectual growth and the prerequisites for a high level of understanding of mathematical material.*

## INTRODUCTION

Modern innovative teaching technologies use the *psychodidactic* approach, which switches the main focus when evaluating teaching effectiveness to changes in pupil's intelligence and personality. Psychodidactics is the area of pedagogy that designs content, forms, and methods of teaching based on integrated psychological, didactic, methodological, and subject-matter knowledge while focusing specifically on the mental patterns of personality development as a basis for organizing the teaching process and general learning environment.

The psychodidactic approach may be used in schools in many ways: by using "didactical situations", including metaphor and emotional context [Broussau, 1997]; through focusing on learning and conceptualization by selecting mathematical tasks and hypothesizing about how each one influences the learning process (*Hypothetical Learning Trajectory*, or *HLT*) [Simon, Tzur, 2004]; by using basic cognitive actions such as recognizing, building, and constructing (*RBC model*) as a foundation for conceptual teaching based on pupils' own experience [Hershkowitz, Schwarz, Dreyfus, 2001; Bikker-Ahsbahs, 2004]; by developing pupils' creative thinking [Burke, Williams, 2008]; and by using "realistic" situations in the learning process [Van den Heuvel-Panhuizen, Drijvers, 2014].

Our research is carried out within the problem of psychodidactics of school textbooks (more widely, psychodidactic requirements for educational texts) [Gelfman, Kholodnaya, 2006; Kholodnaya, Gelfman, 2016*a*; 2016*b*]. We believe that the actual content of school subjects is essential to pupils' intellectual development. It is therefore very important to set the requirements applied to the educational materials. This includes school textbooks that could be used to implement the psychodidactic approach. A text-


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book should not be structured as a reference book. It should rather be a learner-focused teaching book since mathematical knowledge can only have a developmental effect when it is in harmony with the patterns of the pupils' mental development, both intellectual and personal.

We developed the *Development-Focused Educational Texts (DET) Technology* as a part of the *Mathematics. Psychology. Intelligence (MPI)* pedagogical project, for use in middle-school mathematics teaching (grades 5 to 9). DET Technology focuses on pupils' intellectual development based on the design of *mathematics content* and on the development of special-purpose *educational texts*, in particular.

## EDUCATIONAL TEXTS AS A WAY TO FOSTER PUPILS' INTELLECTUAL DEVELOPMENT

When assessing the role of education content as a whole and the role of educational texts, in particular, one must appreciate the key fact that, from a psychological perspective, intellectual development is only possible through learning, processing, and producing diverse subject content, ranging from trivial everyday knowledge to scientific hypotheses about the structure of the universe. The richer the subject-matter environment (physical, social, and educational) of a preschooler or a schoolchild and the more actively they interact with this environment, the greater their intellectual abilityes.

Therefore, *the content of school education* is a key factor in the development of pupils' intelligence. Its minimal component is *an educational text,* which shapes the way pupils interact with various content environments. However, we emphasize that not every format of educational content and not every type of educational text provide the effects of intellectual development of students and a high level of understanding of educational material.

The special role of texts in personal intellectual development is noted by many scholars. They see the text as "a thinking structure" (V.V. Ivanov), "a model of thought adventures" (L.E. Gendenshtein), and "a conversation partner" (M.M. Bakhtin). Texts are a natural medium for intellectual development throughout a person's lifetime.

In school education, texts are always the focus of attention because of their essential role in effective teaching, particularly in the context of *reader-oriented theory*, which suggests that readers actively construct meanings (concepts) as they read, which is also true for mathematics textbooks [Weinberg, Wiesner, 2011].

Therefore, the use of special development-focused educational texts is a promising way to encourage intellectual development. This means that development-focused educational texts do not merely present formal mathematical knowledge; they also facilitate the development of psychological mechanisms for productive intellectual activity by changing the design of educational material and the ways of its presentation in accordance with certain psychological requirements (in the context of the study, we focus on the enrichment of the main components of the mental experience of students).

## ENRICHMENT OF MENTAL EXPERIENCE AS THE PSYCHOLOGICAL BASIS FOR PUPILS' INTELLECTUAL DEVELOPMENT

We believe that the psychological basis for intellectual development should be the enrichment of pupils' mental experiences while learning. The structural model of intelligence in terms of the architecture of a person's mental experience outlines four levels of mental experience, each with its own purpose [Kholodnaya, 2002]:

1. *Cognitive experience* refers to the mental structures ("cognitive schemes") responsible for presenting, recognizing, storing, and sorting information. Their main role is immediate information processing.

2. *Conceptual experience* refers to mental structures ("concepts") which generalize and transform information through abstraction, idealization, and interpretation. Their main purpose is to identify meaningful properties and to reproduce regular and consistent features of the environment.

3. *Metacognitive experience* refers to mental structures ("metacognitions") which allow involuntary and voluntary regulation of information processing. Their main aim is to control the intellectual activity and status of personal intellectual resources.

4. *Intentional (emotional and evaluative) experience* refers to the mental structures ("intentions") underlying individual cognitive dispositions. Their main purpose is to form subjective preferences in selecting subject areas, ways of solving problems, information sources, etc.

"Enrichment" of pupils' mental experiences here includes, first, the development of every of the above four key mental experience components as a foundation for nurturing their intellectual abilities, and, second, creation of the conditions for pupils to demonstrate their individual cognitive styles.

The basic directions of the enrichment of mental experience are as follows:

- ***Enrichment of cognitive experience***. Here we should seek to develop different *ways of information encoding* (verbal/symbolic, visual, substantive/practical, sensory/emotional) and to widen the range of *declarative and procedural cognitive schemes* for mathematical concepts and activity methods and to increase their flexibility.
- ***Enrichment of conceptual experience***. Here we aim to improve students' understanding of mathematical language *semantics* and to expand the semantic fields pertaining to mathematical concepts, differentiating and integrating *categorical structures* (and focus on helping students to identify *substantial conceptual features, links between concepts* from different generalization levels, main *phases of concept formation* such as motivation, categorization, enrichment, transfer, and crystallization). This direction relies *on hypothesizing, interpretation,* and *creating texts independently.*
- ***Enrichment of metacognitive experience***. Here we should help to develop *voluntary and involuntary controls* of intellectual activity, including the abilities to plan, evaluate, predict, and self-check. The aim is to increase *metacognitive awareness*, refer-

ring to the student's understanding of how academic knowledge is organized and differences between learning methods. We encourage an *open cognitive position*, readiness to absorb "impossible" information, accept an alternative point of view, and properly react to discrepancies.

- ***Enrichment of intentional (emotional and evaluative) experience***, which implies offering students *a choice* of how to study educational materials. This direction relies on *the pupil's personal and intuitive experience* (pupils are encouraged to share doubts, guesses, beliefs, "anticipatory" ideas, and emotional evaluations); it involves *play elements* and the *value-based approach* to educational materials and should include adopting multiple *individual cognitive styles*, which reflect personal preferences and dispositions.

## THE TYPOLOGY OF DEVELOPMENT-FOCUSED EDUCATIONAL TEXTS

Based on the structural model of intelligence, different *development-focused educational texts* were designed for the school mathematics courses (grades 5 to 9). Each text type addressed a specific component of the mental experience framework with the aim of facilitating its development [Gelfman, Kholodnaya, 2016*a*; Kholodnaya, Gelfman, 2016*b*]. *The text typology* is presented in Table 1.

Table 1

**The typology of development-focused educational texts**

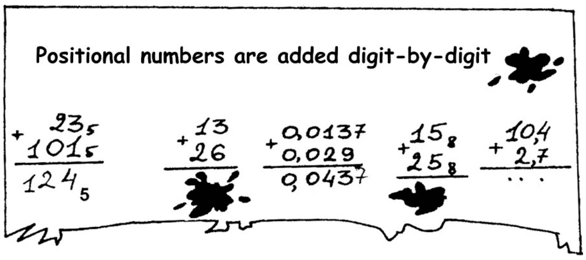
| Components of mental experience | Learning activity | Types of educational texts |
|---|---|---|
| ***Cognitive experience*** | | |
| Information encoding ways | Verbal/symbolic | • Learning mathematical symbols<br>• Finding a formula<br>• Drafting definitions |
| | Visual | • Developing a normative image<br>• Image classification<br>• Image evolution<br>• New image motivation<br>• Conversion from verbal/symbolic to visual encoding initiation of personal imaginative experience |
| | Practical information | • Laboratory work<br>• Situation in practice |
| | Sensory/emotional | • Emotional impression<br>• Metaphor<br>• Play |
| Declarative cognitive schemes | Schemes of mathematical concepts | • Introduction of focus example<br>• Framework<br>• Summary |
| Procedural cognitive schemes | Cognitive schemes of mathematical activity methods | • Algorithm (procedure)<br>• Operation |

End of Table 1

| Components of mental experience | Learning activity | Types of educational texts |
|---|---|---|
| ***Conceptual experience*** | | |
| Semantic structures | Mathematical language semantics | • Term meaning<br>• Systematization of term meanings<br>• Translation from the native language to the mathematical language |
| Categorial structures | Identification of category features, establishment of links between categories and construction of concepts | • Identification of concept features<br>• Selection of concept features<br>• Establishment of links between concepts<br>• Concept motivation<br>• Verbal and visual categorization<br>• Enrichment of conceptual content<br>• Transfer of a concept to a new situation<br>• Crystallization of conceptual content |
| Conceptual structures | Hypothesizing, interpretation, and creation of texts | • Search for and generalization of regularities<br>• Modeling<br>• Micro-essay<br>• Unassisted composition of a text<br>• Invitation to a project |
| ***Metacognitive experience*** | | |
| Involuntary and voluntary control | Planning | • Program<br>• Goal selection<br>• Planning |
| | Predicting | • Development of a hypothesis<br>• Prediction in an uncertain situation<br>• Prediction of operation results |
| | Self-checking | • Self-checking methods<br>• Selection of a self-checking method<br>• Search for mistakes |
| Metacognitive awareness | Reflection on own intellectual activity | • Reflection on solution methods<br>• Self-assessment of personal knowledge and skills<br>• Educational self-monitoring<br>• Psychological commentary |
| Open cognitive position | Readiness to work with inconsistent information | • Problematization<br>• Alternative<br>• Collision of contrasting opinions<br>• Impossible situation |
| ***Intentional (emotional and evaluative) experience*** | | |
| Preferences | Selection of learning method | • Selection of activity methods<br>• Selection of cognitive position<br>• Individual cognitive styles |
| Beliefs | Use of intuitive experience | • Conjecture<br>• Creative work |
| Attitudes | Value-based treatment of educational material | • Mathematics in the world around us<br>• Key directions in mathematics development<br>• History of mathematics |

Let us give an example of a development-focused educational text of the “text—search for mistakes” type.

Misha and Oleg found a piece of paper with records.




It was not known who it belonged to; neither the author nor the grade appeared on the paper.

“Well, this is not an ‘F’ grade,” — said Misha. — “I see a correctly solved task.”

“The grade couldn't be excellent,” — said Oleg. — “I see an incorrectly solved task.”

Friends began to check everything themselves. However, they found that the answers in these two tasks were different in writing, but equal in value.

The task is to restore the records on the piece of paper and specify *the rules of addition of natural numbers and decimal fractions*.

What words and tasks seem the most important for the addition of natural numbers and decimal fractions?

## CONCLUSIONS

Noting the importance of educational texts in school mathematics, we can push the limits of the popular view that teaching mathematics simply means teaching pupils how to solve mathematical problems. We believe that teaching mathematics actually means teaching children how to interpret the meanings and implications of mathematical concepts and operations. Thus, the problem of understanding mathematical knowledge (mathematical objects, actions, situations) comes to the forefront [Godino, 1996; Simon, 2017]. How can we operationalize the idea of mathematical understanding?

The system of development-focused educational texts provides a tool which encourages pupils’ intellectual development by offering a variety of topics within the school mathematics course, facilitating the enrichment of the main components in pupils’ mental experience. Mathematics teaching based on the prolonged and systematic use of all types of development-focused educational texts promote the growth of pupils’ intellectual resources and, as a result, a higher level of understanding of educational mathematical material [Budrina, 2009; Gelfman, Kholodnaya, 2006; Gelfman, Podstrigich, 2006; Gelfman et al., 2013; 2009].

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# EYE MOVEMENTS DURING COLLABORATIVE GEOMETRY PROBLEM SOLVING LESSON

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*This study examines student eye movements during collaborative geometry problem solving. Specifically, we are analysing the differences in fixation durations when working on paper versus working with GeoGebra. We examined eight students' fixation durations in four classrooms. We found that use of technology influenced the duration of gaze fixations. The use of GeoGebra resulted in slightly more short fixations, less medium length fixations, and clearly more long fixations. A more detailed analysis suggested that the long fixations are related both to instrument manipulation and cognitive processes.*

## INTRODUCTION

Eye-tracking is a method to get information on student cognition as it happens. Hartmann and Fischer [2016] compare eye-tracking information to mind-reading: the target of a fixation (maintaining of the visual gaze on a single location) tells what we think about and the fixation duration corresponds with processing time. Glöckner and Herbold [2011] summarize research evidence to suggest that gazes related to more automatic processes would have shorter fixations (below 250 ms) and more elaborated information processing generally requires long fixations of more than 500 milliseconds. However, there is much evidence that sometimes the connection breaks and the fixation and thoughts are not aligned (e.g., [Schindler, Lilienthal, 2017]).

Our research will focus on examining student eye movements in the context of collaborative problem solving in geometry. This study will address three under-examined research areas. First, while eye-tracking research is strong in the context of language processing [Rayner, 1998], the method has been far less used in the area of mathematics [Hartmann, Fischer, 2016]. Second, eye-tracking research has mainly been conducted in laboratory situations and studies conducted in real classrooms have only recently started to appear (e.g., [McIntyre, Mainhard, Klassen, 2017]). And third, research on multiple persons interacting (e.g., [Rogers et al., 2018]) is so far extremely limited. Although methodologically challenging, moving into these poorly explored areas is important for mathematics education. Mathematical problem solving takes place by social beings in the complexity of the learning environment where multiple modalities are present [Radford, 2008; Arzarello et al., 2009] and multiple goals need



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to be addressed [Hannula, 2006; Goldin et al., 2011]. We need to study mathematical behaviour in ecologically valid ways.

Technology has becoming an important element of mathematics learning environment, and it seems to have positive effects on achievement [Chauhan, 2017; Li, Ma, 2010]. Singer and Alexander [2017] found out that technology influences reading comprehension: people read longer texts and learn smaller details better on print. We have not found eye-tracking studies comparing computers and paper for mathematics, we expect to find a difference. We formulate our research question as follows: Does the selection of tool (paper vs. GeoGebra) have an effect on student fixation durations when students are doing problem solving. Moreover, we will examine the longest fixations when students are using GeoGebra.

## METHOD

### *Participants*

The current study examines fixation durations of grade nine students in a Finnish mathematics class. The teacher and a collaborative group of four target students volunteered to wear gaze-tracking glasses throughout the lesson.

### *Apparatus*

We recorded the actions and conversations of the problem-solving session using audio recording and three stationary video cameras in the classroom. The gaze-tracking device consisted of two eye cameras, a scene camera, and simple electronics attached to 3D-printed frames. The devices and software were self-made (see [Toivanen, Lukander, Puolamäki, 2017]). The camera frame rate depended on lightning conditions, and maximum rate in optimal conditions was 30 frames/second. Data was recorded on laptop computers that were carried in backpacks allowing subjects to move freely in the classroom.

### *Procedure for data collection*

The data was collected during grade nine mathematics lessons in Finnish lower secondary schools. The ethics review has approved our research procedures.

In each class, the students solved a non-routine mathematics problem collaboratively in groups of four students. The problem solving sessions lasted from 32 to 56 minutes. Two of the participating classes worked with GeoGebra software and two solved the task using pen and paper. In each of the four classes, the teacher and a collaborative group of four target students volunteered to wear gaze-tracking glasses throughout the lesson. One student data in the paper and pen setting was lost due to tracker malfunctioning.

To reduce the variation of pedagogical choices, we had scripted the problem solving session for the teachers. However, there was some variation in how the four lessons were constructed, especially with respect to training in GeoGebra before the problem task and whether they used extension tasks.

We identified three types of sequences during the problem solving: (1) sequences happening only during some of the lessons (e.g. GeoGebra training and extension task); (2) sequences that were similar for all lessons and minimally influenced by the choice of tool (GeoGebra or paper); and (3) sequences when the activity was done directly with the tool. The first type of sequences we excluded from further analysis. Then, we used the second type of sequences to identify students whose gaze patterns are sufficiently similar. Finally, we used the third type of sequences to analyse the effect of the tool on gaze durations.

We used the following lesson sequences in our analysis: (2*a*) Teacher gives instructions regarding the lesson structure, (2*b*) Teacher poses the problem and instructs individual work, (3*a*) Individual work, (2*c*) Teacher gives instructions for pair work, (3*b*) Pair work, (2*d*) Teacher gives instructions for group work, (3*c*) Group work, (2*e*) Students go to the board to present their solutions, and (2*f*) Whole class discussion. During the type 3 sequences, the teacher was instructed to provide encouragement and to ask questions that require students to explicate their thinking but to not provide hints on how to solve the problem. When students were working individually, in pairs, or in groups of four, the teacher's activity consisted of roaming in the classroom and stopping for scaffolding one group at a time. The teacher was instructed to provide encouragement and to ask questions that require students to explicate their thinking but to not provide hints on how to solve the problem.

### *Procedure for data analysis*

First we analyzed the descriptive statistics of fixation durations. As the distributions were non-normal, we used non-parametric tests in our consequent analyses. We used the data from comparable sequences (type (2)) to make pairwise Mann-Whitney *U*-tests between individual students. When comparing fixation durations of two samples, we decided to use the Kolmogorov-Smirnov *Z*-test rather than Mann-Whitney's *U*-test, because it has more power to detect changes in the shape of the distributions.

The analyses for the tool effect was then done comparing students selected based on the pairwise comparison. We compared the gaze distribution for both tool-independent and tool-dependent sections using Mann-Whitney *U*-tests and Kolmogorov-Smirnov *Z*-test. To further illustrate the differences in distributions, we used the data from all students for the whole problem solving session to identify ten equally large groups of fixations. We then report frequencies of fixation duration lengths separately for students using GeoGebra and students using paper and pencil.

Finally, we analyse the longest gazes for the students using GeoGebra. For this, we used visual markers placed in the classroom. When a visual marker was identified in the student's eye-tracking video, the location of the fixation was computed in relation to the marker. For each visual marker, we collected related fixations visually in one image. From this summary image of multiple video frames we identified fixation targets. Information about the moment of the fixation was used to find the video segment when the fixation took place for a qualitative examination.

## RESULTS

The results of the pairwise Mann-Whitney comparisons identified eight students who had no statistically significant differences in their gaze durations during type 2 lesson sequences. Three of these students were from classes using paper and pencil and five from classes using GeoGebra.

Our analysis of the fixation durations shows that even the more sensitive Kolmogorov-Smirnov test found no statistically significant difference in the distributions between the two tools during the sequences when the tool was not essential (Table 1). However, the differences were statistically significant during the times when the tool was essential.

To further explore the nature of the difference between paper and GeoGebra for visual attention, we looked at the distributions of different fixation lengths. We see a rather complex pattern of differences between the conditions for using GeoGebra vs. using paper (Fig. 1). When students were working with GeoGebra, they had more fixations in the time range 135 ms to 201 ms and also more long fixations (longer than 936 ms) in comparison to paper condition. On the other hand, students working with paper, had more fixations in the time range 201 to 936 ms compared with GeoGebra.

We then analysed what were the targets of the longest fixations in the GeoGebra condition. During the time from the beginning of individual work until the end of group work the five students had 1124 fixations longer than 936 ms, which was the largest bin in the histogram. Of these fixations 999 (89%) were identified in relation to a visual marker. Most of the long fixations (596; 53%) were on own computer screen and additional 56 (5%) were on peer's screen. The remaining 31% of the long gazes were mostly on peripheral parts of the computer, other people (Fig. 4), or notebooks.

When watching the long fixations as part of the gaze videos, two different types of long fixations were found. Figures 2 and 3 are heatmaps generated based on long fixations on screen. Long fixations were located mostly at the left end of the menu

Table 1

**Tool effect on fixation durations**

| **Lesson sequences** | **Tool used for problem solving** | **Number of fixations** | **Median for fixation durations (ms)** | **Mann-Whitney *U*** | **Kolmogorov-Smirnov *Z*** |
|---|---|---|---|---|---|
| Tool independent phases | Paper and pencil | 2858 | 240 | 7728972 ($p$ = 0.19) | 1.337 ($p$ = 0.06) |
| | GeoGebra | 5504 | 238 | | |
| Tool dependent phases | Paper and pencil | 7828 | 240 | 32449039 ($p$ = 0.02) | 3.159 ($p$ < 0.001) |
| | GeoGebra | 8464 | 237 | | |
| Total | | 24654 | 238 | | |

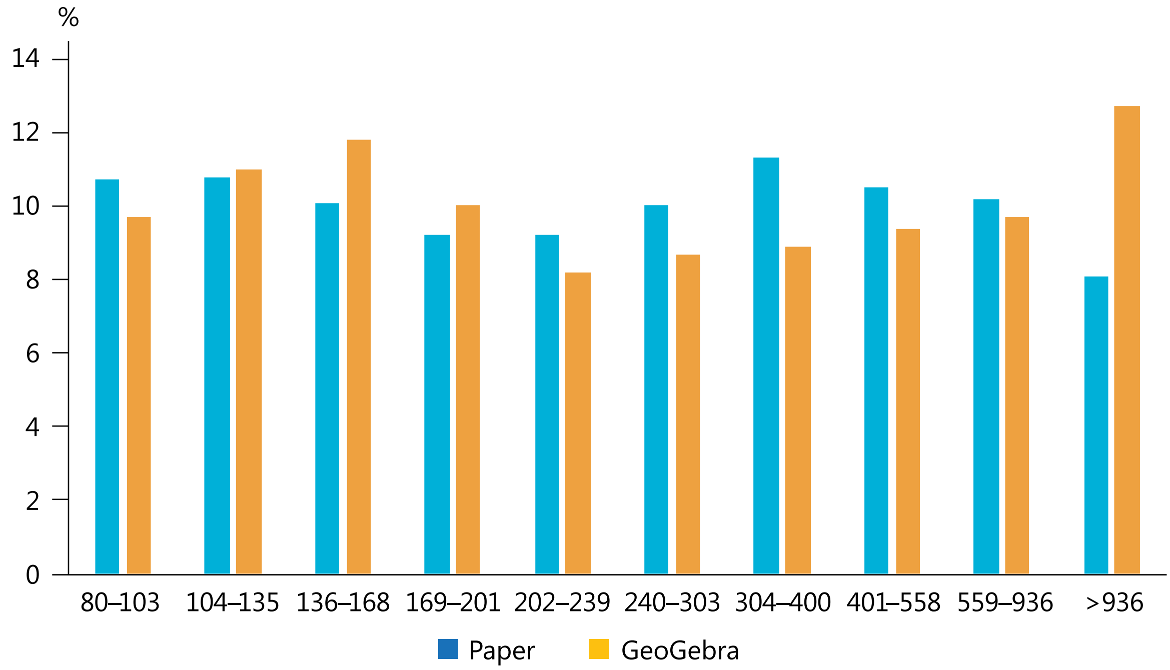


*Fig. 1.* The distribution of fixation durations (ms) when students were using the tools

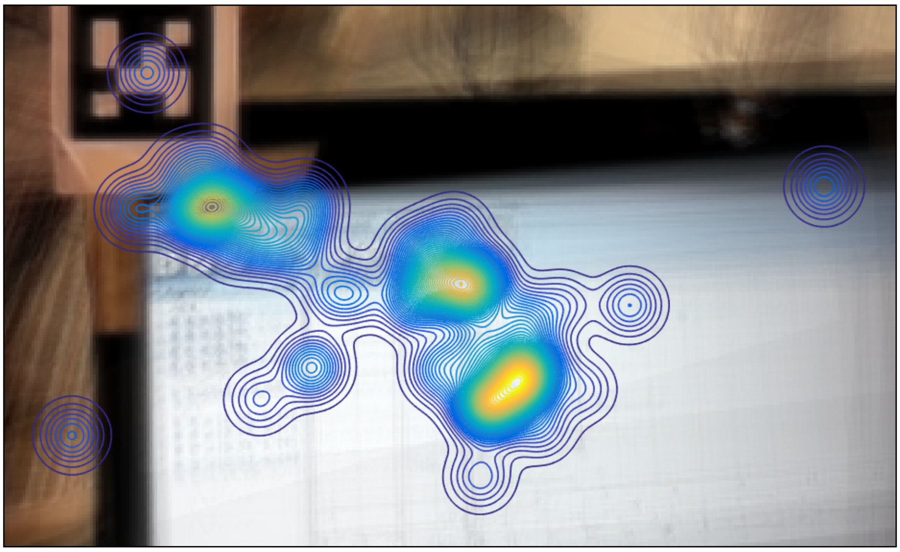

*Fig. 2.* A heatmap of fixations on and near student's own screen. Based on 54 long fixations of one student

bar and on the figure the students drew on the screen. However, the fixations on Fig. 3 were almost exclusively related to the student carefully targeting the mouse at the right place and the student did little progress with the problem. While several fixations on Fig. 3 were also related to the manipulation of the tool, many of these fixations seemed to have more cognitive context. The student seemed to be looking at the figure, alternating the gaze target until he had an insight and was able to produce a better solution.

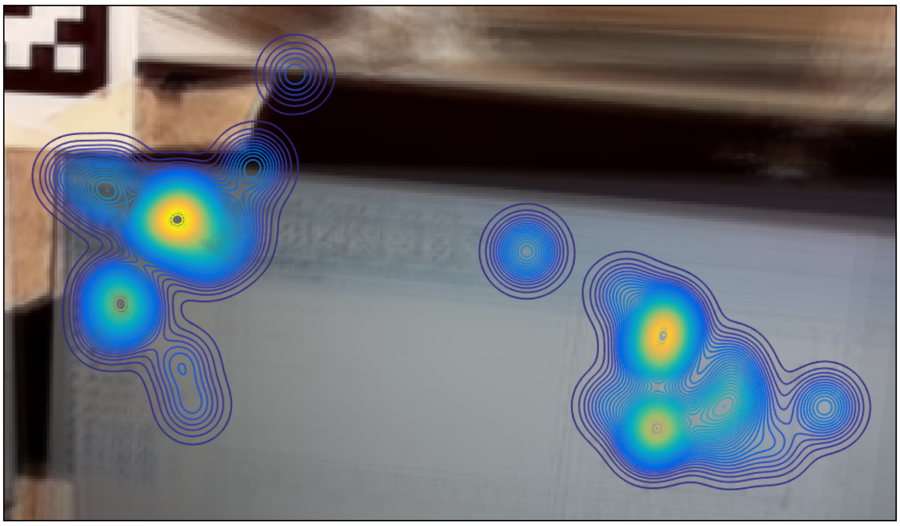

*Fig. 3.* A heatmap of fixations on and near student's own screen.
Based on 41 long fixations of one student

## DISCUSSION

The results show an effect in student fixation durations for the choice between computer and paper as a media to solve a geometry problem. Use of GeoGebra is related to increase in relatively short fixations and also increase of long fixations. These differences are likely to have different explanations. According to Glöckner and Herbold [2011] more automatic processes would have shorter fixations (below 250 ms) whereas more elaborated information processing is associated with long fixations of more than 500 milliseconds. These results suggest that using a digital tool increases the amount of both automatic scanning fixations and long fixations related to more elaborated processing. However, the qualitative examination of the long gazes in context sug-

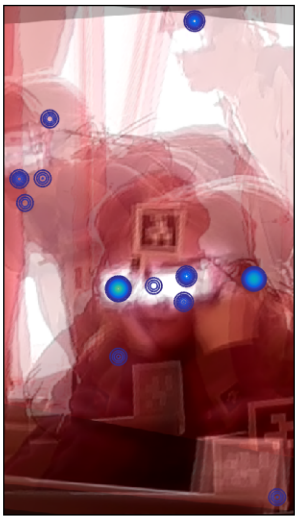

*Fig. 4.* Fixations on and near peer's head.
The shapes behind the peer are teacher

gests that a significant amount of these long fixations are related to interacting with GeoGebra, for example when selecting an option from a drop down menu or using mouse to place an object in the coordinate system.

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# THE EFFECT OF PHONOLOGICAL ABILITY ON MATH ACHIEVEMENT IN ELEMENTARY SCHOOL IS MODULATED BY SOCIOECONOMIC STATUS

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*Current study focused on estimation of the effect of phonological ability on math achievement during first year of schooling and testing the hypothesis that this effect varied depending on students SES. To achieve our aims we used two-waves longitudinal study which were conducted on large sample of first-graders (N = 2948) in Tatar Republic (Russia). The results revealed that phonological ability had a significant positive effect on math achievement even when reading achievement, number identification skills and SES were controlled for. The effect of phonological ability was higher for students with larger number of books at home and who used not only Russian language at home.*

## INTRODUCTION

Many factors contribute to individual differences in math achievement. Both the cognitive and social predictors of math development were extensively discussed in literature. Sociologists and policymakers mostly focused on the relationship between math achievement and socioeconomic status (SES), whereas cognitive and educational psychologists extensively discussed the cognitive predictors of math achievement such as domain-specific (e.g., number sense or spatial ability) and domain-general (e.g., working memory, phonological ability). To optimize mathematical educational and instructional practices, it is necessary to understand how these factors interact with each other during development. In particular, this study aimed to estimate how SES modulate the effect of phonological ability on math performance during the first year of schooling.

Despite a large body of research regarding the relationship between phonological ability and math and the effect of SES on math achievement, little is known about how SES modulates the effect of phonological ability on math achievement and math progress. Meanwhile, this issue might be important for planning of remedial programs for children with mathematical difficulties or for design of developmental programs for children from low socioeconomic families.

## THEORETICAL FRAMEWORK AND OVERVIEW OF LITERATURE

Socioeconomic status (SES) is supposed to be one of the strongest predictors of academic achievement, both in reading and math (e.g., [OECD, 2010; Sirin, 2005]). Ad-



In *Shvarts A.* (Ed.). Proceedings of the PME and Yandex Russian conference:
Technology and Psychology for Mathematics Education. P. 146–152. Moscow, Russia: HSE Publishing House.

vantages of children from families with high SES emerge before the start of schooling; these differences might remain or expand through school years [Bradley, Corwyn, 2002; Caro, McDonald, Willms, 2009]. SES also affected cognitive functions and early precursors of math achievement such as number sense and number competence (e.g., [Ardila et al., 2005; Jordan, Levine, 2009]). SES can also modulate relationships between certain types of cognitive predictors and math achievement (e.g., [Demir, Prado, Booth, 2015]). Particularly, it was demonstrated that SES moderated the relationship between math gains and brain activation in regions related to verbal numerical representations and spatial representations [Demir, Prado, Booth, 2015].

Several studies have demonstrated that certain types of verbal abilities correlate with math achievement (e.g., [Grimm, 2008; Planas, Morgan, Schütte, 2018]). In particular, it has been shown that phonological abilities (individual's sensitivity to the sounds of the language [Wagner, Torgesen, 1987] predict some types of math skills (e.g., [De Smedt et al., 2010; Hecht et al., 2001; Krajewski, Schneider, 2009]).

Phonological ability has a larger effect on math problem solving when retrieval of arithmetic facts strategy is used [De Smedt et al., 2010]. The close relation between phonological processing and arithmetic fact retrieval has been also confirmed in neuroimaging studies [Arsalidou, Taylor, 2011; Simon et al., 2002; Prado et al., 2011]. The predictive role of phonological ability in math has been also demonstrated in the investigation of children with dyscalculia who have the deficit of fact retrieval and suffer from poor phonological ability [Robinson, Menchetti, Torgesen, 2002; DeSmedt, Boets, 2010].

However, some studies failed to find significant correlations between phonological abilities and math (e.g., [Passolunghi, Vercelloni, Schadee, 2007]). Phonological abilities were found to be unique predictors of reading performance but not mathematics [Bryant et al., 1990; Passolunghi, Vercelloni, Schadee, 2007]. Potentially, relations between phonological abilities and mathematics may be mediated by reading achievement. Sometimes children with language deficits demonstrated a low level of math competencies (e.g., [Koponen et al., 2006; Shin et al., 2013]). Jordan, Kaplan and Hanich [2002] found that children who started school with specific reading difficulties were at risk for developing secondary or associated mathematics difficulties.

Most previous studies were conducted on small samples, which could lead to biased estimations of the effects. There existed also a relative deficit of longitudinal studies which aimed to estimate both the effect of phonological ability and SES on math achievement. Our study aims to overcome these restrictions by using a large sample size and longitudinal two-wave design.

This study has three main goals. The first goal is to estimate the effect of phonological ability on math performance controlling for reading achievement and number identification skills in first year of schooling. The second goal of our study is to estimate the effect of SES on math achievement during the first year of schooling. The third aim of our study is to estimate how the effect of phonological ability varies for students with different SES. We hypothesize that children from high SES families tend to more actively involve verbal processing during math problem solving, hence the effect of phonological ability is larger for children with high SES.

## METHODOLOGY

Participants: This study was conducted in Russia (in the Tatar Republic) during the 2017–2018 academic year. The sample of 3450 first-graders was assessed at the beginning of the first grade (Time 1), and the second stage of the assessment was conducted at the end the first grade (Time 2). In the resulting sample only children who participated in both waves and whose parents gave information about SES were saved. The final sample consisted of 2948 first-graders (49% were girls). The mean age was 7.3 years at Time 1.

The parents of the respondents gave their informed consent before the start of the survey. The data were collected anonymously. The Institutional Review Board approved the study, and the data were collected according to the guidelines and principles for human research subjects.

**Instruments and procedure.** The Russian version of the iPIPS (international Performance Indicators in Primary Schools) instrument was used. iPIPS is based on the Performance Indicators in Primary Schools (PIPS) monitoring system that was developed by the Centre for Education and Monitoring at Durham University in the UK [Tymms, Merrell, Wildy, 2015]. The Russian version of the iPIPS assessment was developed from 2013 to 2015 [Ivanova et al., 2016].

This instrument assessed phonological ability, reading performance, and mathematics performance at the beginning and at the end of the first year of schooling. During two assessment cycles, the same sample of children were presented with the same set of items. In order to examine the achievement level of students over time, we applied the IRT technique, specifically, anchor item equating by performing the dichotomous Rasch model [Kolen, Brennan, 2004], using Winsteps software [Linacre, 2006]. Thus, the items were equated such that a continuous scale was used to assess student growth from Time 1 to Time 2. The reliability of the Russian versions of the mathematics and reading baseline and follow-up scales varies from 0.8 to 0.9 (Cronbach's alpha).

For estimation of math achievement word problem solving tasks and two-digit arithmetic tasks were used. In order to assess phonological ability rhyming tasks and word/pseudoword repetition tasks were used. The number-identification skills were assessed with tasks included single-, two- and three-digit numbers. The reading performance scale was constructed based on tasks that included letter recognition, word decoding and reading comprehension.

The information about family's SES was obtained from parents' questionnaires. We used two indicators of family SES: mother's education (1 — mother has higher education; 0 — mother has no higher education) and number of books at home (1 — family has more than 100 books at home; 0 — family has less than 100 books at home). We also included variable "language at home" (1 — family uses only Russian language at home; 0 — family uses both Russian and not Russian language at home).

**Statistical approach.** In order to answer our research questions, we used mixed-effect models in which Time 1 and Time 2 measures were considered as nested in individuals. These models allow us to estimate the effect of predictors on outcomes and time changes in outcome. Mixed-effect model also estimate between-individuals and within-individual variance (random effect). Using mixed-effect models enabled us to estimate the effects of both time-varying and time-invariant predictors, so we were able to estimate both the effect of phonological ability on math achievement and the effect of SES.

The analysis was performed using Stata 13.0 software [StataCorp., 2013].

## RESULTS

Results of mixed-effect analysis are shown at Table 1.

Table 1

**The results of mixed-effect analysis for math achievement as outcome**

| Variables | Model 1 | Model 2 | Model 3 | Model 4 |
|---|---|---|---|---|
| Constant | –1.85*** (0.07) | –1.85*** (0.08) | –1.83*** (0.07) | –1.95*** (0.08) |
| Time | 1.04*** (0.04) | 1.04*** (0.04) | 1.03*** (0.04) | 1.03*** (0.04) |
| Phonological ability | 0.19*** (0.02) | 0.19*** (0.02) | 0.18*** (0.02) | 0.27*** (0.03) |
| Reading achievement | 0.16*** (0.01) | 0.16*** (0.01) | 0.16*** (0.01) | 0.16*** (0.01) |
| Number identification | 0.19*** (0.01) | 0.19*** (0.01) | 0.19*** (0.01) | 0.19*** (0.01) |
| Mother has higher education | 0.30*** (0.05) | 0.30*** (0.05) | 0.30*** (0.05) | 0.30*** (0.05) |
| Number of books at home | 0.12 (0.08) | 0.11 (0.08) | –0.04 (0.10) | 0.12 (0.07) |
| Only Russian language at home | –0.06 (0.07) | –0.06 (0.07) | –0.06 (0.07) | 0.06 (0.08) |
| Gender (girl = 1) | –0.33*** (0.05) | –0.33*** (0.05) | –0.33*** (0.05) | –0.33*** (0.05) |
| *Interaction effect* | | | | |
| Mother's education*Phonology | | 0.002 (0.03) | | |
| Book at home*Phonology | | | 0.09* (0.04) | |
| Language at home*Phonology | | | | -0.09* (0.04) |
| *Random effect* | | | | |
| Between-individuals variance | 0.78 | 0.78 | 0.78 | 0.78 |
| Within-individuals variance | 2.00 | 2.00 | 2.00 | 2.00 |
| Log-likelihood | –11263.05 | –11263.04 | –11259.79 | –11259.99 |
| LR test (Δdf) | | 0.01 (1) | 6.51* (1) | 6.13* (1) |

*** p < 0.001, * p < 0.05

Results revealed that phonological ability had a positive effect on math achievement. There were significant difference in math achievement regarding mother's education: the students from families where mother had higher education had a higher math achievement.

Models with interaction revealed that mother's education did not moderate the effect of phonological ability on math achievement. Number of books at home significantly moderated the effect of phonological ability on math. The effect was higher for children with more than 100 books at home. Interaction between phonological ability and language at home was significant and negative. This indicated that the effect of phonological ability was lower for children from families that used only Russian language at home comparing to children from families with other language at home.

## DISCUSSION AND CONCLUSION

Our study had three main findings. First, we found that phonological ability affected math achievement even when reading achievement and early precursor of math achievement such as number identification skills were controlled for. Second, we found that children with higher educated mother had higher math achievement and larger growth in math.

Third, although number of books at home and language at home did not affect math achievement directly, they might modulate the relationship between phonological ability and math. Particularly, phonological ability had a higher effect on math achievement for children with higher number of books at home. During their development children from families with higher cultural capital might learn to better manipulate verbal representations in general and verbal representations of numerosities specifically comparing to lower SES children. As a consequence, high SES students may more often recruit phonological processing during math problem solving.

The results of our study may have several practical implications. First, our results demonstrate that phonological skills should be taken into account when planning interventions to improve mathematical achievement. Second, the effect of phonological ability varies depending on family SES, meaning that children with low SES were less likely to recruit phonological resources for math problem solving. So, in planning the training programmes for low SES children it could be beneficial to use less verbal instructions and more visual representations of math concepts and tasks. The presentation of math tasks in different formats, verbal and visual, can reduce SES-related difference in math, at least, in elementary school.

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# MATHEMATICS SELF-EFFICACY OF SECONDARY SCHOOL STUDENTS IN THE U.K.: THE ROLE OF PARENTAL SUPPORT AND PERCEIVED TEACHING PRACTICES

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*Drawing on data from a large study of secondary school students in the UK which looked into the relationship between teaching practices and students' mathematics learning outcomes, we investigate mathematics self-efficacy. We are particularly interested in parental support and perceived teaching practice and further explore how these relationships are affected when considering the individual characteristics of students and other affective variables. Our methodological approach includes a validation stage with the use of the Rasch model and a modelling stage with multiple linear regression. Results with the responses of 13643 11 to 16 years old students start illuminating the complex interrelationships of variables with MSE. We discuss these results in regards to their implications for mathematics education practice.*

## INTRODUCTION

In this study, we focus on the mathematics self-efficacy (MSE) of secondary school students in the UK. We focus on Mathematics, because of its importance in formal school curriculum in UK but also internationally. Its eminent significance for students' access to STEM (Science, Technology, Engineering and Mathematics) subjects is acknowledged, and hence for students' educational and socioeconomic life opportunities [Smith, 2017]. The study of students' mathematics dispositions is also vital because it may reveal key influences on their choices and decision-making and hence future engagement with STEM.

Previous studies had also identified a plethora of socio-cultural factors which are significant in shaping students' dispositions and choice-making in education in general, and in STEM subjects and mathematics in particular: gender, class, nationality, ethnicity, parental and peer affects, amongst others. Another important element involves the relationship of teaching practices with affective learning outcomes, such as dispositions, as in previous work where we showed how mathematics dispositions drop throughout secondary school and into post-compulsory education, and this drop is associated with traditional transmissionist teaching practices (e.g., [Pampaka et al., 2012; Pampaka, Williams, 2016]).

However, not many studies focus on the possible influences of the combination of parental/family and school/teaching practices, as well as the interactions between these factors and other attitudinal variables on MSE. So we aim to address this gap.

In *Shvarts A.* (Ed.). Proceedings of the PME and Yandex Russian conference:
Technology and Psychology for Mathematics Education. P. 153–161. Moscow, Russia: HSE Publishing House.

## THEORETICAL FRAMEWORK AND OVERVIEW OF LITERATURE

The theoretical perspective underlying this study, and the construction of the instrument for measuring MSE, is largely based on Bandura's work [1997] who initially contextualized the self-efficacy construct and its two dimensions, i.e. personal self-efficacy and outcome expectancy. Bandura [1997] defined self-efficacy as an individual's self-confidence of his/her capability to organise and control over his/her own actions, "*in order to produce given attainments*" [p. 3]. Perceived self-efficacy "*refers to belief in one's agentive capability*" [p. 382], or in other words, "*is a judgement of one's ability to organise and execute given types of performance…*" [p. 3]. From this it follows that self-efficacy should be contextualised and specific to a discipline (with measurement implications).

In the literature, studies for MSE can be separated into several main themes: applying MSE as a predictor of performance, course and career choices, developing measures for MSE, exploring the sources of MSE, and investigating cultural differences in MSE. In relation to the former, self-efficacy was found as a common non-cognitive predictor of academic performance in a meta-analysis focusing on psychological correlates of college students' academic achievement; in particular, performance self-efficacy was found to be the strongest predictor [Richardson, Abraham, Bond, 2012]. With a specific focus on mathematics, PISA results [OECD, 2013] show that secondary students' MSE is strongly associated with mathematics performance across OECD countries. We should note here, however, that most of these studies, including PISA, do not use contextualised items for measuring MSE, which is another gap we address in this study.

When exploring determinants of MSE, many studies focus on the four sources of self-efficacy (i.e. mastery experiences, vicarious learning, verbal persuasion, physiological and emotional state) put forward by Bandura [1997]. Various theories have emphasised the crucial role of parents on their children's learning. For example, the ecological theory of development [Bronfenbrenner, 1986] emphasises the dynamic interplay of immediate and distal social systems (e.g., family, school) in shaping children's development; the contextual system model [Pianta, Walsh, 1996] emphasised the interconnection of family and school system in shaping students' academic and school outcomes. Furthermore, the academic socialisation model [Taylor, Clayton, Rowley, 2004] combines the above frameworks, emphasising that children's achievement-related attitudes and behaviours can be shaped by parents via academic socialisation practices, such as parental involvement and school transition practices. Thus, in the present study, we are particularly interested in how parents' academic support practices (e.g., checking and helping homework, giving appropriate encouragement) as perceived by UK secondary students can affect their MSE.

Finally, the general argument for the importance of the quality of mathematics teaching on students' learning (outcomes) is well documented (e.g., [Askew et al., 1997]), and many have argued that formative assessment and more dialogical, connectionist pedagogies are required for conceptual, metacognitive, and affective outcomes (e.g.,

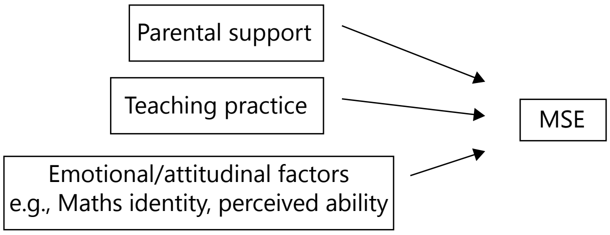


***Fig. 1.*** Proposed theoretical model

[Wilkins, Ma, 2003]). In our previous work we have also shown how more transmissionist (less connectionist) teaching practices are associated with decline in students' mathematics dispositions throughout secondary and in post compulsory education [Pampaka et al., 2012; Pampaka, Williams, 2016].

Some studies have further showed separately that parental involvement [Kung, Lee, 2016] and teachers' support [Chouinard, Karsenti, Roy, 2007] are significantly associated with students' MSE. However, there is a gap in evidence on combining these effects and looking at their interactions with students' emotional/attitudinal variables. Thus, in this study, we aim to shed more light on how factors at home (i.e. perceived parental academic support) and students' perception of the teaching practices they experience (transmissionist teaching) are related to their MSE, and how these variables may be interacting with one another in explaining MSE.

## METHODOLOGY

**Project Design and Sampling.** The results presented here are part of a larger ESRC (Economic Social Research Council) funded study of teaching and learning secondary mathematics in the UK (<www.teleprism.com>). The project was largely based on longitudinal surveys of students and their teachers in all 5 year groups of secondary education, and also included a qualitative case study element (not used here). The project was designed to capture the five years of students' progression in Secondary Education (Year 7 to 11, i.e. students aged 11 to 16) in about one year of data collection: From October 2011 to December 2012. The study employed a varied sampling frame to ensure maximum coverage of the schools of England, approaching over 2200 schools and established collaboration and responses from 40 of them (for more details on sampling see [Pampaka, Wo, 2014; Pampaka, Williams, 2016]). For this analysis we draw on data from the first data point which includes responses from 13 643 students who were in year 7 ($N = 3926$), year 8 ($N = 3039$), year 9 ($N = 2716$), year 10 ($N = 2127$) and year 11 ($N = 1835$).

**Instrumentation.** Data collection took place from October to December 2011 involving a questionnaire covering students' attitudes to mathematics, confidence at various mathematical topics (MSE), future aspirations, and their perceptions of the teaching they encounter. For the measurement of MSE we followed the same approach as described in previous work [Pampaka et al., 2011] with items appropriate for the involved year groups. Students were asked to report their confidence, using a 4-point scale (Not confident at all (1), Not very confident (2), Fairly Confident (3), and Very confident (4)) on solving each of the given mathematical tasks (but emphasising that they should not solve the problems). Some example tasks are provided in Fig. 2.

**Analytical Approach.** Our methodological approach includes a measures validation and a modelling stage. Validation is performed within the Rasch measurement framework, seeking validity evidence through various statistical indices (i.e. fit and category statistics, Differential Item Functioning and person-item maps). The method allows for the construction of continuous measures based on the responses to various ordinal items, if the data fit the model and fulfil its assumptions. This applies to all used measures in this study (as listed in Table 1).

Once the measures' validity is established, we use these new measures in further statistical modelling (General Linear Modelling in R; [Hutcheson, Sofroniou, 1999]) with models where students' MSE is the outcome, and we explore its associations with students' characteristics, parental academic support and their perception of transmissionist teaching (as per Fig. 1). We add such variables in the models as explanatory variables as shown with the regression equation below:

$Y \sim X1 + X2 + X3 + \ldots$, where $Y$ = MSE and $X$s are the explanatory variables. Regression coefficients and model fit statistics are used for model comparison.

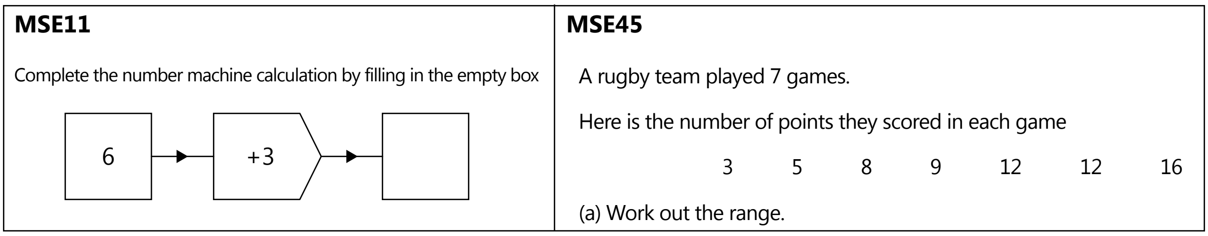


***Fig. 2.*** MSE items used in the surveys for Year 7 (left) and all year groups (right)

Table 1

**Summary of measures used with their construct definitions**

| Measure Name | 'Construct' description |
|---|---|
| Maths disposition | Measure related to expressions of behavioural intention for future engagement with mathematics (the higher the score the more disposed the student is towards further study or engagement with mathematics) |
| Maths identity | Measure related to students' self-concept about maths, constructed with items denoting mainly feelings and preferences towards maths (the higher the score the more positively the student relates/identifies with maths) |
| Maths self-efficacy | Confidence in solving mathematical problems |
| Perceived transmissionist teaching | The higher the score the more 'traditional' or teacher-centred the practices as reported by the students. |
| Perceived parental support | Students' perceived parental involvement/support |

## RESULTS

This section presents the results of the regression models. We had conducted several models in order to understand how the different variables of interest explain students' MSE. We adopted a stepwise procedure to input explanatory variables into models following theoretical assumptions (MSE as the outcome variable in all models):

Step 1: Gender and year group of students (Model 1) as the usual controls.

Step 2: Parental academic support (Model 2).

Step 3: Whether parents have been to higher education (Model 3).

Step 4: Students' perception of transmissionist teaching (Model 4).

Step 5: Mathematics identity (Model 5).

Step 6: Mathematics disposition (Model 6).

Step 7: Perception of mathematics ability (Model 7).

The results of this process with model fit statistics and the coefficients of each model are summarized in Table 2. Further explorations informed the need to include interacting variables as with Model 1, where we found a significant interaction between gender and year group on MSE, in addition to Model 0 (not presented here). Model 2 results show that parental academic support is significantly and positively associated with MSE, when accounting for the effect of gender and year group (with a sustained significant interaction effect). Similarly, Model 3 shows that both parental academic support and whether parents have been to HE are significantly associated with students' MSE, with a slight decrease in the effect of parental support; this suggests that part of the effect of parental academic support on MSE is mediated by whether parents have been to HE or not (this relationship was further explored in [Lei, Pampaka, in press]). Results in Model 4 show that students' perception of transmissionist teaching practices is significantly and negatively associated with students' MSE (when accounting for the rest of the variables). Models 5 to 7 progressively included other emotional aspects (dispositions, identity and perceived ability), which increased the variance explained with the models and showed the expected direction of significant effects. They unexpectedly, however, changed the direction of the effect of the other variables of interest: When maths identity was entered into Model 6, the previously positive association of parental academic support on MSE becomes negative; similarly, the previously negative association of perceived transmissionist teaching on MSE becomes positive. Moreover, these findings remain even after maths disposition and perception of maths ability were entered into the models (Model 6 and 7). Thus, we further analysed these patterns with Model 8 where we included the interaction between maths identity and perceived transmissionist teaching, as well as the interaction between maths identity and parental academic support to explore the potential moderating effects of these variables. The effect plots in Fig. 4 illuminate these complex patterns, while Fig. 3 shows the effects of parental HE background and the interaction between gender and year group on MSE in the same model. It is apparent from Fig. 4 that the effect of

Table 2

**Results of regression models for MSE**

| Exploratory variables | Model 1 | Model 2 | Model 3 | Model 4 | Model 5 | Model 6 | Model 7 | Model 8 |
|---|---|---|---|---|---|---|---|---|
| Constant | 1.37 (0.05) | 1.38 (0.05) | 1.80 (0.05) | 1.83 (0.05) | –0.84 (0.05) | 0.84 (0.05) | 0.10 (0.08) | 0.18 (0.08) |
| Gender (Ref: Male): | –0.33 (0.06)*** | –0.35 (0.06)*** | –0.32 (0.06)*** | –0.31 (0.06)*** | –0.02 (0.06)*** | 0.003 (0.05)*** | 0.03 (0.05)*** | 0.03 (0.05)*** |
| Year Group (Ref: year 7) | | | | | | | | |
| Year 8 | 0.12 (0.07) | 0.13 (0.07)* | 0.16 (0.07)* | 0.17 (0.07)* | 0.43 (0.06)*** | 0.44 (0.06)*** | 0.42 (0.06)*** | 0.42 (0.06)*** |
| Year 9 | 0.41 (0.07)*** | 0.43 (0.07)*** | 0.47 (0.07)*** | 0.48 (0.07)*** | 0.86 (0.06)*** | 0.87 (0.06)*** | 0.86 (0.06)*** | 0.86 (0.06)*** |
| Year 10 | 0.58 (0.07)*** | 0.61 (0.07)*** | 0.62 (0.08)*** | 0.64 (0.08)*** | 1.06 (0.07)*** | 1.06 (0.07)*** | 1.01 (0.07)*** | 1.00 (0.07)*** |
| Year 11 | 0.84 (0.08)*** | 0.89 (0.08)*** | 0.95 (0.08)*** | 0.98 (0.08)*** | 1.43 (0.07)*** | 1.45 (0.07)*** | 1.41 (0.07)*** | 1.40 (0.07)*** |
| Gender (Ref: Male): Year group (Ref: year 7) | | | | | | | | |
| Year 8 | 0.13 (0.09) | 0.15 (0.09) | 0.05 (0.09) | 0.04 (0.09) | 0.05 (0.08) | 0.03 (0.08) | 0.04 (0.08) | 0.04 (0.08) |
| Year 9 | –0.37 (0.09)*** | –0.35 (0.09)*** | –0.40 (0.10)*** | –0.39 (0.10)*** | –0.37 (0.09)*** | –0.37 (0.08)*** | –0.36 (0.08)*** | –0.35 (0.08)*** |
| Year 10 | –0.40 (0.10)*** | –0.38 (0.10)*** | –0.38 (0.10)*** | –0.37 (0.10)*** | –0.28 (0.09)** | –0.28 (0.09)* | –0.21 (0.09)* | –0.20 (0.09)* |
| Year 11 | –0.30 (0.11)** | –0.29 (0.11)** | –0.33 (0.11)** | –0.33 (0.11)** | –0.36 (0.09)*** | –0.39 (0.10)*** | –0.37 (0.09)*** | –0.36 (0.09)*** |
| Parental academic support | | 0.09 (0.02)*** | 0.06 (0.02)*** | 0.04 (0.09)** | –0.12 (0.02)*** | –0.13 (0.02)*** | –0.10 (0.02)*** | –0.09 (0.02)*** |
| Parent have been to HE (Ref: Been to HE) | | | | | | | | |
| Not been to HE | | | –0.59 (0.04)*** | –0.59 (0.04)*** | –0.35 (0.04)*** | –0.36 (0.04)*** | –0.37 (0.03)*** | –0.36 (0.03)*** |
| Don't know | | | –0.84 (0.04)*** | –0.84 (0.04)*** | –0.59 (0.04)*** | –0.59 (0.04)*** | –0.58 (0.03)*** | –0.58 (0.03)*** |
| Perceived Trans teaching | | | | –0.14 (0.04)*** | 0.14 (0.03)*** | 0.20 (0.03)*** | 0.16 (0.03)*** | 0.13 (0.03)*** |
| Maths identity | | | | | 1.32 (0.02)*** | 1.09 (0.03)*** | 0.76 (0.03)*** | 0.73 (0.03)*** |
| Maths disposition | | | | | | 0.17 (0.01)*** | 0.11 (0.01)*** | 0.11 (0.01)*** |
| Perceived maths ability (Ref: poor) | | | | | | | | |
| Average | | | | | | | 0.34 (0.07)*** | 0.30 (0.07)*** |
| Good | | | | | | | 0.83 (0.08) | 0.77 (0.08) |
| Excellent | | | | | | | 1.67 (0.09) | 1.62 (0.09) |
| Maths identity: Perceived Trans teaching | | | | | | | | 0.12 (0.04)** |
| Maths identity: Parental academic support | | | | | | | | –0.06 (0.02)*** |
| **Model Fit Statistic** | **Model 1** | **Model 2** | **Model 3** | **Model 4** | **Model 5** | **Model 6** | **Model 7** | **Model 8** |
| F (degree of freedom) | 50.16 (9,12721) | 48.44 (10,12702) | 82.2 (12,11593) | 77.13 (13,11586) | 361.1 (14, 11585) | 352.6 (15,11584) | 350.1 (18,11540) | 317.3 (20,11538) |
| $R^2$ | 0.03427 | 0.03674 | 0.07842 | 0.07965 | 0.3038 | 0.3135 | 0.3532 | 0.3548 |
| Adjusted $R^2$ | 0.03359 | 0.03598 | 0.07746 | 0.07862 | 0.303 | 0.3126 | 0.3522 | 0.3537 |
| p–value | < 0.001*** | < 0.001*** | < 0.001*** | < 0.001*** | < 0.001*** | < 0.001*** | < 0.001*** | < 0.001*** |

**Note.** Model parameters on the top part of the table are presented as: coefficient (standard error) and significance ($p < 0.001$***; $p < 0.01$**; $p < 0.05$*).

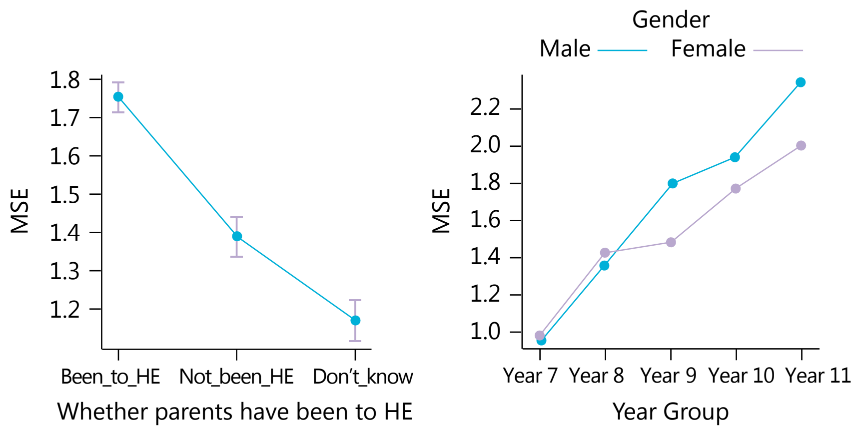


*Fig. 3.* Effect plots (Model 8) for MSE by: whether parents have been to HE (left), and the interaction of gender and year groups (right)

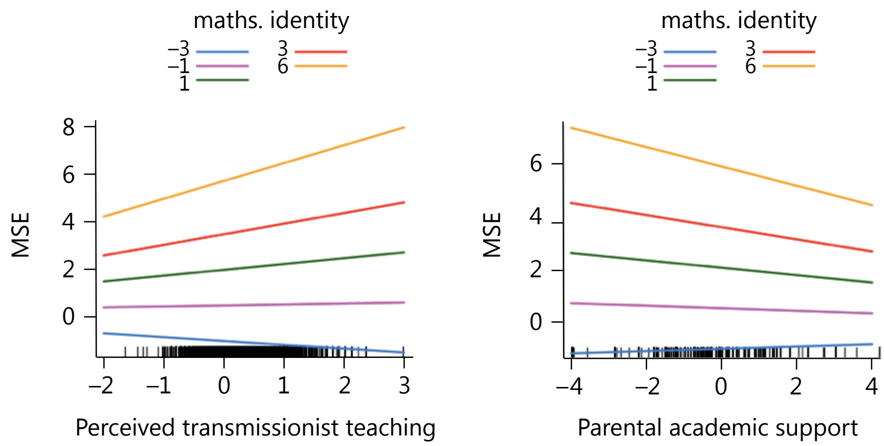


*Fig. 4.* Effect plots (Model 8) of the interactions between: maths identity and perceived transmissionist teaching (left), and maths identity and parental academic support (right)

parental support and perceived teaching practices on MSE are different for lower and higher levels of reported maths identity (lines for math identity denote various trends with low and high values).

## DISCUSSION AND CONCLUSION

In this study we have focused on using previously constructed measures of MSE, maths disposition and identity, parental academic support and perceived transmissionist teaching in regression models for MSE (as the outcome). Results provided evidence that social contextual factors at home (e.g., parental academic support) and school (students' perception of transmissionist teaching) are indeed significant correlates of secondary students' MSE, when we accounted for the effects of students' individual characteristics and other emotional variables.

Our findings suggest that the effects of both parental academic support and transmissionist teaching practice vary depending on students' maths identity. That is, for students who identify more with mathematics, parental support seems to have a negative effect on their MSE. This is consistent with a previous meta-analysis supporting that adolescence processed with dramatic cognitive and self development as an autonomous/efficacious individual needs less direct parental involvement such as homework helping [Hill, Tyson, 2009]. In addition, more transmisionist teaching practice at schools can also help enhance the MSE of these students with high math identity perhaps because it reinforces their strengths and abilities. In contrast, for students who believe that they are not that good at mathematics, more parental academic support and less transmissionist teaching practice at schools can help increase their MSE. Such findings can help teachers and parents adopt approaches with students according to their emotional states for better experiences. Our findings while revealing part of the complexity in these associations also point to the need for further research on MSE and its association with social and contextual factors, especially in light with international comparisons which consider such constructs, e.g. PISA. Investigating such differences in a robust manner points to the need to explore the universal utility of MSE across cultures.

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# CO-MEASURMENT AS THE SPINE OF ARITHMETIC CURRICULUM: TENTATIVE LEARNING PROGRESSION

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*Learning ratios is difficult. There is a strong evidence that math curricula fail to provide appropriate scaffolding for ratio-based concepts. One of the most psychologically adequate Math curricula is based on the idea that number is a ratio between the quantity and its measure. Yet it exploits only part-whole relations. As we revise the measurement paradigm of Davydov's curriculum, we see the potential to put the idea of measurement in the context of co-measurement, i.e. the coordinated measurement of two different values to control the corresponding intrinsic value. Expected benefits in terms of students' goal-oriented actions are discussed. A tentative learning progression based on co-measurement is presented.*

## INTRODUCTION

Proportional reasoning and ratio-concepts are known to cause tremendous difficulties to students from the moment they encounter these concepts in school and throughout all steps of education — both in Mathematics and in Natural Sciences [Nunes, Bryant, 2015; Lamon, 2007]. As literature analysis shows, this is a worldwide problem [Davydov, Tsvetkovich, 1991; Nunes, Bryant, 2015; Dole et al., 2012; Vysotskaya et al., 2017; 2018; Van Dooren, Lehtinen, Verschaffel, 2015].

Regular textbooks in many countries exploit very similar approaches to introduce fractions, proportions, percent, etc. [Alajmi, 2012]. Throughout the whole 6-years regular Math curriculum until the introduction of proportions, only part-whole relationship between values is considered in the classroom. In this aspect state Russian textbooks do not differ from other countries; neither do our students avoid typical mistakes (e.g., [Vysotskaya et al. 2017; 2018; Behr et al., 1984]. Thus, students always deal with one parameter only. When at the end of Arithmetic students come to proportions, they have no other way to treat this special case than to reduce the situation to part-whole task. Unfortunately, it is the same in the Developmental Instruction curriculum. Despite all the work done to deepen students' comprehension of values and numbers, the specific ratio-situation (two different values and their changes have to be considered



In *Shvarts A.* (Ed.). Proceedings of the PME and Yandex Russian conference: Technology and Psychology for Mathematics Education. P. 162–169. Moscow, Russia: HSE Publishing House.

to control the resulting third intensive magnitude formed by their ratio) is still a trap for students and a challenge for teachers.

Here are some important trends that researchers suggest to improve teaching:

- Students should learn to distinguish between different parameters beforehand (e.g., [Maclin et al., 1997]).
- Multiplicative thinking should replace additive thinking: 6 cups with 10 spoons of sugar is equally sweet as 3 cups with 5 spoons, because it is twice as much (the answer: 6 cups and 8 spoons, which is plus 3 units to each parameter, is incorrect). (e.g., [Hilton et al., 2016]).
- The part-whole relationship and counting, that stands behind the number-concept, is not enough for treating number as a ratio (e.g., [Tzur, 1999; Schmittau, 2003]). It is rather a web-like structure [Lamon, 2007].

Our previous findings show [Vysotskaya et al., 2017] that students make mistakes while treating numbers as wholes when the numbers represent ratios — an effect called whole number dominance [Behr et al., 1984]. Thus, these tasks demand a possibility to regard them as ratios — even when they are written as wholes. For example, 75 % means "75 for each 100" or, "five times larger" refers to the ratio: 5 : 1.

Thus, our major question is: "How can a child get such a kind of number-concept that can scaffold proportional reasoning and learning ratio-concepts in future, be it Math, Science, or every-day life? How can we provide it?"

In this article we present our local instruction theory that we have started this year in our experimental class. It is based on our previous work [Vysotskaya et al., 2016; 2017; 2018] with 5–6$^{th}$-graders and our main questions now are: will it work for primary school? What benefits will it bring? What problems thus can be solved? We will describe the innovations we made and discuss ways to test the effectiveness of our curriculum.

## THEORETICAL FRAMEWORK

As we implement Developmental Instruction [Davydov, 2008/1986], we analyze the psychological aspects of learning ratios: what actions can mediate acquisition of the concepts? Thus, we act as curriculum developers, as we devise local instruction theories and carry them out in our experimental classrooms to see whether the promotion is feasible. We strive to provide students with psychologically adequate learning environments (including models, situations, instruments, cultural templates, etc.) that will scaffold concept development through students' activity according to the tentative learning progression that we devised.

The fundamental idea of the Developmental Instruction Theory is: to build robust concepts, they should be extracted from student's own activity as a reflection of the guidelines he used. Thus, his action has to reconstruct meaningful, goal-oriented cultural activity — the one that exploited and gave birth to the concept. The role of curriculum designers is to devise learning environment to scaffold this activity.

## MEASUREMENT-BASED CURRICULUM

Taught since 1960s, Davydov's Math curriculum is built upon the action of measurement. The idea was to introduce the concept of number thus it could evolve into the concept of rational and real number from the start. Here is the sequence of problem-situations that introduces proportions and fractions [Gorbov, Mikulina, Saveljeva, 2002].

**Step 1. Magnitude.** Students compare objects based on various attributes; construct objects with a given attribute (length, area, weight, amount, volume, etc.).

**Step 2. Measure and Number.** Students use a measure as a mediator to reconstruct or compare magnitudes, in case there is no magnitude of the same size available. Students use Numbers as a mean to communicate the result of measurement.

> Task example: "How would you pour the same amount of juice for your friend, if your glasses are of different shape and you have only small similar cups?" (It is prohibited to pour juice between your glasses.)"

**Step 3. Composite unit.** Students use a composite unit, when they have to measure or construct a much bigger magnitude than the measure provided. This situation can develop in several directions: multiplication, place-value principle (when a system of measures is constructed), and fractions.

**Step 4. Fractions.** Fractions appear as a way to write down the result of measurement when a measure is bigger than a magnitude, or when it does not fit in the magnitude precisely (Fig. 1).

**Step 5. Proportions.** This step does not imply any actions with hands-on materials. A special table (see Table 1 below for a task example) is introduced to write down both

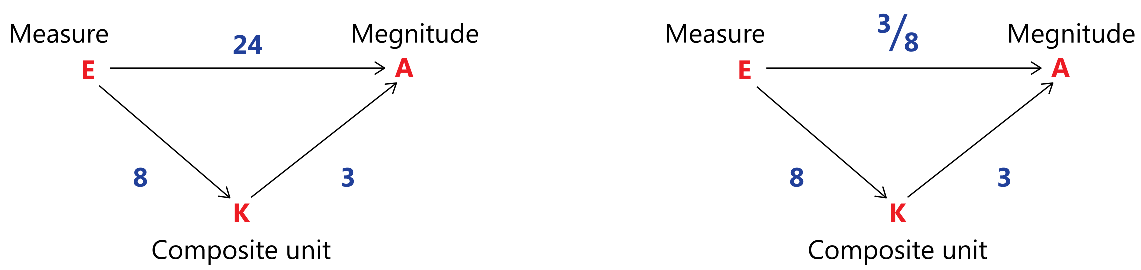



*Fig. 1.* Introducing fractions through the composite unit model

Table 1

**Special table to solve problems on proportions (from Gorbov's textbook)**

| | *Y* (distance) | *X* (time) | *K* (speed) |
|---|---|---|---|
| Pedestrian | | 50 min | 6 (km/h) |
| Bicycle | 63 (km) | 3 (h) | |
| Catching up | | ? | |

Task example: The pedestrian was walking 50 minutes, when the bike started after him. Pedestrian's speed is 6 km/h and bicycle goes at a constant speed, making 63 km in 3 hours. What time will it take the bicycle to overtake the pedestrian?

values and immediately move to the part-whole relations through corresponding calculation of an intrinsic value. Alongside formulas and rules are given.

Unfortunately, the sequence proves to be hard to follow: students' actions should be guided too closely by a teacher, and the only explanations behind the introduced instruments is "comfort" and "convenience". We considered such phenomena as a sign that students feel a lack of sense and meaning in the actions taught.

> For example, when the idea of measure is introduced, most children would prefer to use several different magnitudes that would fit instead of measuring it with only one small measure. The teacher has to insist on using only one magnitude, saying "it is convenient" which is doubtful in real-life situations, when a random measure is most likely not to fit precisely, and to compose magnitude of several different pieces is easier.

As we have observed, students have the same problems with proportions as they do in regular Math curriculum [Vysotskaya et al., 2017]. We considered measurement basis of Davydov's Math curriculum including construction of various magnitudes as a cornerstone to build a proper concept of number as a ratio. We suggest making one step ahead: putting measurement and construction in the context of ratio. Then the construction itself becomes justified by a supreme purpose — to maintain the ratio (see Fig. 2 and Fig. 3).

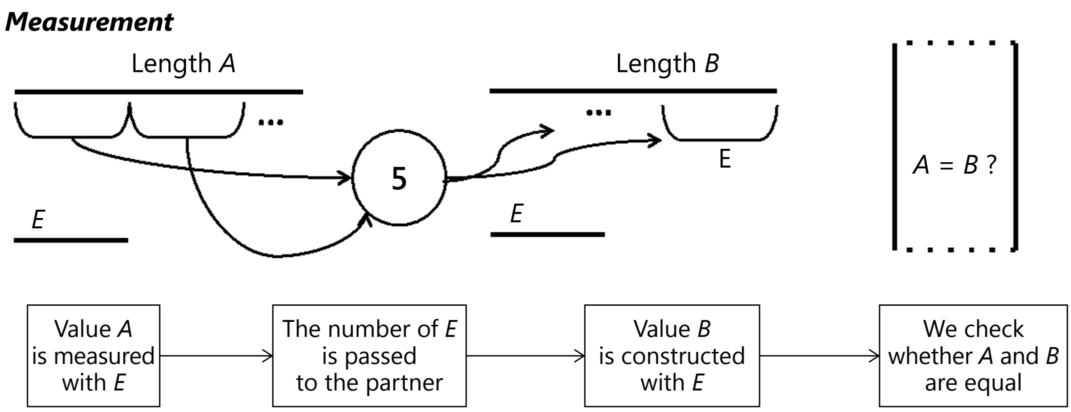



*Fig. 2.* Measurement situation

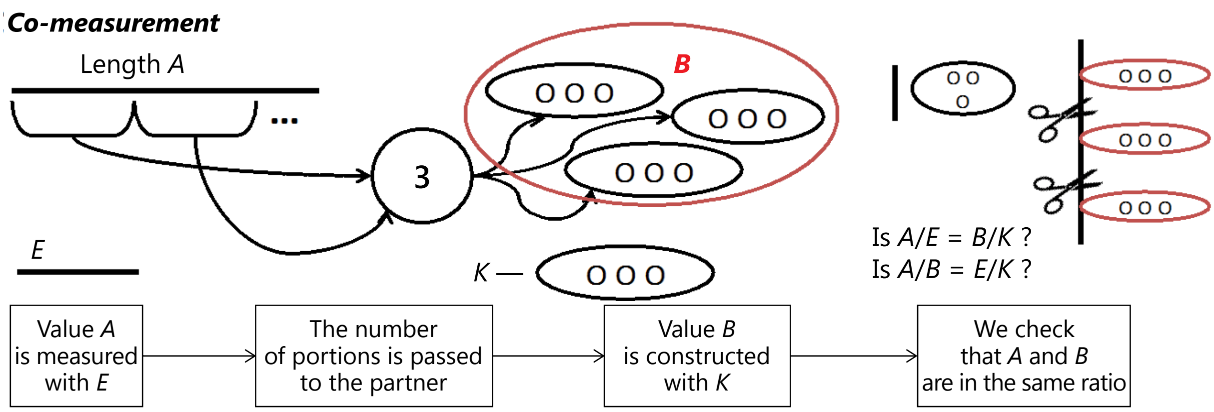



*Fig. 3.* Co-measurement situation

Let's consider the difference in the sense of students' actions. For example, it can be a ladder of the length enough to reach the roof. "I am measuring distance to the roof with some measure *E*, than I pass *E* to my partner and say I need a ladder with the length of 5 *E*. He constructs it (cuts out of paper) and we check, that it fits".

Or a co-measurement situation: "We are making button loops on a doll's dress. A string of "this" length (measure *E*) suits to make button loops for three buttons (measure *K*). If we have a string "this" long (length *A*), how many buttons will we need? I am measuring *A* with *E* to pass the number of portions to my partner. Than he constructs the amount of buttons (*B*) with his measure *K*. After that we can check, laying down our portions of buttons and cutting our string that we indeed have the exact amount of buttons for the buttonloops that we will make with our string.

Later on the ratio can be concentration or speed, but right from the start it is a sense of "fairness", that all children exploit, but they don't have instruments for that yet. It is our major assumption that to bring sense and meaning to what children do, we can exploit co-measurement context starting with the compound measure. It can make the "natural" proportional reasoning tangible and manageable for students. The compound measure is a cultural tool, and co-measurement is the cultural activity, that we want children to reconstruct.

## CO-MEASUREMENT-BASED CURRICULUM

Below we present a tentative learning progression built upon the action of co-measurement (Table 2). Illustrations are extracted from our previous researches on proportions [Vysotskaya et al., 2016; 2017; 2018]. This progression has been devised during these researches, but has never been integrated yet as a part of a basic Math curriculum of 1–6 grades. Outlined is only a part of curriculum that presents the development of the Number concept.

## ASSESSMENT AND DISCUSSION

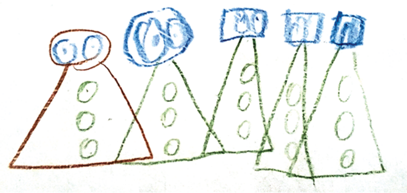
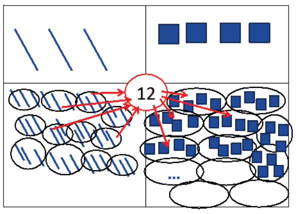
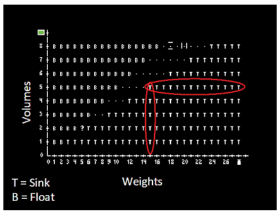
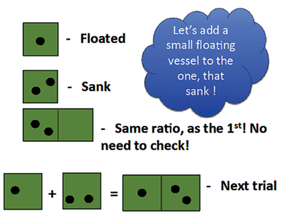
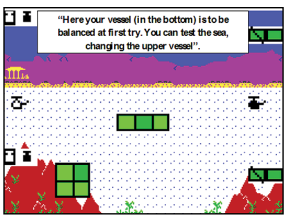
To empirically verify learning progression, that we have outlined, this year we have started 6-year longitude study with one class of 20 children (6–7 years old).

We rely on two indicators to prove effectiveness of the instruction: first and most important for us is the possibility to proceed with students according to the sequence we suggest. Every new step, that students successfully make, encourages us as educators and indicates that we are on the right way.

Second is a number of assessments that we have already tested on 5–6$^{th}$ graders in our previous research [Vysotskaya et al., 2016; 2018]. These assessments include not only paint-mixing and buoyancy problems that are familiar to students but also problems that present different contexts (such as efficiency, etc.) to test whether any transfer happens. See task examples below. We expect that our 3–4$^{th}$ graders will be at least as successful as our former students of middle school were [Vysotskaya et al., 2017; 2018].

Table 2

**Co-measurement based learning progression**

| Step | Students’ actions and ideas |
|---|---|
| 1. Making sets<br> | Students work in pairs — each in charge of his own value. They construct and measure “sets” jointly with a compound measure/ portion. One lays down his measure (3 green buttons) and the other student lays his part (2 blue buttons). Their aim is to have identical sets of buttons for all dolls. The idea is to get the whole picture of operating two parameters at the same time to maintain “similarity” or “fairness”<br>”Button-problem” |
| 2. Number of portions<br> | The situation evolves to the one where the number is introduced as a mean to coordinate the actions of both students. It is necessary to construct one magnitude in accordance to the other when the former one is *delayed or distanced* (for example, the amount of green buttons needed if certain amount of blue is already taken) |
| 3. The 3rd value<br> | The cultural contexts of intrinsic properties, such as: buoyancy/ density, concentration, cost, speed, are introduced to scaffold work between pairs. The aim now is to maintain some value of interest (shade of paint) by passing a recipe (1ml of water and 2 drops of ink) to the other pair of students. Thus, the third value is made tangible through its smallest “portion”<br><br>Photo: “Paint-mixing” [Vysotskaya et al., 2018] |
| 4. Changing the 3rd value<br> | Students (still working in pairs) are changing the value of interest (for example, making a vessel to sink or to float), coordinating the changes, which each of them makes with his parameter. The idea is that the 3rd parameter can be changed both ways: “if we cannot add weights to sink a vessel, we can take away volumes”<br><br>Photo: Graph from “Make it float!” If we know the ship of 5 volumes and 15 weights sinks, all the vessels with either more weights or less volumes can be marked as sinking [Vysotskaya et al., 2016] |
| 5. Working with objects<br> | Students are confronted with a new challenge: parameters cannot be operated separately (for example, you cannot take water or ink away from paint), and thus, they have to work with objects dealing with both parameters at the same time. Another example is adding “sinking” vessel to “floating” to make a “balanced” one<br><br>Figure: A fragment from “Make it float” module |
| 6. Coordinating two ratios<br> | Students have to refer to two ratios to control their objects. A measure for intrinsic property is derived from a compound measure: if the density equals 2.5, it means that the object has 5 weights for two volumes or 2.5 weighs for 1 volume. This appears as a special tool to manage situations where magnitudes are unknown or have some part of unknown that cannot be changed<br>Photo from “Make it float!”: the task is to control buoyancy of a vessel with volumes of unknown weight (light-green) |

Task examples:

1. A log is floating in the water. Then it is cut in two pieces. One of the pieces is 10 times heavier, than the other. Which piece will float and which will sink?

2. Students solved a Math test. Ira was assigned to do 15 problems and she successfully accomplished 10 of them. Jura was assigned to do 20 problems and she successfully accomplished 15 of them. Who performed better?

3. Kids decided to paint school stadium green, but they had only yellow (y) and blue (b) dyes. Each of them took some jars and mixed them in his own bucket as follows:

Kolya: y y y y b b b b b b

Olya: y y b b b

Tolya: y y y b b b b b b

Grisha: y y y b b b b b

Misha: y b b

Jura: y y b b b b

Ira: y y y y b b b b b

Jura started painting, but his dye was not enough. Does someone else have the same shade of green to continue painting? What is the recipe for Jura's paint?

To conclude, we believe, that students' well-known problems with fractions and proportions are rather learning impediments than the complexity of the topics themselves. If so, the issue could be resolved by revising the teaching approach to numbers at the very beginning, and Math curriculum in general. We believe that the design research approach within the Developmental Instruction framework can help make first steps towards this goal.

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# PREPARING ELEMENTARY SCHOOL STUDENT-TEACHERS TO TEACH GEOMETRY WITH GEOGEBRA: COMPARING OUTCOMES FOR RUSSIAN AND AMERICAN PARTICIPANTS

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*This study analysed experiences of elementary school student-teachers participating in a semester-long international collaborative project between Russian and US public universities to integrate GeoGebra into teaching geometry. The analysis of test results showed that Russian student-teachers outperformed Americans in basic geometry knowledge. Survey analysis showed that while American student-teachers demonstrated higher anxiety towards mathematics, they showed significantly stronger beliefs about the value of mathematics than their Russian counter-parts. In addition, American student-teachers demonstrated more positive attitudes towards using computers in teaching. Analysis of lesson videos suggested that participants from both countries developed Technological Pedagogical Content Knowledge (TPACK) at different levels with similar distributions; however, Russian participants mostly used whole class instruction with technology while American participants employed small group instruction with technology.*

## INTRODUCTION

US education system is well known for inquiry-based approaches to teaching and for using technology for student explorations and learning. Russian education system is well known for its rigorous approach to mathematics content starting as early as elementary school, and for strong foundations in mathematics teaching methods. The goals of this project were 1) to combine expertise of university faculty from both countries in order to develop and deliver high quality professional development for elementary school student-teachers to teach geometry with technology, and 2) to compare US and Russian student-teachers' mathematics content knowledge, attitudes towards mathematics, attitudes towards computers, as well as levels of Technological Pedagogical Content Knowledge (TPACK) developed as a result of this project.

In this project, the researchers developed short-term professional development (PD) course on integrating GeoGebra™ into teaching geometry in elementary school. The project specifically focused on geometry knowledge of pre-service teachers and their preparedness to teach geometry. The geometry is a discipline of mathematics where foundations for logical and algorithmic thinking, spatial reasoning, and mathematical literacy can be developed. Consistent introduction of geometry tasks into teach-



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ing mathematics starting at young age results in development of children's geometric thinking, increases their motivation and interest in learning geometry, and leads to deeper understanding of geometric concepts. On the other hand, lack of understanding of definitions and properties of mathematical objects, as well as inability to correctly interpret geometric images leads to misconceptions that children commonly have in elementary school.

This course was taught to a small group of US and Russian elementary school student-teachers using Skype as a platform for synchronous team-teaching. As part of this course student-teachers developed their own geometry lessons with GeoGebra and taught these lessons during student teaching practicum.

## BACKGROUND

### *Mathematics Content Knowledge (MCK)*

The limited mathematical knowledge of elementary school teachers is an international concern. The areas of mathematical difficulties have been well documented, which has led to many universities instituting testing requirements to ensure that preservice teachers have appropriate knowledge of primary school mathematics [Meaney, Lange, 2012; Swars et al., 2007]. Research on mathematics content knowledge (MCK) of elementary school teachers indicates that many teachers do not develop deep understanding of mathematics and that most of elementary school teachers in the United States do not have conceptual understanding of topics they teach [Bransford, Brown, Cocking, 2001]. Multiple studies revealed that pre-service teaches have misconceptions and limited content knowledge [Duatepe-Paksu, Iymen, Pakmak, 2012; Van Steenbrugge et al., 2014; Wilkie, 2014]. An attempt to learn from various international experiences to prepare elementary teachers to teach mathematics led to studies that analysed and compared MCK of elementary teachers in different countries [Schmidt, Houang, Cogan, 2012; Blömeke, Suhl, Döhrmann, 2013]. The Teacher Education and Development Study in Mathematics (TEDS-M) was the first large-scale assessment that compared the knowledge of primary and lower-secondary teachers in the 16 participating countries at the end of their training. The significant differences between countries in primary-level MCK showed that in some countries primary-level teachers lack some of the basic mathematics knowledge that is commonplace among future primary teachers in other countries [Tatto et al., 2012]. The findings of TEDS-M revealed that the preparation of elementary teachers to teach mathematics in the United States and Russian Federation is in the middle of the international distribution, along with other countries such as Germany and Norway. At primary level, the United States and Russian Federation had appropriate courses/content arrangements and, from the view of future teachers, met their needs. These two countries demonstrate good examples of programs with well-organized curricula. The Russian Federation prepares generalists at the primary level (up to the fourth grade) and specialists in mathematics at the upper primary and lower secondary levels. The United States was similar to the Russian Federation, the only exception being that there was a mix of generalists and specialists at the Grade 4–5 levels [Hsieh et al., 2011].

### Attitudes Towards Mathematics

The research into how students' attitudes affect learning of mathematics-related subjects has been one of the core areas of interest by mathematics educators. Improving the attitudes toward mathematics of pre-service elementary teachers is an important concern for university education courses in order to facilitate positive mathematics attitudes in future elementary students [Sherman, Christian, 1999]. Some pre-service elementary teachers have developed negative attitudes toward mathematics because of their weak mathematical background, their experiences with mathematics, lack of support from their families, and effect of their previous mathematics classes [Tsao, 2004]. Pre-service teachers experience higher levels of mathematics anxiety than other university undergraduate students, with the incidence of mathematics anxiety significantly higher among elementary education students [Swars et al., 2007].

### TPACK Framework

Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning. TPACK was developed as a conceptual framework for inclusion of technological knowledge into Shulman's [1986] framework of "Pedagogical Content Knowledge (PCK)". Mishra and Koehler [2006] expanded Shulman's framework by adding the knowledge of technology as a separate domain and defined TPACK as the nature of knowledge that is required by teachers to teach with technology while addressing the complex nature of teacher knowledge for specific subject areas (e.g. mathematics or science) and grade levels. TPACK is identified with knowledge that relies on the interconnection and intersection of content, pedagogy (teaching and student learning), and technology [Mishra, Koehler, 2006; Niess, 2005].

## METHODS

The study followed explanatory mixed methods research design and was guided by the following research questions:

1. Did the project have effect on pre-service elementary school teachers' geometry test scores, mathematics attitudes, and attitudes towards computers?
2. How do geometry test scores, mathematics attitudes, attitudes towards computers, and TPACK levels compare for Russian and American pre-service elementary school teachers?

### Participants

The project involved ten elementary school student-teachers from each university. All participants were females. Comparison of demographics between the groups from two countries revealed that US participants represented more diverse group (Table 1).

Majority of American student-teachers were the first to go to college in their families (8 out of 10), while only three out of ten Russian student-teachers indicated that their parents did not have a college degree.

Table 1

**Participants demographics**

| | | USA | Russia | Total |
|---|---|---|---|---|
| Age | Mean | 22.1 | 20.5 | 21.3 |
| | Min | 21 | 19 | 19 |
| | Max | 28 | 21 | 28 |
| Ethnicity | White | 5 | 10 | 15 |
| | Hispanic | 5 | 0 | 5 |
| Marital status | Single | 8 | 9 | 17 |
| | Married | 2 | 1 | 3 |
| Employment | Full-time | 1 | 0 | 1 |
| | Part-time | 5 | 2 | 7 |
| | Do not work | 4 | 8 | 12 |

The two universities have similar program plans to prepare elementary school teachers. Typical course of study includes 8 semesters, consists of liberal arts courses and education courses, and concludes with student teaching practicum. However, the content covered in mathematics courses is different for these two programs. Only half of mathematics courses in US program focus on foundations for the topics in elementary school curriculum and another half focuses on topics in finite mathematics, college algebra and trigonometry. On contrary, in Russian program all mathematics content courses focused on foundations for the topics in elementary school curriculum.

*Procedure*

Prior to the teaching practicum, the researchers developed 10-hour professional development course for student-teachers on how to use GeoGebra application to effectively teach geometry in elementary school. This course was team-taught over four consecutive Saturdays to all student-teachers using Skype as a platform for synchronous delivery and translation. As part of course requirements, in small collaborative groups, student-teachers developed lesson plans that integrated GeoGebra. There were three groups of student-teachers in each University according to grade level of assigned school classrooms. During the final Saturday session all groups micro-taught their lessons to others and engaged in discussions providing each other feedback. The project therefore produced six distinct lessons that were then taught in 20 elementary school classrooms in New York and Vladimir. The teaching occurred within 3 weeks after completion of the course. Lesson plans, videos of teaching, student artefacts, project evaluations, reflections about teaching, and journals were collected shortly after student-teachers taught their lessons. Focused interviews were held at the end of semester as well as 1–2 years after the project.

*Instruments*

Quantitative data were collected using the following instruments: (1) Geometry Test — 21 multiple-choice and short response questions measuring basic level of geometry knowledge, developed from released questions from 1992–2013 NAEP Grade 4 test

(<http://nces.ed.gov/nationsreportcard/>) — administered before and after the course, (2) Mathematics Attitudes Scale — 68-items Likert scale survey adapted from Fennema and Sherman (1976) to measure the attitudes and beliefs of student-teachers. The survey assessed confidence, anxiety, value, enjoyment, motivation, and teacher expectations as defined by Tapia and Marsh [2004] — administered before and after the course, (3) Pre-service Teacher's Attitudes toward Computers — 105 questions with 10 subscales: interest, comfort, accommodation, interaction, concern, utility, perception, absorption, significance, and adoption. The subscales were created from a collection of 14 various attitude-related scales, where each scale was tailored to focus on items most directly related to computers [Christensen, Knezek, 2000] — administered before and after the course. (4) TPACK Levels Rubric measures teachers' TPACK level based on the TPACK model for teacher growth through five progressive levels [Recognizing (1), Accepting (2), Adapting (3), Exploring (4), to Advancing (5)] [Lyublinskaya, Tournaki, 2012] — used to assess lesson plans and videos of taught lessons.

### *Quantitative Data Analysis and Results*

Analysis of geometry test scores revealed that there was no difference in participants' performance on geometry pre- and post-tests. Both groups demonstrated solid knowledge of elementary school geometry from the beginning of the project. Russian participants significantly outperformed Americans on both, pre- and post-tests ($p < 0.001$). However, the mean difference for Russian participants reduced from 4.6 on pre-test to 3.9 on post-test. American participants had wider distribution of scores compared to Russian PSTs. The difference in scores was not surprising as Russian students had stronger mathematics preparation than Americans.

Analysis of scores on Mathematics Attitudes Scale revealed that there was no difference in participants' performance on pre- and post-surveys. Strong correlations between pre- and post-scores ($r$-values ranging between 0.504 and 0.868, $p < 0.05$) were observed for all variables except enjoyment. Comparison of attitude scores between American and Russian participants indicated that there was a strong positive correlation between Russian participants' enjoyment of mathematics and motivation to pursue additional experiences in mathematics ($r = 0.839$, $p < 0.005$), as well as between perceived expectations they had from their mathematics teachers and confidence in their ability to successfully mathematics tasks ($r = 0.747$, $p < 0.05$). On contrary, there were no significant correlations between different categories of mathematics attitudes for Americans. Moreover, American participants had significantly higher anxiety towards mathematics ($t(18) = -3.080$, $p < 0.01$) and were significantly more concerned about their teacher's expectations of them ($t(18) = -2.212$, $p < 0.05$) than Russians. Nevertheless, Americans had significantly stronger beliefs about the value of mathematics than their Russian counterparts ($t(18) = -2.575$, $p > 0.05$). These results may be explained by the differences approaches to mathematics education in the two countries. The perception of teacher's expectations in Russian students seems to lead to more confidence, while in American students — to more anxiety. At the same time recent changes in accountability and high stakes tests in mathematics in USA could have led to higher values American student-teachers placed on mathematics compared to Russians.

Analysis of computer attitudes scores revealed that there was no significant difference in participants' performance on pre- and post-surveys. American student-teachers had much stronger attitudes than Russians on six out of ten scales of the instrument — accommodation, interaction, utility, perception, significant, and adoption ($p < 0.005$). The use of technology in Russian schools is limited by the health regulations. In elementary schools, students are allowed to be on computer 10–15 minutes per lesson depending on the their age, and that time is usually used by computer science teachers. It is not surprising that Russian student-teachers did not see as much value of computers for their classrooms as Americans.

Lesson plans and videos of lessons delivered by participants to elementary school students after completion of the PD course were analysed using TPACK levels rubric. The analysis revealed that development of TPACK was different for individual participants with similar patterns for both groups. The levels ranged from Recognizing (1) to Exploring (4) with American participants ($M = 2.9$) slightly outperforming Russians ($M = 2.6$).

### *Project Outcomes*

The value of the project for Russian and America student-teachers goes beyond the scores on the tests and surveys. As researchers complete the analysis of qualitative data that is currently in progress, they will be able to better evaluate the impact that this project had on teaching and learning. The reflections from participants in both countries indicated the value of the project for them:

> This project opened my eyes on how I can use technology in math- something that is completely overlooked. Before this project I always thought of math being best to learned through the use of pencil and paper. My outlook on how to teach math and use technology changed as I now feel more confident knowing the many ways I can use GeoGebra within my future math lessons (US participant).
>
> I had an interesting experience of communicating and exchanging ideas with international colleagues, learning how to work with GeoGebra, learning how to teach with GeoGebra (Russian participant).

Based on reflections and interviews, student-teachers gained confidence in using technology and in teaching mathematics in general. The researchers followed up with the participants of the project two years later to learn that most of them continued to use GeoGebra for teaching in their own classrooms:

> GeoGebra has given me the opportunity to make learning math a fun and interactive task rather than repeated, boring, individual computations. GeoGebra on both school and home devices ensures that students can learn here at school, and on their own time (US participant).
>
> We used GeoGebra with children outside of regular classes. Many children who did not understand the topic before now understand better how to find the area and perimeter of rectangles. They use it at home and they really like it (Russian participant).

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# LEARNING INTRANSITIVITY: FROM INTRANSITIVE GEOMETRICAL OBJECTS TO "RHIZOMATIC" INTRANSITIVITY

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*A new class of intransitive objects — geometrical and mathematical constructions forming intransitive cycles A > B > C > A — are presented. In contrast to the famous intransitive dice, lotteries, etc., they show deterministic (not probabilistic) intransitive relations. The simplest ones visualize intransitivity that can be understood at a qualitative level and does not require quantitative reasoning. They can be used as manipulatives for learning intransitivity. Classification of the types of situations in which the transitivity axiom does and does not work is presented. Four levels of complexity of intransitivity are introduced, from simple combinatorial intransitivity to a "rhizomatic" one. A possible version of the main educational message for students in teaching and learning transitivity-intransitivity is presented.*

## INTRODUCTION

In decision making, many researchers consider the transitivity axiom (if *A* > *B* and *B* > *C* then *A* > *C*, where ">" means "is preferable to") as a key component of rational thinking. The authors of the Comprehensive Assessment of Rational Thinking (CART) declare that if "you have violated the transitivity axiom, … you are not instrumentally rational. The content of A, B, and C do not matter to the axiom" (Five Minutes with Keith E. Stanovich, Richard F. West, and Maggie E. Toplak, 2016). "Any claim of empirical violations of transitivity by individual decision makers requires evidence beyond a reasonable doubt", according to Regenwetter et al. [2011]. These statements are contrary to numerous studies in an adjacent area — math research of various intransitive objects and the intransitive cycles between them. The intransitive cycle of superiority is characterized by such binary relations between *A*, *B*, and *C* that *A* is superior to *B*, *B* is superior to *C*, and *C* is superior to *A* (i.e., *A* > *B* > *C* > *A*, in contrast to transitive relations *A* > *B* > *C*). Various sets of intransitive objects (intransitive dice, lotteries, playing cards, etc.) have been invented and many studies of intransitive cycles emerging between such objects have been conducted (see e.g., [Conrey et al., 2016; Gardner, 1970; 1974; Grime, 2017; Pegg, 2005; Trybuła, 1961]). They show that, contrary to the CART authors' opinion, the content of *A*, *B*, and *C* does matter (some *A*, *B*, and *C* are in transitive relations of superiority, some others are in intransitive ones, and it depends on their content). While choosing between intransitive dice, one should prefer dice *A* to dice *B* in the pair *A*-*B*, *B* to *C* in pair *B*-*C*, and *C* to *A* in pair *A*-*C*. Currently, numerous educational videos can be found via Internet searches for the terms intransitive dice and non-transitive dice (e.g., [Lawler, 2017]). Various problems ranging in com-



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plexity are designed to promote intransitivity understanding in students of various ages and educational levels (from secondary to higher school settings) in different areas including not only math but also biology, sociology, etc. [Beardon, 1999/2011; Scheinerman, 2012; Stewart, 2010; Strogatz, 2015]. One should agree with T. Roberts, who writes:

> Transitivity and intransitivity are fascinating concepts that relate both to mathematics and to the real world we live in. A couple of lessons devoted to this topic are almost certain to interest and engage students of almost any age, as they seek to discover which relationships are transitive, and which are not, and further to try to discover any general rules that might distinguish between the two [Roberts, 2004].

The only clarification that can be made is that a couple of lessons may be enough for students' primary engagement in the topic, but hardly enough for its detailed analysis — see, for example, the analysis of intransitive dice by Fields medalist T. Gower [2017] in the pages of his Polymath project. If P. C. Fishburn's [1991] analogy between an advanced understanding of intransitivity and non-Euclidean geometry is right (we agree with it), the levels of complexity of the issue can very high. Yet the initial levels, even related to exact reasoning, can be (unexpectedly) simple. Let us consider this in more detail.

All of the intransitive math objects presented in math studies and in problems for students deal with numbers, mostly with probabilities which are not evident and must be counted. In this article we present geometrical and mechanical constructions in intransitive relations of superiority. From a mathematical view, it is a new class of intransitive objects. From an educational view, they can be considered in the framework of the Vygotskian theory of cultural tools (e.g., [Erickson, 1999]) including manipulatives. "Manipulatives are tools students use to support meaningful learning" and to "construct new insights" [Cramer, Wyberg, 2009]. Our manipulatives, intransitive geometrical and mechanical constructions, show deterministic (not probabilistic) intransitive relations in an evident way. The objects vary in complexity from very simple to advanced. The simplest ones demonstrate such intransitivity that can be understood at a qualitative level and does not require quantitative reasoning.

A note on terminology: in the math literature, the terms "intransitive" and "non-transitive" (e.g., "intransitive dice" and "non-transitive dice") are used as synonyms in spite of some difference between the logical terms "intransitive relation" and "non-transitive relation". In this article we will use the term "intransitive" as explicitly related to the concept of intransitive cycles.

## DESCRIPTION OF INTRANSITIVE GEOMETRICAL AND MECHANICAL CONSTRUCTIONS

All of the objects are designed as Condorcet-like compositions, in correspondence with the structure of the Condorcet paradox (or the voting paradox; [Beardon, 1999/2011]).

Our geometrical interpretation of the paradox is that we use chains of geometrical elements ordered like elements in the Condorcet paradox (originally, voters' preferences, but it does not matter here): *ABC*, *BCA*, *CAB*. One can see that the first element of any set moves to the last position in the next set and moves all the other elements one position to the left without changing the sequence.

As an example, let us consider such counter-intuitive objects as intransitive double gears (or friction wheels). The notation of elements will be the following: *X* is a larger gear (a larger wheel), *Y* is a smaller gear (a smaller wheel), and *Z* is an empty part of a shaft (without any gear or wheel on it).

Then, in correspondence with the Condorcet paradox:

- the first double-gear (*A*) will have the element sequence *X*, *Y*, *Z*;
- the second double-gear (*B*) will have the element sequence *Z*, *X*, *Y*; and
- the third double-gear (*C*) will have the element sequence *Y*, *Z*, *X*.

Figure 1(*c*) shows that, if joined in pairs, *A*'s rotational speed is higher than *B*'s in the pair *A*-*B*; the rotational speed of *B* is higher than that of *C* in the pair *B*-*C*; but the rotational speed of *C* is higher than the rotational speed of *A* in pair *A*-*C* [Poddiakov, 2010; Poddiakov, Valsiner, 2013].

The same principle of design is applied to other objects. Let us consider three geometrical blocks modeling tractors with different shapes of towing couplers (see Fig. 1*d*). Tractor *A* has a triangle lug at the front to be coupled as a trailer by another tractor, and a square hole from behind to couple another tractor as a trailer. Tractor *B* has a square lug at the front to be coupled as a trailer by another tractor, and a circle hole from behind to couple another tractor as a trailer. Tractor *C* has a circle lug at the front to be coupled as a trailer by another tractor, and a triangle hole from behind to couple another tractor as a trailer. A driver stands near the tractors. Which tractor should the driver choose as a leading one to sit in it if s/he has an aim to bring:

- — couple *A*-*B*;
- — couple *B*-*C*;
- — couple *A*-*C*

to a destination point?

One can see that the driver should choose *A* in couple *A*-*B*, *B* in couple *B*-*C*, and *C* in couple *A*-*C*. This model of intransitive relations does not require quantitative comparisons, counting, an understanding of probability, or other operations required to understand more complex intransitive objects like intransitive dice or playing cards. Distinction and comparison of geometrical shapes is all that is necessary here (besides an understanding of the task statement).

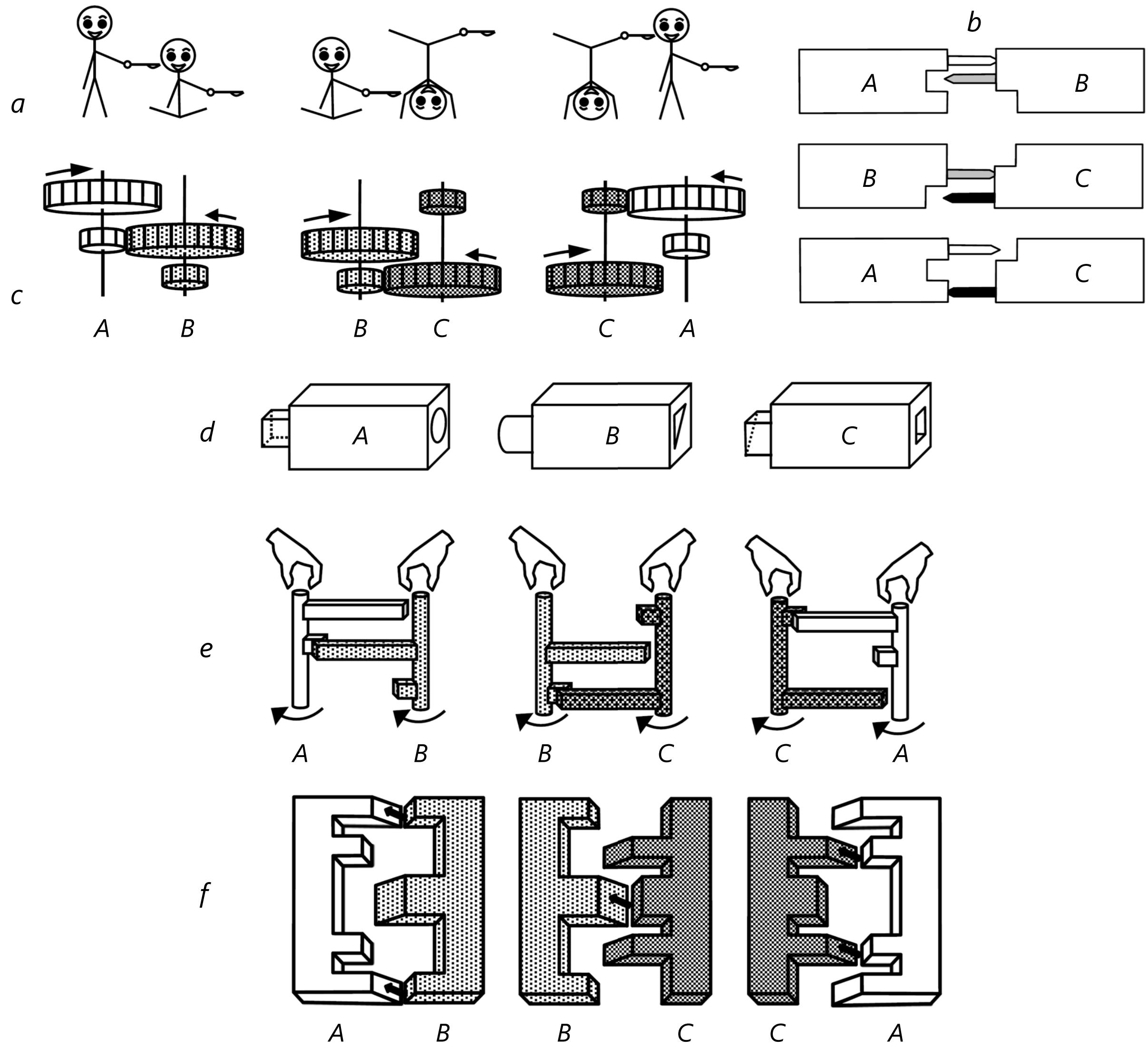


***Fig. 1.*** Examples of intransitive geometrical Condorcet-like compositions:
(*a*) toy Monkeys feeding one another;
(*b*) stylized plastic Mobile Assault Towers marking one another with inserted felt-tip pens;
(*c*) Intransitive Double Gears with intransitive speeds of rotation;
(*d*) stylized Tractors with intransitive towing couplers;
(*e*) Intransitive Double Levers (with the same rotation force applied to the shaft, Lever *A* will overpower Lever *B*, Lever *B* will overpower Lever *C* and Lever *C* will overpower Lever *A*);
(*f*) stylized Combs with Intransitive Ramps (Comb *A* can serve as a ramp for Comb *B* and lift it but not vice versa, Comb *B* can lift Comb *C* but not vice versa, and Comb *C* can lift Comb *A* but not vice versa).

## DISCUSSION

In spite of a rich tradition of math studies of various intransitive objects, there is no appropriate tradition of studies of understanding (misunderstanding) intransitive objects in cognitive and educational psychology. Owing to the brilliant Piagetian works in the area of cognitive and developmental psychology, the main trend is related to studies of abilities to make transitive inferences (if $A > B$ and $B > C$ then $A > C$) about transitive options (e.g., lengths of sticks) [Andrews, Halford 1998; Andrews, Hewitt-Stubbs, 2015; Camarena et al., 2018; Mou, Province, Luo, 2014; Shultz, Vogel, 2004]. Naturally, for such options violations of transitivity are a fallacy. Before math studies of intransitivity, this approach could have seemed universal. Even opponents of Piaget like Trabasso [Bryant, Trabasso, 1971] questioned not the status of transitivity as a normative rule and its violations as fallacies, but only age and conditions in which, for example, children already demonstrate that they can master transitivity and not violate it. Yet how can people understand mathematical intransitive objects? How are they solving intransitivity problems designed by math educators? More generally: how are abilities to reveal non-evident intransitive relations and to make inferences about objective intransitivity (e.g., about intransitivity of intransitive dice, athletes' teams, game strategies, etc.) developing in different domains (or as a general complex)? How are these abilities related to abilities to make "classical" transitive inferences? These questions have not yet been answered.

A possible theoretical framework which can include both 1) beliefs about the transitivity of superiority as an axiom with a ban on its violations and 2) beliefs about intransitivity as an objective property of complex (systems) interactions between multi-variable objects involves the distinction of four types of situations [Poddiakov, 2010].

(1) Relations of superiority between objects are objectively transitive (e.g., in case of three sticks), and a problem solver makes correct conclusions about their transitivity.

(2) Relations are objectively transitive, but a problem solver wrongly considers them as intransitive. Most studies are conducted in this paradigm.

(3) Relations of superiority between objects are objectively intransitive (e.g., relations between three or more sets, each of which contains three or more sticks having lengths equal to numbers on sides of intransitive dice, are intransitive — in contrast to the situation of a comparison of just three sticks), and a problem solver makes correct conclusions about their intransitivity.

(4) Relations of superiority between objects are objectively intransitive, but a problem solver wrongly considers them as transitive (e.g., because of taking the transitivity axiom for granted).

Here one can roughly distinguish between four levels of complexity of intransitive relations of superiority. This classification is not exhaustive and serves to mark some reference points.

(a) Simple combinatorial intransitivity between non-interacting objects (e.g., in intransitive dice sets, intransitive sets of sticks, etc.). Each object can be exactly described by a few parameters (like numbers on the sides of dice). The parameters are additive, without interactions: the sticks' intransitive sets do not interact with one another, only the sticks' lengths are compared, and comparisons are possible even without immediate touching. Information about the objects is complete.

(b) Interactive intransitivity without qualitative transformations of the objects participating in the intransitive relations. Information about the objects and their interactions is complete. An example is the intransitivity between interacting geometrical and mechanical objects described above. The intransitive gears are rotating at different speeds as a result of the intransitive interactions, but there are no qualitative transformations of the gears.

(c) Interactive intransitivity with qualitative transformations of the objects participating in the intransitive relations. Information about the objects and their interactions is complete. This intransitivity can be observed between pieces' positions in strategy games like chess. Position *A* for White is preferable to Position *B* for Black (i.e., when offered a choice, one should choose *A*), Position *B* for Black is preferable to Position *C* for White, which is preferable to Position *D* (Black) — but the latter is preferable to Position *A* (White) [Poddiakov, 2017]. The positions qualitatively transform after each move.

(d) Interactive "rhizomatic" (multiple, intertwining) intransitivity of superiority in real complex systems. A body of biological studies is devoted to the complex intransitive competitions of various species and individuals in ecological niches; for a review see Permogorskiy [2015]. Such competition transforms participants. Information about the participants, their features and interactions is incomplete for the participants and for observers (researchers) because of complexity and the multiplicity of interactions and permanent changes of the participants themselves and their strategies.

## CONCLUSION

Let us get back to the statement that "if you have violated the transitivity axiom, you are not instrumentally rational, and the content of *A*, *B*, and *C* do not matter" in an educational context. The main message for students in teaching and learning transitivity-intransitivity can be more multi-dimensional and not so straightforward. In complex and multi-variable situations, intransitive choices are perfectly rational because the choice options are in intransitive relations of superiority (like intransitive dice). That is, transitive choices of intransitive options are a fallacy. Here any attempts of linear, transitive ordering of options lead to a loss. By contrast, in situations of objective transitivity, any intransitive cycle of choices of options ends in a loss, and one must solve problems related to the building of linear hierarchies. Various educational tools can be used to support students' understanding of these different types of situations. The aim of our future research will be testing opportunities that use of some of the objects described above to support intransitivity understanding.

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# INVESTIGATING MATH TEACHERS' PROFESSIONAL COMPETENCIES

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*The article describes the results of a research of professional competencies of mathematics teachers. Problems of readiness for professional activities were identified. The components of the professional competence of a mathematics teacher are highlighted. We also describe levels of mastery and teaching tools.*

## INTRODUCTION

The changes taking place in the modern school assume the readiness of mathematics teachers to accept the challenges of modern times. This requires the identification of problems in their acquisition of professional competencies demanded by the modern education system. The concept of "competence" includes knowledge and understanding (theoretical knowledge in the academic field, the ability to know and understand), knowledge as an action (practical and operational application of knowledge in a particular situation), and knowledge as the basis of value relationships in professional activities (as an integral part of perception and life in social contact).

But usually teachers are trained and take advanced training courses without taking into account individual gaps and difficulties in their professional training. And more often, only their mathematical preparation is checked, the first component of competence, not the ability to solve methodological and pedagogical tasks, which is clearly not enough to identify the problems of a particular teacher in his professional training.

So the main goal of the research is to map current level of mathematics teachers' professional competencies. It is necessary and important to offer teachers adequate professional development program and thus to support their mathematics teaching in modern schools according to state educational standards.

## BACKGROUND

The study of readiness of mathematics teachers to carry out professional activities abroad was devoted to the research projects: TEDS-M (Martina Döhrmann, Gabriele Kaiser, Sigrid Blömeke, 2012), and NorBA, — a comparative study of mathematical education in the North Baltic countries. As part of this study, a questionnaire was developed that aimed at exploring the beliefs of primary school teachers about effective teaching and learning of mathematics [Lepik, Pipere, Hannula, 2011]. The main difference between this questionnaire and the TEDS-M questionnaire is in its orientation


In *Shvarts A.* (Ed.). Proceedings of the PME and Yandex Russian conference: Technology and Psychology for Mathematics Education. P. 186–193. Moscow, Russia: HSE Publishing House.

to teacher's practices (studies of beliefs related directly to the teaching activity), while TEDS-M examines beliefs about the nature of mathematics and the process of teaching mathematics. In addition to the above studies, the following can be highlighted: TALIS [Teaching and Learning International Survey, 2013] and SABER — Teachers. The TALIS study focused on the following main areas: teacher installation, rules and practices, teacher assessment and feedback, and school leadership; it did not set a special goal to assess the professional competence of mathematics teachers. The method of data collection is questionnaires in a paper or an online version. A review of research to identify various types of teacher knowledge that is necessary for teaching mathematics is made in the article by Chapman [2013]. In the study by Koponen et al. [2016], the relationship between teachers' knowledge and their education was explored. The researchers developed a 72-item survey to measure teacher educators' and graduated teachers' perceptions of what graduated teachers have learned well or poorly during teacher education. In the field of study of teachers' knowledge assessment, the Learning Mathematics for Teaching form (LMT-PR: [Learning Mathematics for Teaching, 2007]) is also of interest. This instrument was used in the study of Orrill and Cohen A.S. for teaching Proportional Reasoning and included 73 unique items. They were created to measure common content knowledge and specialized content knowledge [Orrill, Cohen, 2016]). Among Russian studies are research projects conducted in four countries. The researchers used the questionnaire of NorBA (Nordic-Baltic comparative research in mathematics education [Kardanova, Ponomareva, 2014]). In 2015, the approbation of tools for the study of subject competencies of teachers of the Russian language and mathematics was conducted at a conference "National Studies of the Quality of Education: Results and Prospects" [2015].

Thus, the fundamental difference in our study from previous ones is: 1) in the formulation of research objectives, which were identifying competencies necessary for the professional activities of the teacher, teachers' difficulties arising in the process of solving professional tasks building personalized advanced training routes focused on the identified "pain points", as well as on the teacher's "points of growth"; and 2) in the toolkit we used, which included a questionnaire to identify contextual information, the test aimed at diagnosing a basic level of competence, three methodological tasks, a professional task and a video recording of a lesson.

### *Research questions*

This study attempts to answer the following questions:

What are the levels of professional competencies of mathematics teachers as they deal with subject, methodological and professional problems?

How effective is the developed toolkit for identifying problems in teacher training?

## METHODOLOGY, METHOD AND TECHNIQUES

The assessment and measurement of a math teacher's knowledge is a complicated process, since professional teacher knowledge is multidimensional. It includes knowledge of the content of different sections of mathematics, pedagogical knowledge, as well as the skills to teach students (e.g., [Ball, Thames, Phelps, 2008; Manizade, Mason, 2011; Shulman, 1986; Silverman, Thompson, 2008]). In this case, it is more expedient to talk about competencies. While Shulman's [1986] notion of different types of knowledge permits us to identify a set of competencies that characterize professional activity, it also allows solving the problems arising in professional activities. Therefore, methodological and professional tasks are the main tool for assessing the professional competence of teachers.

The ability to solve professional problems offered in diagnostic work, and professional tasks within the framework of the lesson demonstrated, allows us to make a conclusion about the teacher's ability to solve professional problems arising unreal situations of professional pedagogical activity using knowledge, professional and life experience, and personal and professional values. In the conducted research, an understanding of the essence of the professional task already formed in pedagogical science was used (the Herzen State Pedagogical University of a Russia and NFPK, 2002)

We have identified three levels of the competence.

| Level of competence | Characterization of the level |
|---|---|
| Level I | The ability to solve problems of professional activity with a predetermined condition (without taking into account the variability of contexts arising in a real situation). |
| Level II | The ability to solve tasks of professional activity in a changing situation, reflecting various real (not planned) contexts (conditions), to propose and, accordingly, to choose various means of solving them. |
| Level III | The ability to solve problems of professional activity, acting in a situation of uncertainty, which involves not only taking into account the developing and increasingly complex contexts (conditions) of real activity, and the variability of the means of solution, but also the use of new resources to solve them. |

To determine the level of professional competence of mathematics teachers, ensuring the achievement of students' results in mathematics in accordance with the state standards, we developed a model that allows us to identify gaps in the professional training of a mathematics teacher. The model assumes realization of two components: performing extended diagnostic work and conducting lessons with video shooting.

We have identified three main components of professional competencies possessed by the teacher — subject, methodology, professional activity, — and each is based on the development of the previous one(s). At the same time, in each component there is a basic part, the mastering of which is necessary for the transition to the next component.

Diagnostic work contains three parts in accordance with the selected components.

When conducting the research, a demonstration version was available to the participants. We give examples of some tasks in each part.

*Variant 1*

**Part 1.** Mastery of mathematics teachers by subject-methodological competencies (includes two blocks). Means of verification: 12 tests. The tasks are designed in the unity of the informative and activity components.

Block 1. Mathematical preparation. To verify the subject preparation, teachers are offered tasks covering the main content of the academic subject, with several variants of each task at different levels of complexity.

Block 2. Subject-methodological competencies of math teachers. We control knowledge of the general and private methods of teaching mathematics.

Tasks.

1. In the geometry course, the concept "trapezoid" is inductively introduced. Establish a sequence of the teacher's methodological actions with this approach of introducing the concept.
   - recalling the use of the term "trapezoid" in everyday life;
   - actualization of students' ideas about the form of objects, in the naming of which the term "trapezoid" is used;
   - selection of the essential properties of a trapezoid as a geometric figure;
   - selection of the minimum set of properties of the trapezoid that are necessary and sufficient to distinguish it from a variety of other geometric objects;
   - formulation of the definition of a trapezoid.
2. In solving the problem: "Find a natural number $x$ at which $2\frac{5}{9} < \frac{x}{9} < 3\frac{7}{9}$" the student received an erroneous answer $x = 2\frac{6}{9}$. The reason for his error is: a) his ignorance of the order of numbers in the natural row (series) of numbers; b) his inability to convert a mixed number into an improper irregular fraction; c) misconceptions about the location of fractional numbers on the number axis; d) his inability to compare ordinary fractions; or e) a misunderstood task.

**Part 2.**

Block 3. Methodological problems. Give an example scenario for solving a methodological problem and comments on the evaluation of a detailed response.

Task statement

1. Evaluate the student's solution of the *C*4 problem student in accordance with the criteria, and justify your assessment.
2. Offer one auxiliary problem aimed at finding a solution to this problem.

*C*4. In the acute triangle *ABC* the height *BH* is drawn. From the point *H* the perpendiculars *HK* and *HM* to the sides *AB* and *BC* respectively are drawn.

- Prove that the triangle *MBK* is similar to the triangle *ABC*.
- Find the ratio of the area of the triangle *MBK* to the area of the quadrilateral *AKMC* if $BH = 3$, and the radius of the circle described around triangle *ABC* is 4.

The student's solution of this *C*4 problem, and a table with criteria for evaluating the response, are proposed.

**Part 3.** Solving a professional problem (for completion within three days).

Each professional problem contains: 1) the formulation of the conditions of a professional task (a description of the situation of the teacher's professional activities; a description of the possible context that reveals the degree of uncertainty of the professional situation; and the formulation of the problem in general terms, without regard to the context). For an example "Working with your students, you are faced with a pressing problem for today's teenagers — the low level of development of communication skills. You are planning a lesson in the form of a game. What elements of the subject environment can be used to promote the development of children's communication skills in the process of conducting this lesson in mathematics?" 2) a list of steps-tasks, the fulfilment of which should demonstrate to the expert the process of solving the problem by the teacher; this list of steps covers the field of evaluation of all aspects of the teacher's professional activities. The teacher chooses to solve one problem at his or her discretion.

All developed tasks meet the state educational standards, are built according to a single structure, so there is no typology of tasks in this part of the diagnostic work.

The study was conducted in 13 regions of Russia, and 2.253 mathematics teachers participated. As for the participants of this research, the requirements for the sample of research participants (mathematics teachers working in grades 5–11) were formulated:

take into account the age structure of the respondents (from 20 to 30, from 31 to 40, from 41 to 50, from 51 and older — in equal share participation);

take into account the qualifications of the respondents (qualification groups: teachers who do not have a category, teachers of the highest category, and teachers of the first category — in equal share participation).

According to the selected parameters of the research participants, an analysis of the dependence of the results of the diagnostic work on these parameters was conducted.

## RESULTS AND DISCUSSION

The results of the performing part 1 (test) of diagnostic work by teachers are presented in Fig. 1. The results allowed us to identify problems and directions of improving the first component of the professional competencies of mathematics teachers. The diagram allows making a preliminary conclusion about a lower level of methodological training compared to the subject level.

We also considered the dependence of the results of performing part 1 of the diagnostic work on the selected parameters of the research participants, in particular, on the experience of work as a mathematics teacher.

The results of the implementation of the methodological tasks (MT) (Part 2) are presented in Fig. 2.

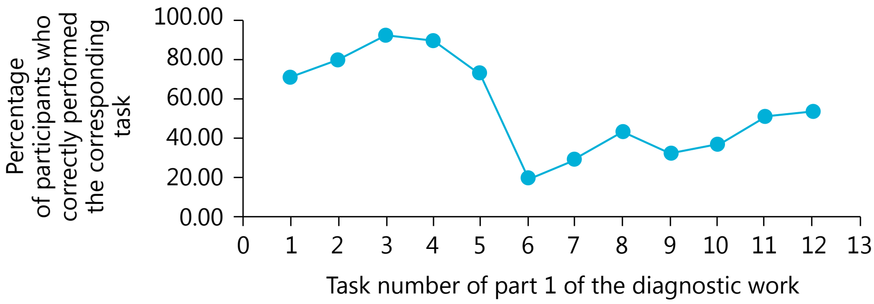


***Fig. 1.*** The results of solving tasks of part 1

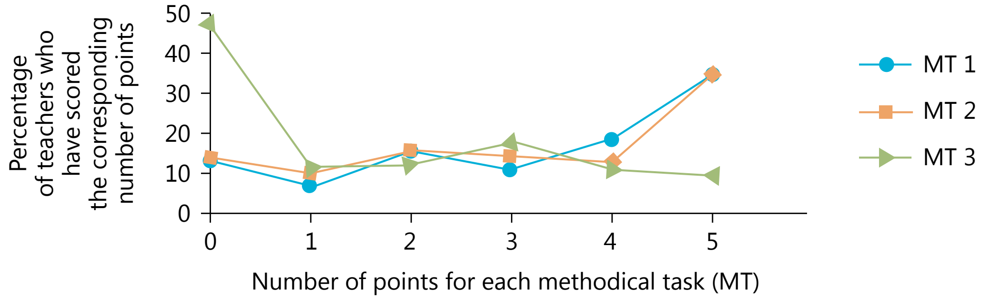


***Fig. 2.*** The results of solving methodological tasks 1–3

Analysis of the results of solving MT 1 by the research participants allows identifying the following problems:

1) Half of the participants (49%) failed to correctly solve the proposed mathematical problem of increased complexity;

2) Less than half of the participants (45%) were able to compose a sequence of questions and tasks for students who seek to find a solution to this problem, this suggests that not enough attention is paid to working with the task as a component of educational material.

Analysis of the results of MT 2 showed: 14% of the participants were able to point out all the errors and their causes, but could not show how to organize work to eliminate them, 44.7% were able to partially or fully propose an option to eliminate them.

The analysis of the results of MT 3 showed a low percentage of participants who successfully completed it (19.6%): the participants could not correctly evaluate the work of the students.

The step-by-step analysis of the presented solutions of general professional tasks (Part 3) showed: a) the majority of teachers showed a fairly high willingness to work with substantive content; b) the majority of teachers, instead of the context of the situation of professional activity, cited general considerations revealing the significance of the problem posed; c) some teachers misunderstand the "real professional context".

Table 1

**Correlating the number of points scored with the level of competence**

| Task or lesson | Maximum score | Level I (point) | Level II (point) | Level III (point) |
|---|---|---|---|---|
| Part 1 | 12 point | > 9 (not included in total points) | | |
| Part 2 | MT 1 (5 point) | 5 (MT 1 is solved completely) | | |
| | MT 2 (5 point) | ≥ 3 | 5 (MT 2 is solved completely) | |
| | MT 3 (5 point) | ≥ 3 | ≥ 4 | 5 |
| Part 3 | 40 point | ≥ 21 | 22–31 | 32–40 |
| Lesson | 36 point | 18–23 | 24–31 | 32–36 |
| Total | Sum of points | 50–59 | 60–76 | 79–91 |

The matrix of points scored according to the results of performing diagnostic work and conducting a lesson and relating the number of points scored from the level of competence is presented in Table 1.

## CONCLUSION

With the help of the developed tools the level of competencies of every mathematics teacher was determined on the basis of performing the diagnostic work (Parts 1, 2 and 3) by him and conducting lessons with video recordings. Approximately 24% of study participants showed professional competencies at the first level, 44% — at the second, 9% — at the third level and 23% did not pass the level assessment. Professional difficulties arising in the process of solving methodological and professional tasks were caused by the inability to solve practical tasks of building the educational process (organizing work with a mathematical task, searching its solution, evaluating students' activities, ...) and implementing an educational program requiring consideration of the real conditions of professional activity that support student progress. The identified problems will allow offering adequate professional development program to every teacher. The results will be discussed in detail at PME conference.

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# CONCENTRATED TEACHING AND ITS APPLICATIONS IN THE DESIGN OF THE UNIVERSITY MATHEMATICAL COURSE

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*The new method of the concentrated teaching of mathematics is proposed and considered. It includes requirements related both to the construction of the course as a whole (preparation, anticipation, repetition and deepening) and to the development of a separate topic (combination of functions, linkage). Requirements related to the arrangement of the content and to the impact on various channels of perception by students are also important. The implementation of this method in the course "Algebra and Number Theory" at the pedagogical university is described.*

## 1. INTRODUCTION

The theoretical framework of this paper is the theory of genetic teaching of mathematics and also the Vygotskian framework of educational psychology based on the concept of activity [Vygotsky, 1996]. Furthermore, various approaches using concentric ways of teaching are taken into account (e.g., spiral curriculum by [Bruner, 1960]).

The genetic approach to mathematics teaching [Safuanov, 2005; 2007] integrates the educational and philosophical ideas of Leibnitz [1880], Diesterweg [1850] a.o., the psychological discoveries of Piagetian and Vygotskian schools as well as rich experience of practice in mathematical education.

The principle of genetic approach in mathematics teaching requires that the method of teaching a subject should be based, as much as possible, on natural ways and methods of knowledge inherent in the science. The teaching should follow ways of the development of knowledge. That is why we say "genetic principle" and "genetic method".

German theologist and educator F.W. Lindner (1779–1851) was probably the first to use the genetic principle as a (historical genetic) method of teaching [Lindner, 1808; Schubring, 1978]. Prominent German educator F.W.A. Diesterweg (1790–1866) used the expression "genetic teaching" in his 1835 "Guide to the education of German teachers": "…The formal purpose requires genetic teaching of all subjects that admit such teaching because that is the way they have arisen or have entered the consciousness of the human …" [Diesterweg, 1850].

Genetic approach should be accompanied by several means constituting the method that we name "concentrated teaching". It uses ideas connected with concentric teaching and multiple effect on students.

The term "concentrism" (concentric teaching) is not new; it was introduced in works of 19th century German educators, and various authors have attached different



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meanings to it. The most common understanding is the following: the study subject is divided into parts, and the later ones repeat and deepen previous ones to some extent.

As a principle of epistemology, concentrism goes back to Hegel [1959] who understood the process of knowledge acquisition not as a series of concentric circles but rather as a spiral, the subsequent circles of which return to the previous one, but at a deeper level. Thus, concentrism as a principle of teaching is in a certain sense connected with the genetic approach (since, by Hegel's understanding of the process of knowledge, concentric learning follows the natural paths of this process).

The most famous version of concentrism in the modern theory of education is the "spiral program" by J. Bruner [1960; 1967]; see also [Buchter, 2014].

Wittmann [1997] wrote about ideas similar to concentrism. He argued that the principle of a spiral is one of the most important didactic principles. He Wittmann deduced two principles from the principle of the spiral construction of a curriculum:

> "1) The principle of anticipation.
>
> The study of the subject should not be postponed until it becomes possible to consider it in its final form; it is necessary to introduce it in a simplified form at earlier stages.
>
> 2) The principle of continuity.
>
> The choice and consideration of the topic in a certain place of the curriculum should not be made arbitrarily, but in such a way that it will be possible to develop the subject at a higher level. It is necessary to avoid those didactic decisions that may result later in the need of reconsidering in a different way" [p. 86].

## 2. THE METHOD OF THE CONCENTRATED TEACHING

We propose the new method of concentrated teaching that should not be confused with mere concentric way of teaching. Our method will consist of nine substantial elements and only first two of them relate to the concentrism.

Taking into account all above-mentioned ideas, we consider expedient the spiral arrangement of the subject, and, in our opinion, the requirement of the spiral arrangement of the subject can be concretized in the form of two important substantial elements:

1) preparation (anticipation) and 2) repetition at a deeper level and increase. Consider more concretely the first two elements of concentrated learning.

Preparation is an extremely important element both in teaching and in various kinds of art. This element is well known by professional writers and theatrical directors. For example, A.P. Chekhov wrote: "If in the first act a gun hangs on the wall, in the last it must shoot."

Diesterweg [1850] suggested rules close to these elements: "Distribute a material in such a manner that at each stage a pupil would be in a position to guess or definitely expect the next stage… Indicate at each stage some elements or parts of the following material and, not making essential breaks, cite certain elements from the future themes in order to excite inquisitiveness of pupils, not satisfying it, however, fully… Distribute and arrange a material in such way that (where possible) at the following stage during studying new things the previous elements were repeated.” He noticed that mathematics teaching could benefit from the concentric method.

Slightly modifying the classification of Zholkovsky and Shcheglov [1977], one can consider three types of preparation: a) the presentation; b) the anticipation; and c) the refusal.

As an example of the “refusal” element, a problematic way of studying a theme can be helpful when students are presented with the fact of an absence of the theory for the solution of the problem, and then the required is constructed in some way.

The most interesting and fruitful of these elements is, in our view, “anticipation”. Indications of this element can be found in many classics of mathematics education. The above-cited rules of Diesterweg directly state this requirement. The demand to teach pupils to guess is put forward by Polya [1965].

Concerning repetition, note that one can speak not only about the repetition of those or other elements of a material, but also about repetitions of the relations between objects at various levels in a mathematical discourse.

For example, the relations between objects in the theory of finite-dimensional vector spaces in many respects are repeated in the general theory of linear spaces. Likewise, the relations between objects in analytical geometry on a plane in many respects are repeated in analytical geometry in a space. And the relations between objects in the elementary number theory are in many respects repeated in the theory of polynomials. Even more interesting are the repetitions of relations between objects at the higher levels of abstraction, such as in abstract algebra. For example, categories and functors are in the relation similar to the relation between algebraic systems and homomorphisms; the composition of morphisms in a category is similar to the partial algebraic operation.

Preparation (anticipation) and repetition will be first two elements of the method of concentrated teaching. Further, we will add several elements connected with selection and arrangement of the subject and also with the multiplicity of means of influence on students. The following three important requirements serve as the manifestation of the psychological principles of integrity, continuity and enrichment.

For the efficiency of the concentric study, anticipation should be based on a very deep study of the fundamentals of a subject. The thorough and slow study of the foundations requires the economical and considerate selection of the most necessary material. It is possible to describe the requirement of a deep and strong study of the carefully selected foundations of a discipline as follows (we continue the numeration started at the beginning of this section:

3) The requirement of fundamentality.

Diesterweg wrote: “Delay mainly on studying the basics!

This rule refers to thoroughness and determines true success. Whoever does not lay the proper foundation, is condemned to fix the gaps forever, or must fear the destruction of the entire building. Any superficiality and frivolous attitude to the real fundamentals of the subject inevitably avenge themselves” [Diesterweg, 1850].

4) Combination of functions (the same methods, tasks and ideas are applied in different places of the course or are considered from different angles, and they carry different educational functions) and economy. The combination of functions is successfully used in art, when, for example, the same objects simultaneously carry several meanings: both substantial and compositional, and moreover, for example, symbolical.

This requirement leads to the careful selection of examples illustrating various concepts. Thus, among groups, one of the best examples is the symmetric group $S_3$, consisting of six different permutations of three elements (with the composition of permutations as a group operation). This group is an example of a finite group, an example of a non-Abelian group, an example of a group with elements that are not numbers. This group has several subgroups, including those that are not normal (that is, the right cosets of such subgroups do not coincide with the left ones). Finally, because it is composed of a small number of elements, this group is capable of illustrating complex concepts and is not very difficult to learn. Its properties can be viewed using the Cayley table (the “multiplication table” of a group). This example may be, and moreover should be, introduced in the course of algebra early on, since permutations are necessary (for example, in linear algebra for studying determinants).

5) Linkage. This method, developed in 19th century German didactics (Verzahnung), is mentioned in Shokhor-Trotsky’s book “The Didactics of Arithmetic” (1915). It consists of an interweaving of a subject from one topic into another. So, for example, when considering the continued fraction with all elements equal to 1, one can acquaint students with Fibonacci numbers and the Golden section. Considering arithmetic operations on congruence classes modulo $m > 1$, we can mention groups and rings. Note that this method is connected with anticipation.

Thus, the requirements of concentrated teaching are related both to the construction of the course as a whole (preparation, anticipation, repetition and deepening) and to the development of a separate topic (combination of functions, linkage).

The method of concentrated teaching is also connected with the multiplicity of means of influence on students.

Teaching a mathematical discipline is a complex, multi-component activity. A significant educational effect is achieved in the mathematical course (or in the textbook, and even at the micro level of a single lesson or lecture) by means of not a single tool, but using several or many actions directed to the same goal.

By analogy with the creation of works of art and literature (novels, plays, paintings, operas, etc.), we can speak here about composition. In this case, we refer to the composition of a mathematical course or textbook, and at a more local level, of a course section or separate lecture (or, e.g., a textbook chapter).

Note also important means related to the arrangement of the content:

6) Variation: the explanation of a theoretical concept by a series of different examples, or, more importantly, the consideration of all major aspects of the subject; such a comprehensive consideration was required by Diesterweg [1850, p. 266];

7) Splitting the content into smaller pieces: the requirement to break the material into small finished parts was also expressed by Diesterweg [p. 256];

8) Contrast: for example, the less studied method, consisting of the use of very fruitful questions of the form "How does Concept (the figure, etc.) A differ from Concept B?"

There are also several requirements related to the impact on various channels of perception of the subject by students during the process of teaching: concretization; enlargement; three ways of receiving information about the external (enactive, visual-pictorial and verbal-symbolical), according to Bruner [1967]; multiple representations; individual style; surprise and humor. We unite these requirements under one name:

9) Multiple effect.

## 3. APPLICATIONS IN THE MATHEMATICAL COURSE DESIGN AT THE UNIVERSITY

Thus, the principle of concentrated teaching includes many elements: anticipation, repetition, fundamentality, and a combination of functions, linkage, variation, splitting, contrast and other requirements related to multiple effect. As these elements (separately or together) have been effectively used in art, literature and also in mathematics education [Kosmodemyanskiy, 1975; Safuanov, 1995], we think the proposed method of concentrated teaching will is capable to improve mathematics teaching. We implemented it for several years in teaching various mathematical courses for future mathematics teachers.

As the pedagogical activity in its nature carries a creative character, it is important, that the future teachers would change their beliefs, gain a wider view of the nature of the purposes and methods of mathematics teaching. And for this purpose, alongside with the appropriate construction of the curricula of didactical courses (e.g. by inclusion in them of all modern progressive theories and methods of teaching), it is necessary to teach students of mathematical faculties of pedagogical institutes so that they would effectively acquire not only knowledge and ways of activity (i.e. ways of thinking and acquisition of new knowledge) in the field of mathematics, but also modern views of mathematical education. Extremely important for this purpose is to try to apply new theories of mathematics teaching in standard mathematical courses at universities and pedagogical institutes. Note that many researchers argue that the methods of teaching mathematical disciplines in pedagogical universities should serve for the students — future teachers as a source of didactical ideas, helping them to acquire modern didactical beliefs and skills, and in some sense as a sample for building their future professional activity. This idea ascends to Diesterweg: "The instructor of prospective teachers should not use any methods of teaching except those that can be applied in their (prospective teachers') future work at school" [Diesterweg, 1850].

Very promising seems the idea of intertwining didactical component with mathematical courses. This idea was suggested by several authors: Wittmann [1992], Reichel [1992] and earlier by Polya [1965]. These authors indicate that such integration of didactical elements into courses in higher mathematics should be implicit rather than explicit.

Thus, we aimed not only at improving mathematical knowledge of our students but also at changing their views of mathematics and mathematics and mathematics education.

Consider how these elements are used in the construction of the program of our course, "Algebra and Number Theory". Note that preparation and implementation of concrete lectures have been similar to those using genetic method [Safuanov, 2005].

Anticipation begins in the first hours of a course. At the introductory lecture, we give a brief survey of the whole course. The concepts of a matrix, a system of linear equations, and determinants of the 2nd and 3rd order are given. Thus, the basic themes of the "Linear algebra" unit are anticipated. In the initial lecture, students are first acquainted with the concept of a group, which anticipates the study of algebraic systems throughout almost the entire course. Further in the introduction, the elementary concepts of the theory of sets and logic notation are considered. Thus, the introduction anticipates more serious study of both elements of the theory of sets and mathematical logic in the future. Many concepts and objects introduced in the beginning of the course are later repeated in various situations. For example, throughout the whole course an important role is played by such concepts as mapping, equivalence relations and binary algebraic operations. Also repeated are the concepts of permutation, groups, rings, fields, integral domains and congruence classes. In our course, the means of combinations of functions is also used, which promotes economy in the selection of content. We try to use the same substantial and fruitful examples in various situations during the study of various concepts and ideas. The following examples (in brackets are the sections in which those examples occur) are used throughout the whole course: triangular numbers (proof by induction, number-theoretical functions and perfect numbers, Pascal's triangle); Fibonacci numbers (Pascal's triangle, continuous fractions and Golden section); congruence classes (partitions, equivalence classes, right and left cosets by a subgroup, cosets of a normal subgroup, of an ideal in a ring); groups, rings and fields of congruence classes (finite examples of these algebraic systems, ideals, quotient groups and quotient rings), permutations (transformations of finite sets, determinants, symmetrical polynomials), symmetrical groups, especially $S_3$ (finite groups, non-Abelian groups, subgroups, Cayley's theorem, normal subgroups and subgroups that are not normal, cosets of normal subgroups and of subgroups which are not normal); the roots of the unit (complex numbers, cyclical groups and subgroups, solving equations of the 3rd degree); matrices (non-Abelian groups, subgroups, normal subgroups, noncommutative rings, subrings, ideals, linear algebra), etc.

The next means, “linkage”, is connected to anticipation and also promotes multiple effect. We will show some cases of application of means linkage, when the content of one topic is intertwined with the content of other topics. So, during the study of number-theoretical functions, triangular and perfect numbers are considered. It anticipates some properties of Pascal’s triangle. During the study of congruence classes modulo $m > 1$, groups and rings are introduced, although the appropriate topics are considered later. Studying vector spaces with inner product and orthogonal systems of vectors in them, we take as an example the space of functions continuous on a segment with a system of orthogonal (trigonometrical) polynomials as a base, the expansion of functions in a Fourier series, leading up to practical applications such as in the study of waves and musical sounds. Considering continuous fractions, for the strengthening of the effect we consider also infinite continuous fractions and their applications for rational approximations, expansion of a square root into a continuous fraction, the Golden section, and also various practical applications of continuous fractions. Variation is exhibited when considering an object from different sides, illustrating a theoretical item on a series of various examples, and considering various proofs of the same statement.

Both in oral teaching and in writing textbooks, the individual style of the teacher (author) and also elements of surprise and humor are important.

## 4. CONCLUDING REMARKS

In this paper we have described how to apply a system of concentric teaching in the design of a mathematical discipline. So far we successfully realized this system (combining it with the genetic approach) in the Algebra and Number Theory course at the pedagogical university.

Our experience has shown that not only mathematical knowledge of students taught by new method has been improved but their mathematical beliefs seriously changed and became more progressive and appropriate for modern teaching.

Of course, the concentrated teaching can be applied for the teaching of other mathematical topics and mathematical disciplines.

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# MATHEMATICS LEARNING IN LANGUAGE INCLUSIVE CLASSROOMS*

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*In the U.S., English Language learners (ELs) score lower in elementary mathematics compared with their English Proficient peers (EPs). To provide information on strategies for enhancing learning opportunities for ELs and EPs, we document the efficacy of Learning Mathematics through Representations (LMR), a 19-lesson curriculum unit on integers and fractions. LMR features the number line as a representational context and the use of embodied representations to support students as they explore mathematical ideas, construct arguments, and elaborate explanations. The lessons were implemented in 11 LMR classrooms and 10 matched comparison classrooms. Both theory and empirical results support the value of LMR as a math intervention benefitting both EL and EP students in language inclusive classrooms.*

Students classified as English Learners (ELs) in the United States show lower test scores in mathematics relative to English Proficient (EP) students at fourth and eighth grades on both national assessments and state assessments [Hemphill, Vanneman, 2011]. The EL-EP achievement gap points to persistent inequities in learning opportunities for ELs, and educators are only beginning to understand how to address these concerns [Hakuta, Santos, 2012]. This paper reports on the efficacy of a 19-lesson experimental curriculum unit about hard-to-learn and hard-to-teach ideas in integers and fractions in language inclusive classrooms (classrooms containing both EL and EP students). The unit, *Learning Mathematics through Representations* (LMR), supports learning opportunities for both ELs and EPs through the use of the number line as a principal representation [Saxe et al., 2015*a*]. In this paper, we disaggregate prior analyses that documented the efficacy of LMR [Saxe, Diakow, Gearhart, 2013] to focus on the achievement of ELs as distinct from the achievement of EPs.

## *LMR*: EXPECTED SUPPORT FOR BOTH EL AND EP STUDENTS

The product of design-based research [Saxe et al., 2015*b*], LMR's use of the number line provides continuity of ideas, supporting students' as they build on insights from prior lessons in subsequent lessons (for related treatment of linear representations, see [Davydov, Tsvetkovich, 1991]). In the early integers lessons, students engage with activities about positive integers as units and multiunits on the number line; in later integers lessons, students extend these ideas to numbers to the left of zero on the number line. In the fractions lessons, activities begin with the idea of fractions as splitting integers into subunits (equal parts of a unit) on the number line; in later frac-

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tions lessons, students extend these ideas to multiplicative relations between fraction numerators and denominators.

LMR's design-based research approach has some important features. The LMR project began with grounded conjectures about the value of the number line for a strong instructional treatment of integers and fractions. Early *classroom studies* revealed the diversity of students' reasoning with the number line (e.g., [Saxe et al., 2013]). Subsequent *interview studies* probed patterns of student reasoning about integers and fractions in different representational contexts (e.g., [Ibid.]). *Tutorial studies* enabled the design and validation of productive learning trajectories when students are provided with visual, definitional, and embodied representational supports [Saxe et al., 2010]). Results from these studies led back to the classroom to partner with teachers in developing lesson sequences [Saxe et al., 2015*b*]. Finally, we conducted an efficacy study that provided quantitative evidence of LMR's effectiveness [Saxe, Diakow, Gearhart, 2013], and qualitative analyses of effective classroom practices [Saxe et al., 2015*a*].

Each LMR lesson consists of a 5-phase structure, as depicted in Fig. 1. The structure supports teachers' efforts to build upon student thinking. Lessons begin with non-routine *opening problems* that provide a focus for the opening discussion and serve as formative assessments. The task featured in Fig. 1, for example, presents an opening problem that contains a number line with only the numbers 6 and 7 labelled, and students are asked to label a third number at the leftmost position. In the *opening discussion* students explain their thinking about opening problems, and the teacher reviews a mathematical principle to support the resolution of conflicting ideas. For the lesson illustrated in Figure 1, the teacher introduces the definitions of interval as "the distance between any two numbers on the number line" and unit interval as "the distance from 0 to 1 or any distance of 1," and encourages actions on the line such as displacing a unit interval from one position to another. During *partner work*, students apply insights from the opening discussion as they solve problems that are sequenced in difficulty. In the *closing discussion*, the teacher encourages students to communicate ideas, and guides the class to resolve disagreements. The lesson concludes with *closing problems* that provide teachers an assessment of student thinking and progress.

The design of the LMR lesson structure and the 19-lesson sequence builds upon many mathematics educators' thinking about high quality mathematics education for both EP and EL students. We review principal dimensions below.

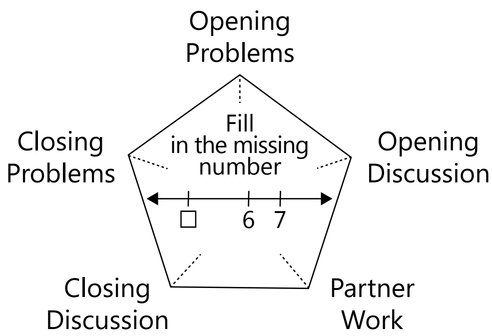


***Fig. 1.*** Five phase lesson structure with an example of a non-routine problem

### Mathematical communication, argumentation, and problem solving

Many mathematics educators argue that K-12 mathematics education should emphasize communication and problem solving (e.g., [National Council of Teachers of Mathematics, 2000]), and their argument is consistent with established theoretical treatments of cognitive development (e.g., [Sfard, 2008; Vygotsky, 1986]). The idea is that all students, including ELs, develop mathematical ability through participation in discourse practices such as presenting arguments, responding to others, and explaining solutions [Moschkovich, 2002]. The emphasis on communication and problem solving is a marked departure from common practices in classrooms serving ELs, especially low-income ELs of Latino descent; these classrooms often emphasize learning lower-level skills such as computation and rote memorization [Darling-Hammond, 2007]. In contrast, LMR professional development and lesson guides introduce instructional strategies that enhance students' opportunities for mathematical communication, as we noted in our review of Fig. 1.

### Resources that afford students' productive use of material, embodied, and linguistic representations

Many education scholars agree that high quality math lessons should encourage students to coordinate resources such as visual/physical materials, actions, and linguistic representations [Hakuta, Santos, 2012; Moschkovich, 2002]. Consistent with these views, LMR lessons engage students with visual and physical representations through use of the number line and Cuisenaire™ rods to represent linear distances. In early integers lessons, for example, students use rods to measure distances on number lines with only 0 labelled (e.g., locating the integer "3" as a distance of three red rods from 0). In so doing, students engage in the actions of placing, iterating, and partitioning intervals as they work to quantify linear distances. ELs efforts to make sense of the emergent representational environments rooted in their actions may be particularly useful for ELs' mathematical development [Bustamante, Travis, 1999; Piaget, 1970]. In later integers lessons, the rods become means for students to measure the distance between labelled points. In still later lessons, students investigate ideas like equivalent fractions by using rods to split a marked unit interval into "subunit" intervals (e.g., four subunits for fourths).

Linguistic representations, coordinated with other representational forms, are also important features of LMR lessons. LMR's core mathematical vocabulary, termed "number line principles/definitions," are progressively recorded (with diagrammatic support) on a classroom poster to provide all students access to foundational ideas, like unit, multiunit, and subunit. LMR's lesson guides encourage teachers to create opportunities for students to use definitions to resolve conflicts and to support argumentation (one recommended technique is to engage students in correcting the incorrect reasoning of a hypothetical person, and justifying their correction with reference to number line definitions [Saxe et al., 2015*a*].

### Productive norms and routines

Classroom norms that value participation and argumentation are regarded by many educators as a key feature of high-quality mathematics instruction (e.g., [Yackel, Cobb, 1996]). Ramirez and Bernard [1999] suggest that ELs' mathematical learning opportunities may suffer in lecture-based and textbook-centered classrooms (see also [Fuson, Smith, Lo Cicero, 1997]). LMR classrooms support the norms that students should (a) reference definitions to support argumentation, (b) offer conjectures and explanations during classroom discussion, and (c) carefully listen and perhaps respond to their peers' ideas in discussions and partner work.

## FINDINGS FROM THE CURRENT STUDY

Previously reported findings showed the efficacy of LMR lessons [Saxe, Diakow, Gearhart, 2013], but, to date, no analyses have been conducted on differential achievement gains for EP vs. EL students. For reasons reviewed above, we expected that LMR affords learning opportunities for ELs beyond the standard integers and fractions curriculum. We therefore conducted new analyses that included all participants in the Saxe, Diakow, Gearhart [Ibid.] original study: 571 4th and 5th grade students from three urban and suburban school districts in urban language inclusive classrooms. Forty-four ELs participated in LMR classrooms (11 classrooms), and 51 ELs participated in Comparison classrooms (10 classrooms). Assignment of classrooms to treatment condition followed a stratified random assignment procedure: Teachers were matched on three indicators: greatest terminal degree, years of teaching experience, and previous professional development, and then assigned to LMR and Comparison groups; assignment was modified so that teachers from different groups did not work at the same school (for more information on the procedure, see [Ibid.]).

For reasons described above, we expected the implementation of LMR would support learning opportunities for ELs as well as EPs. Accordingly, we assessed students' developing understanding of integers and fractions using a set of three linked tests. The assessment was administered on four occasions — pretest in September, interim test in October (LMR only), posttest in December, and final test (identical to the posttest) in May. A set of 18 common items, used in all three assessments, allowed us to link student scores from the different time points using item response models [Adams, Wilson, Wang, 1997]. Each assessment contained an additional 11–14 unique items that were intended to assess specific forms of learning: The unique items were easier at pretest and harder at final test, and the content emphasis shifted over time from integers to fractions. Item response modelling yielded scores in units of logits, an interval-level unit of measurement in which all assessments were linked to a common scale. The assessments contain items adapted from a wide range of sources, and item formats balanced number line representations vs. other representations (e.g., numbers only, area models).

We had four specific expectations about EL and EP learning gains. We articulate each below, and we report analyses that provide empirical support for each.

(*1*) *LMR efficacy For ELs.* We expected that ELs who participate in LMR will show greater gains in mathematics than ELs in Comparison classrooms. Figure 2 reveals, as expected, different patterns of growth for ELs in LMR and those in comparison classrooms:

ELs in LMR classrooms show steady growth over time, though less growth during the 5-month gap between posttest and final test; in contrast, ELs in the comparison classrooms showed less growth from pretest to posttest but sharper growth from posttest to final test. Statistical analysis confirmed trends observable in Fig. 2. Performance of the LMR and comparison EL classrooms was comparable at pretest ($p$ = 0.736), but at posttest the achievement of ELs who participated in LMR was 1.52 logits higher than the achievement of comparison group ELs ($ES$ = 1.14, $p$ < 0.001). Performance on the final test showed that LMR ELs maintained their advantage; LMR EL achievement was 0.90 logits higher than comparison EL achievement ($p$ = 0.011, $ES$ = 0.68). The growth spurt between post and final assessments for comparison ELs may be due to the observation that the comparison text spends greater time with integers and fractions in the Spring.

(2) *LMR efficacy for supporting equivalent learning opportunities for ELs and EPs.* We expected that the rates of learning for EL and EP students in LMR classrooms would be similar; thus, while we expected that EPs would outperform ELs at pretest (evidence of the pre-existing achievement gap), we would find that both groups would show strong gains, and thus the gap between ELs and EPs would not widen at interim, post, and final tests. Figure 3 confirms this expectation, revealing similarity in the growth trajectories for LMR ELs and EPs. Indeed, we found between-group differences in gains over the year were not statistically significant ($p$ = 0.937). The achievement differences were stable over time of testing: At pretest, the achievement gap between ELs and EPs was 0.47 logits ($p$ = 0.002), and the EL-EP gaps were stable on the interim (0.45 logits), post (0.58 logits) and final assessments (0.46 logits). We take these findings as evidence of strong learning gains for ELs and EPs in LMR classrooms and that LMR supported similar rates of growth; however, though the growth rates were similar, the gap between ELs and EPs achievement in LMR classrooms were not attenuated.

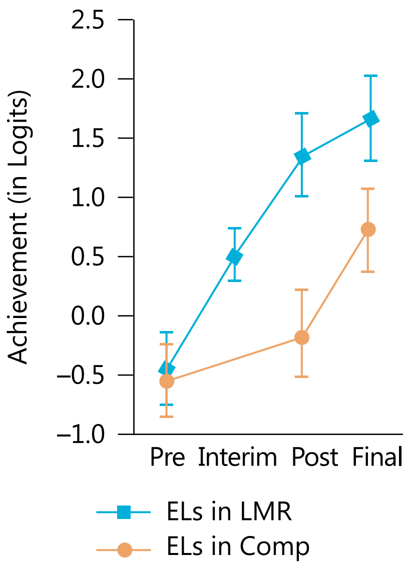


*Fig. 2.* Comparison of ELs in comparison classrooms

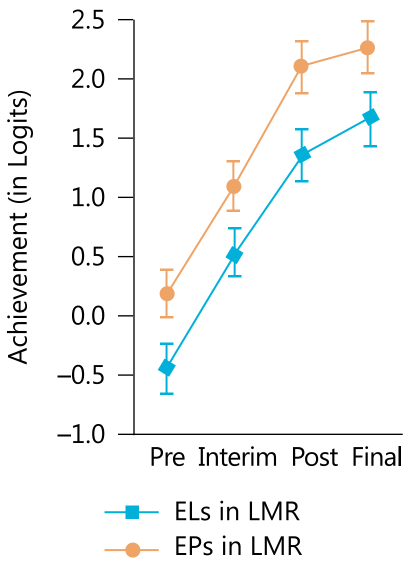


*Fig. 3.* Comparison of ELs in the LMR classrooms with EPs in LMR classrooms

(3) *LMR efficacy in attenuating the EL-EP mathematics achievement gap.* We expected that the use of the LMR curriculum would attenuate or eliminate the achievement gap between ELs and EPs when ELs in LMR classrooms were contrasted with EPs in comparison classrooms. Figure 4 reveals our findings. At pretest, there was a statistically significant pre-intervention EL-EP achievement gap estimated at 0.60 *SD*, with ELs demonstrating lower achievement than EP students ($p = 0.005$). In the first part of the academic year, the achievement of ELs in LMR grew sharply relative to the achievement of EP students in the comparison group. By the post assessment, the EL-EP achievement gap had reversed, with ELs in LMR showing greater estimated achievement than EP students in the comparison group ($p = 0.027$; $ES = 0.52$). The reverse achievement gap narrowed between posttest and final test, and, at the end of the school year, the final test showed no statistical difference between ELs in LMR and EP students in the comparison group ($p = 0.608$). Thus, the findings indicate that the EL-EP achievement gap in integers and fractions achievement was effectively eliminated.

(4) *LMR efficacy for non-number line assessment items.* As noted earlier, we included items on the linked tests that included and did not include number lines. The reason for inclusion of both item types was to determine whether student gains were limited to the central representational context of instruction, number lines. Because the curriculum was engineered to provide support for conceptual understanding, we expected that ELs would gain proficiency in solving both kinds of problems, though with some advantage for number line items, given the non-linguistic support that the lines could provide in an assessment context. Figure 5 reveals that our expectations were confirmed. The figure shows strong gains for ELs' and EPs' on both item types over the course of the school year. For EL students, the difference between number line

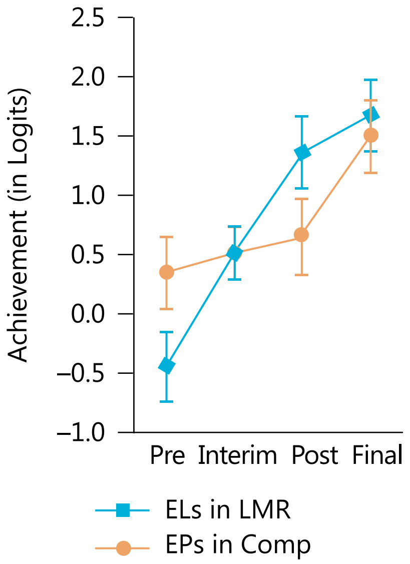


*Fig. 4.* Comparison of ELs in the LMR classrooms with EPs in comparison classrooms

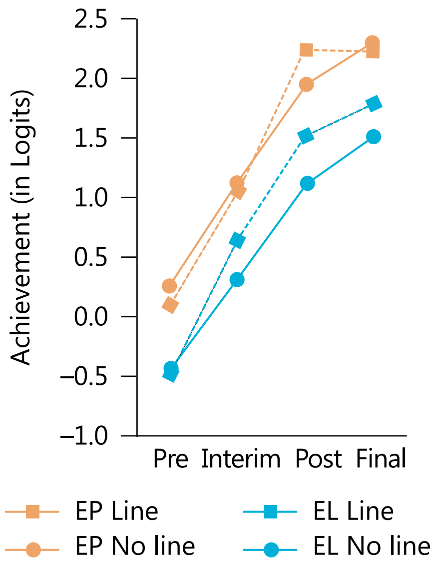


*Fig. 5.* Comparison of EL and EP students' performance on items containing and not containing number line representations who participated in LMR classrooms

items and no line items was not statistically significant at pretest ($p$ = 0.807), but ELs scored higher on the set of line items than the no line items on the interim ($p$ = 0.015, $ES$ = 0.25), post ($p$ = 0.003, $ES$ = 0.30), and final ($p$ = 0.042, $ES$ = 0.21) assessments.

## DISCUSSION

How did LMR support the mathematics strong learning gains of EL students while at the same time the strong gains of their EP peers? We expect that features of LMR afforded ELs better use of their partial mastery of the English language to build upon their prior mathematical understandings and intuitions in ways that are not be possible with more traditional curricula. In particular, the coordinated support for ELs use of visual representations (number lines), verbal and written representations (mathematical definitions supported by number line diagrams), and sensorimotor representations (manipulation of linear representations with Cuisenaire rods) is distinctive in the LMR lesson sequence. Further, the five-phase lessons afford teachers opportunities to assess and integrate student reasoning in discussions and to adapt their instruction as students with diverse understandings and linguistic proficiencies reason publicly with varied representational formats. The gains of EP students indicate that they reaped similar advantages in what generally are regarded as "hard-to-teach" and "hard-to-learn" mathematical ideas.

Of course, some of the distinctive LMR features may have played a stronger role than others, and these roles may have varied across classrooms and students. Our current study cannot identify specific features of LMR that supported gains in achievement for ELs or EPs. Indeed, we treated LMR and comparison groups as "packaged variables" since it was not possible to isolate distinctive features constitutive of LMR. However, in light of the documented gains of both ELs and EPs in LMR classrooms, what we can assert is that the emergent environments in LMR engaged ELs in ways that provided more equitable learning opportunities, though the particular interplay of LMR features in student activities may well have varied over students.

In closing, we regard our design-based research methods involving interview, tutorial, and classroom studies rooted in a developmental framework as potentially important resources for mathematics education researchers and professionals. We expect that the methodology would be useful for developing instructional approaches that engage all children with rich learning opportunities regardless on mathematical domain.

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# ORIENTATION TOWARDS GIVEN KNOWLEDGE: CONCEPTUAL BASIS AND RESEARCH PERSPECTIVES

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*There are two groups of factors which influence effective learning (EL): psychological and pedagogical. Based on the Activity Approach designed by Russian psychologists, we consider that the main predictor of EL is a child's goal-oriented activity, which should be organized. The teacher should choose actions, give students* the method*, encourage them to use it, and organize formation. What about a student who learns effectively? To what extent does he or she use these factors? We suppose that the important predictor of EL is orientation towards given knowledge (OGK) — the inclusion of the given knowledge into new actions. Our main aim is to discuss the nature of OGK, to suggest methods for it's diagnostics and to present a research project of the differences between* 4th graders *from different educational systems.*

## INTRODUCTION

Educational standards currently implemented in the Russian Federation assign a significant role to metacognitive educational outcomes, among which are universal learning skills [Federal'nyi…, 2014]. All of these skills are about what any student should do to learn effectively. The ability to learn (learning-to-learn, L2L) is one of the eight key competencies recommended by the European Parliament and the Council of the European Union in 2006 to implement the concept of "lifelong learning" [Steffens, 2015]. Our study of L2L is based on the Russian activity approach in educational psychology, designed by A.N. Leontyev, P.Ya. Galperin and V.V. Davydov, among others. According to this approach, the main factor of effective learning is the child's goal-oriented activity, which should be organized. So, the teacher should choose appropriate actions, give students the method for their actions, and encourage them to use such a method, in addition to organizing the action's formation stage-by-stage [Galperin, 1966]. The child's activity can be organized in various ways. **Orientation basis of action (OBA)** is the specific term of Russian Activity Approach. It is the student's method of action. There are 3 types of OBA [Ibid.]. Teacher, who forms 1st type of OBA, does not offer the student any method, only demonstrates the action and/or it's result. So students have to look for it independently. When the teacher forms 2nd or 3rd types of OBA he or she not only gives to students the method of action, but also encourages them to use it (in case if 2nd type its only a algorithm of specific action, in case of 3rd type its general method for some actions). According to Galperin, it is essential to form the 2nd and the 3rd types of OBA instead of the first one. But what about students, who used to



In *Shvarts A*. (Ed.). Proceedings of the PME and Yandex Russian conference: Technology and Psychology for Mathematics Education. P. 210–217. Moscow, Russia: HSE Publishing House.

be learning in a way of 2nd or even 3rd type of OBA? Is it possible to say that they can use given information for solving tasks instead of finding solution without it? We call such ability the orientation towards given knowledge (OGK). Why is it important for us to start the study with this particular component? What is the orientation towards given knowledge, in general: a property of individual actions or a holistic skill? The preliminary answers to these questions are given and the proposed research project is suggested.

*Orientation towards given knowledge: Concept and research*

OGK is the characteristics of student's OBA, it is the inclusion knowledge, given for learning, in the method of solving tasks and transformation these knowledge in that direction. Our position is very close to position of L. Radford [2013], who differentiates knowledge and knowing. According to Radford knowledge is a "historically and culturally codified fluid form of thinking and doing'', it exists independently from human's mind. But real knowledge ("knowing") exists only in human's actions. In the Russian literature we can find several studies in which the importance of forming an OGK is indicated. The first example of such research is represented by the early works of Galperin and Talyzina [Galperin, Talyzina, 1957], devoted to the concept's formation in the recognition action. This research showed the importance of using the given signs as early as the first realization of the action. Children who, as a result of the formation experiment, acquired the general recognition action, then considered any new concept from the point of view of the future action of the recognition — would the given signs allow recognition to take place? Talyzina distinguished 3 operations in the recognition action: a) identifying identification signs; b) the establishment of their presence (or absence) in each presented object; c) the conclusion about the belonging (or non-belonging) of an object to this class of objects in accordance with the logical rule of a recognition action. In this sense, the recognition action is an action that directly supports the formation of the OGK. Similar results were shown in a study of the formation of artificial concepts that was conducted on the material of figures from the method of Vygotsky and Sakharov [Teplenkaya, 1968]. In this work, preschoolers didn't demonstrate the stage of mental development described by Vygotsky, because they had not only all of the signs for a recognition action but also an ensured reliance on these signs in solving problems. Nevertheless, even in the Vygotsky-Sakharov methodology, the concepts were given (although in an implicit form), which lead us to the conclusion that the solution to these problems wasn't understood as an independent search for guidelines in a situation of direct interaction with objects. It is a cultural and social process in which an adult plays an important role. "The aim of learning as a specifically social process," writes Talyzina, "is not to induce the child to rediscover this long-ago-open system of signs, but using them as a model to '**look at**' **objects**... **from the side that is presented in this concept**" [Talyzina, 1975]. Mandatory OGK when performing actions is also supported in Galperin's theory by encouraging students to announce what he or she is doing (speech form of action). This makes sense because due to pronunciation it is easy to "not lose" the OBA when the action changes to a mental form or to make a generalization. It is shown that generalization is based on signs, which became part of the OBA [Talyzina, 1975]. Traditionally, teachers use the method when students receive signs to which they should orient (through definition), but real orientation on them is not provided. Thus, it makes sense to distinguish 1) the assignment of concepts and their signs in learning (explicitly or implicitly), and 2) ensuring orientation toward them. If the first is somehow present in school, the second is not practically provided. Concerning the 3rd type of OBA, teaching was organized in a way that the signs proposed in the OBA scheme were not just part of the method of problem solving, but the students also realized the need to use only these and not any other signs. This was due to the students' conscious assimilation of the function of these units in the whole system. The search for such a function was the purpose of a specific *orientation action*, which is called a **learning task** later (in the theory of Developmental education of Elkonin and Davydov). For example, in the studies of Aydarova [1968], learning to distinguish a morpheme was based not only on its formal features (signs of a prefix, suffix and so on), but on their "functional" meaning — the transmission of a specific message. Later these and other results were summarized by Galperin, who said that the main purpose of teacher who tries to use the 3rd type of OBA is transferring the function of a set of objects expressed by the concept [Galperin, 1966]. Authors of other experiments emphasized similar things: we need to give children the orientation meaning of features on which it is necessary to rely in order to properly perform an action [Venger, 1969; Podolsky, 1987].

In cognitive psychology, the problem of students' orientation toward given knowledge can be described in terms of the relation between declarative and procedural knowledge and their use in solving problems. There are many studies that show that even with clear procedural knowledge in the text, students have difficulties in using them for solving problems, which researchers attribute to a lack of certain metacognitive components in such students [Kendeou et al., 2014; Duke, Pearson, 2009]. A more complicated situation is the possibility of transforming declarative knowledge into procedural knowledge, the opportunity to see the actions as connected with concepts. Interestingly, the establishment of semantic links in the text is significantly improved in a situation where students are invited to ask questions about this text (independently or on the basis of a generalized list of questions) [King, 1994; Oleynikova, 2012]. In fact, asking questions prompts students to find actions for which the knowledge given in the text will become indicative, which contributes to understanding. Thus, students' lack of learning skills in working with texts (highlighting the main point, systematization, summarizing, etc.) can be understood as a lack of OGK, including the target attribution to the perception of knowledge as orientation and the ability to choose actions that are adequate to a given knowledge. Another area of research of similar phenomena is the study of conceptual changes [Posner et al., 1982; Vosniadou, 2013]. Resistance to conceptual changes can also be associated precisely with the inability of students to relate the knowledge gained to their own actions. Often this is associated with the formation of meta-learning — student's theories about their learning and approaches to learning [Entwistle, 2000]. Thus, with a superficial approach, the student perceives teaching as a memorization of facts and their reproduction, referring to the given knowledge as something that needs to be simply reproduced. With a deep approach, the text is understood from the point of view of the presence of connections in it — a certain system. Accordingly, the student has an opportunity to discover the actions and their elements in the text.

Thus, according to our assumptions, a student who knows how to learn effectively will a) rely on the given knowledge in the process of solving problems, b) in the process of working directly with the given knowledge (reading texts, working with lesson materials, listening), connect such knowledge with possible actions, the guidelines of which can be extracted from it. A complete orientation to a given knowledge, according to our assumption, should include both of these. For us also it is interesting to answer the question, if students don't use given knowledge, why they don't it? Is it because they can't or because they havn't such a goal? In other words — is OGK only ability or also an intention? Another question — is there any differences between OGK in different educational systems — Developmental Education (DE), designed by Elkonin and Davydov and Traditional Education (TE)? DE implies a change in the education' content, which means including to content general methods of action and theoretical concepts (it's close to third type of OBA). We use 2 groups of DE: DE-2 is different from DE-1 only in aspect of special attention in reading comprehension. So, our tasks was as follows: (1) Choosing the appropriate diagnostic methods for the OGK in direct (OGK-D) and indirect (OGK–I) instruction; 2) Comparison of OGK among students of different systems (TE, DE-1, DE-2).

### *Hypotheses*

1. OGK-D will be significantly better then OGL-I. 2. Reading comprehension will positively correlate to both OGK-D and OGK-I. 3. DE students (all 2 groups) will demonstrate higher levels of OGK then TE. 4. DE-1 students will demonstrate higher levels of OGK then DE-2 students

## METHOD

### *Participants*

Participants were 63 fourth graders drawn from one public school in Moscow, 33 boys and 30 girls (average age was 10.7). All participants were divided into 3 groups: 22 students educated in the TE, 20 students, educated in the DE-1 and 21 students, educated in the DE-2 (with special attention to formation reading skills).

### *Methods of OGK' diagnostics*

For our purpose we chose a concept's recognition tasks. The general principles of such diagnosing are described in the works of Talyzina [1975]. The participants were offered 2 groups of tasks; each group was built on the same structure: students were given definition of concept and then objects that need to be recognized (these objects were given verbally, through a description or with the help of pictures). For each concept, ten tasks are proposed: 1) 5 tasks with all necessary and sufficient conditions, 2) 1 task with all necessary and sufficient but also redundant conditions, 3) 1 task with some necessary conditions, 4) 1 task that lacks of some necessary conditions and has the presence of excess, and 5) 1 task, in which description and picture didn't match. It was not the correct result that was evaluated, but rather whether this result was justified by the given knowledge. Suggested concepts for recognition were: "straight line", and "mammal". For "straight line" it was an indirect instruction (after definition it was given the task — "choose '+', '–' or '?'"). For "mammal" it was a direct instruction (after definition it was instruction ("You should help to a little girl to find mammal among other animals using short texts") and task — "choose '+', '–' or '?'"). In total, the student solves 10 problems in condition of indirect instruction and 10 in condition of direct instruction.

### *Method of reading comprehension' diagnostic*

The technique was developed specifically for students in grades 4–5 based on a modification of the method "Choice of main sentences" intended for high school students [Ilyasov, Malskaya, Mozharovsky, 1984]. The student is offered a text and is given the task "Underline the main sentences of this text." The text is composed in such a way that 8 out of 16 sentences are significant (describe the basic definitions, facts and their explanations) and 8 are not (statements, similar in form to the present definitions, dates, statements about the significance, etc.). The authors of the methodology developed it as a technique that diagnoses the ability to find out the most important sentences in the text. From our point of view, the ability to find out the main idea is connected with the fact that the student perceives the text as an orientation for solving a specific task — the task of presenting and explaining the facts. The evaluation of the quality is carried out on a scale from 0 to 8.

## RESULTS

Correlations between OGK-D, OGK-I and reading comprehension, the mean values and standard deviations are presented in Table 1.

As we can see reading comprehension positively correlates with OGK with direct instruction ($k = 0.352$, $p < 0.01$). Also OGK with indirect instruction was more difficult for students then indirect ($p < 0.001$, Wilcoxon Signed Rank Test).

Table 1

**Means, standard deviations and correlations between OGK-D, OGK-I and reading comprehension**

| | Mean | SD | Reading comprehension | OGK — I | OGK — D |
|---|---|---|---|---|---|
| **Reading comprehension** | 4.00 | 0.97 | | 0.109 | 0.352 (*) |
| **OGK — I** | 0.86 | 1.19 | | | 0.203 |
| **OGK — D** | 3.77 | 3.11 | | | |

Note: * $p < 0.01$, two tailed, $N = 58$.

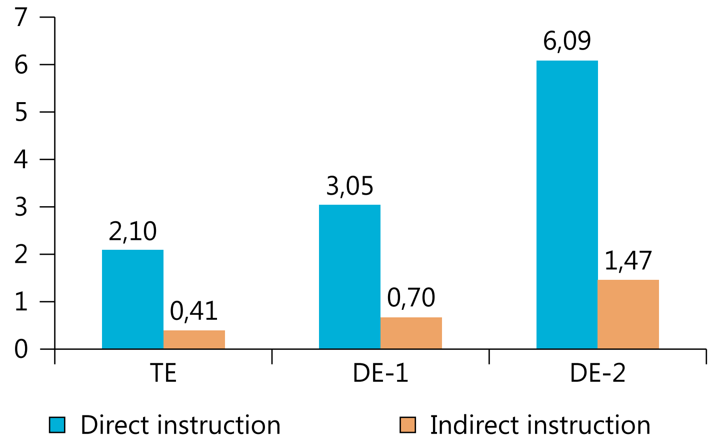


***Fig. 1.*** Average number of tasks, where participants were demonstrated OGK in case of direct and indirect instruction

Differences between three groups of students (TE, DE-1, DE-2) are presented in Fig. 1.

As we can see both groups of DE were demonstrated higher levels of OGK (in case of both types of instruction) then TE ($p < 0.05$ in case of direct instruction and $p < 0.01$ in case of indirect instruction, Test Mann-Whitney). But DE-2 group demonstrates higher results then DE-1 ($p < 0.05$ in case of direct instruction and $p < 0.01$ in case of indirect instruction, Mann-Whitney Test). Interesting, that there is no differences between results of diagnostics of reading comprehension between DE-1 and DE-2.

## DISCUSSION

When we were planning this study, we first suggested that the results of OGK-D' diagnostics will be significantly better then the results of OGK-I' diagnostics. This assumption was confirmed. So, we can say that if even students don't use given knowledge it doesn't mean that they can't do it. They can, but they are not used to using it. Thus, children can work with text very well, they can find any required information in any text if teacher ask them. But text doesn't have orientation function for problem solving. Perhaps the ability to perceive the text as orientational and to solve problems based on a given knowledge represents two different aspects of the OGK. So we should think about further research, perhaps a more detailed study (for example, using texts and concepts from another subject area, etc.). Results of 2st hypothesis' checking confirm partially: reading comprehension very positively correlates with OGK-D and has no correlation with OGK-I. Probably it can be explained by the same reason as previous. Our third hypothesis is confirmed partially. DE students (all two groups) were demonstrated higher levels of OGK then TE students. But, as we suggested, DE-2 students were demonstrated higher levels of OGK-D and OGK-I then DE-1 students. Probably it can be explained by differences in DE educational programs (as we said, in DE-2 teachers pay special attention to formation of reading skills). Anyway it will be necessary for us to refer to the real lessons and to analyze them precisely from the point of view of how much the OGK is maintained in them. In this paper we can't discuss all results and, that more important, all limitations of our study (for example, influences of given concepts' specifics or number of respondents). It was the one of the possible way for investigation of OGK and it requires future modifications. Actually, we would like to emphasize significance of the research problem of orientation towards given knowledge.

## ACKNOWLEDGEMENTS

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# EXAMINING SECONDARY MATHEMATICS TEACHERS' NOT-KNOWING IN THE PROCESS OF PROBLEM SOLVING

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*Not-knowing is an underexplored concept defined by an individual's ability to be aware of what they do not know as a means to plan and more effectively face complex situations. This qualitative study focuses on analyzing students' ability to express their "not-knowing" while completing tasks and reflecting periodically. It becomes evident rather quickly that the students have difficulty expressing their not-knowing. Through transcription analysis, reflection coding, and interviews, four recurring themes emerge that could possibly determine why students have difficulty expressing their not-knowing. These four themes are deflection, student pressure, lack of heuristic sense, and fractured knowledge. Each one of these themes will be discussed addressing challenges in relation to students' ability to express not-knowing.*

## INTRODUCTION

Not-knowing is the first step to understanding, carrying an important value in learning. Mason and Spence [1999] claim, "awareness of knowing and of not knowing is crucial to successful mathematical thinking." Tahta [1972] uses not-knowing as a means to describe the algebraic process of finding what we do not know with the use of what we do know. Shah (1968) quotes ancient wisdom according to which not-knowing is a critical state because, from it, knowing can follow. However, a little research has been done to understand the not-knowing phenomenon. In this study, we are examining the following research questions: how do students express what they do not know? and, what challenges do they face in externalizing the not-knowing?

## FRAMEWORK

The theory of unconscious thought [Dijksterhuis, Nordgren, 2006] is closely related to the main construct and the unit of analysis of this study. Addressing the theory, Funke [2017] elaborates, "the basic idea is that the quality of decision-making depends on conscious and unconscious thought simultaneously. The term conscious thought is understood to mean a mental state that encompasses a person's rational awareness, whereas the term unconscious thought refers to the underlying influence, of which one is typically unaware and which has an impact on one's behavior. Unconscious


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thought tends to outmatch conscious thought, especially in complex and untransparent situations. Unconscious thought takes place when conscious attention is directed elsewhere [Funke, 2017]. Analysis of conscious attention, such as that of awareness of not-knowing [Mason, Spence, 1999], can be considered as an important step in a student's decision-making process in complex problem-solving situations.

Being aware of one's own not-knowing is a spark that could potentially ignite inert knowledge. Renkl, Mandl, and Gruber [1996] suggest that "the problem of inert knowledge is surely of major educational importance." If awareness of not-knowing is a potential solution to inert knowledge, it is a topic that needs to be further explored.

Along with recommendations to effective problem solving, scholars [Carlson, Bloom, 2005; Perkins, 2000] identify some common barriers to successful problem solving such as an inability to identify and connect the relevant knowledge (e.g. in the Results section we refer to this challenge as "fractured knowledge"), a lack of experience with how to go about solving problems (e.g. lack of heuristic sense), a false belief that you either know how to do something or you don't and that if you don't there's not much point trying beyond a few minutes (e.g. pressure), seeing errors as indicating incompetence rather than just an inevitable and necessary part of learning, going with the first solution approach which comes to your mind rather than trying to determine what the possible options might be (e.g. deflection), etc.

Furthermore, Carlson and Bloom [2005] claim that a good problem solver exhibits flexibility as well as powerful mathematics related processes to arrive at their solution. They also state that those who solve the problems do not solely rely on heuristics. The awareness of not-knowing through understanding the barriers to successful problem solving opens the doors to becoming more flexible at solving problems while less dependent on heuristics — an ability which Simon [1996] details through the construct of "transformational knowledge" as a way to assess a given situation and select the best possible outcome. This line of thinking could potentially be achieved if students become aware of their not-knowing and use it as a means to understand a situation.

Therefore, not-knowing could be considered as the first step to effective problem-solving. This is one of the key motivations for conducting the study, as it aims to uncover how not-knowing can help students understand their own thinking and develop more effective problem-solving strategies.

## METHODOLOGY

This qualitative study focused on students' articulation of not-knowing and the challenges they faced during this process. Ten students were selected for the study at a university in the southwestern border region of the United States. These students were enrolled in the course on Geometric Reasoning, which focused on problem-solving using a tangram (a seven-piece puzzle) to construct squares with a different number of pieces. This setting was used to collect data and analyze student externalization of their not-knowing. Data sources consisted of audio recording, reflections, and interviews described below.

### Recordings and Transcriptions

At four different points during the course, students paired up and audio-recorded each other depending on who was attempting to create a given square using the tangram pieces. The task was to attempt to complete the square using the given pieces while vocally expressing what they knew and did not know at that specific moment. First, the participants recorded the seven-piece attempt, followed by a second attempt at the seven-piece, then the six-piece, and finally the five-piece.

The audio recording of the participants while attempting the task should bring forth their thought process while focusing on voicing what they knew and didn't know at the time. This was the reasoning behind the audio recordings, as it was thought to be a way of encapsulating students' not-knowing as a data source for analysis.

### Reflections

Two major reflections were assigned: post-activity reflections and post-lesson planning reflections. These reflections contained three and five questions respectively. The participants were tasked to reflect on certain ideas discussed as well as their knowing and not-knowing. Since these reflections asked the participants what they didn't know directly, their answers could be used to analyze how they expressed their not knowing.

### Interviews

Two students out of ten were selected for semi-structured interviews conducted at the end of the course to analyze student reflective thoughts on the course. The interview consisted of thirteen questions focused on extracting student not-knowing in a reflective fashion, which was audio recorded. The audio recordings were then transcribed and analyzed.

### Data Analysis

In order to analyze the transcriptions, reflections, and interviews, meaning coding, meaning condensation and interpretation techniques were used [Kvale, Brinkmann, 2009] as the main methods of analysis. The first set of analysis consisted of transcriptions from the participants' attempts at creating the seven, six, and five-piece squares using the tangram while voicing their knowing and not knowing. There were two methods of analysis used for this part. First, the transcriptions of each individual participant were separated and analyzed to examine how well the participant expressed their not knowing at different points in time during the course. Second, an analysis of the transcriptions as a whole was conducted, in an effort to encompass key similarities between them. The second set of analysis was two reflections in which students reflected on what they knew and did not know at the given time based on the course. Meaning coding technique was used to interpret their ideas and make connections to their transcriptions. Finally, the last part of the analysis included the two interviews. The interviews are meant to provide a closer look into the thoughts of the two participants. The questions are aimed at evoking not-knowing reflectively and active not-knowing while the interview is being conducted. Meaning interpretation and meaning condensation were the primary tools in analyzing this data to make connections between other forms of not-knowing expressed throughout this study.

## FINDINGS

The data analysis clearly demonstrated that the participants had difficulties expressing their not-knowing. The analysis shows that there are several recurring statements made by the participants. These statements were categorized and the following four major themes emerged that will be described below.

### *Deflection*

Deflection of not-knowing can be identified as avoidance of challenge when an individual shifts the focus of their not-knowing somewhere else besides themselves. Below is an example of the participant's response, which demonstrates deflection of not-knowing:

> Student: I know as far as formulas and all that they are not going to help me at all. It's more of a pattern thing and if students tried to do this it would be the same thing for them…

The participant makes it a case that she believes formulas will not help her at arriving at a solution, inferring that she does not know how to arrive at the solution. Her thought of not-knowing takes a shift stating that if students tried, it would be the same for them. Why is it that when asked about her own not-knowing she deflected? Instead of being aware of her not-knowing as a first step to finding the answer, the individual deflects to what she believes others don't know. A total of five out of the ten participants deflected at one point or another throughout the analysis.

### *Pressure*

Participants demonstrated pressure through direct vocalization, frustration, or sense of urgency. Every one of the participants demonstrated pressure at different points during the transcription and reflection sections. Look at these two statements below:

> Student 1: Makes no sense to me. Jesus… Ok, so it doesn't make sense… Jeez. It's almost something. Oh, God. Can I make a rectangle?
>
> Student 2: This is ridiculous [laughs]. Putting these squares together. It's a lot harder than I thought it would be. It's destroying my idea of what a square is… Maybe if I … no. Oh my God, this is so much harder than I thought it would be. I think you did that. Then we can put a little triangle here. I think that's what you had isn't it? No. Argh, this is so frustrating.

These two students demonstrate pressure, which may be a factor impeding awareness of not-knowing. The first student demonstrates clear frustration throughout his thought process at his inability to make sense of the situation. The second student directly expresses her frustration, derived from expecting the task to have been easier than expected.

### *Lack of Heuristic Sense/Trial-and-Error*

Every single participant attempted the first task through trial and error as demonstrated by his or her transcriptions. We hypothesize that in seeking the solution, students may become "tunnel visioned" in the process of trial and error, clouding their awareness of not-knowing. Below is part of a transcription, where a participant demonstrated "tunnel vision."

> Student: I am going to start with the parallelogram just because it has the oddest shape. And… I'm going to try to make the sides even, and I'm just trying to add from there but it doesn't fit. So I don't know how to get the sides to be straight without having any leftovers. Ummm… ok. No, I'm going to start again. I'm going to start with the big pieces now. I'm going to put the two triangles together. Okay, I'm going to put the two big triangles and try to make everything fit in the middle. Okay, so I'm putting some and they don't fit but I'm kind of getting the shape, kind of not.

Most of the participants, even with new knowledge, stuck to trial and error to the very end. Perhaps, inability to evoke awareness of their not-knowing was a factor that led to not solving the tangram.

### *Fractured Knowledge*

Fractured knowledge is present when a participant may have knowledge gaps within the given topic, have misunderstandings of said topic, or simply lack the prior knowledge required for the given topic. We argue that if an individual has fractured knowledge, it will directly impede their ability to use not-knowing as a means to gather knowledge that simply is not there or is "fractured." Below is a representation of a participant with fractured knowledge:

> Student: Ok, so that might be too long. So, I think it has to be smaller than 2 and square root 2. Maybe it can be 3? I will try for 3.

While attempting to find the side length of a square that must be constructed, the participant makes the revealing statement above. There is a clear misunderstanding of the number 2 ("two square root of 2") in comparison to the number 3. The participant believes that 3 is smaller than 2 and carries on without a second thought, guiding her down the wrong path.

These four themes frequently emerged throughout participant transcriptions, reflections, and interviews. Even though some of these themes emerged less than others, they all hold importance, as they reflected challenges in participants' ability to express not-knowing.

### *Connections to Learning*

The data gathered from the students clearly demonstrated the difficulty faced when reasoning with the tangram activity. The audio recordings displayed growth in understanding from all students, some better than others. Once the lesson was over, two students stood out who displayed almost complete mastery of the tangram. While all the first audio recordings displayed difficulty in student ability to express their not knowing in the moment, these two students demonstrated to be very active thinkers as they attempted the task. Even though they also faced difficulty clearly expressing not knowing, they demonstrated more effective attempts as opposed to counterparts seemed to have contributed to their learning. These two students were interviewed as a means to better understand the progress made. Their interviews gave insight into their thinking during the attempt to construct the tangram. Both students expressed their "hate" for having to express what they didn't know in the moment. One student stated that the question of "what don't you know?" seemed like a really simple question, but in fact found it to be really hard to answer. It appeared that these students disliked the question because it was difficult to articulate what they didn't know at the moment. The important factor is that these students made the attempts to express their not knowing. Let's take a look at these two excerpts below:

Student 1: So I have this big triangle in the corner and I keep putting it there because it seems like it makes sense there but I don't know if this is the right way to go I wonder if there is more than one way to do it.

Student 2: It can't be two square root of two, I tried that already. So it can't be two square root of two because it's the largest we can have which will give me an area of eight. So it has to be less than two square root of two and I tried side length of 2, but it limits the amount of pieces I can use. I am wondering if this is even possible.

Student 1 attempt may have been flawed, but her ability to step out of the problem and question if there are different possible ways to go about it is a line of thinking that may have led her to her correct answers later on. Student 2 struggles with constructing a square with 6 pieces, however, her not knowing leads to think about what she does know, arriving at her conclusion.

These two students responses and performance contrast directly with other participants who made little to no attempts at expressing their not knowing or simply lacked the ability to do so. Those who made little attempts of expressing not knowing throughout the lessons seemed to have struggled more extensively. While both parties attained new knowledge from the course, the results indicate that those students who more actively sought to express their not knowing gained a better understanding of the concept.

## DISCUSSION AND CONCLUSION

Individuals "deflecting" their not-knowing should not come as a surprise, as not many individuals are fond of admitting their lack of knowledge or understanding. Nevertheless, the presence of deflection in the analysis shows how someone may shift their not-knowing onto someone else, rather than accepting their not-knowing and use it as a means to find a solution to a given problem.

The theme of "pressure" did not come as a surprise either. It may be closely related to Krashen's affective filter hypothesis (1985), which details students' abilities to learn based on what other thoughts might be on their mind. In this study, we saw pressures caused by time, frustration, and even fear of being judged by others.

The “heuristic sense” theme observed in the study is one deeply rooted in students’ minds. Since one of the most fundamental heuristics is trial-and-error it is easily available for anyone to use. Trial-and-error was the guiding force in decision making for many participants from the start of the course, and in some cases, to the end of the course. Students may be diverting to trial-and-error heuristic to ease the cognitive load.

Not-knowing can serve as a spark in creating a plan to solve a problem, but if fractured knowledge exists, it is likely that not-knowing will not cause yield an incorrect answer. Just like in the example concerning fractured knowledge, it can be determined that as long as the individual has “fractured knowledge”, it will impede the initial awareness of not-knowing.

These themes seem to be closely related to Funke’s [2017] theory of unconscious thought. All recurring themes could potentially be unconscious thoughts. Not being aware of such thoughts could prove more difficult for individuals to overcome.

Mason and Spence [1999] make it clear how important not-knowing is as a first step for knowing to occur. This study focused on finding out how students are expressing their not-knowing. Results demonstrated that the participants had difficulty expressing their not-knowing. Four recurring themes came to light from analyzing participants’ transcriptions, reflections, and interviews. Examples of deflection, pressure, heuristic sense, and fractured knowledge were discussed to demonstrate how each affected not-knowing awareness. Understanding these themes can better help in minimizing the problem while maximizing the potential of not-knowing as a lead to knowledge and understanding. The findings might serve as a stepping stone to further research of not-knowing.

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# SEMANTIC ALIGNMENT IN MATHEMATICAL WORD PROBLEMS: EVIDENCE FROM THE RUSSIAN EDUCATION*

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*Solutions of mathematical word problems are moderated by the semantic alignment of real-world relations with mathematical operations. The current study examined evidence for semantic alignments in Russia in comparison to the USA and South Korea. Textbook analyses revealed semantic alignments for arithmetic word problems, but not for rational numbers. However, Russian college students showed semantic alignments both for arithmetic operations and for rational numbers. Since Russian students exhibit semantic alignments for rational numbers in the absence of exposure to examples in school, such alignments likely reflect people's everyday experience with and natural understanding of mathematical representations of real-world situations.*

## THEORETICAL FRAMEWORK

When applying mathematics to real-world situations, students must understand how to construct mathematical representations of the situations they encounter. Educators try to teach this process by approximating real-world situations with word problems. Previous research has shown that the process by which people coordinate situation and mathematical models is often guided by *semantic alignment* (e.g., [Bassok, Chase, Martin, 1998]).

Semantic alignment is a process of analogical mapping between semantic relations implied by the objects in the problem situation, and potential mathematical relations [Ibid.]. For example, flowers and vases evoke the semantic relation *contain* [*flowers, vases*], which is asymmetric (because vases normally *contain* flowers and not vice versa). This semantic relation aligns structurally with the mathematically asymmetric relation *divide* [*dividend, divisor*]. The semantic alignment for the objects tulips and roses naturally evoke their shared categorical superset relation, *both-flowers* [*tulips*, *roses*]. Unlike the *tulips-vases* pair, tulips and roses play symmetric roles in the "both-flow-

* The results of Experiments 1 and 2 were already published in *Tyumeneva Y.A., Larina G., Alexandrova E., DeWolf M., Bassok M., Holyoak K.J.* Semantic alignment across whole-number arithmetic and rational numbers: Evidence from a Russian perspective. Thinking and Reasoning. 2017. <https://doi.org/10.1080/13546783.2017.1374307>.



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ers” relation, which aligns structurally with the symmetric mathematical relation *add* [*addend 1*, *addend 2*].

A different type of semantic alignment, based on the distinction between discrete and continuous quantities, has also been observed. Bassok and Olseth (1995) found that the discreteness versus continuity of the entities described in a problem (e.g., salary increases versus increases in the value of a coin in $/year) affects the way people represent problem structures, and therefore impacts transfer of learned solutions. The same alignment process affects how people choose a format for rational numbers (fractions versus decimals) to represent discrete versus continuous entities [Rapp et al., 2015]: college students and textbook writers show a preference for representing relations between discrete or countable entities with fractions (e.g., 3/4 of the marbles), and for representing magnitudes or measures of continuous entities with decimals (e.g., 75 L of water). Thus, alignment not only influences the generation of concrete instantiations of mathematical representations (the focus of the present paper), but also the generation of mathematical representations to match given concrete situations.

The nature of semantic alignment highlights a critical theoretical question: whether people’s preferred semantic alignments reflect a basic understanding of mathematical representations as analogical models of real-world situations, or reflect a history of specific learning experiences and therefore correlate with instructional practices. Although that question was addressed in [Lee et al., 2016; Rapp et al., 2015], the semantic alignments in both nations appeared in textbooks; hence that question still remains unclear.

### *Cross-national difference in curricula*

The goal of the present paper is to address this question by expanding the cross-national exploration of semantic alignment to the Russian Federation, where the math curriculum differs radically from that found in either the U.S. or South Korea.

Two major features of the Russian math curriculum may influence the degree to which students’ mathematical thinking is guided by semantic alignment. The first is abstractions. Russian children are taught the beginnings of algebra as early as elementary school, including skills such as building and solving equations. Although abstract thinking in math is also promoted in the U.S. school system, this emphasis is nowhere near as strong as in Russia. This focus on abstraction in the Russian curriculum might be expected to diminish the impact of semantic alignment, which depends on the more concrete properties of the objects involved in mathematical problems.

The second distinguishing feature of the Russian math curriculum is the strong focus on magnitude and measurement which would seem likely to promote general dominance of continuous over discrete magnitudes, thereby diminishing semantic alignment of discrete magnitudes with fractions and continuous magnitudes with decimals. In addition to the focus on measurement, fractions and decimals are introduced simultaneously within the Russian curriculum (in contrast to the U.S., where fractions are typically introduced to students at least a year prior to decimals). Coupled with the strong focus on measurement, which unifies continuous and discrete entities, simultaneous learning of the two notations for rational numbers might further reduce any selective semantic alignment of number format with entity type (for a full discussion of the math curriculum in the Russian Federation see [Davydov, 1990]).

Thus, in our study we examined whether the patterns of semantic alignments for basic arithmetic operations and for rational numbers, previously found in the textbooks and in the performance of students in the U.S. and in South Korea, will also be observed in Russia. We examined whether the focus on abstraction would diminish the magnitude of semantic alignment in textbook word problems and in subsequent students' performance.

## ARITHMETIC TEXTBOOK ANALYSIS

In order to determine whether Russian textbooks show semantic alignment for basic arithmetic problems (addition: symmetrical, division: asymmetrical), we conducted a textbook analysis of $4^{th}$ and $5^{th}$ grade textbooks. We analyzed two textbooks for Grades 4 and 5 [Moro et al., 2011; Vilenkin, Zhokhov, 2008], which have a large market share and are widely used across Russia. We analyzed all word problems involving addition/subtraction ($n = 419$) or else division/multiplication ($n = 321$), a total of 740 problems.

In result, almost all of the addition and subtraction problems (99%) involved symmetric object pairs, whereas the great majority of the division and multiplication problems (88%) were asymmetric. There was a significant relation between arithmetic operation and semantic structure ($\chi^2(1) = 578.797$, $p < 0.001$). This finding replicates the pattern of results observed in American textbooks, in which 97% of addition problems involved symmetric relations and 94% of division problems involved asymmetric relations ([Bassok, Chase, Martin, 1998]; Experiment 3).

## EXPERIMENT 1. ALIGNMENT WITH ARITHMETIC IN UNDERGRADUATES

Participants were 77 undergraduate students from the Faculty of Psychology, NRU "Higher School of Economics" (72 females and 5 males). The experiment was conducted at the beginning of the school year and only 1st-year students were selected, thus minimizing the influence of university education on students' performance. The materials were adapted from Bassok et al., Experiment 2 (1998). Participants were randomly assigned to receive an addition ($N = 40$) or a division ($N = 37$) booklet. Three types of object pairs were used: symmetric (e.g. boys-girls), subset-set (e.g. boys-children) and asymmetric (e.g. boys-school). They were asked to create math word problems involving addition or else division for each of the six object pairs in the booklet. The allotted time was limited to 20 minutes. In total, participants constructed 240 addition word problems and 222 division word problems.

The generated problems were coded in the following four categories based on the equation required for the problems' solution: Mathematically direct (MD, the equation related the given sets directly by the required arithmetic operation), Complex mathematically direct (complex MD, the equation related the given sets directly but included further computation), Semantic escape (SE, problems where some requirements were not fulfilled), or Other.

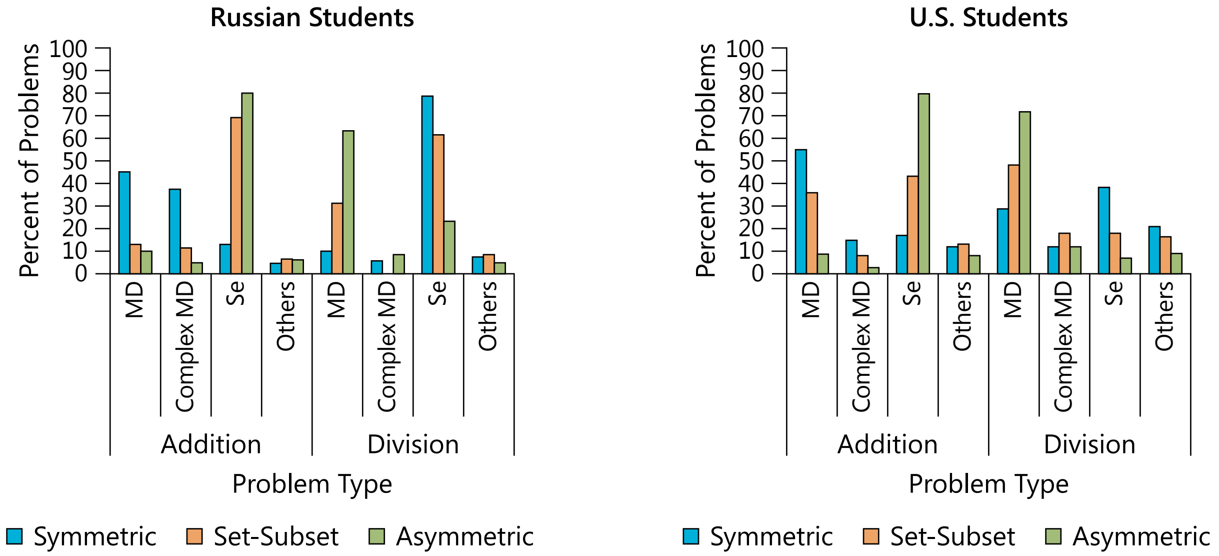


*Fig. 1.* Percentages of MD, Complex MD, SE, and Other problems constructed in addition/subtraction and division/multiplication conditions for each of semantic relations (U.S. sample: data from [Bassok, Chase, Martin, 1998, p. 117])

Figure 1 shows percentages of MD, complex MD, SE and Other problems constructed for each pair type by participants. In the Addition condition, the relative frequency of MD problems was higher for aligned symmetric pairs (45%) than for misaligned asymmetric pairs (10%), with the frequency for subset-set pairs falling in between (13%). The pattern for generating Complex MD problems generally matched that for MD problems, being higher for symmetric pairs (37%) than asymmetric pairs (4%), with subset-set pairs intermediate (11%). The opposite pattern was observed for SE problems, where the percentage was higher for asymmetric pairs (80%) than for symmetric pairs (13%), with subset-set pairs intermediate (69%).

These patterns generally reversed in the Division condition. The percentage of MD problems was far higher for aligned asymmetric pairs (63%) than for misaligned symmetric pairs (10%), with the frequency for subset-set pairs falling in between (31%). Complex MD problems were seldom generated in the Division condition (8%, 0% and 5% for symmetric, subset-set, and asymmetric pairs, respectively). The relative frequency of SE problems in the Division condition was higher for misaligned symmetric pairs (78%) than aligned asymmetric pairs (23%), with subset-set pairs intermediate (61%).

In general, the results obtained from the Russian sample closely matched those obtained using the U.S. sample tested by [Bassok, Chase, Martin, 1998, p.17]. In the case of word problems based on natural number arithmetic, Russian educators use arithmetic problems that are consistent with semantic alignment in a manner similar to their use by U.S. educators. Similarly, Russian adults show the same pattern of alignment with addition and division problems as U.S. adults.

## RATIONAL NUMBER TEXTBOOK ANALYSIS

We examined three textbooks for grades 5, 6 and 7 [Vilenkin et al., 2005; 2008; Makarychev, Mindjuk, Meshkov, 2008]. Books by these authors (for different grades of secondary and higher school) are recommended by the Russian Ministry of Education. They have a very large circulation, and are chosen by great number of Russian teachers. Only word problems containing fraction or else decimal numbers were analyzed. Problems consisting of several parts or containing several fraction/entity pairs were coded separately as different problems. In total, 476 problems were examined, of which 216 included decimals and 260 included fractions.

The great majority (91%) of the decimal problems used continuous entities. However, in stark contrast to findings in comparable analyses of similarly popular textbooks used in the U.S. and South Korea (described by [Rapp et al., 2015; Lee et al., 2016], respectively), most fraction problems (87%) also used continuous entities. A test of independence between number and object type showed that the two factors were not reliably associated, $\chi^2(1) = 2.55$, $p = 0.11$. Thus, the findings from the textbook analysis indicate that students in Russia are *not* exposed to any systematic alignment of formats for rational numbers with types of entities.

## EXPERIMENT 2. ALIGNMENT WITH RATIONAL NUMBERS

Sixty-four undergraduates (mean age 20 years; 42 females and 22 males) from the Department of Computer Sciences, NRU 'Higher School of Economics', were asked to take part in the experiment in lieu of their regular class. The instructions given to the participants were exactly the same as those used in Experiment 1 from [Rapp et al., 2015] and Experiment 1 from [Lee et al., 2016]. Each of them was given a sheet of paper containing three examples of simple word problems with whole numbers. First, the constructed problems were divided into decimals or fraction problems; then, they were classified as continuous or discrete depending on the type of entity used in problem. All the generated problems were coded using criteria based on the study by [Rapp et al., 2015].

As seen in Fig. 2, Russian college students more often used continuous entities with decimals (74%) compared to using continuous entities with fractions (51%). Converse-

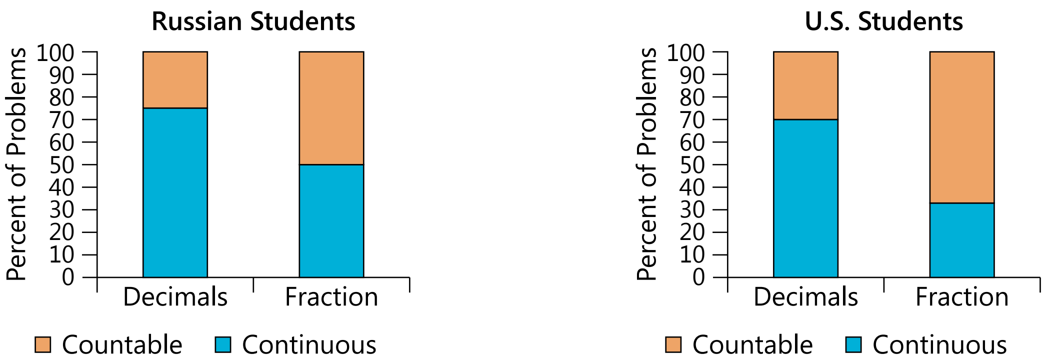


*Fig. 2.* Percentages of decimal and fraction problems constructed by Russian college students in continuous or countable entity conditions. (U.S. sample: data from Rapp et al., 2015, p. 50)

ly, they used discrete entities more often with fractions (49%) than with decimals (26%). A test of independence between number and object type confirmed that number type and continuity were significantly associated, $\chi^2(1) = 6.76$, $p = 0{,}009$; Phi = 0.224. This pattern of alignment is strikingly similar to that found with American students tested by [Rapp et al., 2015] and as well as by South Korean students tested by [Lee et al., 2015]. In summary, Russian college students, like their counterparts in the U.S. and South Korea, and unlike Russian math textbooks, tend to use decimals to represent continuous entities and fractions to represent discrete entities.

We might expect that a rapid transition from concrete objects to abstract algebra in the Russian curriculum would attenuate natural alignment patterns. However, despite the absence of such alignments in math textbooks in Russia, the results showed that Russian college students have aligned the type of rational numbers with the type of entities. In other words, the outcome of a markedly different curriculum in Russia is much the same as the outcome in the United States and South Korea. It seems that people naturally anchor their mathematical reasoning (at least with rational numbers) in properties of concrete entities, even against the curriculum.

However, it is also possible that the curriculum itself was effective enough to change this natural alignment pattern, although for a shorter time period. Indeed, our participants had been exposed to not-aligned word problems over several years. What if the curriculum had affected the natural alignment pattern, but subsequent everyday activity annulled the effect?

## EXPERIMENT 3. ALIGNMENT IN EIGHTH-GRADERS

We hypothesized that the effect of non-aligned word-problems in Russian textbooks is to be found in younger students who had just completed their basic drilling in modeling discrete and continuous objects with fractions and decimals. Therefore the goal of the study was to examine the accumulated effect of not-aligned textbook problems in a sample of eighth-graders. Mostly, Russian eighth-graders have already finished studying word-problems and continue dealing with fractions in algebra where no real objects are supposed to be used.

Thirty-six eighth-grade students from a public school in Moscow (*M*age = 14), were asked to take part in the experiment during a regular lesson. The instructions given to the participants exactly replicated the instructions in Experiment 2 with college students. The constructed problems were coded using the procedure and criteria from Experiment 2.

Generally 72 word problem were produced. As it seen in Fig. 3 the eighth-graders did not show any clear tendency either towards of continuous or countable objects: they modeled with decimals or fractions with almost equal probability (in 60% of cases continuous objects are associated with decimals and in 50% — with fractions). A test of independence between number type and object type showed that these two factors were not significantly associated, $\chi^2(1) = 0.839$, $p = 0.36$; Phi = 0.115.

The absence of semantic alignment in eighth-graders was consistent with the textbook analysis results, and contrasted with the strong alignment pattern of college

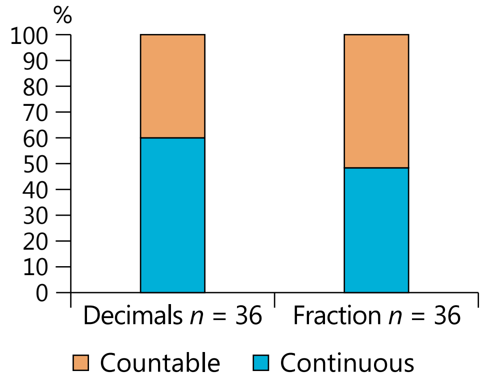


*Fig. 3.* Percentage of decimal and fraction problems which were countable or continuous (Experiment 3)

students. These findings support our hypothesis that the curriculum can change the natural alignment pattern of pupils, which, however, could return after some time (at least in non-mathematicians).

## GENERAL DISCUSSION

The primary goal of the study was to compare patterns of semantic alignment in different educational approaches to teaching basic arithmetic and rational numbers. We examined whether the strong focus on abstraction and magnitude and measurement, which is the distinguishing feature of the Russian math curriculum, would diminish the semantic alignment in textbook word problems and in the subsequent performance of students. We found that in the case of arithmetic Russian educators use aligned arithmetic problems in textbooks in a manner similar to their use by U.S. and South Korea educators. Similarly, in dealing with arithmetic problems, Russian adults showed the same pattern of alignment as adult Americans and Koreans.

However, in the case of rational numbers, Russian textbooks demonstrate no alignment in modeling discrete and continuous objects relative to rational number format. Moreover, only a very small proportion of the total problems in textbooks involve discrete objects (< 5%). Both fractions and decimals are highlighted as tools for continuous measurement. This was not very surprising given that the concepts of magnitude and measurement play a key role in teaching math in Russia. The issue of the developmental effects of the non-aligned curriculum is more interesting. The eighth-graders, who had recently been exposed to extensive practice with non-aligned word problems, didn't show alignment in solving word problems with rational numbers. But college students, who had finished with that part of curriculum years before, did. Russian alignment patterns were actually very similar to the ones the U.S. and South Korea students showed, even though latter groups were exposed to drastically different curriculum.

The first question which arises is what is going on with the understanding of rational numbers when they are taught as dependent on or independent of the object they model. Possibly, the Russian emphasis on continuous measurement, independent of rational number format, may provide an advantage for magnitude assessments of rational numbers — especially those of fractions, which students over the world struggle with more than decimals (e.g., [Lee et al., 2016]). On the other hand, emphasis on magnitudes may hamper relational reasoning with fractions, since a fraction also represents the relation between the cardinalities of two sets.

Another question is whether acquiring semantic alignments is in fact desirable. For a mathematical concept it doesn't matter which kind of concrete objects to substitute. Examples aligned with concepts might help students at the initial stages of learning. However, it remains unclear whether there are any negative implications of semantic alignment, e.g. whether they do not artificially limit the range of concrete objects that a mathematical concept can describe. In any case, to improve our knowledge of these issues, alignment patterns should be explicitly stated by educators and highlighted for students.

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# THINKING WITH OBJECTS IN MATHEMATICS: THE CASE OF GEOMETRY

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*The emphasis laid upon the skills of numerical calculations and algebraic transformations, which dominated the pre-computer era, is gradually losing its significance. Today mathematical education is viewed as aiming at the formation of scientific thinking skills. The paper suggests an approach to mathematical education based on the concept of object-oriented mediation. The characteristic feature of this approach consists in that the student is placed in conditions similar to those that historically led to the formation of new sections of mathematics.*

Computerization, which during the past decades changed dramatically the outlook of modern scientific and educational culture, has also led to the need of rethinking the aims of school education, especially in mathematics. In the pre-computer era, mathematical education was mainly aimed at mastering the skills of numerical calculations and symbolic (algebraic) transformations required in various fields of applied science and industry. The main efforts of teachers were directed to this objective. Currently, the widespread use of computer technology has made this goal obsolete, at the same time posing new challenges for school teachers and methodologists. Besides the mastering of the subject-matter, one expects from modern students the achievement of goals that go beyond mathematics itself. In particular, a greater emphasis is laid nowadays upon the development of scientific thinking by means of mathematics.

The distinctive character of scientific thinking consists in that it reveals essential connections between various aspects of reality that remain hidden to ordinary consciousness. In solving problems, the recognition of these connections consists in finding "intermediate situations" (one or many), which are somehow "in between" the two opposites — the source data and the desired result. In philosophical parlance, such "situations" are called mediations (Vermittlungen). Teaching the art of mediation by solving mathematical problems may be considered as an effective means of forming scientific thinking skills [Bytchkov, Zaytsev, 2015].

A special role in developing these skills belongs to geometry. This fact has been long recognized by the theorists of mathematical education. Thus, Hans Freudenthal, among others, insisted that geometry has a special potential for the development of creative thinking since it can be taught — at least at the earlier stages — in terms of operations with real tangible objects in the real three-dimensional space [Freudenthal, 1972].


In *Shvarts A.* (Ed.). Proceedings of the PME and Yandex Russian conference: Technology and Psychology for Mathematics Education. P. 234–240. Moscow, Russia: HSE Publishing House.

The main idea of this paper consists in that the special role of geometry in education is due to the diagrammatic character of its content, rather than the axiomatic-deductive form of its exposition. This means that practical operations with geometrical objects are valuable not only as prolegomena to the study of logical structure of the corresponding theory. Their pedagogical value is determined by the fact that "thinking with objects" compels students to solve problems, that is, find "intermediates" between what is done and what should be found by drawing new auxiliary constructions consisting of points, lines or figures, and not by selecting appropriate propositions from a fixed list of axioms [Bytchkov, 2014].

Let us first give an example of how an object-oriented mediation may be accomplished using a rather simple subject "the area of triangle." The usual practice of mastering this topic — let us say it frankly — consists in memorizing the formula $S = ½\ ah$ (where $a$ is the base and $h$ the height of the triangle) and repeatedly applying it for solving problems. Needless to say, that such an approach can contribute neither to the understanding of the rationale behind the formula, nor to the formation of creative thinking.

The situation is different when the student, asked to find the area of a triangle, is obliged to solve the problem by drawing an appropriate diagram. The resolution of a problem consists — to use a popular truism — in reducing it to the problems that have already been solved. This means — in terms of object-oriented mediation — that in order to calculate the area of triangle a figure should be found to which triangle can be "reduced" by means of practical operations and for which the problem of finding the area has already been solved.

The easiest way to verify the correctness of the formula $S = ½\ ah$ is by inspecting the following diagram (Fig. 1) [Lockhart, 2009]:

When the figures are drawn and the reduction of the triangle to the half of the rectangle accomplished, the solution of the problem becomes obvious. But how can one guess that a rectangle should be described around the triangle and the height dropped from its vertex? In order to answer this question, let us turn to the history of the concept of area [Bytchkov, Zaytsev, 2006]. What geometrical figure is — from a historical point of view — at the origin of the notion? Such a figure is a rectangle whose area is reduced to the sum of the areas of (unit) squares. The school course of geometry follows essentially the same strategy: the area of rectangle is calculated as the sum of the areas of unit squares. This means, that one should link the triangle with a certain rectangle. It is obvious, that the rectangle described around the triangle is the one which is "most closely connected" with the triangle in question. The idea of dropping the height from its vertex is somewhat more complicated, but a targeted use of object-oriented mediation can help in this case too. What figure constitutes the "link" between the two figures — triangle and rectangle? It is clear, that such a figure will be right-angled triangle. Since the finding of its area by means of the described rectangle is rather easy, one should only guess how to reduce the case of arbitrary given triangle to that of the right-angled one. This is not very difficult. Thus, the entire chain of me-

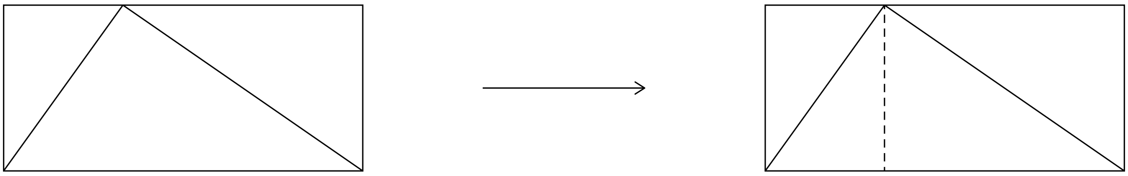

*Fig. 1*

diation, displayed in Fig. 1, is restored. In describing the steps leading to the solution, we have paid special attention to the targeted nature of the search for links between the given data and the desired result. Being truly creative, such a search would eventually contribute to the formation of scientific thinking skills.

Two educational strategies — the one basing on formal-analytical inference and the other on object-oriented mediation — have their counterparts in the real history of mathematics. In the historical development of almost all of its branches object-oriented procedures always preceded the formal-analytical and axiomatic-deductive ones [Zaytsev, 2011; 2018]. The example of algebra will help to familiarize with the idea. Nowadays, at the heart of the course of algebra is a formal approach that goes back to F. Viète and R. Descartes. The mastering of this approach essentially consists in memorizing the rules of manipulation of algebraic symbols. The substantial part of the course is devoted to the solution of quadratic equations, which is carried out in a purely formal way: by reducing these equations to the canonical form and applying to it the standard formula for the calculation of roots ("through discriminant"). The only result of this educational strategy consists in that students acquire skills in symbolic transformations and substitutions of parameters by numerical values. Obviously, this is not an approach that could contribute to the formation of scientific thinking skills.

As an alternative, we propose to use in this context an object-oriented construction underlying the formula of the roots. This construction is designed within the framework of the so-called "geometric algebra." Originated in ancient mathematics (Babylonian, Indian and ancient Greek), geometric algebra is a discipline that studies the magnitudes by presenting them as geometrical formations. Within such framework one can establish properties of linear and plane magnitudes, accomplish transformations of complex expressions, and find solutions of linear equations with one unknown. It can also be used to demonstrate certain algebraic identities of the second degree and solve quadratic equations (and systems of equations of the second degree with two unknowns) by interpreting algebraic symbolisms in terms of geometry. The main didactic advantage of geometric algebra consists in that it not only provides the desired result (as symbolic algebra does), but also reveals the reason *why* this result is actually correct. The answer to this question can be read off the accompanying diagram.

Let's look at some examples starting with a well-known geometric demonstration of algebraic identity

$$(a + b)^2 = a^2 + 2ab + b^2. \qquad (1)$$

► Let's represent the magnitudes $a$ and $b$ by line segments. The expression on the left side of the identity (1) is a two-dimensional magnitude, *viz.* the square with the side $a + b$ (Fig. 2). Its area is equal to the sum of the areas of its parts — two squares with the sides $a$ and $b$, respectively, and two equal rectangles with the sides $a$ and $b$. ■

Let us now write down the identity (1) by means of abbreviations that better express the idea of underlying geometric figures: $\square_{(a+b)} = \square_a + 2\square_{a\,b} + \square_b$.

As a second example, let us consider the demonstration of algebraic identity

$$(a + b)^2 = 4ab + (a - b)^2, \text{ provided that } a > b. \qquad (2)$$

► Let us construct a square with the side $a + b$ using four equal rectangles $A = ab$, attached to each other, as shown in Fig. 3. After the construction of the square is complete, one finds in its middle a small square with the side $a - b$.

The diagram in Fig. 3 looks as the stylized propeller, the inner "axis" of which is constituted by the square with the side $a - b$, while the "blades" are four equal rectangles $A$ with the sides $a$ and $b$. From the drawing it follows that the area of the large square is equal to the sum of the areas of the four rectangles $A$ and the small square with the side $a - b$. This may be better expressed by means of the abbreviated notation $\square_{(a+b)} = 4 \cdot A + \square_{(a-b)}$ equivalent to the identity (2). ■

The idea of representing squares as a "propellers," that is as the combinations of four rectangular "blades" and square "axes" can be used as an alternative to solving quadratic equations by means of the usual algebraic formula "with discriminant." The advantage of this diagrammatic method over its formal counterpart consists in that it involves the construction of a square which is used as the mediating link between the given and the sought magnitudes. To begin with, it is necessary to distinguish three canonical types of equations of the second degree, depending on the signs of the coefficients:

$$x^2 + px = A \quad \text{or} \quad \square_{x(x+p)} = A, \qquad (3)$$

$$x^2 - px = A \quad \text{or} \quad \square_{x(x-p)} = A, \qquad (4)$$

$$px - x^2 = A \quad \text{or} \quad \square_{x(p-x)} = A. \qquad (5)$$

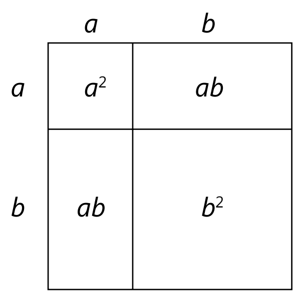


**Fig. 2**

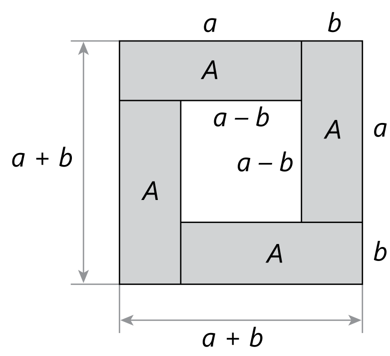


**Fig. 3**

To provide the geometric rationale behind these algebraic equations, let us assume that the unknown $x$ and the coefficient $p$ are linear magnitudes (line segments), while the known term $A$ is a plane figure (rectangle).

In each of the three cases, we should find an unknown magnitude $x$, given the known magnitudes $p$ and $A$.

Let us start with the solution of the equation (3).

► To solve this equation, let us build a square by attaching to each other four equal rectangles $A = \square_{x(x+p)}$, as shown in Figure 4. The "axis" of the propeller will be then a square with the side $p$. It follows from the diagram that the square with the side $2x + p$ is equal to the sum of four rectangles and a square; this is expressed by the formula:

$$\square(2x + p) = 4 \cdot A + \square p. \tag{6}$$

We find the square of $2x + p$ by calculating the sum of four rectangles $A$ and the square with the side $p$. Then, by extracting the root we find $2x + p$, and finally, $x$. ■

Let us now solve equation (4).

► Let us construct a square with the side $2x - p$ by adjusting rectangles $A = \square_{x(x-p)}$, as shown in Fig. 5. The "axis" is the inner square, the side of which is equal to $p$. The decomposition of the larger square into four equal rectangles ("blades") and the smaller square ("axis") can be expressed by the formula:

$$\square_{(2x-p)} = 4 \cdot A + \square_p. \tag{7}$$

Calculating the right part of the identity (7), we will find the square of $2x - p$, then, extracting the root, the magnitude $2x - p$ itself, and, finally, the value of $x$. ■

Finally, let us consider equation (5).

► Let us take as "blades" of the "propeller" four rectangles $A = \square_{x(p-x)}$. Since it is *a priori* not clear which side of the rectangle is larger than the other, two cases should be considered.

Case 1: $x > p - x$, that is $x > p/2$ (Fig. 6).

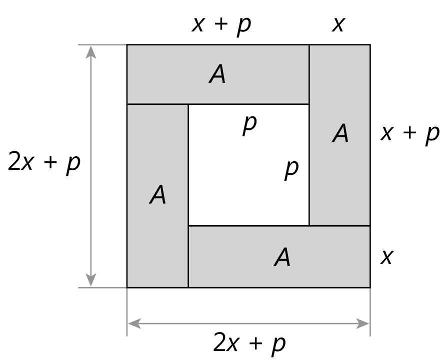


**Fig. 4**

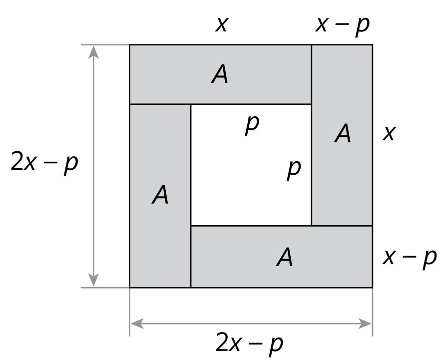


**Fig. 5**

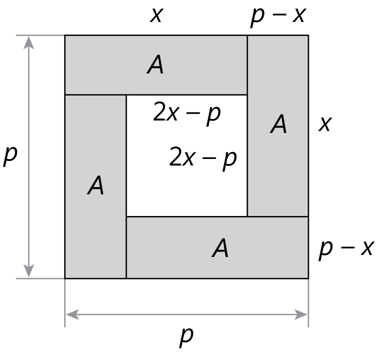


*Fig. 6*

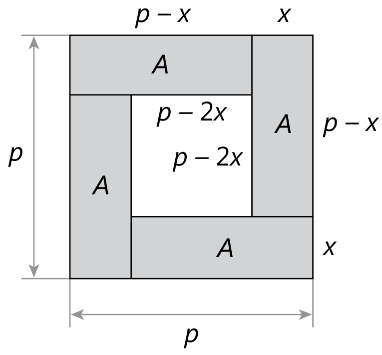


*Fig. 7*

In this case, the large square has the side $p$ and the inner one has the side $2x - p$. The decomposition of the larger square into four equal rectangles and the inner square is expressed by the formula $\square_p = 4\cdot A + \square_{(2x-p)}$, which is equivalent to

$$\square_{(2x-p)} = \square_p - 4\cdot A \tag{8}$$

Calculating the value of the plane magnitude in the right part of the formula (8), we find the square of the unknown magnitude $2x - p$, then the magnitude $2x - p$ itself, and, finally, the value of $x$.

Case 2: $x < p - x$ (Fig. 7).

Now the larger square has the side $p$, and the inner one has the side $p - 2x$. The decomposition of the larger square into four equal rectangles and the inner square gives the formula: $\square_p = 4\cdot A + \square_{(p-2x)}$ which is equivalent to

$$\square_{(p-2x)} = \square_p - 4\cdot A. \tag{9}$$

Calculating the right part of the formula (9), we find the square of the unknown magnitude $p - 2x$, then the magnitude $p - 2x$ itself, and, finally, the value of $x$. ■

In a similar way, by constructing a suitable diagram in form of "propeller," one can obtain diagrammatic solutions of systems of two equations of the second degree. Thus, this method of solution is a rather general one.

Its obvious shortage consists in the separate consideration of the three types of quadratic equations. In addition, it is impossible — within its framework — to obtain negative solutions. Here we have to turn to the usual algebraic technique based upon symbolism.

Given the pros and cons of the approach, we propose to introduce elements of geometric algebra as a propaedeutic complement to the ordinary course of symbolic algebra. The mastering of the diagrammatic reasoning at an earlier stage will allow students better understand the geometric origins of algebraic technique and increase motivation for its study. Geometric solutions of algebraic problems can be found by drawing diagrams with pencils and markers of different colors, or with the help of computer graphics, which will make the study of this technique more attractive for modern students.

In favor of the object-oriented teaching strategy, the following historical arguments can be also adduced [Zaytsev, 2011]. First, history of mathematics attests, that the stage of object-oriented mathematics always preceded its development as an axiomatic-deductive or symbolic system. History also testifies, that this preliminary stage — during which the mathematicians literally thought with objects — was more creative than the periods of formalization (this is certainly true for the main sections of school mathematics). This circumstance can be used in school teaching: putting the student in conditions similar to those that led to the development of mathematics at its earlier stages (of course, these conditions must be competently modeled by teachers), we thereby can create the prerequisites necessary for the development of his or her own creative abilities. Another advantage of object-oriented mathematics is that it usually relies on a very limited set of elementary technical means. This means that in the course of education teachers will not have to spend much time and effort on the preliminary training of formal techniques, but could at once proceed to the exposition of the main subject. Within this approach, the student will be lead not by the "logic" of constantly complicated formal rules, but by the "logic" of the subject itself. Thirdly, the use of object-oriented teaching will help loosen the rigid sequence of study of mathematical disciplines (which is now being criticized). While within the framework of formal approach, learning of subsequent sections requires a solid knowledge of the technique, mastered in the previous stages (for example, symbolic algebra is necessary for the study of mathematical analysis), the object-oriented teaching can ignore this condition. Finally, the use of diagrammatic reasoning can be justified by the fact that with the introduction of a narrow disciplinary specialization (this is the case in Russian schools) a group of students will appear that will not study mathematics at the advanced level, but would prefer the basic, i.e. minimal, one. At this level, emphasis could be placed at the acquisition of basic skills of "thinking with objects," rather than at mastering the application of memorized formal rules. Finally, we assume that "thinking with objects" would help students with disabilities.

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# ORAL COMMUNICATIONS

# ON THE CONTENT AND METHODS OF TEACHING MATH IN ELEMENTARY SCHOOL FOR GAT STUDENTS

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The pedagogical experiment in teaching mathematics which has been implementing over decades in different settings was set in motion under the psychological guidelines of the theory of step by step formation of mental actions and concepts of Prof. P.Ya. Galperin [Galperin, 1985]. This presentation focuses just on teaching in Elementary school and for GAT students.

The goals of the research were to check whether the change of the content and sequence of presenting themes along with interactive way of teaching could cause a substantial turn in motivation of students and make a difference in the development of their math concepts.

There were 2 groups of students, 16 students in each one, who came through this experiment and two other groups of same size are currently continue participation. All students were previously selected before enrolling in 1st grade; the selection of students was based on their general cognitive abilities.

Relying on the 3rd type of education [Galperin, 1976] and rearranging core curriculum completely, we've achieved the following results. The first-grade students have learnt all four arithmetic operations with natural numbers and got acquainted with basic geometric concepts. During the second year they expanded their knowledge of numbers on integers. They knew to draw graphs of functions on integer plane like $y = ax + b$, $y = ax^2 + bx + c$, $y = |x|$ and their compositions. They knew the transformations of graphs. During their third grade, they got acquainted with the language and basic concepts of mathematical logic and set theory, continued to prove theorems in plane geometry, factor polynomials, solve (in integer numbers) systems of linear equations and quadratic. By the completion of their fourth year of studies, they knew integer powers and many other things which do not present currently in math manuals for elementary students. But most important, we can state they met this challenging course with great enthusiasm and that was the major factor which made such accelerated studies possible.

The method can be applied not only far beyond the elementary years and can level the graduate requirements in mathematics at least for AP students to those of sophomores with major in mathematics, but also to other subjects like physics, computer studies, languages and even history.

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# SECONDARY MATHEMATICS TEACHERS' DISPOSITION AND POSITIONAL FRAMING TOWARD ERRORS

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University of Texas, El Paso, USA

This mixed methods study examines how teachers' disposition and positional framing toward errors are intertwined during teacher and students' moments in classroom interaction. The quantitative phase included the Error Orientation Questionnaire to measure secondary mathematics teachers' disposition toward errors. The qualitative phase involved semi-structured interviews and classroom observations of two teachers assessing their positional framing.

This paper focuses on the qualitative phase of the study. Data analysis was conducted using meaning coding technique from the grounded theory to generate themes across teachers' disposition toward errors and their positional framing based on Beyers' [2011] *dispositional functioning* framework along with the construct of *positional framing* [Greeno, 2009].

The initial findings suggest that there are multiple emerging themes in teacher productive disposition toward errors such as error usefulness, error communication, error reasoning. The emerging themes in teacher non-productive disposition are error uselessness, covering errors, error weakness, to name few. The study also found that the activation of different types of dispositions (cognitive, conative, affective) by the participating teachers is aligned with the way in which they position themselves and their students. During an interview, Angela (pseudonym), a teacher with a non-productive disposition toward errors, expressed: "When I make a mistake during my teaching, I point it out immediately and I try to correct it as soon as possible". During her teaching, Angela frames errors by focusing her teaching on the answers rather than the processes and correcting errors by herself. In contrast, Damian (pseudonym) communicated his productive disposition toward errors as saying: "When I make a mistake during the lesson and I tried to stop right there and ask some guiding questions to make them [students] see the mistake. So, instead of just telling them right away oh this is wrong, I try to see if they can figure it out". Accordingly, his positional framing in the classroom is focusing on a systematic connection between error analysis and learning, dedicating time for discussion about errors and involving students in the error correction process. The significance of this study unfolds the relationship between teachers' disposition and positional framing toward errors.

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# EXPLORING FRIEZES AND ROSETTES: AN EXPERIENCE WITH FUTURE TEACHERS

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This study, part of an ongoing project, analyses the performance of future teachers of primary education (3–12 years old) in identifying and constructing symmetries, especially friezes and rosettes, with different resources (paper/pencil; software).

Geometric transformations are one of the most important applications of mathematics in daily life, allowing the establishment of rich connections. It enables students to explore/create patterns, solve problems and think spatially. However, students generally show a low level of learning when geometric transformations are concerned (e.g., [Swoboda, Vighi, 2016]). Research supports the assertion that appropriate technology and media-supported instruction can help with learning in a variety of domains, including geometric transformations [NCTM, 2014]. So, technological tools may be useful to develop visual skills and overcame some difficulties.

To conduct this exploratory study we followed a qualitative approach. The participants were 14 future teachers of primary education. Data was collected during the classes of a Didactics of Mathematics course, through: observation, written productions and photos. They were exposed to the teaching of geometric transformations (translations, rotations, reflections and glide reflections), analysing examples of applications in mathematics and other areas. The students were also motivated to identify/construct friezes and rosettes, using paper and pencil. After a period of appropriation of the processes involved, they were invited to explore the same aspects in a dynamic environment provided by the software *Gecla*. The functions used were: *Search for symmetries*, *Classify/Generate Friezes and Rosettes*.

Preliminary results show that these students easily identify symmetries with both resources and are confortable with the construction/generation of friezes and rosettes, since it is a step-by-step process. These students exhibited difficulties in identifying the motif/module that generates some friezes/rosettes. *Gecla* aloud them to develop an intuition in some of these cases and increased their motivation, however certain students referred that the software could show the composed motif besides the minimum motif.

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# QUALITY OF DIFFERENTIATION WITH TECHNOLOGY IN MATHEMATICS INSTRUCTION BY PRE-SERVICE ELEMENTARY SCHOOL TEACHERS

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Research shows the positive effect of differentiated instruction on engagement and performance in mathematics [Konstantinou-Katzi et al., 2013], and outline tools, technologies, and practices to support differentiation in the mathematics classroom [Cabus, Haelermans, Franken, 2017]. However, there is lack of research that focus on assessment of the quality of differentiated instruction in mathematics classrooms. The purpose of this study is to analyze differentiation skills with technology of elementary pre-service teachers (PSTs) enrolled in a graduate level pedagogy course, focusing on integrating technology in special education and inclusive classrooms.

The authors developed and validated the Differentiated Instruction Rubric (DIR). The DIR uses two categories suggested by [Roy, Guay, Valois, 2013] — Instructional Adaptations (IA), and Academic Progress Monitoring (APM). Content validity and inter-rater reliability were confirmed by two independent raters with Pearson r-values ranging from 0.723 to 1.000 ($p < 0.001$). Cronbach's alpha confirmed internal consistency ($\alpha = 0.830$).

Lesson plans and videos of math lessons taught by 170 PSTs in grades 1–5 were scored using the DIR. The scores in two categories were compared using a paired sample t-test. The results indicated that IA scores were significantly higher than APM scores ($p < 0.001$) with small effect size (Cohen's $d = 0.212$). The scores also indicated that PSTs interpreted differentiation only as adaptation for students with special needs, but demonstrated lack of differentiation necessary to meet the needs of all students. The study is currently in the stage of qualitative data analysis that will shed more light on approaches PSTs used to differentiate mathematics instruction with technology.

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# VIEW ON EQUATIONS–SEQUENTIAL VERSUS RELATIONAL

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The solving of linear equations has been intensively investigated in mathematics education research. In order to solve mathematical problems with linear equations successfully, students need to (1) translate them into a linear equation and (2) use equivalence transformations to solve the equation. We assume that — after a teaching period in algebra — students, who have not yet understood how to apply linear equations would meet obstacles when they are asked to use a representation system that is new to them. Our aim is to identify such obstacles. For our study, we have used the MAL-System (MAL stands for Multimodal Algebra Learning [Reinschlüssel et al., 2018]) as a new representation system; this is a digital system implemented into a tablet where tangibles (or tiles) represent numbers and variables in two versions, (1) purely digital or (2) as a hybrid version of haptic tangibles combined with digital feedback. On the tablet screen the two sides of an equation are represented in a mat. A colored vertical line in the middle offers feedback, yellow says there is no mistake, red means there is a mistake and green indicates that the task is solved. Colors of the tiles represent signs, blue the positive and red the negative sign. Squared tangibles are units and lengthy tangibles variables. In the course of the solving process, equations are also represented by algebraic symbols on the screen. 22 students from a school with highly proficient students of grade 7 are asked to solve five tasks with one of the two systems. Student pairs of similar achievements in algebra used either the digital or the hybrid version. Data consists of video recordings of the solving processes and the students' post descriptions of how they solved the tasks. The video data are analyzed to reconstruct obstacles. The system turns out to be quite intuitively accessible for the students. In most cases, they try to solve the tasks by first translating them into symbolic algebraic expressions in their imagination and then they translate this expression into the new system. Preliminary results show three main obstacles: (1) tiles with a negative sign are difficult to handle, (2) the product of a number and a variable like 2x is often misinterpreted (one tile for x and 2 unit tiles), (3) the students interpret application tasks sequentially, word by word, rather than relationally. The third obstacle seems to cause in-depth difficulties, e.g. all the students made the inverse error and several students tried to solve the equation by manipulating just one side.

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# THE PROBLEM OF CONNECTION BETWEEN PEDAGOGICAL SCIENCE AND TEACHING PRACTICE

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It is well known that numerous studies, both pedagogical and psychological, are completely unclaimed in practice. What is the problem? Is it laziness or illiteracy? Is it the absence of an organization? Is it the incomprehensibility of scientific reasoning? These and many other questions frame two main ones: why is this happening and how can it be fixed? We try to answer these two burning questions.

The problem, as it turns out, is that pedagogical science is in a state of intra-scientific crisis, the protracted nature of which is determined by the specifics of pedagogical activity. The key concept which discover the essence of this crisis is differentiation of the phenomenological and empirical knowledge.

This differentiation is based on the methods of obtaining the knowledge, it is observation in the first case and experiment in the second one. The first method produces phenomenon which is usually structured and described by external (with respect to the object of observation) means appeared to be metaphorical. Phenomenological knowledge describes only "what may occur."

To the contrary, the empirical knowledge describes the object in terms of it's own properties shown in the act of changing of this object and has the strict form "conditions-action-result" which allow to reproduce the same action at the same conditions and obtain the same result.

The problem of the pedagogical science is that it is almost pure phenomenological and can't pass to the empirical state (we call it "phenomenological crisis"). The cause is the simple fact that the pedagogical action is unreproducible because the conditions could never be "the same."

The proposed solution to this problem consists in the constructing the reproducible result not in terms of the acting but just in terms of the thinking. The thinking by means on the schemes, which allow in any particular situation construct the action giving the result we need. It is not a fantastic idea but the practice of the pedagogical education at MSU. Instead of the discussion "what to do" at this or that situation we discuss "what and how to think", and this provide the reproducible results.

Development of pedagogical thinking in our practice at the present time is based on about 30 schemes and principles that allow build an adequate pedagogical action solving almost all typical pedagogical problems (as, for example, the problem of the self-control, motivation, communication, and others).



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# COMPUTER RESEARCH IN TEACHING GEOMETRY AT THE UNIVERSITY

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Many pedagogical studies have theoretically proven and experimentally confirmed that empirical research activities in education has great potential, since students best learn information that they have discovered independently. The challenge of constructing a technique for teaching Bachelors-level Geometry (for students in the field of Mathematics and Computer Science) with elements of computer research and experiments is examined in this paper. The methodology of the research comprises the analysis and generalization of the scientific research results made by Russian and foreign scientists specialized in the sphere of teaching Geometry in the framework of higher education and based on productive, academic and objective approaches. Some peculiarities of the educational process of classical universities in developing the methodology of experimental research were taken into account, including: the study of the fundamentals of science, of science itself in development; the link between the independent work of students with the research work of teachers; and the unity of scientific and educational foundations in the activities of a teacher. The use of computer research and experiments in teaching Mathematics in the framework of higher education allows us to improve the content of academic courses, to increase the number of tasks and exercises for self-study, to develop practical skills for conducting mathematical reasoning and to simulate and illustrate the concepts and objects being studied. These aspects will provide an opportunity to explore certain topics thoroughly, to motivate students and to increase interest in the discipline as a whole. Furthermore, the use of computer research allows us to bring the scientific work of students in Mathematics to a fundamentally new level. The article also touches upon the role of the problem as an invariant of methodological support of teaching Geometry and changes in its structure in conditions of using computer tools. The content and organizational conditions for the implementation of the experimental and research trainings in Geometry are studied, and examples of research problems in Geometry with various degrees of complexity are examined.

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# SCHOOL ENGAGEMENT AND MATH PERFORMANCE: RATINGS FROM STUDENTS AND TEACHERS IN RUSSIA

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School engagement reflects children's investment in learning at their school. Mathematics is core in school curriculum and fundamental for children's scholastic and career success. Critically, it is unclear how school engagement relates to mathematical performance in Russian children. This study aims to determine the degree to which student school engagement and teacher's assessment of school engagement of students can predict math grades in the classroom. To test this hypothesis, 100 Russian school-aged children (ages 6–12 years) and their teachers were chosen using simple random sampling method from schools in Moscow. Students were asked to complete a school engagement scale that addressed emotional, behavioral and cognitive factors associated with school activities. Also, we developed a new scale to evaluate teacher's assessment of student's school engagement. Teachers were asked to complete this scale that addressed emotional, behavioral and cognitive components of student's school engagement. In addition, we collected data on children's classroom performance in math, science, language and reading. Results showed that both teachers' and student's evaluation of cognitive school engagement were predictors of math and science marks, and as expected the teacher's evaluations were stronger. Both teachers' assessment of behavioral school engagement of students and behavioral school engagement of students predicted language and reading marks than other components. Interestingly, behavioral school engagement of students was a stronger predictor of language marks than teachers' assessment of the same component. Furthermore, teachers' assessment of students' behavioral school engagement was a stronger predictor of reading marks than behavioral school engagement. Together these findings suggest a relation between teacher's and student's assessment of school engagement. Theoretically, these findings aid psychologists and educators to design interventions and to develop instruments to track the student performance in different subjects more precisely. Practically, shared assessment can serve as an improved approach for educational assessments.

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# A PRIMARY PRE-SERVICE TEACHER´S INSTRUMENTAL ORCHESTRATION OF ROTATIONAL SYMMETRY

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The study investigates the explanatory power of Artifact-Centric-Activity-Theory (ACAT) framework [Ladel, Kortenkamp, 2016] through its confrontation with a primary pre-service teacher's (JS) teaching episodes, in which JS orchestrates her dynamic geometry concept with 17 children (6 girls, 11 boys) of Grade 4 in a computer based teaching-learning environment. An adapted ACAT model (figure) for this study describes the process of instrumental genesis (Artigue, 2002), the different levels of activity theory in internalization and externalization mediated by dynamic geometry artifacts, and social interaction in primary classroom in view of the instrumental orchestration (Drijvers, Trouche, 2008). JS introduced *GeoGebraClassic* (*Geometry*) dynamic geometry system to the children who are using it for the first time. In the following week, JS orchestrated the "rotational symmetry" episode with the children. Her instrumental orchestration is interpreted through a didactical configuration, an exploitation mode and a didactical performance (Drijvers, 2012). JS used a central screen; the children were working individually on computers following JSs instructions. They engaged themselves with the instrumental genesis using the instrument "*rotate about a point*" of GeoGebra pacing differently corresponding their heterogeneous performance levels. Observation, JSs lesson plan, screen recording of children's work on computers and interview are used to collect data. Analysis of the data suggests that the ACAT framework is productive for analyzing JSs competencies, particularly in combination with the theory of instrumental orchestration perspectives. Some children (S4, S6) had difficulties with instrumentalization. JSs orchestration found to be generally effective, as almost all the children appropriated the instruments to rotate the triangle about a point bonus with the fascination of animating the triangle through 360° illustrated with a windmill rotation.

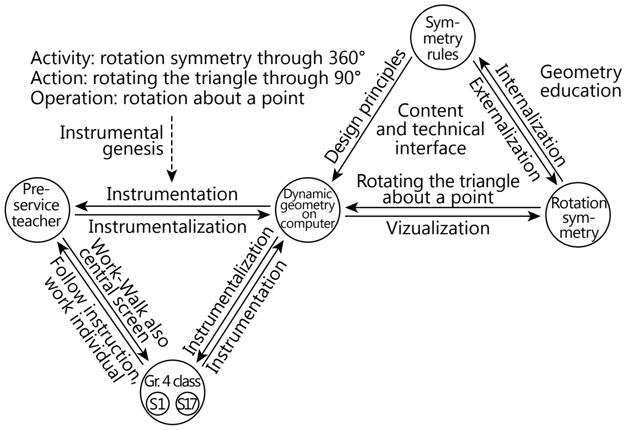


*Fig. 1.* An adapted ACAT model

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# THEORETICAL JUSTIFICATION OF A STRUCTURE MODEL AND THE FORMATION OF MATHEMATICAL THINKING

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In the psychological-pedagogical literature, there is a large number of works describing empirical observations and conclusions about the structure and patterns of the formation of mathematical thinking. But quite rarely, these studies has a theoretical rationale and foundation. The intent of our research — to build a model of the structure of mathematical thinking and to describe the laws of its development, based on methodological and theoretical principles of psychology.

According to J. Bruner, the curriculum must reflect the structure of the studied science, which can even be abstracted. However, the difficulty is that we do not yet know how exactly the basic structures (reference axes) of the discipline should be formed. Answering this question, A.Poincare. R.Tom, M.Minsky suggested that the most of abstract structures (Boolean algebra, topological structures) are always present in an implicit form in the child's psyche. Having traced their genesis, J.Piaget [1995] established the sequence child's appearance of the basic mathematical substructures identified by N.Burbaki.

Based on these and other theoretical positions, we modeled the structure of mathematical thinking. It represents the intersection of five main clusters (substructures of thinking) that are homomorphic to the main mathematical structures: topological, projective, ordinal, metric and algebraic (N.Burbaki, J.Piaget).

It turned out that this mathematical structures play a different role in thinking and each individual has his own hierarchy and a dominant cluster. We have established that the dominant cluster manifests itself in all mathematical actions, and depending on it, each chooses his own individual solution method adequate to his cluster. The process of thinking through the dominant cluster is carried out "automatically", simultaneously and minimized. The teacher's help to the student should be to provide this reformulation. Finally, for the development of mathematical thinking it is enough developing only the dominant cluster. It influences and leads the development of others clusters that are developing, thanks to the “formal effect of learning” [Vygotsky, 1982].

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# ROOT OF 4
# AND MATHEMATICS TEACHER TRAINING

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It has been shown by several studies that while the symbol $\sqrt{4}$ is familiar to students of mathematics starting from the middle school, some believe that the value is 2 as others insist on two values ±2. The concept of square root is used in algebra, calculus and complex analysis. Undergoing didactic transposition in presentation by textbooks and instructors, it may produce students' confusion. Our research question is: how can one improve the situation at the level of teacher training? Our data was collected from 60 pre-service teachers in Canada, 67% of whom had confusion about radicals. While confirming the presence of diverging opinions, we disagree that square root is a concept that "lacks a consensual approach in the mathematical community" [Kontorovich, 2018] and question the proposal to reconsider the labels 'erroneous' and 'correct'. Instead we claim: 1. *The* mathematical community that upkeeps a theoretical basis of modern technology *does* have a consensus on the matter, as consistency is one of the main principles of science. The community distinguishes concepts of *a square root of* 4 (which is 2 or −2) and *the* (*principal, or arithmetic*) *square root of* 4, denoted by $\sqrt{4} = 2$. There is the univalent square root function $y(x) = \sqrt{x}$ and $y(x) = 2$. In algebra, the two roots of the equation $x^2 = 4$ are $\pm\sqrt{4}$. In complex analysis there is the multi-valued root function (inverse to $z \to z^2$) and its principal branch. 2. The inconsistency of presentations by textbooks and teachers and also of students' conceptualization is rather a characteristic of *a* subcommunity where individual meanings of the concept are being constructed. These should be carefully compared against the professional "cultural meaning" and corrected accordingly. 3. The corrections could result from resolution of inconsistencies, a process that mimics the scientific development. Such tasks are appropriate for shaping teachers' content knowledge. In our setting, students contextualized their views by answering more elaborate questions than just a root evaluation. They also made their own questions, such as: If $\sqrt{*}$ is multivalued, what becomes of equation $\sqrt{3 + 2x} = -x$?; By uniqueness, $\lim_{x \to 0}\sqrt{4 - x} = \pm 2$ can not exist, does that agree with the graph (on my gadget) of the function $y = \sqrt{4 - x}$ at $x = 0$? The instructional practices of either telling tricks for exam or letting students keep their misconceptions and inconsistent reasoning are equally irresponsible. As students' survey showed, 90% of them benefited from discussions where the source of contradiction was explicitly articulated and marked as erroneous.

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# PRE-SERVICE MATHEMATICS TEACHERS' TEACHING AND LEARNING CONCEPTIONS

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Teaching and learning conceptions often refer to teachers' beliefs about their preferred ways of teaching and learning. Studying teaching and learning conceptions of pre-service teachers has been vital. Pre-service teachers bring in existing beliefs about the teaching and learning of different subject areas as they enter university. However, these beliefs may not align with those advocated in the curriculum. If misaligned beliefs remain unchallenged, they are likely to be perpetuated among students and thus hinder their conceptual understanding of subject matter.

This study assessed teaching and learning conceptions of 80 pre-service mathematics teachers (PMTs) with the Teaching and Learning Conceptions Questionnaire [Chan, Elliott, 2004] on a 5-point Likert scale (1: Strongly Disagree to 5: Strongly Agree). Parallel analysis of the data found two correlated conceptions, namely the traditional conception (TRAC) with 17 items and the constructivist conception (CONC) with 12 items. Alpha reliabilities for the TRAC and the CONC were 0.880 and 0.825 respectively. On average, the PMTs held a more CONC than a TRAC. Whereas female PMTs were more likely to hold the CONC than male PMTs, the two conceptions were not associated with year of study, number of professional courses completed, and number of teaching practicum completed.

This study offers empirical findings for PMTs to reveal their teaching and learning conceptions and make informed pedagogical decisions, which constitute part of the professional competence for future teachers [Blömeke et al., 2008]. Teacher educators may design learning experiences for PMTs to consolidate or moderate conceptions that align or contradict with existing beliefs. More research is needed to identify other factors that may predict teaching and learning conceptions of PMTs, including mathematical beliefs, mathematics self-efficacy, and mathematics teaching efficacy.

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# CULTURAL TRANSPOSITION AND HYBRID LESSON STUDY

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The Lesson Study (in Mathematics) has become a very popular topic in the research on teacher education in the last twenty years. The model was born in the East, in the so-called Confucian heritage culture area, developing primarily in Japan (jugyokenkyu), then in China (guan mo ke) and later in other east countries. Afterwards several realizations have been developed in many other countries, also in the Western area. One of such realizations considers a perspective of an Hybrid Lesson Study — HLS [Ribeiro et al., 2018]. Nowadays the debate of how much these HLS, spread all over the world, respect or not the "original" nature of the East Lesson Study is very hot. In this scenario, we propose a different perspective in which, according with the Cultural Transposition (CT) framework [Mellone et al., 2019], we look at the contact with education practices coming from different cultural contexts as condition for decentralizing the assumptions rooted in specific cultural paradigms. In this sense, we think that the implementation of HLS in different contexts, when accompanied with a suitable cultural sensitiveness, can represent an opportunity to develop an awareness of the educational intentionality embedded in specific cultural contexts, in particular our own ones. In this communication we will illustrate the CT framework, together with some examples of HLS, inspired in the Chines Lesson Study, implemented in Italy within the scope of the last three years of experimentations. According with the CT framework, we will discuss how the implementation of these HLS let us to recognize some contrast features, assumptions embedded in Chinese and Italian educational culture (e.g., the recognition of the class as public *vs* private space; the important role played by imitation or discovery in the learning process; the care to the short or long term goals of teaching; the hierarchy relationships implicit within the class *vs* a practice guided by the principles of dialogue in democracy).

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# PREPARING MATH TEACHERS FOR SCHOOL DISCOURSE

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Current research in various fields, including in pedagogy, appeals to such concepts as discourse, language personality and language behavior. The emphasis on language and speech seems to be quite promising for research in mathematics teaching methods. In our previous study [Mugallimova, 2015], we singled out the concept of mathematics learning discourse, referring to a set of communication tools in an educational situation that are grouped around mathematical texts and students' subjective experience in their mutual semantic content.

At present, we use the term "educational mathematical discourse" with the following interpretation: educational mathematical discourse is the practice of organizing educational activities based on working with mathematical texts in accordance with accepted rules of mathematical activity and the norms of mathematical culture.

Such an interpretation of the term differs from that adopted in the English-language literature, where discourse refers to dialogue in a lesson, such as a question-response activity. Our understanding of the concept under consideration is close to the formulation proposed by E.A. Kozhemyakin [2009]. Based on this, we understand the discourse space as a metaphorical representation of various teaching tools and tools used in the educational space such as elements of the educational environment, ways of presenting educational information, ways of organizing communication in the learning process, etc. — that all directly or indirectly involved in immersion in educational discourse.

Thus the teacher is a subject carrying mathematical texts. We can formulate the informative characteristics of the discursive competence of a mathematics teacher, implying at the same time an integrative characteristics of professionally significant qualities, which includes:

1) knowing the principles of constructing a mathematical text and the rules of pedagogical communication;

2) the ability to select texts in accordance with the studied material and educational situation;

3) skills of perception, generation and interpretation of educational mathematical texts, and

4) experience in planning and managing communication in an educational situation using mathematical texts.

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# AN INVESTIGATION OF STUDENTS' RECOGNITION OF GEOMETRIC SHAPES IN THE ARTS STUDIO

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Spatial thinking forms the basis for learning mathematics [Clements, 2004] and visual arts [Goldsmith et. al., 2016]. It is seen as an overlap between visual arts and mathematics. However, there is a need for studies with theoretical basis that provide strong evidences of what kind of thinking processes evolved in such an overlap. This study investigates students' ways of recognition of geometric shapes, considered as one of the indicators of spatial thinking, in an arts studio environment. It was designed by the researchers based on artful and studio thinking frameworks [Hetland et al., 2013; Tishman, Palmer, 2006]. Three studio works with spatial content were implemented in this environment. Each studio works consists of three structures: observing geometric artworks, creating artworks, and describing and evaluating artworks. This environment was used as a tool to make students' thinking visible and investigate students' recognition of geometric shapes. The participants of this environment were six seventh grade students in a public middle school. The data sources of the study were interviews, observation notes, and students' documents (sketches, artworks, and notes). The studio works and interviews were audio and video recorded. The data analysis was conducted on the basis of studies on spatial thinking [Clements, 2004]. The findings of the study revealed that students reflected four ways of recognizing shapes in such environment: (1) relating geometric shapes with real life objects, (2) identifying geometric shapes and their properties, and (3) identifying shapes from different perspectives, (4) identifying shapes through disembedding and embedding. Therefore, the study suggests that this environment has a potential for eliciting students' recognition of geometric shapes and other ways of spatial thinking in the future studies.

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# PROSPECTIVE MATHEMATICS TEACHERS' ATTITUDE TOWARDS THE USE OF HISTORY OF MATHEMATICS IN MATHEMATICS TEACHING

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The purpose of this mixed research study was to explore the prospective mathematics teachers' (PMTs) attitude towards the use of history of mathematics (HM) in teaching mathematics. HM is considered as an alternative approach in mathematics teaching based on utilizing mathematics' primary and secondary sources of HM [Tzanakis, Arcavi, 2000]. In the context of using HM, Sullivan [2000] revealed that incorporating history in teaching mathematical concepts had positive effects on pre-service secondary mathematics teachers' attitude. Regarding this issue, this study investigated the PMTs' attitude on the basis of Tapia's [1996] attitudinal four dimensions: value, enjoyment, self-confidence and motivation. Exploring the PMTs' attitude towards the use of HM, researcher utilized sequential explanatory research design. A survey and semi-structured interviews were conducted into two sequential phases by utilizing Attitude Scale and Semi-structured Interview Guidelines respectively. The content and construct validity of the instruments were assured by scrutiny through 5 experts and factor analysis; and reliability was maintained by Cronbach alpha ($\alpha = 0.875$, $N = 305$) to assess the internal consistency of the Attitude Scale items. The Attitude Scale was administered to 305 PMTs in Phase I; and interviews were conducted on 8 PMTs in Phase II. The five point Likert survey data were analysed through the descriptive and inferential statistics; and semi-structured interviews were analysed by coding and categorizing themes. The findings derived from the analysis of both types of data were connected in Phase III and revealed that the PMTs had high attitude towards the use of HM. The findings indicate that PMTs open to use the HM in their future professional life due to their higher attitude towards it. A further study might be conducted among in-service teachers to strengthen the argument of using HM in pre-service programs.

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# FIRST-YEAR STUDENTS' SOCIO-PSYCHOLOGICAL ADAPTATION AT THE MECHANICS AND MATHEMATICS FACULTY

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Higher mathematical education is traditionally one of the more difficult for students due to the high level of material content complexity itself and its large volume. The present study was implemented at Perm State National Research University. According to an analysis of student number dynamics at the Mathematics and Mechanics Faculty, the number of graduates who successfully completed their studies did not exceed 45% of the number of those who entered the first year.

In order to improve the success of the curriculum development, an empirical study was organized and conducted, aimed at exploring students' personal characteristics and academic motivation. We were guided by the data obtained in the studies conducted by Ovcharova [2013] and Trofimova [2013]. The results showed that first-year students have high socio-psychological disadaptation indicators, characterized by a lack of self-awareness and communication skills, as well as low levels of personal activity and social status. These personal features contribute to difficulties in adaptating to student groups, and cause problems in subject mastery.

The results of our study describe those personal characteristics of first-year students that determine difficulties in the academic process. Among them are a low level of subject preparation, unformed motivation, a low level of socialization, and the absence of a clearly understood system of value orientations. A program of psychological and pedagogical support for students that aimed to create motivation was also developed in order to increase their adaptability and academic success. Educational and methodological recommendations were developed that reflect the necessary changes in curricula and applied pedagogical technologies. A program of psychological support for students and teachers in individual and group formats is proposed.

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# TEACHING AND LEARNING MULTIPLICATIVE RELATIONSHIPS: ACTION RESEARCH IN ELEMENTARY SCHOOL

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Traditional curricula in many countries dictate the study of arithmetic operations prior to solving problems of multiplicative structures. In these contexts, in order to understand and solve a problem, students are expected to apply their knowledge of operations. In our 3-years research project, we try to inverse this order. Following and extending Davydov's [2008] idea of quantitative relationships, we designed multiple learning activities. Teacher participants tested and discussed these activities and then implemented them in their classrooms. These activities focus on turning students' attention to the multiplicative relationships between quantities rather than to numbers and key words. The discussion about the meaning of these relationships and their visual representations (schematizing and modelling) took the main bulk of each activity. Rather than using arithmetic operations as a mere tool to understanding a given problem, the identification of arithmetic operations was reached at as a result of sense making of quantitative relationships involved.

Data comprised video-recordings of interviews with teacher-participants and of lessons as well as students' written work on problem solving that was administrated at the end of the third year of the project. Following the use of action research and iterative rounds of analyses of the interview data, the recorded lessons, and students' work our findings show:

1. Teacher-participants witness important changes in their understanding of the underlying principles in the teaching of mathematics. They specifically foreground doing mathematics together with students, listening to students' thinking and reasoning, taking time to discuss students' reasoning, probing students' thinking by provoking a doubt in the obtained answer, and focusing on long-run learning achievements rather than on the immediate success of solving one particular problem.

2. Students demonstrate more inclination to discuss ideas, defend their reasoning, and elaborate on their thinking processes while paying attention to quantitative relationships thus reflecting better understanding in solving problems.

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# DEVELOPMENT OF INITIAL MATHEMATICAL CONCEPTS


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The development of mathematical concepts, regardless of the stage of training, is characterized by the necessity for acquisition of mathematical symbolism.

G. Piaget [2003] viewed the symbolic function as the ability to represent a missing object or event with symbols or signs not directly perceived. As an individual mechanism which is manifested in various systems of representations, it is essential for the emergence of mental interactions between individuals and for acquiring collective meanings.

The world is represented and perceived by man using objects and cultural artifacts. They are represented by means of artificial languages, which are holistic systems. Mathematics has a special language which allows representing special, quantitative relations of "man-the world". The mathematical language is the first scientific language for a child. Scientific language does not record obvious, indisputable objects, phenomena and processes.

With the help of conventional symbols, a mathematical language shows integrity, particularity, functional relations, change of states, time reference and so on. An important part of the mathematical language is the alignment of the system based on a certain measure. Measure performs the same function in mathematics as tone in music or point and line in painting. In this case, the measure acts as a constant value only for a particular system. This arbitrariness gives the mathematical language its flexibility, which makes mathematics the universal language of science. The main task of mathematics education is the development of skills to determine the measure and build a system.

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# THE RELATIONSHIP OF "FIELD INDEPENDENCE/DEPENDENCE" — COGNITNIVE STYLES WITH SUCCESS IN MASTERING MATHEMATICS

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The study aimed to determine the general cognitive abilities that contribute to successfully mastering mathematics by students of different specialties. It was performed over a period of five years at ITMO University.

A significant correlation between the ability to construct mathematical concepts and the general ability to operate mental images is presented in one of the first empirical studies of its kind [Seifert, Osorina, 2015]. Two cognitive styles, "field dependence" and "field independence" were studied to identify differences in personal characteristics [Sternberg, 2009].

The research involved 93 first- and second-year students (average age 18.3 years, 22% girls and 77% boys). Gottschald's figures were used as a research method. The subjects were offered 30 disguised figures, each required to find one of five reference figures and specify it. The results were compared with the success of mastering mathematics as evaluated by mathematics teachers using a quantitative scale from 0 to 20. Correlation analysis revealed a significant correlation between these indicators ($r$ = 0.72, $p$ < 0.001) in the whole sample ($N$ = 93). However, the correlation was insignificant for some groups. A subsequent cluster analysis clarified the situation. The results are interpreted as follows: 1. Since the solving of mathematical problems in most cases requires division of context typical of the field-independent style, followed by its restructuring, intensive study of mathematics contributes to the development of the field-independent style of perception and information processing. 2. Field independence is among those cognitive features of perception and information processing that provide assimilation of mathematical concepts. Therefore the "threshold" value of the field independence factor is required to master mathematics. 3. "Field dependence/or field independence" cognitive style is one of the necessary conditions for mathematical thinking, along with the ability to operate mental images. 4. The results of this pilot study support the hypothesis of field dependence as a cognitive ability that, provide analytical functions of thinking.

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# QUALITATIVE ANALYSIS OF LOWER SECONDARY SCHOOL MATHEMATICS TEACHERS' TOPIC-SPECIFIC CONTENT KNOWLEDGE: CROSS-NATIONAL STUDY

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Conducting cross-national studies allow comparing, sharing, and learning about issues in an international context which in turn helps researchers understand their own context, teaching practice, teacher knowledge, and student learning. The field lacks research that provides an in-depth analysis of the various facets of teacher knowledge at a topic-specific level. This study examines the U.S. and Russian teachers' content knowledge through the lens of topic-specific context — division of fractions.

This interpretive cross-case study aimed at the examination of the U.S. and Russian teachers' topic-specific knowledge of lower secondary mathematics. In total, $N = 16$ teachers (8 — from the U.S., and 8 — from Russia) were selected for the study using non-probability purposive sampling technique. Teachers completed the Teacher Content Knowledge Survey which consisted of multiple-choice items measuring teachers' content knowledge at the cognitive levels of knowing, applying, and reasoning. Teachers were also interviewed on the topic of fraction division using questions addressing their content and pedagogical content knowledge. In order to analyze the qualitative data, we conducted meaning coding and linguistic analysis of teacher narratives as primary methods of analysis.

The study revealed that there are explicit similarities and differences in teachers' content knowledge as well as its cognitive types. One of the key points of similarity was observed in teachers' responses to the question on important objectives of the fraction division at different cognitive levels. On the other side, one of the revealing differences was reported in teachers' use of mathematical vocabulary: Russian teachers were inclined to use more accurate terminology than their U.S. colleagues in explaining fraction division. The most evident difference between two groups of teachers was observed on the question examining meanings of fraction division ($\chi^2 = 9.474$, $p < 0.05$). Thus, results of the study suggest that in the cross-national context teachers' knowledge could vary depending on curricular as well as socio-cultural priorities placed on teaching and learning of mathematics.

The study's main findings contribute to the body of literature in the field of cross-national research on teacher knowledge with a narrow focus on a topic-specific knowledge. It suggests close comparison and learning about issues related to teacher knowledge in the U.S. and Russia with a potential focus on re-examining practices in teacher preparation and professional development.

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# THE INFLUENCE OF MOTIVATION ON STUDENTS' ACHIEVEMENTS WHEN STUDYING MATHCAD

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For every teacher, the interest of his students in the subject of study is important. When the students themselves seek to obtain high-quality knowledge, they do not need additional external motivation. It has been shown that the growth of academic achievements at the beginning of training strongly correlates with the level of students' intelligence, while further growth of academic achievements is more associated with the trainees' motivation [Murayama et al., 2013]. This is partially confirmed by our research [Lazareva, Ustinova, 2018], in which it is shown that the achievements of students in mastering a course may not be related to their initial preparation.

The purpose of this work is to study whether it is possible to influence the achievements of students with the help of learning motivation. The teacher informs students that they will need to use MathCad in another course. The task of the research is to find out whether such motivation influences students' achievements when studying the MathCad package.

For the study we selected first-year students (first group) studying the MathCad system in an informatics course and second-year students (second group) who became acquainted with this system and used it in laboratory work on the theory of probability and mathematical statistics.

Students of both groups passed the same control test on knowledge of MathCad. In accordance with the Mann-Whitney criterion, it was found that the distinction in test scores is significant. The first group of students showed better results than the second. We believe that this can be explained by a higher internal motivation of students studying computer science. This result may also be associated with a greater interest in studying that first-year students have in comparison with second-year students. Our research showed that it is very difficult to increase the internal motivation of students.

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# FOSTERING PROBLEM SOLVING DISCUSSIONS THROUGH A GALLERY WALK

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This paper presents part of a project about the potential of visual strategies in problem solving by preservice teachers. The aim of the study was to identify and understand the contribution of a gallery walk (GW) to foster productive discussions about solving problems, as well as the participants' reaction to the GW.

School mathematics requires effective teaching that engages students in meaningful learning through individual and collaborative experiences, giving them opportunities to communicate, reason, be creative, think critically, solve problems, make decisions, and make sense of mathematical ideas [NCTM, 2014]. Assuming that the tasks used in the classroom are the starting point of students' learning, teachers should orchestrate productive discussions emerging from tasks that allow multiple (re)solution strategies and provide the use of different representations, in particular visual ones (e.g., [Ibid.]). We argue that mathematical learning should lead students, including preservice teachers, to think visually and develop this ability through experiences that require such thinking (e.g., [Presmeg, 2014]). The GW [Fosnot, Dolk, 2002] emerges as an instructional strategy to contemplate in classroom practices, which allows students to share their productions in posters fixed around the classroom and receive feedback, requiring them also to move around the room, and engage in collective discussions

We adopted an exploratory qualitative approach. Data was collected through observations and written productions, regarding the proposed tasks and written comments from a teaching experience carried out in a curricular unit where a GW was implemented to solve problems. The results allowed to identify the strategies used by future teachers, that appeal to visual resolutions, and to verify the potential of the GW for the improvement of the discussions that contributed for the enlargement of their repertoire of strategies and engagement on the GW instructional strategy.

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# USING MEASUREMENT INSTRUCTION TO IMPROVE NUMBER SENSE IN KINDERGARTEN STUDENTS

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## RATIONALE

Children's early understanding of the association between symbolic numbers and corresponding magnitudes ("number sense") plays a key role in math learning. The current study tested a novel way of facilitating number sense in young learners. It examined whether engaging children in measurement instruction improves their understanding of numeric magnitude. Measurement activities provide an excellent opportunity to link spatial and numeric reasoning. We hypothesized that measurement presents a useful context not only for the acquisition of measurement skills but also for the development of numeric magnitude understanding.

## METHOD

The study included 88 kindergarten students from Moscow, Russia (45% girls, mean age: 77 months). All participants took part in a 5-week training. They were randomly assigned to experimental or control condition. In the *Exp. condition*, they engaged in measurement activities that focused on developing the understanding of unit and the relation between the size and number of units. In the *Contr. condition*, children engaged in math activities on topics covered during regular class time. Before and after training, children's understanding of numeric magnitude was assessed using experimental tasks. For example, the Number Distance task required identifying which of the two numbers (e.g., 4 or 9) was closer to the target number (e.g., 6).

## RESULTS AND CONCLUSIONS

On the Number Distance task, there were no pre-test differences between the Exp. and Contr. conditions (32% and 30% correct, respectively). However, after training, the Exp. group performed better than the Contr. group (66% and 53%, respectively). The same pattern of results was obtained across other measures of numeric magnitude. The findings showed that measurement activities focused on conceptual understanding of units provide a powerful tool for developing number sense in kindergarten students.



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# DEVELOPMENT OF SYMBOLIC NUMBER SKILLS IN PRESCHOOL AGE

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The development of the preschoolers' executive functions (EF) is a powerful predictor of the success of mastering early math knowledge and skills as well as the further math achievements in school (e.g., [Bull, Lee, 2014; Fuhs et al., 2014]). However, the various components of EF may correlate differently to mathematical achievements at the preschool age [Yeniad et al., 2013]. This longitudinal study aims to investigate the development of the three main EF components (working memory, inhibition, cognitive flexibility) and the math skills of children throughout preschool age. The first phase of the study has included assessment of 378 children 5–6 years old ($M$ = 5.6 years; 198 boys). On the second phase of the study, 230 children aged 6–7 years were tested again at the end of last kindergarten group (one year later). The study used NEPSY-II diagnostic complex subtests and the DCCS method for assess the level of EF [Almazova et al., 2017]. Five math tasks were administered to measure different aspects of number knowledge: Counting, Number Identification, Number Reading, Number Writing, Number Comparison [Vasylieva et al., 2018].

The results show that children who had a high level of EF coped with all the mathematical tasks more successful then children with low EF level. Based on the linear regression analysis it was concluded that visual working memory and inhibitory control have the greatest influence on the development of the mathematical skills then verbal working memory and cognitive flexibility. Thus, the research has shown the influence of EF components on the various mathematical skills in senior preschool age.

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# A PHENOMENOLOGICAL APPROACH TO MATHEMATICAL LIVED EXPERIENCE: TOWARDS A RADICAL CHANGE OF ATTITUDE TO STUDENT'S (MIS)CONCEPTIONS

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The current research is based on a **case study,** from a course that took place in the academic year 2010–2011, involving 13 students training to be teachers of mathematics in British secondary schools. Having as a starting point the case of Diana (pseudonym), and the student's arduous course towards achieving "mathematical confidence" I sought for the genetic relation between her mathematical intuition and the radical consequences that it had for her long-rooted conceptions on learning and teaching, and for her according attitudes. I followed the effect that the student's mathematical activity had for these conceptions, and I analysed phenomenologically the relation between her initial intuition and the ensuing ones. The findings of this research are encouraging for the better understanding of how a change of attitudes may take place, and its close relation to *intuitions* in Husserl's sense — namely as an *immediate fulfillment* of *intentionality*, *filled with certainty,* without conclusive material at hand (at least not in a formal, significative sense), yet with an apodictic and generalising tension and seal. This approach to intuitions allowed the connection between the student's *mathematical* intuition (which served the purpose of the open task, in the classroom) and the intuitions that learning that derives from one's own perceptions and intuitions, and teaching based on *intuitive* learning, can release new possibilities for her future as a teacher. It also helped her immensely in dealing with the course's non-guiding teaching line, since this experience was a turning point in starting appreciating the course [Zagorianakos, 2013]. The link between the intuitions on learning and teaching mathematics and her mathematical intuition is epitomised in her transition from an attitude that concerned an answer feeding model, as soon as the problem is posed to the students, to an attitude where the students are given space in order to shape their own understandings, before the answer is given. I noticed during the two remaining years of her studies (in order to become a teacher of mathematics) that her new attitudes persisted, during other courses that I was also observing. The phenomenological analysis of Diana's radical change of attitudes allowed a novel view of the teacher's legitimisation of her mathematical intuition: it was the teacher's legitimisation that made possible the student's appreciation of her learning experience and the changes of her attitudes on learning and teaching mathematics that followed, and it calls for a radical change of attitudes towards the students' (mis)conceptions.

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# POSTER PRESENTATIONS

# MATHEMATICS AND STUDENTS: ONE WAY TO FIT

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A multidisciplinary methodology is proposed that details certain aspects of Smart-Education. The methodology includes the structuring of the studied material [Ellenberg, 2014; Propp, 2009] in conjunction with an analysis of the cognitive profiles of students.

The purpose of the study is to determine ways of customizing the material being studied to the method of teaching for a specific group. The cognitive profiles of the students are determined by a method [Linksman, 2013] similar to the Myers-Briggs MBTI test.

The preliminary result at the seminars is achieving a better understanding. We do not want to analyze the components. We will evaluate the graduation thesis of every student using the assessment methodology of the University. It assesses scientific correctness, presentation logic, and originality. We assume that these estimates will be higher with each release, since the students get key competencies.

The new knowledge is a novel method of system analysis of the studied discipline and individual characteristics of the people involved in the learning process. This technique serves as a prototype for creating similar techniques for the mutual adaptation of participants and learning resources.

In the presentation, the methods and results will be discussed in detail.

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# STORYTELLING AND EARLY MATHEMATIC ACHIEVEMENT

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Children are exposed to narratives at an early age, reading stories and co-constructing narratives with adults. The narrative framework is also taught for communicating in school [Feagans, Appelbaum, 1986]. Narratives appear to be interwoven with many important aspects of children's lives, which poses a question about the influence it may exert in the development of other skills. Although narrative ability is a component of general language skill, it has been suggested to be a stronger predictor of school achievement than general language ability itself [Ibid.]. Few studies, however, have investigated the specific relation between narrative ability and mathematical skill. The two studies that have focused on this relation show that narrative ability in childhood is predictive of later mathematical achievement [Ibid.; O'Neill et al., 2004]. The current study seeks to fill the gaps in the literature: first, by examining personal narratives (rather than fictional narratives), which approximate natural storytelling between children and their parents; and second, by examining specific aspects of narrative ability in preschool children that predict specific emergent numeracy skills eight months later.

Twenty-two children aged 3–4 years participated in this study (male = 12). Each child was asked to construct three personal narratives. Eight months later, these children completed tasks measuring their early numeracy skills. After controlling for IQ and general language skill, analyses showed that conjunction use and perspective shift in children's narratives predicted knowledge of counting. In contrast, the total number of words and content words in children's stories predicted performance on an object based arithmetic task. These findings highlight the broad impact of children's growing ability to tell stories, suggesting that storytelling in preschool could be a simple, yet effective, method to encourage early mathematic achievement.

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# WHAT $f'(x)$ / TELLS US ABOUT $f(x)$: A GEOMETRIC AND GRAPHICAL APPROACH

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The derivative is one of the fundamental concepts in the learning of mathematics in university courses, being fundamental in the construction of more advanced concepts. Results from the national and international literature have pointed out that students have presented difficulties in learning this concept, which, for the most part, is related to the lack of conceptual understanding. In this sense, this qualitative research was based on studies about the importance of the use of computational technologies in situations of teaching and learning of Mathematics and Calculus, aimed to develop a pedagogical intervention for the teaching of derivatives through activities developed with the support of software Desmos, in order to verify the possible potential of the use of this tool for the understanding of the derivative. During the study, data were collected via recording responses of the activities carried out by the students (paper), proposal evaluation questionnaire, audio, photos and the researcher's field diary.

The intervention was developed in the form of a Pedagogical Workshop with a group of students that were enrolled in the Degree Course in Mathematics of an Institution of Education of private network of the State of Rio Grande do Sul. The analysis of episodes indicates that the use of a computerized environment can help students to become more active in the teaching and learning processes, feel encouraged to think, experiment and test what is often transmitted to them as a ready and finished knowledge, motivated to participate and share ideas, making socialization of knowledge under construction. The visualization and experimentation provided through manipulation of the graphs, construction of tables, marking and selection of points, which were mediated by the proposed activities, enabled the understanding of the derivative in a more enriching way, in geometric and graphic aspects.

# CRISIS OF MATHS THINKING IN THE 21st CENTURY: MYTH OR REALITY

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In the context of the development of the modern society and a new stage of education reform, urgent is considered the problem of lack of educated people who can work in a team, make decisions quickly enough in changing conditions, those who have conceptual and verbal-logical thinking, adequately understand the real situation and make the right conclusions. A modern and future employer is interested in such an employee who is able to think independently and apply an acquired knowledge to solve certain problems, has mathematical, critical and creative thinking, a rich vocabulary based on a deep understanding of humanitarian knowledge. We must talk about Maths skills of people in the 21st century. Nowadays we have a crisis of Maths education because a lot of Maths skills have been lost.

The development of science and technology, the universal computerisation and promotion of information and communication techniques in our time determine an increasing role of mathematical training of the younger generation. Modern psychological and pedagogical requirements for mental activity of the child are based on the development of his ability to choose and carry out activities using active search actions, correlate actions with the result, strive for the ultimate goal on the basis of forecasting, objectively evaluate the result, comparing it with your own setup. All these skills are directly related to Mathematics and the development of mathematical thinking.

The data presented in the histogram 1 were obtained in a survey conducted by the Levada Center on August 26–29, 2016. The survey involved 1600 adult urban and rural residents from 48 regions of the country (Russia). Answering the question “In your opinion, which school subjects should be given the most attention now?”: 68% of respondents indicated “Russian language”. On the second place (53%) — Mathematics. This is followed by: History (33%), Literature (32%), Foreign languages (27%), Information Technology (23%) [Levada, 2016].

It should be noted that only half of the respondents consider Mathematics as an important subject. Accordingly, if adults (parents, teachers, teachers of Universities and colleges, and others — not mathematicians), think so, and they will convey that point of view to their children and students. And even before trying to understand this issue on their own, children and students will think that Mathematics is one of the most complex educational areas of knowledge and they doubt whether it is necessary to study it. Maths education plays a significant part in intellectual development of children starting with pre-schooling, including children with special needs.

Thus teaching mathematics includes methods that allow you to get not only mathematical knowledge, but also General intellectual development.

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# K-12 NAMIBIAN TEACHERS' VIEWS ON LEARNING DIFFICULTIES IN MATHEMATICS: SOME REFLECTIONS ON TEACHERS' PERCEPTIONS

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This study explores K-12 Namibian mathematics teachers' perceptions of the difficulties associated with learning mathematics and examines the problems that affect Mathematics Learning Difficulties (MLD). In this study, learning difficulties are referred to as obstacles that lead to difficulty learning mathematics [Karagiannakis, Baccaglini-Frank, Papadatos, 2014]. SACMEQ II and III report that Namibian students performed poorly in mathematics, only 6.2% achieved the competent numeracy level. The reports highlight that students perform poorly at both school levels. From a psychological perspective, researches identify causes of MLD, such as poor foundation in learning mathematics. A wide of opinions have been develop about the nature of mathematics based on 'knowledge' of own experiences acquired through teaching practice [Ernest, 1989]. These perceptions, are likely to be associated with teachers' beliefs and their impact on teaching students with MLD. Bearing in mind the importance of teachers' perceptions, this study aims to answer the following questions:

1. To what extend do mathematics teachers define MLD in an inclusive setting?
2. What problems do mathematics teachers identify when teaching students with MLD and what measurements are used to overcome them?

A total of 231 mathematics teachers (100 primary and 131 secondary teachers), who teach in inclusive classrooms, completed the survey. Responses were analysed using Ritchie and Lewis thematic framework. The survey reveals teachers who frame a student's difficulties within a deficit framework and also perceived the student's difficulties to emanate from a cognitive disability and it indicate how beliefs negatively influence teaching practice. This paper examines the potential causes of MLD according to survey participants and suggests alternative measures, and concludes with suggestions, broadening data source as one way the study can be extended.

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# THE MATHEMATICAL SUCCESS DIARY AS A METHOD OF INCREASING SELF-CONFIDENCE AND THE EFFECTIVENESS OF LEARNING MATHEMATICS

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Fundamental studies of a sense of self-efficacy, confidence in one's abilities, level of anxiety in the field of mathematics were conducted at OECD in the 21st century [OECD, 2013]. The problem of increasing self-confidence has been extensively studied in psychology, but we are not aware of pedagogical experiments using the Mathematical Success Diary (MSD) to increase self-confidence, motivation to study mathematics, level of activity in mathematics lessons and mathematics achievements [Ackerman, Ugelow, 2018].

The aim of the research is to increase the efficiency of teaching mathematics through a pedagogical experiment using the MSD. The hypothesis of the experiment is the assumption that as a result of keeping the MSD, pupils of the 5th grade of elementary school, showing a lack of self-confidence and low mathematics achievements, will significantly increase their motivation, mathematical self-esteem, degree of activity on lessons and academic achievements.

In the MSD, which is conducted by the student together with the teacher, positive points are noted in the process of studying mathematics, such as: showing patience, perseverance, endurance and faith in success in solving difficult examples and problems, formulating a question and finding theoretical and practical material in textbook or notebook, asking for help from classmates or a teacher, raising a hand for answer a question, solving an example or problem on the blackboard, correct answers and solutions of examples and problems, improving the control mark on the exam.

The research will be conducted in the 2018–2019 school year among pupils of the 5th grade of elementary school. During the ascertaining stage, we selected pupils participating in the experiment, used observation methods, interviews, questionnaires, self-assessment and peer review, as well as school documentation data.

The expected results of the research would be caused by keeping MSD that marks the pupil's goals, the progress in achieving them, tranquility of the mind, control, confidence and enthusiasm in learning mathematics.

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# MATHEMATICAL PROBLEM SOLVING: BEHAVIORAL AND NEUROIMAGING STUDIES

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The current study has three components: (a) a functional magnetic resonance imaging (fMRI) meta-analyses of past literature on mathematical operations; (b) a behavioral study to validate a math protocol with parametric changes in the difficulty of math problems that use addition, subtraction, multiplication and division; and (c) an fMRI study that examines the brain correlates of mathematical operations (addition, subtraction, multiplication and division) in relation to subjective effort.

This project investigated for the first time the Right-Left-Right hypothesis using functional brain indices related to solving addition, subtraction, multiplication and division problems. The classic ideas of hemispheric dominance (i.e., visual-spatial abilities in the right hemisphere, verbal ones in the left hemisphere) cannot explain the study's findings as the material provided were all numerical. We adopt a hypothesis derived from cognitive development to predict that hemispheric involvement stems from an interaction between an individual's mental-attentional capacity and the mental demand of the task [Pascual-Leone, 1987; Arsalidou, Pascual-Leone, Johnson, 2010]. To test this hypothesis, we adopted a parametric design with several levels of difficulty (easy, within the individual's competence level and above the individual's competence level). Our results provide new insights on the brain correlates of mathematical problem solving as a function of operation and difficulty.

## ACKNOWLEDGMENTS

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# RELATIONSHIP BETWEEN TEACHING PRACTICES AND MATHEMATICAL PROCESSES OUTCOMES IN SECONDARY SCHOOL: BASED ON A LONGITUDINAL TIMSS-PISA STUDY

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The competence of mathematical modelling is well conceptualized, thought a much-debated question is how to develop it in schools. International achievement test PISA consider mathematics achievements from the modelling perspective (formulating, employing and interpreting), and provides us with comprehensive data to analyze a school factors in math results from the comparative perspective. The goal of our study was to estimate the effect of teaching practices on students' achievements in different PISA mathematical processes while controlling prior achievements (TIMSS).

The paper is based on two waves of a unique longitudinal data of Trajectories in Education and Careers study in Russia (TrEC <http://trec.hse.ru>). The first wave of the TrEC was TIMSS 2011 (8th grade), and the second wave was PISA 2012 oversampled for this study (9th grade). The final sample consists of 3555 students and 185 math teachers. Firstly, we used CFA to construct indexes of teaching practices and, secondly, then we used SEM to estimate the relationship between teaching practices and students' math outcomes. We took information about teaching practices, school characteristics and students' background from TIMSS questionnaire and students' math scores from TIMSS and PISA.

Two groups of teaching practices were identified according to the theoretical framework of Bloom's taxonomy of cognitive skills [Anderson, Krathwohl, 2001]: the first group focuses on the lower-order thinking skills, the second one focuses on the higher-order thinking skills. In result, high-order teaching practices (ex., let students decide on their own procedures for solving complex problems) teaching practices have a larger effect on the first two stages of the modelling process — formulating and employing, whereas family background has a larger effect on the last stage of the modelling process — interpreting. Low-order teaching practices are insignificantly related to the modelling processes.

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# MEASURING MATH SELF-CONCEPT AMONG MIDDLE SCHOOLERS: QUESTIONNAIRE DEVELOPMENT AND VALIDATION

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Self-concept forms individual's progress throughout his life. It is important to consider self-concept as a multidimensional construct. It reveals a person from different aspects, describing one's strengths and weaknesses [Marsh, Shavelson, 1985].

Due to the lack of a valid and reliable measurement of math self-concept among Russian middle schoolers, it is necessary to develop a questionnaire. In our paper we set following objectives: (1) questionnaire development, (2) questionnaire factor structure confirmation, (3) questionnaire psychometric characteristics assessment, and (4) validity support.

The questionnaire for measuring math self-concept is based on Marsh and Shavelson's multidimensional model [Ibid.]. The questionnaire consists of 8 items, which are rated on a 5-point Likert scale. The study was a part of Student Achievements' Monitoring (SAM). A total of 316 fifth-graders from three schools (The Republic of Tatarstan, Russian Federation) participated in the study. The mean age of participants was 11 years ($SD$ = 0.33), 56% of them were girls.

The results of confirmatory factor analysis (CFA) provided strong support for one-factor model, given the theory. In order to improve the model, we excluded 3 items with low factor loadings ($\leqslant 0.4$). Based on fit indices, we chose the model that consists of 5 items. The psychometric characteristics of the questionnaire were investigated using Classical Test Theory (CTT). Psychometric properties are acceptable: reliability Cronbach's Alpha = 0.7; mean difficulty = 0.73; item discrimination index = 0.45. Statements are comprehended by boys and girls in the same way (Differential Item Functioning, Mantel-Haenszel criterion). Concurrent validity was accessed through correlation of the questionnaire scores and math results. High self-concept scores are associated with high results on SAM ($r = 0.45$, $p \leqslant 0.001$).

The questionnaire can be useful for career guidance among Russian middle schoolers. It is crucial to measure math self-concept because it associates with academic performance and interest in this subject [Ayodele, 2011].

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# FUNCTIONAL INVESTIGATION OF THE NETWORKS AND WHITE MATTER SUBSTRATES ASSOCIATED WITH THE PROCESSING OF MATHEMATICAL OPERATIONS

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Processing of mathematical operations and solving numerical tasks implicate a distributed set of brain regions. These regions include the superior and inferior parietal lobules that underlie numerical processing such as size judgments, and additional prefrontal regions that are needed for formal mathematical operations such as addition, subtraction and multiplication [Arsalidou, Taylor, 2011]. Critically, little is known about the connectivity between these regions and the association between math performance and the anatomical structure of white matter tracts.

The present study investigates connectivity and white matter tracks associated with networks related to math performance: arcuate fasciculus (AF) and superior longitudinal fasciculus (SLF). Participants performed a computerized task with mathematical operations (addition, subtraction, multiplication, and division) with three levels of difficulty; accuracy and reaction time were recorded. Diffusion tensor imagining (DTI) recordings provided indices on fractional anisotropy (FA) — a measure of the direction of white matter tracks in the brain. The relation between FA and math performance scores is reported.

Results are expected to that math performance is associated with integrity of both AF and SLF. In addition, improved scores on math performance, specifically in relation to reaction time, are related to FA values in AF and SLF.

Concluding, findings will be discussed in terms of the models of mathematical cognition and developmental theories of cognition and education.

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# FORMING IDEAS ABOUT GRAPH ISOMORPHISM IN MASS CONTESTS WITH COMPUTER SUPPORT

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In the report we consider the role of computer tools, when conducting distance competitions in mathematics and computer science. We discuss the differences in holding competitions with and without computers. The experiment involved more than 50 people of different ages and classes. The pupils were offered the task to build a solution on paper. The task is based on the idea of using constellations to describe graph theory concepts. Participants should first create as many non-isomorphic constellations as possible. The second criterion is the total number of constellations (even isomorphic), and the last criterion is the total length of the constellations' segments. This task interface includes means of visual feedback (namely, highlighting objects), a change of which could improve the result. The best solution of each student is saved automatically. The following represent other experimental results, which from our point of view are more important

1. Use of the paper version required the preparation of printed sheets for the participants' work, and manual analysis of the results might reveal non-obvious calculations of the components of the graph with the desired properties, an advantage in case of errors in the calculations (human factor). Thus the complexity and cost of such a format is determined by the number of participants. At the same time, the use of a computer required the creation of a computer model only once, and the remaining stages, such as performance evaluations and comparative analysis of the results, were carried out instantly

2. The use of the model allows the introduction of deep new ideas into the composition of the concepts being studied, such as isomorphism, which, upon initial introduction, require basic concepts that students should receive from the outside world. At the same time, for the concept of isomorphism, there are no representations that could have been formed spontaneously as a result of operating objects in the external world. Thus, the simulated virtual reality allows students to provide the ideas necessary for the formation of abstract mathematical concepts

## CONCLUSION

After plotting the graph on paper, they do the same task on a computer. While performing the computer implementation of the task, 32 people improved their decisions, the optimal solution was achieved by 29 participants, 2 people showed the same results, and 1 participant worsened their decision. Some features of the software tool provide the participant with an improved understanding of the task.

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# PHOTOGRAPHY AND MATHEMATICS STUDENTS' GAZE

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To make connection from objects in everyday life to mathematics is essential for giving meaning in math class. Using a camera expands perception for objects otherwise unnoticed [Barasch et al., 2017]. Herein, we explore the influence of a photography activity on mathematics teachers' visual behaviour.

Using a design-based approach with 43 in-service teacher students, we have developed a photography-based learning activity in three cycles. Based on their own photograph and discussions students designed and conducted a problem-solving task with their own math class. In the second cycle visual attention of two students were collected [Meier, Hannula, Toivanen, 2018], showing an increased number of fixations and longer dwell time on those objects photographed and discussed. In the current, third cycle we explore how photography and discussion influence students' fixation duration and occurrence. To study students' visual behaviour, four students wore mobile eye-trackers. We compared two conditions: during a walk with a camera and a task, and during a walk without. Between the two walks students discussed the mathematical content of their pictures and how to use them to design a problem-solving task for their students. Analysis of fixation durations revealed significant differences for the two walks. For all participants, the median duration was higher for the walk with photography. Although the effect size was small, the results suggest more conscious looking and interest when taking photographs. In the poster presentation, further results and examples of student's work will be reported in detail.

Visual attention in natural environments is a new area of educational research. This study provides new information on how the activity of taking photographs and discussing photos influences mathematics teacher students gaze behaviour and their teaching.

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# UTILITARIAN MATHEMATICAL DISCOURSE ON SOCIAL MEDIA

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The paper tries to highlight an aspect of mathematics teaching through social media (in Indian context). Mathematics has marked its presence on social media and platforms like YouTube and Facebook have helped spread mathematical discourse informally. A part of such mathematical discourse also emphasises mathematics as a subject to be dealt with formulas, short tricks and quick calculations. Social media has got flooded with videos on mathematics by several persons and includes some math coaches which have millions of followers and views on social media.These also make money based on the number of views on their video. The video is aimed at youth, especially school pass outs who prepare for competitive examinations leading to employment opportunities. These contents are made catchy with captions and thumbnails like "*Solve within 3–4 seconds*", "*Quickest Method*" etc. The viewer's generally respond to such content in an excited way by sometimes applauding the trainer for his trick or sometimes treating the trainer as 'God' of mathematics. Such discussions or discourse are common on social media. For example: A trainer on YouTube teaches to multiply two two-digit numbers quickly. In his trick he shares some steps to place numbers and get the result. However, it lacks mathematical explanation and also it is helpful when the numbers are close to 100 or 50. For this video, a viewer comments, "Sir, you are god of mathematics." Another viewer comments, "Sir, your mind is computer."

Sam [1999] in his doctoral study finds out that majority of the adult held utilitarian view on mathematics. Some comments by viewers also reflect similar images of mathematics. In a populated country like India, cracking examinations for jobs requires fast calculations. There is evidence that students who speed through content without developing depth of understanding are the very ones who tend to drop out of mathematics when they have the chance (Boaler, cited in [Larson, 2017]). These job seekers may form the image of mathematics as a subject to be dealt with tricks and formulas. They go through a process of sprint. Perhaps, mathematical learning doesn't seem their goal and it can put threat for mathematics education.

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# ASSOCIATIONS WITH THE WORD "MATHEMATICS": THE EMOTIONAL VALUE COMPONENT

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The role of the first-year teacher in forming a positive attitude towards mathematics is important. Therefore, it is relevant to perform a scientific search for mechanisms for building an emotional value component in the preparation of future teachers.

An "auction" of mathematical terms is enacted that should quickly update and systematize multiple mathematical concepts in the mind of freshmen. Students have a task to write 20 words they associate with math in five minutes. The students then call out the words in turns, defining the content of the term and naming related concepts. For example, a "circle" is a planar geometric figure, like a square and a trapezium. Already named terms are marked. A student who names the last unmarked word "wins".

It is possible to repeat at a fast pace: types of numbers; arithmetic operations; one-dimensional, two-dimensional and three-dimensional geometric figures; functions; sections of mathematics; mathematicians; and so on.

Analysis of the accumulated empirical material demonstrates the appearance of words describing feelings (both positive and negative). Students experience these feelings when confronted with math. The qualities that mathematics has, according to students, are also indicated. Two examples are "fear" and "coldness". During a conversation, the teacher should update the experience of positive emotions related to maths and specify the reasons for negative ones. It should be realized that emotions are born inside a person in interaction with some object: a book, a picture, music, and equally with math.

We have found that people associate mathematics with color, including cases of synesthesia. Sometimes there are bright visual images of the named objects, as in, for example, "a mountain of figures". It is also found that concepts associated with mathematics among sociological students are socially oriented.

We link the further direction of research with a more detailed construction of frames of mathematical concepts reflecting the experience of a particular person (people of different professions). The dynamics and stability of the neural network (the connection between the old and recently learned mathematical concepts) of a particular person is also interesting for research.

At the same time, we assume that the involvement of a large number of aspects (visual images, emotions, nonmathematical concepts valuable to a person in their relationship with mathematical concepts) can play an important role in forming a harmonious (non-negative) mathematical picture of the world in a child. Correction of the emotional image of mathematics from a negative one to a positive one or fixing it in a positive area is important for future teachers of mathematics.



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# LOCAL INTERPRETATIONS OF GLOBAL COMPARISONS: MEDIA CONTENT ANALYSIS IN TURKEY

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Countries have been benefiting from the results of international assessments, e.g., TIMSS and PISA, to analyze the level of their education systems globally and to implement educational policies and reforms. Media analysis has expanded researchers' understanding of how, why and under what circumstances international assessments are used across altering national contexts [Saraisky, 2015].

In 2016, Dr. Pizmony-Levy conducted a study called "Ranking Storm Project" which includes two parts: Public Opinion Survey and Media Analysis. Here, in this presentation, only Turkish media will be analyzed. The research questions are:

- How does the media present the results?
- Which interpretations and voices are common?

The sample from Turkey comprises 22 articles from 3 newspapers with different political leanings. These articles were published in early December 2016, in the first two weeks following the release of the results for both TIMSS and PISA. The researcher analyzed the data by using the codebook developed by Pizmony-Levy [2018].

Media analysis shows that educators and experts attribute Turkey's results to a multitude of factors, such as educational gaps between different types of schools, recent changes in the educational system in Turkey, money spent per student, memorization rather than construction of knowledge, etc. Furthermore, this media analysis reveals that the political leaning dictates the tone of the articles.

Since OECD will release the PISA 2018 results in December 2019, a broader study may give better insight about the changes of public discourse over time by the political changes and educational reforms in Turkey.

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# BRIDGING TWO THEORIES: THE THEORY OF DIDACTICAL SITUATIONS AND THE THEORY OF DEVELOPMENTAL INSTRUCTION

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In different parts of the world, researchers question the teaching and learning of mathematics and look at it from various perspectives. These efforts unfold different processes of theorization and produce diverse sets of concepts. The theory of didactical situations, developed in France by Brousseau [1998] between 1970–90 has become central to the study of didactics of mathematics in French-speaking countries. Concurrently, in Russia, Davydov [2008] and his colleagues built a theory of developmental instruction. Based on the Vygotskian idea of the cultural-historical nature of teaching and learning, Davydov's theory has informed research and practice in Russia and other countries worldwide. While the teaching and learning of mathematics is the subject matter for both theories, the philosophies behind the two theories are distinct thus producing distinct sets of concepts. Rather than treating these theories as categorically different, we ask what conceptual links can be drawn between the two theories. What would be some potential affordances in highlighting the similarities between the two theories to mathematics education? The theory of didactical situation operates the concepts such as *didactical situation*, *learning obstacle*, *didactic contract, fundamental situations*. The theory of developmental instruction speaks about *empirical thinking* and *theoretical thinking*, *learning activity, and learning tasks*. While both theories contribute to our understanding of teaching/learning process and both propose important ideas about mathematics education, we turn attention to whether and how both theories can be considered simultaneously. We thus highlight tenets and principles from both theories to show the complexity of the subject matter and to contribute to the need for further theoretical work in connecting different bodies of knowledge in order to understand the teaching/learning phenomena in mathematics.

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# THE IMPACT OF GENETIC TEACHING ON PRE-SERVICE TEACHERS' VIEWS OF MATHEMATICS AND ITS TEACHING

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The theoretical framework of this paper is theory of mathematical beliefs (see e.g., [Pehkonen, Toerner, 1996; Pehkonen, Safuanov, 1996]) of pre-service mathematics teachers. Pehkonen [1994] indicated the importance of changing teachers' beliefs for the effectiveness of their teaching. Extremely important for this purpose is to try to apply new theories of mathematics teaching in standard mathematical courses at universities and pedagogical institutes. Note that many researchers argue that the methods of teaching mathematical disciplines in pedagogical universities should serve for the students — future teachers as a source of didactical ideas, helping them to acquire modern didactical beliefs and skills, and in some sense as a sample for building their future professional activity.

During 2015–2018 academic years, Mathematics majors in their first year of study at the Moscow City University have been divided into two groups. One of the groups has been taught Algebra and Number theory course using genetic method. Another group has been taught the same course using traditional (reproductive) methods. The aim of this study is to reveal the impact of such teaching on students' beliefs about mathematics and mathematical education.

The questionnaire of 29 questions on mathematics teaching was administered in these groups twice: prior to the course and upon the finishing the course.

The results of the questionnaire administered prior to the course indicated that among students the conviction in effectiveness of explanatory-illustrative and reproductive methods of teaching still predominated.

The questionnaire administered upon the finishing our experience of teaching mathematics by genetic method (taking also into account affective and emotional aspects of teaching) has shown that pre-service teachers' views of mathematics teaching seriously changed and became more progressive and appropriate for modern teaching.

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# INFLUENCE OF VERBALIZING GEOMETRY RULES ON THEIR TRANSFER IN PRIMARY SCHOOLERS

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Inductive reasoning involves making predictions about novel objects or situations based on existing knowledge. This kind of reasoning is crucial for mathematical learning, where the ability to transfer acquired mathematical knowledge to analogical tasks or situations is substantial. One of the possible factors that contribute to transferring found solutions to similar tasks is verbalization, i.e. the process of saying task solution out loud. Our research showed that adult participants who verbalized solutions for induction tasks transferred found solutions to similar tasks faster than participants who did the tasks tacitly. The present study aims to test the hypothesis that verbalization will similarly influence on transfer performance in primary schoolers. We will conduct the experiment where 7–10-year-old children will solve simple geometry tasks called "Bongard problems" [Bongard, 1970]. Each Bongard problem consists of two groups of geometrical objects, and participants will have to induce the basis for the classification by finding out what feature of the objects is relevant. The experimental group will explicitly verbalize the solution for set of Bongard problems, whereas the control one will solve the tasks tacitly. In the test phase we will measure and compare how students from both groups solve the same problems but with new objects. The results and comparison of children's and adults' performance will be presented and discussed at the conference.

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# DIRECTIONS AND TRENDS IN THE IMPROVEMENT OF MATHEMATICAL EDUCATION

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Currently, the trend in education is the introduction of innovative educational technologies [Ustinova, Podberezina, Shefer, 2017; Tarbokova, Ustinova, Rozhkova, 2018]. The article considers the following techniques: fishbone, flipped classroom, project-based learning. “Fishbone” technology was used by us in mathematics classes in the presentation of the material of one of the most complex topics of mathematical analysis “Indefinite integral.” The project method was applied in the lessons of Linear algebra and analytical geometry. Each student of our course was offered to implement the project “Analytical geometry of straights and planes”. We used the technology of flipped classroom in the course of linear algebra and analytic geometry.

The aim of this work is to compare different innovative educational technologies in teaching mathematical disciplines. In addition, our task was to study the impact of innovative educational technologies on the achievements of students.

For the study were selected first-year students studying the course of linear algebra and analytical geometry using the method of projects (the first group), with the use of the technology of flipped classroom (the second group) and students (the third group), which with the use of “Fishbone” technology studied the topic of indefinite integral of course of mathematical analysis. As a result of entrance control of knowledge it has been established that the level of proficiency in mathematical skills in controlled groups was approximately identical. Students of three groups passed the control test on knowledge of a subject which was estimated by identical quantity of points. The null hypothesis of equality of average scores in the control groups was tested using the analysis of variance. The highest average score was in the third group. On the basis of statistics it was found that the use of these technologies can significantly increase the level of new information assimilation (approximately 19%).

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# RATIO-CONCEPTS THROUGH JOINT ACTIONS

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As we approach the problem of learning ratio-concepts [Dole et al., 2012] from the position of Activity Theory, we have to search for special students' actions, in which these concepts can be "loaded"/embedded. In ratio-concepts, we have to consider both: operating separate parameters and coordinating their change. The latter is the most important for learning ratios — thus, we have to make it a special subject for children.

We assume that distributing the work over the ratio between two students is a feasible way to learn ratios. This way, ratio-concepts derive from students' attempts to coordinate parameters' change. The means of this coordination (multiplicative thinking — laying portions after portions) is the essence of the concept of ratio.

The computer simulation was devised so it would scaffold joint actions. In it students are to construct a vessel that will sink, float or stay balanced. Each in pair is given control (through the shared keyboard) over only one parameter — either the size of the vessel, or its' weigh. The tasks can also be done individually.

Task progression allows changing the coordination scenario. The progression is as follows. 1) Students are "playing against" each other, working to "fix", what the other has done, as his changes cannot be undone: for example, he made the vessel sink, putting more weighs. Now his partner cannot take them away, but can add volumes. 2) Students are working together to make the vessel balanced. So they have to coordinate and consider the changes, which they make to their parameter. 3) Due to restrictions of the tasks students have to plan their coordination beforehand.

Two groups of students of 3–4$^{th}$ grade (9–10 years of age), 20 students in each, — participated in the study. One group was doing the tasks individually (each student could change any parameter), while students of the other group were working in pairs changing the parameters jointly. Post-test showed, that those, who learned through joint actions, did better.

Joint actions, as they coordinate changes of two parameters, bring students to using the compound measure (2 volumes and 5 weighs). In its turn the compound measure is the first step to introducing fractions.

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# EXAMINATION OF THE PERCEPTIONS OF THE PRE-SERVICE ELEMENTARY MATHS TEACHERS TOWARD THE INTEGRAL

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Many researchers have revealed the difficulty in understanding the integral concept [Sealey, 2008; Orton, 1983]. It can be thought that the difficulties faced by the students regarding the integral concept may change their perceptions of integral. In the light of this idea, the purpose of this research is to examine the perceptions of the pre-service elementary maths teachers (pre-service EMTs), who attended a teaching experiment in which real life problems were used, toward integral concept within the context of anxiety, attitude, and daily life usage.

In this research, qualitative and quantitative data were collected simultaneously. While the quantitative data of the research were collected via one group pretest-posttest experimental design, the qualitative data were collected via case study. The study group of the research consists of 28 pre-service EMTs who attend Faculty of Education in a state university located in the Marmara Region of Turkey. A teaching experiment enriched with 8 real-life problems created by using Riemann sums and definite integral was implemented to the pre-service EMTs. While the quantitative data of the research were obtained from the scale for the concept of integral that was developed by the researchers and whose validity and reliability studies were conducted, the qualitative data of which were obtained from the semi-structured interview form.

It was seen that while the pre-service EMTs' anxiety levels (Mean = 26.86, SD = 5.45) for the integral concept were high, their attitudes (Mean =21.14, SD = 4.27) and awareness levels about its daily life usage (Mean = 18.82, SD= 2.80) were low before the teaching experiment. It has been concluded that the anxiety levels of the Pre-service EMTs toward the integral concept decreased (Mean = 24.393, SD = 5.20) and their attitude (Mean = = 23.93, SD = 4.77) and awareness levels about its daily life usage (Mean = 20.29, SD = = 3.18) increased after the teaching experiment enriched with real-life problems. It was found that there is a statistically significant mean difference between the pretest and posttest scores of pre-service EMTs at $p = 0.05$ level in terms of anxiety, attitude and awareness levels. This situation supports the findings obtained from the interviews.

As a result of the research, it has been seen that the teaching experiment enriched with the real-life problems decreased the anxiety levels of pre-service EMTs toward integral and increased the attitude and the awareness levels regarding its usage in daily life.

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