Inertial Manifolds and Limit Cycles of Dynamical Systems in $\mathbb{R}^n$

L.A. Kondratieva[1], **A.V. Romanov**[2]

[1] Moscow Aviation Institute (National Research University), Russia

[2] National Research University Higher School of Economics, Moscow, Russia,

Email adress: av.romanov@hse.ru

Abstract. We show that the presence of a two-dimensional inertial manifold for an ordinary differential equation in $\mathbb{R}^n$ permits reducing the problem of determining asymptotically orbitally stable limit cycles to the Poincare–Bendixson theory. In the case $n = 3$ we implement such a scenario for a model of a satellite rotation around a celestial body of small mass and for a biochemical model.

2010 *Mathematics Subject Classification*: 34C07, 34C45.

Keywords: ordinary differential equation, limit cycle, inertial manifold.

1. Introduction

We consider ordinary differential equations

$$\dot{x} = -Ax + F(x), \quad x \in \mathbb{R}^n, \quad n \geq 3, \tag{1.1}$$

where A is a symmetric $n \times n$ matrix with eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and the function F belongs to $C^{1+\alpha}(\mathbb{R}^n, \mathbb{R}^n)$ for some $\alpha \in (0, 1)$. We let $F'(x)$ denote the Jacobi matrix of the mapping F at a point x, and $\|\cdot\|$ and $\|\cdot\|_2$ denote the Euclidean norm in $\mathbb{R}^n$ and the Euclidean norm of matrices, respectively. If one of the two conditions

$$\|F(x) - F(y)\| \leq K \|x - y\|, \ \ \|F'(x)\|_2 \leq K, \quad x, y \in \mathbb{R}^n, \tag{1.2}$$

that are equivalent in this situation is satisfied, then equation (1.1) generates a C^1-smooth phase flow $\{\Phi_{t\in\mathbb{R}}\}$ in $\mathbb{R}^n$. Everywhere below we identify linear operators on $\mathbb{R}^n$ with their matrices. Let $f = -A + F$ be a vector field of (1.1), then we call $x_s \in \mathbb{R}^n$ a *singular point* if $f(x_s) = 0$. By a *cycle* we mean a closed trajectory. A *stable limit cycle* is a cycle that is asymptotically orbitally stable as $t \to +\infty$.

The theory of inertial (that is, invariant and globally exponentially attracting) manifolds was developed in the 1980s as a tool for studying the final (at large times) dynamics of semilinear parabolic equations with a vector field structure of the form (1.1) in an *infinite-dimensional* Hilbert space X (see [1, Ch. 8], [2] and the references therein). In this case, as usual, it is assumed that A is an unbounded self-adjoint positive linear operator in X with a compact resolvent. In such a situation, the presence of an m-dimensional inertial manifold (IM) permits describing the final dynamics of an infinite-dimensional evolutionary system by an ordinary differential equation (ODE) in $\mathbb{R}^m$. Here we demonstrate the usefulness of inertial manifolds in the finite-dimensional case $X = \mathbb{R}^n$. Namely, the existence of a two-dimensional IM ($m = 2$) allows one to reduce studying the final dynamics of equation (1.1) to solving the corresponding problem in $\mathbb{R}^2$ and, in several cases, to prove the presence and to discover the localization of a stable limit cycle without using the bifurcation technique or some rather complicated topological constructions. We stress that, in contrast to the bifurcation theory, our approach proves the existence of stable self-sustained oscillations of a "large amplitude".

2. Inertial manifolds

A set $\Lambda \subseteq \mathbb{R}^n$ is said to be invariant if $\Phi_t\Lambda = \Lambda$, $t > 0$. Let P_m and Q_m be orthogonal projection operators in $\mathbb{R}^n$ on the subspaces X_m and X_{n-m} corresponding to the eigenvalues $\lambda_1, \dots, \lambda_m$ and $\lambda_{m+1}, \dots, \lambda_n$, $\lambda_m < \lambda_{m+1}$, of the matrix A.

Invariant manifold of the form

$$H_m = \{x \in \mathbb{R}^n : \ x = u + h(u), \ \ u \in X_m\} \tag{2.1}$$

with the function $h \in \operatorname{Lip}(X_m, X_{n-m}) \bigcap C^1(X_m, X_{n-m})$ we call inertial, if for each trajectory $x(t)$, there exists a trajectory $\overline{x}(t) \subset H_m$ such that

$$\|x(0) - \overline{x}(0)\| \leq M_1 \|Q_m x(0) - h(P_m x(0))\|, \tag{2.2}$$

$$\|x(t) - \overline{x}(t)\| \ \leq \ M_2 e^{-\gamma t} \|x(0) - \overline{x}(0)\| \tag{2.3}$$

for $t > 0$, where $M_1, M_2, \gamma > 0$. If a set $E \subset \mathbb{R}^n$ is bounded, then the Lipschitzian function $h : X_m \to X_{n-m}$ is bounded on the bounded set $P_m E$ and for everyone $x(0) \in E$ we have $\|Q_m x(0) - h(P_m x(0))\| \leq M$ with $M = M(E)$. It follows from (2.3) that

$\|x(t) - \overline{x}(t)\| \leq M_1 M_2 M e^{-\gamma t}$ for $x(0) \in E$, $t > 0$, which means H_m exponentially and uniformly attracts E. Let $\Lambda \subset \mathbb{R}^n$ be a compact invariant set and $y \in \Lambda$. If $x(0) = \Phi_{-t} y$, then $x(0) \in \Lambda,\ x(t) = y$, and

$$\|x(t) - \overline{x}(t)\| = \|y - \overline{x}(t)\| \leq \overline{M}(\Lambda) e^{-\gamma t}.$$

Since $t > 0$ is arbitrary, $\overline{x}(t) \in H_m$ and the set H_m is closed, then $y \in H_m$ and $\Lambda \subset H_m$. In this way, the inertial manifold contains all compact invariant sets (including the singular points and cycles) of the dynamical system.

It is well known [3, 4] that if the *exact spectral gap condition*

$$\lambda_{m+1} - \lambda_m > 2K \tag{2.4}$$

is satisfied, then there is such a manifold with $h \in \mathrm{Lip}\,(X_m, X_{n-m})$ and the factor 2 on the right-hand side of (2.3) cannot be decreased in general. Later, it was shown [2], that condition (2.4) also provides the existence of a C^1-smooth inertial manifold. Estimate (2.2) means that $\|x(0) - \overline{x}(0)\|$ is small if the initial point $x(0)$ is close to H_m. Estimate (2.3) reflects the exponential tracking of the initial trajectory $x(t)$ by the trajectory $\overline{x}(t) \subset H_m$.

By the reduction principle [4, Lemma 1], the compact invariant sets Λ of equation (1.1) and $P_m \Lambda$ of the ODE

$$\dot{u} = -Au + P_m F(u + h(u)), \qquad u = P_m x, \tag{2.5}$$

in $X_m \simeq \mathbb{R}^m$ are simultaneously asymptotically stable or unstable. The dynamical system generated by (2.5) is topologically conjugate to the restriction of the original dynamical system (1.1) to H_m. This means that the final (for $t \to +\infty$) regimes of the original equation in $\mathbb{R}^n$ are fully described by some ODE in space of smaller dimension, which in many cases simplifies their research. Essentially, we highlight the $m < n$ "defining" degrees of freedom of a n-dimensional dynamical system. In addition, if t is sufficiently large then every solution $x(t)$ of equation (1.1) is completely determined by its projection $u(t) = P_m x(t)$ onto the subspace X_m and is reconstructed by the formula $x(t) = \psi(u(t))$ with $\psi(u) = u + h(u)$.

Splitting the right-hand side of equation (1.1) into linear and nonlinear components, of course, is not unique. Right choice matrix A in (1.1) can help to satisfy the condition

(2.4). On the other hand, condition (2.4) can sometimes be ensured by using a nondegenerate linear change of variables; the topology of the phase portrait of the dynamical system does not change in this case. Such a method is used below in Section 4 to study a mathematical model of cell processes.

Remark 2.1. The existence of a two-dimensional inertial manifold allows one to assert that the union of all singular points and cycles (if any) has the form of a Lipschitz graph over a certain plane $X_2 \subset \mathbb{R}^n$.

It should be noted that, under condition (2.4), the inertial manifold H_m does not inherit the smoothness of the nonlinearity F; for example, the condition that F is real analytic in $\mathbb{R}^n$ does not even imply that $H_m \in C^2$.

Definition 2.1. A domain $D \subset \mathbb{R}^n$ is strictly positive invariant if $\Phi_t \overline{D} \subseteq D$, $t > 0$.

In particular, this means that the boundary ∂D does not contain singular points.

Remark 2.2. Even under a weaker condition $\Phi_t D \subseteq D$, $t > 0$, the continuity of the mapping $x \to \Phi_t x$ for $x \in \mathbb{R}^n$ guarantees the inclusion $\Phi_t \overline{D} \subseteq \overline{D}$, $t > 0$, for the closure $\overline{D}$.

The strict positive invariance of D is ensured if the vector field $f(x) = -Ax + F(x)$ of equation (1.1) on the boundary ∂D is directed inside the interior of D. If the domain $D \subset \mathbb{R}^n$ is strictly positive invariant, then the domain $P_m D \subset X_m$ has the same property with respect to the ODE (2.5).

Remark 2.3. The closure of the union of all cycles contained in the strictly positive invariant domain D does not contain points of ∂D.

This is a consequence of the continuity of the phase flow $\{\Phi_t\}$ with respect to $x \in \mathbb{R}^n$.

Consider the quadratic form $V(x) = \|Qx\|^2 - \|Px\|^2$ with an arbitrary orthogonal projection operator P in $\mathbb{R}^n$ and $Q = \mathrm{Id} - P$. Assume that, for some $\lambda, \varepsilon > 0$, any two solutions $x(t)$ and $y(t)$ of (1.1) satisfy the following relation holds with $t > 0$:

$$\frac{d}{dt} V(x(t) - y(t)) + 2\lambda V(x(t) - y(t)) \leq -\varepsilon \|x(t) - y(t)\|^2 . \tag{2.6}$$

This condition is known in the theory of inertial manifolds as the *strong cone condition*.

Remark 2.4 (see [2, Lemma 2.21; 4, Lemma 4]). Condition (2.4) implies (2.6) with $P = P_m$, $\lambda = (\lambda_{m+1} + \lambda_m)/2$ and $\varepsilon = (\lambda_{m+1} - \lambda_m)/2 - K$.

Recall the well-known (see [5]) estimate $\mathcal{T} \geq 2\pi/K_1$ of the periods $\mathcal{T} > 0$ of periodic solutions (1.1), where $K_1 = \lambda_n + K$ is the Lipschitz constant of the vector field $f =$

$-A+F$. For $\tau = \pi/K_1$, we set $U_\tau(x) = x - \Phi_\tau x,\ x \in \mathbb{R}^n$. The zeros of the vector field U_τ are precisely the singular points of equation (1.1). A point x_s is said to be asymptotically unstable if the spectrum $\sigma(f'(x_s))$ contains an eigenvalue with $\operatorname{Re}\lambda > 0$. In this case, $\sigma(U'_\tau(x_s)) = \{1\} - \exp(\tau\sigma(f'(x_s)))$.

Theorem 2.1. *Assume that the following conditions are satisfied for equation* (1.1):

(i) *there exists bounded convex strictly positive invariant domain* $D \subset \mathbb{R}^n$ *containing a unique singular point* x_s*, this point is asymptotically unstable and satisfies* $\det f'(x_s) \neq 0$;

(ii) *the function* F *is real analytic in* D;

(iii) $\lambda_3 - \lambda_2 > 2K$.

Then at least one stable limit cycle is localized in the domain D.

Proof. We use condition (iii) to reduce the final dynamics of (1.1) to the two-dimensional inertial manifold $H_2 \ni x_s$. By Remark 2.4, the estimate (iii) implies relation (2.6) for the quadratic form V with $P = P_2,\ \lambda = (\lambda_3 + \lambda_2)/2$ and $\varepsilon = (\lambda_3 - \lambda_2)/2 - K$. Assume that $\operatorname{Re}\kappa_1 \geq \operatorname{Re}\kappa_2 \geq \cdots \geq \operatorname{Re}\kappa_n$ for $\kappa_i \in \sigma(f'(x_s))$. If we consider the matrix $f'(x_s)$ as a perturbation of the matrix $-A$, then condition (iii) implies the inequality $\operatorname{Re}\kappa_3 < -\lambda < 0$. It follows from condition (i) that the vector field U_τ with $\tau = \pi/K_1$ has a unique zero x_s in $\overline{D}$.

Since the domain D is convex and $\Phi_\tau\overline{D} \subset D$, then according to [6, Theorem 21.5] the vector field U_τ is not is degenerate (0 does not belong to $\sigma(U'_\tau)$) on ∂D and the rotation of U_τ on ∂D is equal to 1. By the hypothesis (i) of the theorem the vector field U_τ is not degenerate at the point x_s, therefore from [6, Theorem 20.6] and [6, Theorem 21.6] we successively find that $\operatorname{ind} x_s = 1$ and $\operatorname{ind} x_s = (-1)^\beta$, where ind is the Poincare index and β is an even sum multiplicities of the real $\lambda > 1$ in $\sigma(\Phi'_\tau(x_s))$. At the same time, β is the sum multiplicities of positive $\kappa \in \sigma(f'(x_s))$. So, since $\operatorname{Re}\kappa_3 < 0$ and $\operatorname{Re}\kappa_1 > 0$, then $\operatorname{Re}\kappa_2 > 0$.

Thus, taking (i), (ii), and Remark 2.3 into account, we see that the assumptions in [7, Corollary 6.1] are satisfied, and hence the domain D contains at most finitely many cycles. One can see that the point $P_2 x_s$ is an unstable focus or an unstable knot of equation (2.5) in the plane $X_2 \subset \mathbb{R}^n$. By the Poincare–Bendixson theory [8, Sect. 2.8], this equation has finitely many embedded cycles in the strictly positive invariant domain

$P_2 D \subset X_2$ and at least one of them, Γ, is stable. Then $\psi\Gamma$ is a stable limit cycle of the original equation (1.1). $\Box$

Theorem 2.1 gives us a method for determining stable limit cycles of ODEs in $\mathbb{R}^n$. In what follows we refer to this method as to the "spectral gap method". In fact, notion similar to that of inertial manifold has been used successfully by R.A. Smith (see [7, 9, 10] and the references therein) in his studies of cycles of ODEs. This author worked with Lipschitz invariant manifolds of the form (2.1), attracting (not necessarily exponentially) all trajectories for $t \to +\infty$ and containing all bounded invariant sets. He did not use the simple and convenient condition (2.4) but directly considered[1] the condition of type (2.6) with an arbitrary quadratic form $V(x)$ of the signature $(0, n-2, 2)$. Formally, assumption (2.6) is weaker than (2.4) and does not mean that the vector field of the equation splits into linear and nonlinear parts. At the same time, the spectral gap condition (2.4) can be verified significantly simpler.

On the other hand, the method proposed in [3] guarantees the existence of an inertial manifold of dimension $m < n$ for equations of the form (1.1) with an *arbitrary* linear part $-A$ if, for some $\lambda > 0$, the spectrum $\sigma(A)$ has m values (with multiplicity taken into account) in the half-plane $\operatorname{Re} z < \lambda$, the straight line $\operatorname{Re} z = \lambda$ lies in the resolvent set $\rho(A)$, and $\|(A - \lambda - i\omega)^{-1}\|_2 < 1/K$, $\omega \in \mathbb{R}$. Such a technique was independently used to determine stable limit cycles in [10]. The author believes that the revival of this approach is rather perspective.

It should be noted that the technique of this paper (as well as papers [7, 9, 10]) only detect ODE cycles lying on invariant 2D-manifolds of the Cartesian structure (2.1).

In the following two sections we illustrate the spectral gap method with examples from two distinct areas of natural science.

3. Satellite motion model

The problems of the periodic dynamics of the satellites of celestial bodies extensive literature is devoted (see, for example, [11] and references therein). In particular, the dynamics of a artificial satellite flying around a celestial body of small mass was studied

[1] See, e.g., [10, Theorem 3].

in [12]. We consider here this model as a successful mathematical application of our method for detecting stable limit cycles. Let (r, φ) be the polar coordinates in the plane of the motion $r = r(t)$, $\varphi = \varphi(t)$ of a flying vehicle. According to [12], the radial and transverse control forces act on the satellite, depending on the positive parameters μ_1, μ_2, μ_3 and some smooth function $g(\dot{\varphi})$. The goal is to determine the values μ_1, μ_2, μ_3 and the function g so as to ensure the existence of a stable periodic motion in coordinates $(r, \dot{r}, \dot{\varphi})$. We set $x_1 = \dot{r} + \mu_2 r$, $x_2 = r$, $x_3 = \dot{\varphi}$. In these new coordinates, the satellite dynamics can be described by the system of equations (slightly different from the system in [12])

$$\begin{aligned} \dot{x}_1 &= -\mu_1 x_1 + g(x_3), \\ \dot{x}_2 &= -\mu_2 x_2 + x_1, \\ \dot{x}_3 &= -\mu_3 x_3 + x_2 \end{aligned} \tag{3.1}$$

with control parameters $\mu_1, \mu_2, \mu_3 > 0$ and the "admissible" nonlinear function $g \in C^{1+\alpha}(\mathbb{R})$. We define the class of admissible smooth functions g in (3.1) by conditions

$$0 < g(x_3) < M, \quad -1 \leq g'(x_3) < 0 \tag{3.2}$$

for $x_3 \in \mathbb{R}$. The choice of such a class will allow us to apply Theorem 2.1 under certain conditions on the parameters μ_1, μ_2, μ_3. A similar mathematical model was studied in [10, Sect. 7] from a different standpoint. System (3.1) takes the form (1.1) if we set

$$A = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad F(x) = \begin{pmatrix} g(x_3) \\ x_1 \\ x_2 \end{pmatrix}.$$

This decomposition of a vector field (3.1) is natural from the point of view of condition (iii) of Theorem 2.1, so as the matrix A is symmetric, and the Lipschitz constant of nonlinearity F easy to appreciate.

Due to the second condition in (3.2), system (3.1) generates a C^1 phase flow $\{\Phi_t\}$ in $\mathbb{R}^3$.

Lemma 3.1. *The convex domain*

$$D = \{x \in \mathbb{R}^3 : \ 0 < x_1 < \frac{M}{\mu_1}, \ \ 0 < x_2 < \frac{M}{\mu_1 \mu_2}, \ \ 0 < x_3 < \frac{M}{\mu_1 \mu_2 \mu_3}\}$$

is strictly positive invariant and contains a unique singular point.

Proof. The search of the singular points of the system (3.1) reduces to solving the scalar equation $g(x_3) = \mu_1\mu_2\mu_3 x_3$. Since according to conditions (3.2) we have $0 < g < M$ and $g' < 0$, then this equation has a unique solution $x_3 = \nu > 0$. So there exists a unique singular point in $\mathbb{R}^3$:

$$x_s = (\mu_2\mu_3\nu, \mu_3\nu, \nu) = \left(\frac{g(\nu)}{\mu_1}, \frac{g(\nu)}{\mu_1\mu_2}, \frac{g(\nu)}{\mu_1\mu_2\mu_3}\right).$$

Note that $x_s \in D$.

We first show that $\Phi_t D \subseteq D$, and hence $\Phi_t \overline{D} \subseteq \overline{D}$ for $t > 0$. Consider the solution $x(t) = (x_1(t), x_2(t), x_3(t)$ with $x(0) \in D$. On the faces $x_1 = 0$ and $x_1 = M/\mu_1$ of the parallelepiped D, we have $\dot{x}_1 = g(x_3) > 0$ and $\dot{x}_1 = -\mu x_1 + g(x_3) < 0$ respectively, so $0 < x_1(t) < M/\mu_1$ for $t > 0$. On the faces $x_2 = M/(\mu_1\mu_2)$ and $x_2 = 0$, we have $\dot{x}_2 < 0$ and $\dot{x}_2(t) = x_1(t) > 0$ respectively, and hence, $0 < x_2(t) < M/(\mu_1\mu_2)$ for $t > 0$. On the faces $x_3 = M/(\mu_1\mu_2\mu_3)$ and $x_3 = 0$, we have $\dot{x}_3 < 0$ and $\dot{x}_3(t) = x_2(t) > 0$ respectively, so that $0 < x_3(t) < M/(\mu_1\mu_2\mu_3)$ for $t > 0$.

We write $\Pi = \{x \in \partial D : \Phi_t x \in D,\ t > 0\}$ and $\Pi_0 = \partial D \backslash \Pi$. We see that $\Pi_0 \subseteq l_1 \bigcup l_2 \bigcup \{0\}$, where $l_1 = \{x \in \partial D : x_1 = 0,\ x_2 = 0,\ x_3 > 0\}$ and $l_2 = \{x \in \partial D : x_1 > 0,\ x_2 = 0,\ x_3 = 0\}$. On l_1 and l_2, we respectively have $\dot{x}_1 > 0$ and $\dot{x}_2 > 0$, and hence $\Phi_t x \in D$, $t > 0$, on $\Pi_0/\{0\}$. Because $\Phi_t 0 \neq 0$, we have $\Phi_t 0 \in D$, $t > 0$. Thus, $\Pi_0 = \phi$, $\Pi = \partial D$, and $\Phi_t \overline{D} \subseteq D$ for $t > 0$. □

Clearly,

$$F'(x) = \begin{pmatrix} 0 & 0 & g'(x_3) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (F'(x))^* \cdot F'(x) = \operatorname{diag}(1, 1, (g'(x_3))^2),$$

and $\|F'(x)\|_2 = 1$ for all $x \in \mathbb{R}^3$. Let $\lambda_1, \lambda_2, \lambda_3$ stand for the parameters μ_1, μ_2, μ_3 permutated by nondecreasing order. We have $K = 1$ and the spectral gap condition (2.4) becomes

$$\lambda_3 - \lambda_2 > 2. \tag{3.3}$$

We linearize the vector field of the system (3.1) at the singular point x_s. Note that the Routh–Hurwitz criterion gives the condition of asymptotic instability of x_s by the inequality

$$-g'(\nu) + \lambda_1\lambda_2\lambda_3 > (\lambda_1+\lambda_2+\lambda_3)(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)\,. \tag{3.4}$$

In addition, $\det(F'(x_s)-A) = g'(\nu)-\lambda_1\lambda_2\lambda_3 \neq 0$. Estimates (3.3), (3.4) determine a nonempty open set Ω in the positive octant $\mathbb{R}^3_+$ of the parameters λ_1, λ_2, λ_3. In particular, the domain Ω contains points of the form $(\delta,\ \delta,\ 2+2\delta)$ for all sufficiently small $\delta>0$. If the function g in (3.2) is real analytic for $0<x_3<M/(\mu_1\mu_2\mu_3)$, then by Theorem 2.1, system (3.1) with $(\lambda_1,\lambda_2,\lambda_3)\in\Omega$ has a stable limit cycle $\Gamma\subset D$.

As an admissible nonlinear function in (3.1) we can, for example, take

$$g(x_3) = \operatorname{arccot}(x_3-\nu), \quad \nu = \frac{\pi}{2\mu_1\mu_2\mu_3}.$$

This function satisfies conditions (3.2) with $g'(\nu)=-1$ and $M=\pi$.

In similar constructions [12], the real analyticity of the function g in (3.1) is not required, but it is only necessary to prove the existence of an orbitally stable periodic trajectory on which at least one different trajectory is "winding" as $t\to+\infty$.

4. A model of cell processes

Another example illustrating the spectral gap method is related to the complex dynamics in cell processes [13]. Consider the following the system of equations

$$\begin{aligned} \dot{x} &= -kx + R(z), \\ \dot{y} &= x - G(y,z), \\ \dot{z} &= -qz + G(y,z), \end{aligned} \tag{4.1}$$

where

$$R(z) = \frac{1}{1+z^4}, \quad G(y,z) = \frac{Ty(1+y)(1+z)^2}{L+(1+y)^2(1+z)^2}$$

and $k,q,T,L>0$ are constants. Here x, y, and z are dimensionless concentrations of the matters S_1, S_2, and S_3, where S_1 is the initial product, S_2 is the intermediate product, and S_3 is the final product; k and q are constants of the rate of variation in S_1 and S_3. We have

$$R_z = -\frac{4z^3}{(1+z^4)^2}, \quad G_z = \frac{2TLy(1+y)(1+z)}{(L+(1+y)^2(1+z)^2)^2}\,,$$

$$G_y = \frac{2TLy(1+z)^2}{(L+(1+y)^2(1+z)^2)^2} + \frac{T(1+z)^2}{L+(1+y)^2(1+z)^2},$$

$R_z(z) < 0$ for $z > 0$, and $G(y,z) < T$, $G_y(y,z) > 0$, $G_z(y,z) > 0$ for $y, z > 0$. Since the first derivatives of the functions R and G are uniformly bounded in $z \in \mathbb{R}$ and $(y,z) \in \mathbb{R}^2$, we see that system (4.1) generates a smooth flow $\{\Phi_t\}$ in $\mathbb{R}^3$. We fix the values $T = 10$ and $L = 10^6$ that are physically meaningful from the standpoint of the authors of [13] and try to determine pairs of free parameters $(k,q) \in \mathbb{R}^2_+$ for which this system satisfies the conditions of Theorem 2.1 and hence admits a stable periodic regime.

Everywhere below we restrict ourselfs to the simple case when $kT > 1$ and $k > q$. By $p(x,y,z)$ we denote points in $\mathbb{R}^3$.

4.1. Positive invariant domain and a singular point. We note that $G(+\infty, 0) = T$ and $G(0,z) = 0$ for $z > 0$. Since $kT > 1$, we can uniquely determine the value $y_0 > 0$ from the relation $G(y_0, 0) = 1/k$. In what follows we set $x_0 = 1/k,\ z_0 = T/q$.

Lemma 4.1. *The convex domain* $D = \{p \in \mathbb{R}^3 : 0 < x < x_0,\ 0 < y < y_0,\ 0 < z < z_0\}$ *is strictly positive invariant and contains a unique singular point.*

Proof. Equating the right-hand side of (4.1) to zero we obtain the relations $x = qz$ and $kqz = R(z)$ which are satisfied for a unique pair of values $x_s,\ z_s > 0$. Another scalar equation $\varphi(y) = 0$ with $\varphi(y) = qz_s - G(y, z_s)$, $\varphi' < 0$, has a unique solution $y_s > 0$. So system (4.1) has a unique singular point $p_s = (x_s,\ y_s,\ z_s)$ in $\mathbb{R}^3_+$. Since the function R decreases in $z > 0$, it follows that $z_s = (kq)^{-1}R(z_s) < (kq)^{-1} < z_0$ and $x_s = k^{-1}R(z_s) < x_0$. Taking into account that G is an increasing function with respect to each variable $y > 0$ and $z > 0$, from the relation $x_s = G(y_s, z_s)$ we derive that $x_s = G(y_s, z_s) < x_0 = G(y_0, 0)$, and hence $y_s < y_0$ and $p_s \in D$.

First, we show that $\Phi_t D \subseteq D$, and hence $\Phi_t \overline{D} \subseteq \overline{D}$ for $t > 0$. We consider the solution $p(t) = (x(t), y(t), z(t))$ with $p(0) \in D$. On the faces $z = 0$ and $z = z_0$ of the bar D, we have $\dot{z} = G(y, 0) > 0$ and $\dot{z} = -T + G(y, z_0) < 0$, respectively, and hence $0 < z(t) < z_0$ for $t > 0$. On the faces $x = 0$ and $x = x_0$, we have $\dot{x} = R(z) > 0$ and $\dot{x}(t) = -1 + R(z(t)) < 0$ for $p(t)$, respectively, and hence $0 < x(t) < x_0$ for $t > 0$. On the faces $y = 0$ and $y = y_0$, we respectively have $\dot{y}(t) = x(t) - G(0, z(t)) = x(t) > 0$ and $\dot{y}(t) = x(t) - G(y_0, z(t)) < x_0 - G(y_0, 0) = 0$ for $p(t)$, whence $0 < y(t) < y_0$ for $t > 0$.

We write $\Pi = \{p \in \partial D : \Phi_t p \in D,\ t > 0\}$, $\Pi_0 = \partial D \setminus \Pi$, and $p_0 = (x_0, y_0, 0)$. We see that $\Pi_0 \subseteq l_1 \bigcup l_2 \bigcup l_3 \bigcup \{p_0\}$, where $l_1 : \{x = x_0,\ 0 \le y < y_0,\ z = 0\}$, $l_2 : \{x = 0,\ y = 0,\ 0 \le z \le z_0\}$, and $l_3 : \{x = x_0,\ y = y_0,\ 0 \le z \le z_0\}$. On l_1, l_2, and l_3, we respectively have $\dot z > 0$, $\dot x > 0$, $\dot x < 0$, and hence $\Phi_t p \in D$, $t > 0$, on $\Pi_0 \setminus \{p_0\}$. Since $\Phi_t p_0 \ne p_0$, we see that $\Phi_t p_0 \in D$, $t > 0$. Thus, $\Pi_0 = \phi$, $\Pi = \partial D$, and $\Phi_t \overline{D} \subseteq D$ for $t > 0$. □

4.2. Inertial manifold. In the natural decomposition $f = -A + F$ of the vector field f of system (4.1) into the linear and nonlinear parts, we have

$$A = \begin{pmatrix} -k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -q \end{pmatrix}, \qquad F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R(z) \\ x - G(y,z) \\ G(y,z) \end{pmatrix}.$$

This decomposition with symmetric matrix A is chosen in order to best provide condition (iii) of Theorem 2.1. For the matrix A we have $\lambda_1 = 0$, $\lambda_2 = q$, $\lambda_3 = k$. The change $u = y + z$ takes (4.1) to the form

$$\dot x = -kx + R(z), \quad \dot u = x - qz, \quad \dot z = -qz + G(u - z, z) \tag{4.2}$$

in the variables (x, u, z) with the vector field decomposition $f_1 = -A + F_1$, where $F_1 : (x, u, z) \to (R(z),\ x - qz,\ G(u - z, z))$. In this case,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C\begin{pmatrix} x \\ u \\ z \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The nonlinear part F_1 in (4.2) is simpler than the nonlinear part F in the original system (4.1), which allows us to sharpen the estimate of $K = K(k, q)$ for the norm of its Jacobi matrix in the spectral gap condition $\lambda_3 - \lambda_2 > 2K$. The domain $C^{-1}D$ is strictly positive invariant for (4.2). We put

$$K = \max_{C^{-1}D} \left\| F_1'(\overline{p}) \right\|_2 = \max_{D} \left\| (F_1' C^{-1})(p) \right\|_2, \quad F_1' C^{-1} = \begin{pmatrix} 0 & 0 & -R_z \\ 1 & 0 & -q \\ 0 & G_y & G_z - G_y \end{pmatrix}, \tag{4.3}$$

where $\overline{p} = (x, u, z)$. The condition (2.4) of existence of the inertial manifold means that (1.2) is satisfied for the function F_1 on $\mathbb{R}^3$. In this connection, it is useful to consider a $C^{1+\alpha}$ extension of F_1 from the domain $C^{-1}D$ to $\mathbb{R}^3$ with the same value of K. To this end, consider the functions $\overline{R}$ and $\overline{G}$ defined as follows. The function $\overline{R}$ satisfies $\overline{R}(0) = R(0)$ and its derivative $\overline{R}_z$ is an even $2z_0$-periodic extension of R_z from $[0, z_0]$ to $\mathbb{R}$. Similarly, $\overline{G}$ satisfies $\overline{G}(0,0) = G(0,0)$ and its derivatives $\overline{G}_y$ and $\overline{G}_z$ are even, with respect to both y and z, and $(2y_0, 2z_0)$-periodic extensions of G_y and G_z, correspondingly, from $[0, y_0] \times [0, z_0]$ to $\mathbb{R}^2$. If we now put $F_2 : (x, u, z) \to (\overline{R}(z),\ x - qz,\ \overline{G}(u - z, z))$, then the function F_2 yields the sought extension of F_1 from $C^{-1}D$ to $\mathbb{R}^3$. Clearly, the phase dynamics of system (4.2) in the domain $C^{-1}D$ remains the same when F_1 is replaced by F_2.

Let $\Theta = \{(k, q) \in \mathbb{R}^2_+,\ k - q > 2K(k, q)\}$. Then $\lambda_3 - \lambda_2 = k - q$ and, for $(k, q) \in \Theta$, the system of equations

$$\dot{x} = -kx + \overline{R}(z), \quad \dot{u} = x - qz, \quad \dot{z} = -qz + \overline{G}(u - z, z) \tag{4.4}$$

admits a two-dimensional inertial manifold. The same is also true for the system

$$\dot{x} = -kx + \overline{R}(z), \quad \dot{y} = x - \overline{G}(y, z), \quad \dot{z} = -qz + \overline{G}(y, z), \tag{4.5}$$

which inherits the phase dynamics of (4.1) in the domain D.

Remark 4.1. If $(k_0, q_0) \in \Theta$, then $(k, q) \in \Theta$ for $k \geq k_0$, $q \geq q_0$, $k - q \geq k_0 - q_0$.

Indeed, since the strictly positive invariant domain D decreases as k and q increase, it follows that the constant $K = K(k, q)$ in (4.3) does not increase and the inequality $k - q > 2K$ still holds. We see that systems (4.1) and (4.5) demonstrate the two-dimensional final dynamics in the vast domain Θ of the parameters (k, q).

4.3. Instability of the singular point. The singular points of systems (4.1) and (4.4) are simultaneously stable or unstable. The Jacobi matrix $f'(p_s)$ of the vector field of system (4.1) at the singular point $p_s = (x_s, y_s, z_s) \in D$ has the form

$$\begin{pmatrix} -k & 0 & -b \\ 1 & -c & -d \\ 0 & c & d - q \end{pmatrix}$$

with $b = -R_z(z_s)$, $c = G_y(y_s, z_s)$, and $d = G_z(y_s, z_s)$. By the Routh–Hurwitz criterion, this point is asymptotically unstable if $a_1 < 0$ or $a_1 a_2 - a_3 < 0$ or $a_3 < 0$, where

$$a_1 = c - d + k + q, \quad a_2 = k(c - d) + qc + kq, \quad a_3 = (kq + b)c\,.$$

Because $a_3 > 0$, the point p_s is unstable under the condition $a_2 < 0$. We have $\det f'(p_s) = c(b - kq)$.

4.4. Stable limit cycle. The complicated character of nonlinearity in (4.1) requires the use of computational tools (Maple package) for estimating the Lipschitz constant $K(k, q)$ and analyzing the instability of p_s. As an example, we take two pairs of parameters $k > q$ and estimate the norms for the points $p \in D$. The square numerical matrices B satisfy the inequality $\|B\|_2 \le \sqrt{\|B\|_\infty \cdot \|B\|_1}$, where $\|B\|_\infty$ and $\|B\|_1$ are the norms of the linear operators corresponding to B in $\mathbb{R}^n_\infty$ and $\mathbb{R}^n_1$.

For $k = 3$ and $q = 0.1$, we have:

$$y_1 \approx 186,\ x_s \approx 0.117,\ y_s \approx 49.653,\ z_s \approx 1.167,\ b - kq \approx 0.480,\ a_2 \approx -0.05,$$

$$\left\|(F_1' C^{-1})(p)\right\|_\infty \le 1.209,\ \ \left\|(F_1' C^{-1})(p)\right\|_1 \le 1.166,\ \ \left\|(F_1' C^{-1})(p)\right\|_2 \le K = 1.187.$$

For $k = 2.5$ and $q = 0.1$, we have:

$$y_1 \approx 204,\ x_s \approx 0.123,\ y_s \approx 49.558,\ z_s \approx 1.230,\ b - kq \approx 0.438,\ a_2 \approx -0.01,$$

$$\left\|(F_1' C^{-1})(p)\right\|_\infty \le 1.209,\ \left\|(F_1' C^{-1})(p)\right\|_1 \le 1.166,\ \left\|(F_1' C^{-1})(p)\right\|_2 \le K = 1.187.$$

The vector field of system (4.4) is real analytic in the strictly positive invariant domain $C^{-1}D$, and this domain contains a unique singular point. In both cases $a_2 < 0$, $\det f'(p_s) = c(b - kq) \ne 0$, and $k - q > 2K$, so that by Theorem 2.1, system (4.4) admits a stable limit cycle $\Gamma \in C^{-1}D$ for the chosen values of k and q. It is easy to trace the continuous dependence of the quantities $K = K(k, q)$, $b = b(k, q)$, and $a_2 = a_2(k, q)$ on their arguments, and thus, the system admits stable periodic regimes for the parameters (k, q) in sufficiently small neighborhoods of the points $(3, 0.1)$ and $(2.5, 0.1)$. This implies that, for the same values of (k, q), the original system (4.1) has a stable limit cycle localized in the domain D.

5. Conclusion

The spectral gap method is based on the presence of a natural self-adjoint linear component $-A$ of the vector field of ODE with dominating third eigenvalue, $\lambda_3(A) > \lambda_2(A)$, which somewhat restricts the range of applications. The advantages of the method are the transparency of statements and the relative simplicity of its use. The problems solved by this method are technically reduced to careful estimation of the Lipschitz constant in the nonlinear component of the equations and determination of a strictly positive invariant domain in the phase space that contains a unique (asymptotically unstable) singular point. In general, the proposed method can well complement the list of well-known approaches to the problem of determining stable limit cycles of ordinary differential equations in $\mathbb{R}^n$, lying on invariant 2D-manifolds of the Cartesian structure.

Existence of an inertial manifold of dimension greater than 2 is also of interest. For example, the presence of such manifolds of dimension 3 guarantees, that all invariant tori (if any) of the dynamical system lie on the invariant three-dimensional C^1-manifold of the form (2.1). In the most common spectral gap condition (2.4) allows us to state that the union of all bounded invariant sets lies on the smooth invariant m-dimensional manifold of the Cartesian structure.

References

[1] G. Sell and Y. You, *Dynamics of Evolutionary Equations*, Springer, New York, 2002.

[2] S. Zelik, Inertial manifolds and finite-dimensional reduction for dissipative PDEs, *Proc. Roy. Soc. Edinburgh, Ser. A* **144** (2014), No. 6, 1245–1327.

[3] M. Miklavcic, A sharp condition for existence of an inertial manifolds, *J. Dyn. Differ. Equations* **3** (1991), No. 3, 437–456.

[4] A. V. Romanov, Sharp estimates of the dimension of inertial manifolds for nonlinear parabolic equations, *Izv. Math.* **43** (1994), No. 1, 31–47.

[5] J. A. Yorke, Periods of periodic solutions and Lipschitz constant, *Proc. Amer. Math. Soc.* **22** (1969), No. 2, 509–512.

[6] M. A. Krasnosel'skii and P. P. Zabreiko, *Geometrical Methods of Nonlinear Analysis*, 2nd ed., Springer, New York, 2011.

[7] R. A. Smith, Certain differential equations have only isolated periodic orbits, *Ann. Mat. Pura Appl.*, **137** (1984), No. 1, 217–244.

[8] R. Reissig, G. Sansone and R. Conti, *Qualitative Theorie Nichtlinearer Differentialgleichungen*, Edizioni Cremonese, Roma, 1963.

[9] R. A. Smith, Poincare index theorem concerning periodic orbits of differential equations,

Proc. London Math. Soc. (3) **48** (1984), No. 2, 341–362.

[10] R. A. Smith, Orbital stability for ordinary differential equations, *J. Differ. Equations* **69** (1987), No. 2, 265–287.

[11] V. V. Sidorenko and A. Celletti, "Spring–mass" model of tethered satellite systems: properties of planar periodic motions, *Celestial Mechanics and Dynamical Astronomy*, **107** (2010), No. 1–2, 209—231.

[12] I. A. Galiullin and L. A. Kondratieva, Satellite inertial manifolds and boundary cycles, *Cosmonautics and Rocket Engineering* (2011), No. 3, 73–76 (in Russian).

[13] C. Suguna, K. K. Chowdhury and S. Sinha, Minimal model for complex dynamics in cellular processes, *Phys. Rev. E* **60** (1999), No. 5, 5943–5949.