On curves with the Poritsky property

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To Professor Claude Viterbo on the occasion of his 60th birthday

Abstract

Reflection in planar billiard acts on oriented lines. For a given closed convex planar curve γ the string construction yields a one-parameter family Γ_p of nested billiard tables containing γ for which γ is a *caustic:* the reflection from Γ_p sends each tangent line to γ to a line tangent to γ. The reflections from Γ_p act on the corresponding tangency points, inducing a family of *string diffeomorphisms* $\mathcal{T}_p : \gamma \to \gamma$. We say that γ has the *string Poritsky property,* if it admits a parameter t (called the *Poritsky string length*) in which all the transformations $\mathcal{T}_p$ with small p are translations $t \mapsto t + c_p$. These definitions also make sense for germs of curves γ. The Poritsky property is closely related to the famous Birkhoff Conjecture. Each conic has the string Poritsky property. Conversely, *each germ of planar curve with the Poritsky property is a conic* (H.Poritsky, 1950). In the present paper we extend this result of Poritsky to curves on surfaces of constant curvature and to outer billiards on all these surfaces. For curves with the Poritsky property on a surface with arbitrary Riemannian metric we prove the following two results: *1) the Poritsky string length coincides with Lazutkin parameter up to additive and multiplicative constants; 2) a*

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germ of C^5-smooth curve with the Poritsky property is uniquely determined by its 4-jet. In the Euclidean case the latter statement follows from the above-mentioned Poritsky's result on conics.

Contents

1 Introduction and main results

Consider the billiard in a bounded planar domain $\Omega \subset \mathbb{R}^2$ with a strictly convex smooth boundary. The billiard dynamics T acts on the space of oriented lines intersecting Ω. Namely, let L be an oriented line intersecting Ω, and let A be its last point (in the sense of orientation) of its intersection with $\partial\Omega$. By definition $T(L)$ is the image of the line L under the symmetry with respect to the tangent line $T_A\partial\Omega$, being oriented from the point A inside the domain Ω. A curve $\gamma \subset \mathbb{R}^2$ is a *caustic* of the billiard Ω if each line tangent to γ is reflected from the boundary $\partial\Omega$ again to a line tangent to γ; in other words, if the curve formed by oriented lines tangent to γ is invariant under the billiard transformation T. In what follows we consider only *smooth caustics* (in particular, without cusps).

It is well-known that each planar billiard with sufficiently smooth strictly convex boundary has a Cantor family of caustics [14]. An analogous statement for outer billiards was proved in [2]. Every elliptic billiard is *Birkhoff caustic integrable,* that is, an inner neighborhood of its boundary is foliated by closed caustics. The famous Birkhoff Conjecture states the converse: the only Birkhoff caustic integrable planar billiards are elllipses. The Birkhoff Conjecture together with its extension to billiards on surfaces of constant curvature and its outer and projective billiard versions (due to Sergei Tabachnikov) are big open problems, see, e.g., [7, 8, 12, 24] and references therein for history and related results.

It is well-known that each smooth convex planar curve γ is a caustic for a family of billiards $\Omega = \Omega_p$, $p \in \mathbb{R}_+$, whose boundaries $\Gamma = \Gamma_p = \partial\Omega_p$ are given by the p-th string constructions, see [23, p.73]. Namely, let $|\gamma|$ denote the length of the curve γ. Take an arbitrary number $p > 0$ and a string of length $p + |\gamma|$ enveloping the curve γ. Let us put a pencil between the curve γ and the string, and let us push it out of γ to a position such that the string that envelopes γ and the pencil becomes tight. Then let us move the pencil around the curve γ so that the string remains tight. Moving the pencil in this way draws a convex curve Γ_p that is called the *p-th string construction,* see Fig. 1.

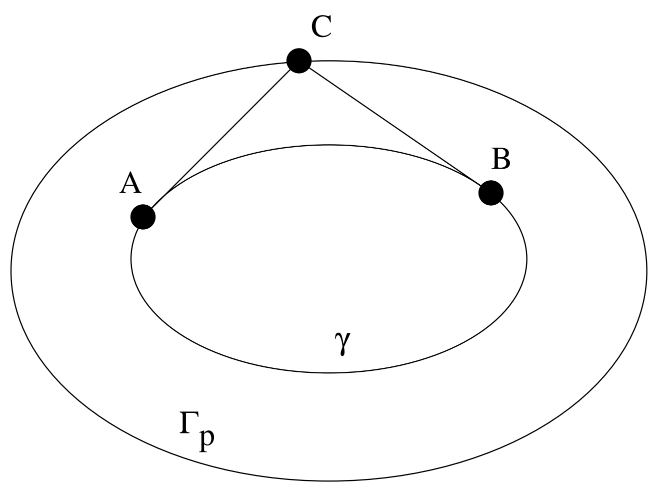

Figure 1: The string construction.

For every $A \in \gamma$ by G_A we denote the line tangent to γ at A. If γ is oriented by a vector in $T_A\gamma$, then we orient G_A by the same vector. The billiard reflection T_p from the curve Γ_p acts on the oriented lines tangent to γ. It induces the mapping $\mathcal{T}_p : \gamma \to \gamma$ acting on tangency points and called *string diffeomorphism.* It sends each point $A \in \gamma$ to the point of tangency of the curve γ with the line $T_p(G_A)$.

Consider the special case, where γ is an ellipse. Then for every $p > 0$ the curve Γ_p given by the p-th string construction is an ellipse confocal to γ. Every ellipse γ admits a canonical bijective parametrization by the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ equipped with a parameter t such that for every $p > 0$ small enough one has $\mathcal{T}_p(t) = t + c_p$, $c_p = c_p(\gamma)$, see [23, the discussion before corollary 4.5]. The property of existence of the above parametrization will be called the *string Poritsky property,* and the parameter t will be called the *Poritsky–Lazutkin string length.*

In his seminal paper [20] Hillel Poritsky proved the Birkhoff Conjecture under the additional assumption called the *Graves* (or *evolution*) *property*: for every two nested caustics γ_λ, γ_μ of the billiard in question the smaller caustic γ_λ is also a caustic of the billiard in the bigger caustic γ_μ. His beautiful geometric proof was based on his remarkable theorem stating that in the Euclidean plane only conics have the string Poritsky property [20, section 7].

In the present paper we extend this theorem by Poritsky to billiards on simply connected complete surfaces of constant curvature (Subsection 1.1 and Section 4) and prove its version for outer billiards and area construction on these surfaces (Subsection 1.2 and Section 5). All the results of the present paper will be stated and proved for germs of curves, and thus, in Subsection 1.1 (1.2) we state the definitions of the Poritsky string (area) property for germs. We also study the Poritsky property on arbitrary surfaces equipped with a Riemannian metric. In this general case we show that the Poritsky string length coincides with the Lazutkin parameter

$$t_L(s) = \int_{s_0}^{s} \kappa^{\frac{2}{3}}(\zeta)d\zeta \tag{1.1}$$

introduced in [14, formula (1.3)], up to multiplicative and additive constants (Theorem 1.15 in Subsection 1.3, proved in Section 6). Here κ is the geodesic curvature. This explains the name "Poritsky–Lazutkin length".

Recall that the billiard ball map acting on the space of oriented geodesics preserves the canonical symplectic form (see the background material in Subsection 7.1). The above-mentioned Theorem 1.15 concerns the family of reflections from the string curves Γ_p, which is a family of symplectomorphisms having a common invariant curve: the curve of geodesics tangent to γ. In Section 7 we extend Theorem 1.15 to a more general class of symplectic maps: families of the so-called "weakly billiard-like maps" with a converging family of invariant curves (Theorem 7.10 stated in Subsection 7.2).

In [14], for a given curve $\gamma \subset \mathbb{R}^2$ V.F.Lazutkin introduced remarkable coordinates (x, y) on the space of oriented lines, in which the billiard ball map given by reflection from the curve γ takes the form

$$(x, y) \mapsto (x + y + o(y), y + o(y^2));$$

the x-axis coincides with the set of the lines tangent to γ;

$x = t_L(s)$ on the $x-$axis up to multiplicative and additive constants.

For the proof of Theorem 7.10 we use analogous coordinates for weakly billiard-like maps and prove Lemma 7.13 on asymptotic behavior of orbits in these coordinates (Subsection 7.3). We retrieve Theorem 1.15 (for C^6-smooth curves) from Theorem 7.10 at the end of Section 7.

R.Melrose and S.Marvizi [15] have shown that the billiard ball map given by a C^∞-smooth curve coincides with a unit time flow map of appropriately "time-rescaled" smooth Hamiltonian vector field, up to a flat correction.

For curves on arbitrary surface equipped with a C^6-smooth Riemannian metric we show that a C^5-smooth germ of curve with the string Poritsky property is uniquely determined by its 4-jet (Theorem 1.19 stated in Subsection 1.4 and proved in Section 8). This extends similar property of planar conics.

Theorem 1.3 in Subsection 1.1 (proved in Section 3) states that if a metric and a germ of curve γ are both C^k, then the string curve foliation is tangent to a $C^{[\frac{k}{2}]-1}$-smooth line field on the closed concave side from γ.

In Section 2 we present a Riemannian-geometric background material on normal coordinates, equivalent definitions of geodesic curvature etc. used in the proofs of main results.

1.1 The Poritsky property for string construction

Let Σ be a two-dimensional surface equipped with a Riemannian metric. Let $\gamma \subset \Sigma$ be a smooth curve (a germ of smooth curve at a point $O \in \Sigma$). We consider it to be *convex:* its geodesic curvature should be non-zero. For every given two points $A, B \in \gamma$ close enough by C_{AB} we will denote the unique point (close to them) of intersection of the geodesics G_A and G_B tangent to γ at A and B respectively. (Its existence will be proved in Subsection 2.1.) Set

$$\lambda(A, B) := \text{ the length of the arc } AB \text{ of the curve } \gamma,$$

$$L(A, B) := |AC_{AB}| + |BC_{AB}| - \lambda(A, B). \tag{1.2}$$

Here for $X, Y \in \Sigma$ close enough and lying in a compact subset in Σ by $|XY|$ we denote the length of small geodesic segment connecting X and Y.

Definition 1.1 (equivalent definition of string construction) Let $\gamma \subset \Sigma$ be a germ of curve with non-zero geodesic curvature. For every $p \in \mathbb{R}_+$ small enough the subset

$$\Gamma_p := \{C_{AB} \mid L(A, B) = p\} \subset \Sigma$$

is called the *p-th string construction,* see [23, p.73].

Remark 1.2 For every $p > 0$ small enough Γ_p is a well-defined smooth curve, we set $\Gamma_0 = \gamma$. The curve γ is a caustic for the billiard transformation acting by reflection from the curve Γ_p: a line tangent to γ is reflected from the curve Γ_p to a line tangent to γ [23, theorem 5.1]. In Section 3 we will prove the following theorem.

Theorem 1.3 *Let $k \geq 2$, Σ be a C^k-smooth surface equipped with a C^k-smooth Riemannian metric, and let $\gamma \subset \Sigma$ be a germ of C^k-smooth curve at $O \in \Sigma$ with positive geodesic curvature. Let $\mathcal{U} \subset \Sigma$ denote a small domain adjacent to γ from the concave side. For every $C \in \mathcal{U}$ let $\Lambda(C) \subset T_C\Sigma$ denote the exterior bisector of the angle formed by the two geodesics through C that are tangent to γ. Then the following statements hold.*

1) The one-dimensional subspaces $\Lambda(C)$ form a germ at O of line field Λ that is C^{k-1}-smooth on $\mathcal{U}$ and $C^{r(k)}$-smooth on $\overline{\mathcal{U}}$,

$$r(k) = [\frac{k}{2}] - 1.$$

2) The string curves Γ_p are tangent to Λ and $C^{r(k)+1}$-smooth. Their $(r(k) + 1)$-jets at base points C depend continuously on $C \in \overline{\mathcal{U}}$.

Definition 1.4 We say that a germ of oriented curve $\gamma \subset \Sigma$ with non-zero geodesic curvature has the *string Poritsky property,* if it admits a C^1-smooth parametrization by a parameter t (called the *Poritsky–Lazutkin string length*) such that for every $p > 0$ small enough there exists a $c = c_p > 0$ such that for every pair $B, A \in \gamma$ ordered by orientation with $L(A, B) = p$ one has $t(A) - t(B) = c_p$.

Example 1.5 It is classically known that

(i) for every planar conic $\gamma \subset \mathbb{R}^2$ and every $p > 0$ the p-th string construction Γ_p is a conic confocal to γ;

(ii) all the conics confocal to γ and lying inside a given string construction conic Γ_p are caustics of the billiard inside the conic Γ_p;

(iii) each conic has the string Poritsky property [20, section 7], [23, p.58];

(iv) conversely, *each planar curve with the string Poritsky property is a conic*, by a theorem of H.Poritsky [20, section 7].

Two results of the present paper extend statement (iv) to billiards on surfaces of constant curvature (by adapting Poritsky's arguments from [20, section 7]) and to outer billiards on the latter surfaces. To state them, let us recall the notion of a conic on a surface of constant curvature.

Without loss of generality we consider simply connected complete surfaces Σ of constant curvature 0, ± 1 and realize each of them in its standard model in the space $\mathbb{R}^3_{(x_1,x_2,x_3)}$ equipped with appropriate quadratic form

$$< Qx, x >, \ Q \in \{\text{diag}(1,1,0), \text{diag}(1,1,\pm 1)\}, \ < x, x >= x_1^2 + x_2^2 + x_3^2.$$

- Euclidean plane: $\Sigma = \{x_3 = 1\}$, $Q = \text{diag}(1,1,0)$.
- The unit sphere: $\Sigma = \{x_1^2 + x_2^2 + x_3^2 = 1\}$, $Q = Id$.
- The hyperbolic plane: $\Sigma = \{x_1^2 + x_2^2 - x_3^2 = -1\} \cap \{x_3 > 0\}$, $Q = \text{diag}(1,1,-1)$.

The metric of constant curvature on Σ is induced by the quadratic form $< Qx, x >$. The *geodesics* on Σ are its intersections with two-dimensional vector subspaces in $\mathbb{R}^3$. The *conics* on Σ are its intersections with quadrics $\{< Cx, x >= 0\} \subset \mathbb{R}^3$, where C is a real symmetric 3×3-matrix, see [11, 28].

Proposition 1.6 *On every surface of constant curvature each conic has the string Poritsky property.*

Theorem 1.7 *Conversely, on every surface of constant curvature each germ of C^2-smooth curve with the string Poritsky property is a conic.*

Proposition 1.6 and Theorem 1.7 will be proved in Section 4.

Remark 1.8 In the case, when the surface under question is Euclidean plane, Proposition 1.6 was proved in [20, formula (7.1)], and Theorem 1.7 was proved in [20, section 7].

1.2 The Poritsky property for outer billiards and area construction

Let $\gamma \subset \mathbb{R}^2$ be a smooth strictly convex closed curve. Let $\mathcal{U}$ be the exterior connected component of the complement $\mathbb{R}^2 \setminus \gamma$. Recall that the *outer billiard map* $T : \mathcal{U} \to \mathcal{U}$ associated to the curve γ acts as follows. Take a point $A \in \mathcal{U}$. There are two tangent lines to γ through A. Let L_A denote the right tangent line (that is, the image of the line L_A under a small clockwise rotation around the point A is disjoint from the curve γ). Let $B \in \gamma$ denote its tangency point. By definition, the image $T(A)$ is the point of the line L_A that is central-symmetric to A with respect to the point B.

It is well-known that if γ is an ellipse, then the corresponding outer billiard map is *integrable:* that is, an exterior neighborhood of the curve γ is foliated by invariant closed curves for the outer billiard map so that

γ is a leaf of this foliation. The analogue of Birkhoff Conjecture for the outer billiards, which was suggested by S.Tabachnikov [24, p.101], states the converse: if γ generates an integrable outer billiard, then it is an ellipse. Its polynomially integrable version was studied in [24] and recently solved in [10]. For a survey on outer billiards see [22, 23, 27] and references therein.

For a given strictly convex smooth curve Γ there exists a one-parametric family of curves γ_p such that γ_p lies in the interior component Ω of the complement $\mathbb{R}^2 \setminus \Gamma$, and the curve Γ is invariant under the outer billiard map T_p generated by γ_p. The curves γ_p are given by the following *area construction* analogous to the string construction. Let $\mathcal{A}$ denote the area of the domain Ω. For every oriented line ℓ intersecting Γ let $\Omega_-(\ell)$ denote the connected component of the complement $\Omega \setminus \ell$ for which ℓ is a negatively oriented part of boundary. Let now L be a class of parallel and co-directed oriented lines. For every $p > 0$, $p < \frac{1}{2}\mathcal{A}$, let L_p denote the oriented line representing L that intersects Γ and such that $Area(\Omega_-(L_p)) = p$. For every given p, the lines L_p corresponding to different classes L form a one-parameter family parametrized by the circle: the azimuth of the line is the parameter. Let γ_p denote the enveloping curve of the latter family, and let T_p denote the outer billiard map generated by γ_p. It is well-known that the curve Γ is T_p-invariant for every p as above [23, corollary 9.5]. See Fig. 2.

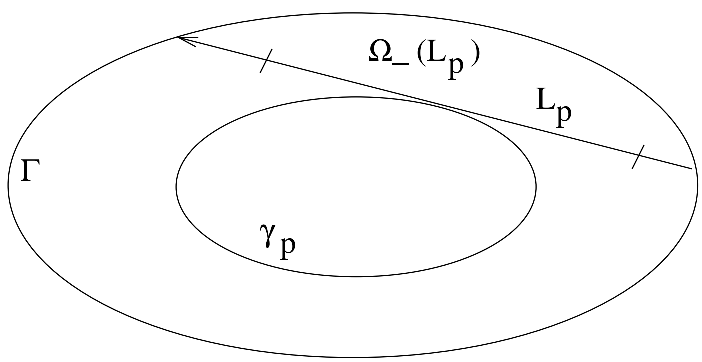

Figure 2: The area construction: $Area(\Omega_-(L_p)) \equiv p$.

Remark 1.9 For every $p > 0$ small enough the curve γ_p given by the area construction is smooth. But for big p it may have singularities (e.g., cusps).

For Γ being an ellipse, all the γ_p's are ellipses homothetic to Γ with respect to its center. In this case there exists a parametrization of the curve

Γ by circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ with parameter t in which $T_p : \Gamma \to \Gamma$ is a translation $t \mapsto t + c_p$ for every p. This follows from the area-preserving property of outer billiards, see [26, corollary 1.2], and T_q-invariance of the ellipse γ_p for $q > p$, analogously to the arguments in [20, section 7], [23, the discussion before corollary 4.5]. Similar statements hold for all conics, as in loc. cit.

In our paper we prove the converse statement given by the following theorem, which will be stated in local context, for germs of smooth curves. To state it, let us introduce the following definition.

Definition 1.10 Let Σ be a surface with a smooth Riemannian metric, $O \in \Sigma$. Let $\Gamma \subset \Sigma$ be a germ of smooth strictly convex curve at a point O (i.e., with positive geodesic curvature). Let $U \subset \Sigma$ be a disk centered at O that is split by Γ into two components. One of these components is convex; let us denote it by V. Consider the curves γ_p given by the above area construction with $p > 0$ small enough and lines replaced by geodesics. The curves γ_p form a germ at O of foliation in the domain V, and its boundary curve $\Gamma = \gamma_0$ is a leaf of this foliation. We say that the curve Γ has the *area Poritsky property,* if it admits a local C^1-smooth parametrization by a parameter t called the *area Poritsky parameter* such that for every $p > 0$ small enough the mapping $T_p : \Gamma \to \Gamma$ is a translation $t \mapsto t + c_p$ in the coordinate t.

Proposition 1.11 *(see [25, lemma 3] for the hyperbolic case; [26, lemma 5.1] for planar conics). On every surface of constant curvature each conic has the area Poritsky property.*

Remark 1.12 (S.Tabachnikov) The area Poritsky property for conics on the sphere follows from their string Poritsky property and the fact that the spherical outer billiards are dual to the spherical Birkhoff billiards [22, subsection 4.1, lemma 5]: the duality is given by orthogonal polarity. Analogous duality holds on hyperbolic plane realized as the half-pseudo-sphere of radius -1 in 3-dimensional Minkowski space [5, section 2, remark 2].

Theorem 1.13 *Conversely, on every surface of constant curvature each germ of C^2-smooth curve with the area Poritsky property is a conic*[1].

[1]For *planar* curves with the area Poritsky property the statement of Theorem 1.13 for C^4-smooth curves was earlier proved by Sergei Tabachnikov (unpublished paper, 2018) by analytic arguments showing that the affine curvature of the curve should be constant. In Section 5 we present a different, geometric proof, analogous to Poritsky's arguments from [20, section 7] (which were given for Birkhoff billiards and string construction), which works on all the surfaces of constant curvature simultaneously.

1.3 Coincidence of the Poritsky and Lazutkin lengths

Everywhere in the subsection Σ is a two-dimensional surface equipped with a C^3-smooth Riemannian metric.

Definition 1.14 Let $\gamma \subset \Sigma$ be a C^2-smooth curve, let s be its natural length parameter. Let $\kappa(s)$ denote its geodesic curvature. Fix a point in γ, let s_0 denote the corresponding length parameter value. The parameter

$$t_L := \int_{s_0}^{s} \kappa^{\frac{2}{3}}(\zeta) d\zeta \tag{1.3}$$

is called the *Lazutkin parameter.* See [14, formula (1.3)].

Theorem 1.15 *Let $\gamma \subset \Sigma$ be a germ of C^3-smooth curve with positive geodesic curvature κ and the string Poritsky property. Then its Poritsky string length parameter t coincides with the Lazutkin parameter (1.3) up to additive and multiplicative constants. That is, up to constant factor one has*

$$\frac{dt}{ds} = \kappa^{\frac{2}{3}}(s). \tag{1.4}$$

A proof of Theorem 1.15 will be presented in Section 6. It is based on the following theorem on asymptotics of the function $L(A, B)$ and its corollaries on string diffeomorphisms, also proved in the same section.

Theorem 1.16 *Let $\gamma \subset \Sigma$ be a C^3-smooth curve with positive geodesic curvature. For every $A \in \gamma$ let s_A denote the corresponding natural length parameter value. Let $L(A, B)$ denote the quantity defined in (1.2). One has*

$$L(A, B) = \frac{\kappa^2(A)}{12} |s_A - s_B|^3 (1 + o(1)), \tag{1.5}$$

uniformly, as $s_A - s_B \to 0$ so that A and B remain in a compact subarc in γ. Asymptotic (1.5) is also uniform in the metric running through a closed bounded subset in the space of C^3-smooth Riemannian metrics.

Corollary 1.17 *Let $\gamma \subset \Sigma$ be a germ of C^3-smooth curve with positive geodesic curvature. For every small $p > 0$ let $\mathcal{T}_p : \gamma \to \gamma$ denote the corresponding string diffeomorphism (induced by reflection of geodesics tangent to γ from the string curve Γ_p and acting on the tangency points). For every points B and Q lying in a compact subarc $\widehat{\gamma} \Subset \gamma$ one has*

$$\kappa^{\frac{2}{3}}(B)\lambda(B, \mathcal{T}_p(B)) \simeq \kappa^{\frac{2}{3}}(Q)\lambda(Q, \mathcal{T}_p(Q)), \text{ as } p \to 0, \tag{1.6}$$

uniformly in $B, Q \in \widehat{\gamma}$.

Corollary 1.18 *In the conditions of Corollary 1.17 one has*

$$\kappa^{\frac{2}{3}}(B)\lambda(B,\mathcal{T}_p(B)) \simeq \kappa^{\frac{2}{3}}(\mathcal{T}_p^m(B))\lambda(\mathcal{T}_p^m(B),\mathcal{T}_p^{m+1}(B)), \text{ as } p\to 0, \quad (1.7)$$

uniformly in $B\in\widehat{\gamma}$ *and those* $m\in\mathbb{N}$ *for which* $\mathcal{T}_p^m(B)\in\widehat{\gamma}$.

A symplectic generalization of Theorem 1.15 to families of the so-called weakly billiard-like maps of string type will be presented in Section 7.

1.4 Unique determination by 4-jet

The next theorem is a Riemannian generalization of the classical fact stating that each planar conic is uniquely determined by its 4-jet at some its point.

Theorem 1.19 *Let* Σ *be a surface equipped with a* C^6*-smooth Riemannian metric. A* C^5*-smooth germ of curve with the string Poritsky property is uniquely determined by its 4-jet.*

Theorem 1.19 will be proved in Section 8.

Remark 1.20 In the case, when Σ is the Euclidean plane, the statement of Theorem 1.19 follows from Poritsky's result [20, section 7] (see statement (iv) of Example 1.5). Similarly, in the case, when Σ is a surface of constant curvature, the statement of Theorem 1.19 follows from Theorem 1.7.

2 Background material from Riemannian geometry

We consider curves γ with positive geodesic curvature on an oriented surface Σ equipped with a Riemannian metric. In Subsection 2.1 we recall the notion of normal coordinates. We state and prove equivalence of different definitions of geodesic curvature. One of these definitions deals with geodesics tangent to γ at close points A and B and the asymptotics of angle between them at their intersection point C. In the same subsection we prove existence of two geodesics tangent to γ through every point C close to γ and lying on the concave side from γ; the corresponding tangency points will be denoted by $A=A(C)$ and $B=B(C)$. We also prove an asymptotic formula for derivative of azimuth of a vector tangent to a geodesic (Proposition 2.7). In Subsection 2.2 we prove formulas for the derivatives $\frac{dA}{dC}$, $\frac{dB}{dC}$, which will be used in the proofs of Theorems 1.3, 1.7, 1.15. In Subsection 2.3 we consider a pair of geodesics issued from the same point A and their points $G(s)$,

$Z(s)$ lying at a given distance s to A. We give an asymptotic formula for difference of azimuths of their tangent vectors at $G(s)$ and $Z(s)$, as $s \to 0$. We will use it in the proof of Theorem 1.19.

2.1 Normal coordinates and equivalent definitions of geodesic curvature

Let Σ be a two-dimensional surface equipped with a C^3-smooth Riemannian metric g. Let $O \in \Sigma$. Let γ be a C^2-smooth germ of curve at O parametrized by its natural length parameter. Recall that its geodesic curvature $\kappa = \kappa(O)$ equals the norm of the covariant derivative $\nabla_{\dot\gamma}\dot\gamma$. In the Euclidean case it coincides with the inverse of the osculating circle radius.

Consider the exponential chart $\exp : v \mapsto \exp(v)$ parametrizing a neighborhood of the point O by a neighborhood of zero in the tangent plane $T_O\Sigma$. We introduce orthogonal linear coordinates (x, y) on $T_O\Sigma$, which together with the exponential chart, induce *normal coordinates* centered at O, also denoted by (x, y), on a neighborhood of the point O. It is well-known that in normal coordinates the metric has the same 1-jet at O, as the standard Euclidean metric (we then say that its *1-jet is trivial* at O.) Its Christoffel symbols vanish at O.

Remark 2.1 Let the surface Σ and the metric be C^{k+1}-smooth. Then normal coordinates are C^k-smooth. This follows from theorem on dependence of solution of differential equation on initial condition (applied to the equation of geodesics) and C^k-smoothness of the Christoffel symbols. Thus, each C^k-smooth curve is represented by a C^k-smooth curve in normal coordinates.

Proposition 2.2 *For every curve γ as above its geodesic curvature $\kappa(O)$ equals its Euclidean geodesic curvature $\kappa_e(O)$ in normal coordinates centered at O. If the normal coordinates (x, y) are chosen so that the x-axis is tangent to γ, then γ is the graph of a germ of function:*

$$\gamma = \{y = f(x)\}, \quad f(x) = \pm\frac{\kappa(O)}{2}x^2 + o(x^2), \ \text{as } x \to 0. \tag{2.1}$$

Proof Proposition 2.2 follows from definition and vanishing of the Christoffel symbols at O in normal coordinates. □

Proposition 2.3 *Let the germ $(\gamma, O) \subset \Sigma$ be the same as at the beginning of the subsection, and let γ have positive geodesic curvature. Let $\mathcal{U} \subset \Sigma$ be a small domain adjacent to γ from the concave side: γ is its concave* *boundary. Let $\widehat{\gamma} \subset \gamma$ be a compact subset: an arc with boundary. For every $C \in \mathcal{U}$ close enough to $\widehat{\gamma}$ there exist exactly two geodesics through C tangent to γ. In what follows we denote their tangency points with γ by $A = A(C)$ and $B = B(C)$ so that AC is the right geodesic through C tangent to γ.*

Proof The statement of the proposition is obvious in the Euclidean case. The non-Euclidean case is reduced to the Euclidean case by considering a point $C \in \mathcal{U}$ close to $\widehat{\gamma}$ and normal coordinates (x_C, y_C) centered at C so that their family depends smoothly on C. In these coordinates the curves $\gamma = \gamma_C$ depend smoothly on C and are strictly convex in the Euclidean sense, by Proposition 2.2. The geodesics through C are lines. This together with the statement of Proposition 2.3 in the Euclidean case implies its statement in the non-Euclidean case. □

Let us consider that Σ is a Riemannian disk centered at O, the curve γ splits Σ into two open parts, and γ has positive geodesic curvature. For every point $A \in \gamma$ the geodesic tangent to γ at A will be denoted by G_A.

Proposition 2.4 *Taking the disk Σ small enough, one can achieve that for every $A \in \gamma$ the curve γ lies in the closure of one and the same component of the complement $\Sigma \setminus G_A$, $\gamma \cap G_A = \{A\}$.*

Proposition 2.4 follows its Euclidean version and Proposition 2.2.

Proposition 2.5 *For every two points $A, B \in \gamma$ close enough to O the geodesics G_A and G_B intersect at a unique point $C = C_{AB} \in \mathcal{U}$ close to O.*

Proof Let H denote the geodesic through B orthogonal to $T_B\gamma$. It intersects the geodesic G_A at some point $P(A, B) \in \mathcal{U}$. The geodesic G_B separates $P(A, B)$ from the punctured curve $\gamma \setminus \{B\}$, by construction and Proposition 2.4. Therefore, G_B intersects the interval $(A, P(A, B))$ of the geodesic G_A. This proves the proposition. □

Proposition 2.6 *For every $A, B \in \gamma$ close enough to O let $C = C_{AB}$ denote the point of intersection $G_A \cap G_B$. Let $\alpha(A, B)$ denote the acute angle between the geodesics G_A and G_B at C, and let $\lambda(A, B)$ denote the length of the arc AB of the curve γ. The geodesic curvature $\kappa(O)$ of the curve γ at O can be found from any of the two following limits:*

$$\kappa(O) = \lim_{A,B \to O} \frac{\alpha(A, B)}{\lambda(A, B)}; \tag{2.2}$$

$$\kappa(O) = \lim_{A,B \to O} 2\frac{\text{dist}(B, G_A)}{\lambda(A, B)^2}. \tag{2.3}$$

Proof In the Euclidean case formulas (2.2) and (2.3) are classical. Their non-Euclidean versions follow by applying the Euclidean versions in normal coordinates centered respectively at C and A (or at the point in G_A closest to B), as in the proof of Proposition 2.3. □

For every point $A \in \Sigma$ lying in a chart (x, y), e.g., a normal chart centered at O, and every tangent vector $v \in T_A\Sigma$ set

$$\mathrm{az}(v) := \text{ the azimuth of the vector } v: \text{ its Euclidean angle with the } x\text{−axis},$$

i.e., the angle in the Euclidean metric in the coordinates (x, y). The azimuth of an oriented one-dimensional subspace in $T_A\Sigma$ is defined analogously.

Proposition 2.7 *Let $A \in \Sigma$ be a point close to O and $\alpha(s)$ be a geodesic through A parametrized by the natural length parameter s, $\alpha(0) = A$.*

1) Let $\kappa_e(s)$ denote the Euclidean curvature of the geodesic α as a planar curve in normal chart (x, y) centered at O. For every $\varepsilon > 0$ small enough

$$\kappa_e(s) = O(\mathrm{dist}(\alpha, O)), \text{ as } A \to O, \text{ uniformly on } \{|s| \leq \varepsilon\}, \tag{2.4}$$

$$\mathrm{dist}(\alpha, O) := \text{ the distance of the geodesic } \alpha \text{ to the point } O.$$

2) Set $v(s) = \dot{\alpha}(s)$. One has

$$\frac{d\,\mathrm{az}(v(s))}{ds} = O(\mathrm{dist}(\alpha, O)) = O(\angle(v(0), AO)\,\mathrm{dist}(A, O)) \text{ as } A \to O, \tag{2.5}$$

uniformly on the set $\{|s| \leq \varepsilon\}$. The latter angle in (2.5) is the Riemannian angle between the vector $v(0)$ and the Euclidean line AO.

Proof In the coordinates (x, y) the geodesics are solutions of the second order ordinary differential equation saying that $\ddot{\alpha}$ equals a quadratic form in $\dot{\alpha}$ with coefficients equal to appropriate Christoffel symbols of the metric g (which vanish at O), and $|\dot{\alpha}| = 1$ in the metric g. The derivative in (2.5) is expressed in terms of the Christoffel symbols. This derivative taken along a geodesic α through O vanishes identically on α, since each geodesic through O is a straight line in normal coordinates. Therefore if we move the geodesic through O out of O by a small distance δ, then the derivative in (2.5) will change by an amount of order δ: the Christoffel symbols are C^1-smooth, since the metric is C^3-smooth (hence, C^2-smooth in normal coordinates). This implies the first equality in (2.5). The second equality follows from the fact that the geodesics through A issued in the direction of the vectors $\overrightarrow{AO}$ and $v(0)$ are respectively the line AO and α, hence, $\mathrm{dist}(\alpha, O) = O(\angle(v(0), AO)\,\mathrm{dist}(A, O))$. This proves (2.5).

Let s_e denote the Euclidean natural parameter of the curve α, with respect to the standard Euclidean metric in the chart (x, y). Recall that $\kappa_e(s) = \frac{d\,\mathrm{az}(v(s))}{ds_e}$. For $\varepsilon > 0$ small enough and A close enough to O the ratio $\frac{ds_e}{ds}$ is uniformly bounded on $\{|s| \leq \varepsilon\}$. This together with (2.5) implies (2.4). The proposition is proved. □

2.2 Angular derivative of exponential mapping and the derivatives $\frac{dA}{dC}$, $\frac{dB}{dC}$

In the proof of main results we will use an explicit formula for the derivatives of the functions $A(C)$ and $B(C)$ from Proposition 2.3. To state it, let us introduce the following auxiliary functions. For every $x \in \Sigma$ set

$$\psi(x, r) := \frac{1}{2\pi} \text{ (the length of circle of radius } r \text{ centered at } x).$$

Consider the polar coordinates (r, ϕ) on the Euclidean plane $T_x\Sigma$. For every unit vector $v \in T_x\Sigma$, $|v| = 1$ (identified with the corresponding angle coordinate ϕ) and every $r > 0$ let $\Psi(x, v, r)$ denote $\frac{1}{r}$ times the module of derivative in ϕ of the exponential mapping at the point rv:

$$\Psi(x, v, r) := r^{-1} \left| \frac{\partial \exp}{\partial \phi}(rv) \right|. \tag{2.6}$$

Proposition 2.8 *(see [6] in the hyperbolic case). Let Σ be a complete simply connected Riemannian surface of constant curvature. Then*

$$r\Psi(x, v, r) = \psi(x, r) = \psi(r) = \begin{cases} r, & \text{if } \Sigma \text{ is Euclidean plane,} \\ \sin r, & \text{if } \Sigma \text{ is unit sphere,} \\ \sinh r, & \text{if } \Sigma \text{ is hyperbolic plane.} \end{cases} \tag{2.7}$$

Proof The left equality in (2.7) and independence of x and v follow from homogeneity. Let us prove the right equality: formula for the function $\psi(r)$. In the planar case this formula is obvious.

a) Spherical case. Without loss of generality let us place the center $x = O$ of the circle under question to the north pole $(0, 0, 1)$ in the Euclidean coordinates (x_1, x_2, x_3) on the ambient space. Since each geodesic is a big circle of length 2π and due to symmetry, without loss of generality we consider that $0 < r \leq \frac{\pi}{2}$. Then the disk in Σ centered at O of radius r is 1-to-1 projected to the disk of radius $\sin r$ in the coordinate (x_1, x_2)-plane. The length of its boundary equals the Euclidean length $2\pi \sin r$ of its projection.

b) Case of hyperbolic plane. We consider the hyperbolic plane in the model of unit disk equipped with the metric $\frac{2|dz|}{1-|z|^2}$ in the complex coordinate z. For every $R > 0$, $R < 1$ the Euclidean circle $\{|z| = R\}$ of radius R is a hyperbolic circle of radius

$$r = \int_0^R \frac{2ds}{1-s^2} = \log\left|\frac{1+R}{1-R}\right|.$$

The hyperbolic length of the same circle equals $L = \frac{4\pi R}{1-R^2}$. Substituting the former formula to the latter one yields

$$R = \frac{e^r - 1}{e^r + 1}, \ L = 2\pi \sinh r,$$

and finishes the proof of the proposition. □

Proposition 2.9 *Let $\gamma \subset \Sigma$ be a germ of C^2-smooth curve. Let s be the length parameter on γ orienting it positively as a boundary of a convex domain. Let $\mathcal{U} \subset \Sigma$ be a small concave domain adjacent to γ, see Proposition 2.3. For every $C \in \mathcal{U}$ let $A(C)$, $B(C) \in \gamma$ be the corresponding points from Proposition 2.3, and let $s_A = s_A(C)$, $s_B = s_B(C)$ denote the corresponding length parameter values as functions of C. Set*

$$L_A := |CA(C)|, \ L_B := |CB(C)|.$$

For every $Q = A, B$ let $w_Q \in T_Q\gamma$ be the unit tangent vector of the geodesic CQ directed to C. Let $\zeta_Q \in T_C\Sigma$ denote the unit tangent vector of the same geodesic at C directed to Q. For every $v \in T_C\Sigma$ and $Q = A, B$ one has

$$\frac{ds_Q}{dv} = \frac{v \times \zeta_Q}{\kappa(Q) L_Q \Psi(Q, w_Q, L_Q)}; \ v \times \zeta_Q := |v| \sin \angle(v, \zeta_Q), \qquad (2.8)$$

where $\angle(v, \zeta_Q)$ is the oriented angle between the vectors v and ζ_Q: it is positive, if the latter vectors form an orienting basis of the space $T_C\Sigma$.

Proof Let us prove (2.8) for $Q = A$; the proof for B is analogous. As $A = A(C)$ moves by ε along the curve γ to the point A_ε with the natural parameter $s_A + \varepsilon$, the geodesic G_A tangent to γ at A is deformed to the geodesic G_{A_ε} intersecting G_A at a point converging to A, as $\varepsilon \to 0$. Let $\alpha(\varepsilon)$ denote their acute intersection angle at the latter point. One has

$$\alpha(\varepsilon) \simeq \kappa(A)\varepsilon. \qquad (2.9)$$

Both above statements follow from (2.2) and definition. One also has

$$\mathrm{dist}(C, G_{A_\varepsilon}) \simeq \alpha(\varepsilon) L_A \Psi(A, w_A, L_A) \simeq \varepsilon\kappa(A) L_A \Psi(A, w_A, L_A), \qquad (2.10)$$

by the definition of the function Ψ and (2.9).

Without loss of generality we consider that v is a unit vector. Let us draw a curve c through C tangent to v and oriented by v. Let τ denote its natural parameter defined by this orientation. Let C_ε denote the point of intersection of the geodesic G_{A_ε} with c, see Fig. 3. Consider $\tau = \tau(C_\varepsilon)$ as a function of ε: $\tau = \tau(\varepsilon)$. One has

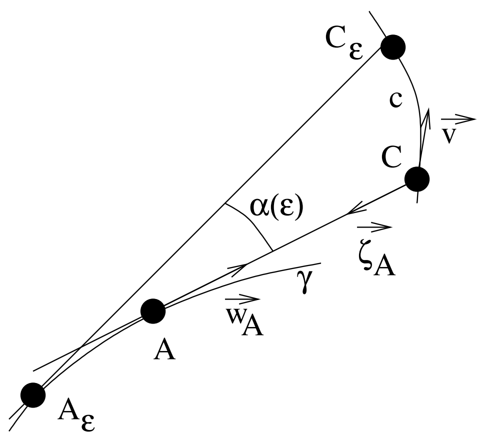

Figure 3: The tangent geodesics to γ at the points A and A_ε. The angle between them is $\alpha(\varepsilon) \simeq \kappa(A)\varepsilon$. One has $\mathrm{dist}(C, A_\varepsilon C_\varepsilon) \simeq \alpha(\varepsilon) L_A \Psi(A, w_A, L_A)$, $L_A = |AC|$, and $\lambda(C, C_\varepsilon) \simeq \mathrm{dist}(C, A_\varepsilon C_\varepsilon)/\sin\angle(v, \zeta_A)$.

$$\frac{ds_Q}{dv} = \left(\frac{d\tau}{d\varepsilon}(0)\right)^{-1}, \ \tau(C_\varepsilon) - \tau(C) \simeq \frac{\mathrm{dist}(C, G_{A_\varepsilon})}{\sin\angle(v, \zeta_A)} \simeq \varepsilon\frac{d\tau}{d\varepsilon}(0) = \varepsilon\left(\frac{ds_Q}{dv}\right)^{-1},$$

as $\varepsilon \to 0$, by definition. Substituting (2.10) to this formula yields (2.8). □

2.3 Geodesics passing through the same base point; azimuths of tangent vectors at equidistant points

Proposition 2.10 *Let the metric on Σ be C^3-smooth. Let $G_t(s), Z_t(s) \subset \Sigma$ be two families of geodesics parametrized by the natural length s and depending on a parameter $t \in [0, 1]$. Let they be issued from the same point*

$A_t = G_t(0) = Z_t(0)$. *Let* A_t *lie in a given compact subset (the same for all* t*) in a local chart* (x, y) *(not necessarily a normal chart). Set*

$$\phi_t = \mathrm{az}(\dot G_t(0)) - \mathrm{az}(\dot Z_t(0)).$$

One has

$$\mathrm{az}(\dot G_t(s)) - \mathrm{az}(\dot Z_t(s)) \simeq \phi_t, \ \text{as } s \to 0, \ \text{uniformly in } t \in [0,1]. \tag{2.11}$$

Proof A geodesic, say, $G(s)$ is a solution of a second order vector differential equation with a given initial condition: a point $A \in \Sigma$ and the azimuth $\mathrm{az}(v(0))$ of a unit vector $v(0) \in T_A\Sigma$. Here we set $v(s) = \dot G(s)$. It depends smoothly on the initial condition. The derivative of the vector function $(G(s), \mathrm{az}\, v(s))$ in the initial conditions is a linear operator (3×3-matrix) function in s that is a solution of the corresponding linear equation in variations. The right-hand sides of the equation for geodesics and the corresponding equation in variations are respectively C^2- and C^1-smooth. Let us now fix the initial point A and consider the derivative of the azimuth $\mathrm{az}(v(s))$ in the initial azimuth $\mathrm{az}(v(0))$ for fixed s. If $s = 0$, then the latter derivative equals 1, since the initial condition in the equation in variations is the identity matrix. Therefore, in the general case the derivative of the azimuth $\mathrm{az}(v(s))$ in $\mathrm{az}(v(0))$ equals $1 + u_A(s)$, where $u_A(s)$ is a C^1-smooth function with $u_A(0) = 0$. This together with the above discussion and Lagrange Increment Theorem for the derivative in $\mathrm{az}(v(0))$ implies (2.11). □

2.4 Geodesic-curvilinear triangles in normal coordinates

Everywhere below in the present subsection Σ is a two-dimensional surface equipped with a C^3-smooth Riemannian metric g, and $O \in \Sigma$.

Proposition 2.11 *Let* $A_uB_uC_u$ *be a family of geodesic right triangles lying in a compact subset in* Σ *with right angle* B_u*. Set*

$$c = c_u = |A_uB_u|, \ b = b_u = |A_uC_u|, \ a = a_u = |B_uC_u|, \ \alpha = \alpha_u = \angle B_uA_uC_u.$$

Let $b_u, \alpha_u \to 0$*, as* $u \to u_0$*. Then*

$$b \simeq c, \ b - c \simeq \frac{a^2}{2c} \simeq \frac{1}{2}c\alpha^2 \simeq \frac{1}{2}a\alpha, \ \angle B_uC_uA_u = \frac{\pi}{2} - \alpha + o(\alpha). \tag{2.12}$$

Proof Consider normal coordinates (x_u, y_u) centered at A_u (depending smoothly on the base point A_u). The coordinates

$$(X_u, Y_u) := \left(\frac{x_u}{c_u}, \frac{y_u}{c_u}\right)$$

are normal coordinates centered at A_u for the Riemannian metric rescaled by division by c_u. For the rescaled metric one has $|A_uB_u| = 1$. In the rescaled normal coordinates (X_u, Y_u) the rescaled metric has trivial 1-jet at 0 and tends to the Euclidean metric, as $u \to u_0$: its nonlinear part tends to zero, as $u \to u_0$, uniformly on the Euclidean disk of radius 2 in the coordinates (X_u, Y_u). One has obviously $|A_uB_u| \simeq |A_uC_u|$ in the rescaled metric, since $\alpha_u \to 0$. Rescaling back, we get the first asymptotic formula in (2.12).

Let S_u denote the circle of radius $|A_uB_u|$ centered at A_u, and let D_u denote its point lying on the geodesic A_uC_u: $|A_uB_u| = |A_uD_u|$; the arc B_uD_u of the circle S_u is its intersection with the geodesic angle $B_uA_uC_u$. In the rescaled coordinates (X_u, Y_u) the circle S_u tends to the Euclidean unit circle. Thus, its geodesic curvature in the rescaled metric tends to 1. The geodesic segment B_uC_u is tangent to S_u at the point B_u, and $\angle B_uC_uA_u \to \frac{\pi}{2}$. The two latter statements together with Proposition 2.2 (applied to $O = B$ and $\gamma = S_u$) imply that in the rescaled metric one has $|B_uC_u| \simeq \alpha$,

$$|D_uC_u| = |A_uC_u| - |A_uB_u| \simeq \frac{|B_uC_u|^2}{2} \simeq \frac{1}{2}\alpha^2 \simeq \frac{1}{2}|B_uC_u|\alpha.$$

Rescaling back to the initial metric, we get the second, third and fourth formulas in (2.12). The fifth formula follows from Gauss–Bonnet Formula, which implies that the sum of angles in the triangle $A_uB_uC_u$ differs from π by a quantity $O(Area) = O(|A_uB_u||B_uC_u|) = O(\alpha|A_uB_u|^2) = o(\alpha)$. □

Proposition 2.12 *Consider a family of C^3-smooth arcs $\gamma_u = A_uB_u$ of curves in Σ (lying in a compact set) with uniformly bounded geodesic curvature (from above) such that $|A_uB_u| \to 0$, as $u \to 0$. Let $\lambda(A_u, B_u)$ denote their lengths. Let α_u denote the angle at A_u between the arc γ_u and the geodesic segment A_uB_u. One has*

$$\lambda(A_u, B_u) = |A_uB_u| + O(|A_uB_u|^3), \ \ \alpha_u = O(|A_uB_u|). \qquad (2.13)$$

Proof The proposition obviously holds in Euclidean metric. It remains valid in the normal coordinates centered at A_u with the geodesic A_uB_u being the x-axis. Indeed, the length of the arc γ_u in the Euclidean metric in the normal chart differs from its Riemannian length by a quantity $O(|A_uB_u|^3)$, since the difference of the metrics at a point $P \in \gamma_u$ is $O(|PA_u|^2) = O(|A_uB_u|^2)$ and the curvature of the arcs γ_u is bounded. □

Proposition 2.13 *Consider a family of curvilinear triangles $T_u := A_uB_uC_u$ in Σ where the side A_uB_u is geodesic and the sides A_uC_u, B_uC_u are arcs of*

C^3-smooth curves with uniformly bounded geodesic curvature. Let the side A_uC_u be tangent to the side A_uB_u at A_u. Set

$$\varepsilon := |A_uB_u|, \ \theta := \frac{\pi}{2} - \angle A_uB_uC_u.$$

Let the triangles T_u lie in a compact subset in Σ, and $\varepsilon, \theta \to 0$, as $u \to 0$. Then

$$\lambda(A_u, C_u) - |A_uB_u| = O(\varepsilon^3) + O(\varepsilon^2\theta). \tag{2.14}$$

Proof One has $|B_uC_u| = O(\varepsilon^2)$, by construction and since $\theta \to 0$. Hence, $|A_uC_u| \simeq \varepsilon$. Let D_u denote the point closest to C_u in the geodesic A_uB_u: the points A_u, C_u, D_u form a triangle Δ_u with right angle at D_u. One has

$$|C_uD_u| = O(\varepsilon^2), \ \lambda(A_u, C_u) - |A_uC_u| = O(\varepsilon^3), \tag{2.15}$$

by definition and (2.13),

$$|A_uC_u| - |A_uD_u| = O(\frac{|C_uD_u|^2}{|A_uB_u|}) = O(\varepsilon^3), \tag{2.16}$$

by (2.15) and (2.12) applied to Δ_u. Let us show that

$$|A_uD_u| - |A_uB_u| = |B_uD_u| = O(\varepsilon^2\theta) + O(\varepsilon^3). \tag{2.17}$$

In the right triangle $\widehat{\Delta}_u$ with vertices B_u, C_u, D_u one has $\angle D_uB_uC_u = \frac{\pi}{2} - \theta + O(\varepsilon^2)$. Indeed, the latter angle is the sum (difference) of the two following angles at B_u: the angle $\frac{\pi}{2} - \theta$ of the triangle T_u; the angle between the geodesic B_uC_u and the curved side B_uC_u in T_u, which is $O(|B_uC_u|) = O(\varepsilon^2)$, by (2.13). This implies the above formula for the angle $\angle D_uB_uC_u$, which in its turn implies that in the triangle $\widehat{\Delta}_u$ one has $\angle B_uC_uD_u = O(\theta) + O(\varepsilon^2)$ (the last formula in (2.12)). The latter formula together with (2.15) and (2.12) imply (2.17). Adding formulas (2.15), (2.16), (2.17) yields (2.14). □

3 The string foliation. Proof of Theorem 1.3

3.1 Finite smoothness lemmas

Everywhere below in the present section we are dealing with a function $f(x, y)$ of two variables (x, y): the variable y is scalar, and the variable x may be a vector variable. The function f is defined on the product

$$Z = \overline{U} \times V$$

of closure of a domain $U \subset \mathbb{R}^n_x$ and an interval $V = (-\varepsilon, \varepsilon) \subset \mathbb{R}_y$.

The following two lemmas will be used in the proof of Theorem 1.3.

Lemma 3.1 *Lef a function f as above be C^k-smooth on Z, $k \geq 2$, and let*

$$f(x,y) = a(x)y^2(1+o(1)), \text{ as } y \to 0, \quad \text{uniformly in } x \in \overline{U}; \quad a > 0. \quad (3.1)$$

Then the function $g(x,y) := \operatorname{sign}(y)\sqrt{f(x,y)}$ is C^{k-1}-smooth on Z.

Lemma 3.2 *Let a function $f(x,y)$ as at the beginning of the section be C^k-smooth on Z and even in y: $f(x,y) = f(x,-y)$. Then $g(x,z) := f(x,\sqrt{z})$ is $C^{[\frac{k}{2}]}$-smooth on Z, and its restriction to $Z \setminus \{y=0\}$ is C^k-smooth.*

In the proof of the lemmas for simplicity without loss of generality we consider that the variable x is one-dimensional; in higher-dimensional case the proof is the same. We use the following definition and a more precise version of the asymptotic Taylor formula for finitely-smooth functions.

Definition 3.3 Let $l, m \in \mathbb{Z}_{\geq 0}$. We say that

$$f(x,y) = o_l(y^m), \text{ as } y \to 0,$$

if for every $j, s \in \mathbb{Z}_{\geq 0}$, $j \leq l$, $s \leq m$ the derivative $\frac{\partial^{j+s} f}{\partial^j x \partial^s y}$ exists and is continuous on $\overline{U} \times V$ and one has

$$\frac{\partial^{j+s} f}{\partial^j x \partial^s y}(x,y) = o(y^{m-s}), \text{ as } y \to 0, \text{ uniformly in } x \in \overline{U}. \quad (3.2)$$

Proposition 3.4 *Let $f(x,y)$ be as at the beginning of the section, and let f be C^k-smooth on Z. Then for every $l, m \in \mathbb{Z}_{\geq 0}$ with $l+m \leq k$ one has*

$$f(x,y) = f(x,0) + \sum_{j=1}^{m} a_j(x)y^j + R_m(x,y), \; a_j(x) = \frac{1}{j!}\frac{\partial^j f}{\partial y^j}(x,0) \in C^l(\overline{U}),$$

$$R_m(x,y) = o_l(y^m), \text{ as } y \to 0, \text{ uniformly in } x \in \overline{U}. \quad (3.3)$$

Proof The first formula in (3.3) holds with

$$R_m(x,y) = \int_{0 \leq y_m \leq \dots \leq y_1 \leq y} \left(\frac{\partial^m}{\partial y^m} f(x,y_m) - \frac{\partial^m}{\partial y^m} f(x,0)\right) dy_m dy_{m-1} \dots dy_1,$$

by the classical asymptotic Taylor formula with error term in integral form. The latter R_m is $o_l(y^m)$, whenever $f \in C^k$ and $k \geq l+m$. □

Proposition 3.5 *One has*

$$y^{-s} o_l(y^m) = o_l(y^{m-s}) \text{ for every } m, s \in \mathbb{Z}_{\geq 0}, \; m \geq s. \quad (3.4)$$

The proposition follows from definition.

Proof of Lemma 3.1. The function $g(x,y) = \text{sign}(y)\sqrt{f(x,y)}$ is well-defined, by (3.1). It is obviously C^k-smooth outside the hyperplane $\{y=0\}$. Fix arbitrary $l,m \in \mathbb{Z}_{\geq 0}$ such that $l+m \leq k-1$. Let us prove continuity of the derivative $\frac{\partial^{l+m}g}{\partial x^l \partial y^m}$ on Z.

Case $m=0$; then $k \geq l+1$. The above derivative is a linear combination of expressions

$$\text{sign}(y) f^{\frac{1}{2}-s}(x,y) \prod_{j=1}^{s} \frac{\partial^{n_j} f(x,y)}{\partial x^{n_j}}, \ s \in \mathbb{N}, \ n_j \geq 1, \ \sum_{j=1}^{s} n_j = l. \tag{3.5}$$

The partial derivatives in (3.5) are C^1-smooth, since f is C^k-smooth and $n_j \leq l \leq k-1$. One has

$$\text{sign}(y) f^{\frac{1}{2}-s}(x,y) \simeq a^{\frac{1}{2}-s}(x) y^{1-2s}, \tag{3.6}$$

by definition. If $s=1$, then $y^{1-2s} = y^{-1}$, the expression (3.5) contains only one derivative, and this derivative is asymptotic to y times a continuous function in x, as $y \to 0$, by smoothness and since $f(x,0) \equiv 0$. Therefore, the expression (3.5) is continuous. If $s \geq 2$, then $n_j \leq l-1 \leq k-2$. Hence, each derivative in (3.5) is C^2-smooth, has vanishing first derivative in y at $y=0$ and is asymptotic to y^2 times a continuous function in x. Then (3.5) is again continuous, by (3.6).

Case $m=1$ is treated analogously with the following change: one of the derivatives in (3.5) will contain one differentiation in y and will be asymptotic to y times a continuous function in x.

Case $m \geq 2$. Then $k \geq m+l+1 \geq l+3$. One has

$$g(x,y) = a^{\frac{1}{2}}(x) y \sqrt{w(x,y)}, \tag{3.7}$$

$$w(x,y) = 1 + \sum_{j=3}^{m+1} a^{-1}(x) a_j(x) y^{j-2} + \frac{o_l(y^{m+1})}{y^2}, \quad a, a_j \in C^l(\overline{U}), \tag{3.8}$$

by (3.3) applied to the function $f(x,y)$ and m replaced by $m+1$. The derivative $\frac{\partial^{l+m-1}g}{\partial x^l \partial y^{m-1}}$ exists and continuous for small y, by (3.8) and since $\frac{o_l(y^{m+1})}{y^2} = o_l(y^{m-1})$, see (3.4). Now it remains to prove the same statement for the derivative $h := \frac{\partial^{l+m}g}{\partial x^l \partial y^m}$. Those terms in its expression that include the derivatives of the function $\frac{o_l(y^{m+1})}{y^2} = o_l(y^{m-1})$ with differentiation in y of orders less than m are well-defined and continuous, as above. Each term in h that contains a derivative $\frac{\partial^{j+m}}{\partial x^j \partial y^m}(\frac{o_l(y^{m+1})}{y^2})$ contains only one such derivative, and it comes with the factor y from (3.7). On the other hand, the latter derivative is $\frac{o_{l-j}(y)}{y^2} = o(\frac{1}{y})$, by (3.4). Thus, its product with the above factor y is a continuous function, as are the other factors in the term under question. Continuity of the derivative h is proved. Lemma 3.1 is proved. □

Proof of Lemma 3.2. Fix $l, m \in \mathbb{Z}_{\geq 0}$ such that $l + m \leq [\frac{k}{2}]$. Then $l + 2m \leq k$, and one has

$$f(x, y) = \sum_{j=0}^{m} a_j(x) y^{2j} + o_l(y^{2m}),$$

where the functions $a_j(x)$ are C^l-smooth, by (3.3) and evenness. Set $z = y^2$. The derivative $\frac{\partial^{l+m}}{\partial x^l \partial z^m}$ of the above sum is obviously continuous, since the sum is a polynomial in z with coefficients being C^l-smooth functions in x. Let us prove continuity of the derivative of the remainder $o_l(y^{2m})$. One has

$$\frac{\partial}{\partial z} = \frac{1}{2y} \frac{\partial}{\partial y}.$$

Therefore, the above $(l + m)$-th partial derivative of the remainder $o_l(y^{2m})$ is $o(1)$, see (3.4). This proves continuity and Lemma 3.2. □

3.2 Proof of Theorem 1.3

The fact that the exterior bisector line field Λ is tangent to the string construction curves is well-known and proved as follows. Consider the value $L(A, B) = |AC| + |CB| - \lambda(A, B)$ as a function of C: here $A = A(C)$ and $B = B(C)$ are the same, as in Proposition 2.3. Its derivative along the string construction curve Γ_p through C should be zero. Let $v \in T_C\Sigma$ be a unit vector. Let α and β be respectively the oriented angles between the vector v and the vectors ζ_A and ζ_B in $T_C\Sigma$ directing the geodesics G_A, G_B from C to A and B respectively. The derivative of the above function $L(A(C), B(C))$ along the vector v is equal to $-(\cos\alpha + \cos\beta)$. Therefore, it vanishes if and only if the line generated by v is the exterior bisector $\Lambda(C)$ of the angle $\angle ACB$. Therefore, the level sets of the function $L(A(C), B(C))$, i.e., the string construction curves are integral curves of the line field Λ.

It suffices to prove only statement 1) of Theorem 1.3: C^{k-1}-smoothness on $\mathcal{U}$ and $C^{r(k)}$-smoothness on $\overline{\mathcal{U}}$ of the line field Λ. Statement 2) on $C^{r(k)+1}$-regularity of its integral curves (the string construction curves) and continuity then follows from the next general fact: *for every C^r-smooth line field the*

$(r+1)$-jets of its integral curves at base points A are expressed analytically in terms of r-jets of the line field, and hence, depend continuously on A.

Fix a C^k-smooth coordinate system (s,z) on Σ centered at the base point O of the curve γ such that γ is the s-axis, $s|_\gamma$ is the natural length parameter of the curve γ and $\mathcal{U}=\{z>0\}$. For every $\sigma\in\mathbb{R}$ small enough let $G(\sigma)$ denote the geodesic tangent to γ at the point with length parameter value σ. For every $\sigma,s\in\mathbb{R}$ small enough let $A(\sigma,s)$ denote the point of intersection of the geodesic $G(\sigma)$ with the line parallel to the z-axis and having abscissa s. The mapping $(\sigma,s)\mapsto A(\sigma,s)$ is C^{k-1}-smooth, since so is the family of geodesics $G(\sigma)$ (by C^k-smoothness of the metric) and by transversality. Set

$$z(\sigma,s):=z(A(\sigma,s)).$$

Proposition 3.6 *The function*

$$y(\sigma,s):=\operatorname{sign}(\sigma-s)\sqrt{z(\sigma,s)}$$

is C^{k-2}-smooth on a neighborhood of zero in $\mathbb{R}^2$ and C^{k-1}-smooth outside the diagonal $\{\sigma=s\}$. The mapping

$$F:(\sigma,s)\mapsto(s,y(\sigma,s)) \tag{3.9}$$

is a C^{k-2}-smooth diffeomorphism of a neighborhood of the origin onto a neighborhood of the origin, and it is C^{k-1}-smooth outside the diagonal. It sends the diagonal to the axis $\{y=0\}$.

Proof For every point $Q\in\Sigma$ lying in a smooth chart (s,z) let $u(Q)$ denote the orthogonal projection of the vector $\frac{\partial}{\partial z}\in T_Q\Sigma$ to the line $(\mathbb{R}\frac{\partial}{\partial s})^\perp$. Set $\mu(Q):=||u||^{-1}$. Recall that $\kappa(s)>0$. One has

$$z(\sigma,s)=\frac{1}{2}\mu(s,0)\kappa(s)(s-\sigma)^2+o((s-\sigma)^2),\ \text{as } \sigma\to s, \tag{3.10}$$

uniformly in small s, by (2.3). This together with C^{k-1}-smoothness of the function z and Lemma 3.1 implies the statements of the proposition. □

Let us now return to the proof of statement 1) of Theorem 1.3. Consider the mapping inverse to the mapping F from (3.9):

$$F^{-1}:(s,y)\mapsto(\sigma,s).$$

The function $\sigma=\sigma(s,y)$ is C^{k-2}-smooth, by Proposition 3.6, and it is C^{k-1}-smooth outside the axis $\{y=0\}$. Recall that the geodesic $G(\sigma(s,y))$

passes through the point $A = (s, z) = (s, y^2) \in \mathcal{U}$. For every s and y let $v = v(s, y) \in T_A\Sigma$ denote the unit tangent vector of the geodesic $G(\sigma(s, y))$ that orients it in the same way, as the orienting tangent vector of the curve γ at $\sigma(s, y)$. The vector function $v(s, y)$ is C^{k-2}-smooth in (s, y). For a given point $A = (s, z)$, set $y := \sqrt{z}$, the unit vectors $v(s, y), v(s, -y) \in T_A\Sigma$ direct the two geodesics through A that are tangent to γ, by construction. Their sum $w(s, y) = v(s, y) + v(s, -y)$ generates the line $\Lambda(A)$ of the line field Λ, by definition. The vector function $w(s, y)$ is even in y, C^{k-2}-smooth in both variables, and $|w| = 2|v| = 2$, whenever $y = 0$. Thus, w is $C^{[\frac{k}{2}]-1}$-smooth in (s, z) and C^{k-1}-smooth outside the curve $\gamma = \{z = 0\}$, by Proposition 3.6 and Lemma 3.2. Finally, w induces a vector field generating Λ that is $C^{[\frac{k}{2}]-1}$-smooth on $\overline{\mathcal{U}}$ and C^{k-1}-smooth on $\mathcal{U}$. Theorem 1.3 is proved.

4 Billiards on surfaces of constant curvature. Proofs of Proposition 1.6 and Theorem 1.7

In Subsection 4.1 we prove Proposition 1.6. The proof of Theorem 1.7, which follows its proof given in [20, section 7] in the Euclidean case, takes the rest of the section. In Subsection 4.2 we prove the following coboundary property of a curve γ with the string Poritsky property: for every $A, B \in \gamma$, set $C = C_{AB}$, the ratio $|AC|/|BC|$ equals the ratio of values at A and B of some function on γ. In Subsection 4.3 we deduce Theorem 1.7 from the coboundary property by planimetric arguments using Ceva's Theorem.

4.1 Proof of Proposition 1.6

We re-state and prove Proposition 1.6 in a more general Riemannian context. To do this, let us recall the following definition.

Definition 4.1 [1, p. 345] (implicitly considered in [20]) Let Σ be a surface equipped with a Riemannian metric, $\gamma \subset \Sigma$ be a (germ of) curve with positive geodesic curvature. Let Γ_p denote the family of curves obtained from it by string construction. We say that γ has *evolution* (or *Graves*) *property,* if for every $p_1 < p_2$ the curve Γ_{p_1} is a caustic for the curve Γ_{p_2}.

Example 4.2 It is well-known that each conic on a surface Σ of constant curvature has evolution property, and the corresponding curves Γ_p given by string construction are confocal conics. In the Euclidean case this follows from the classical fact saying that the caustics of a billiard in a conic are confocal conics (Proclus–Poncelet Theorem). Analogous statements hold in non-zero constant curvature and in higher dimensions, see [28, theorem 3].

Proposition 4.3 *Let Σ be a surface equipped with a C^4-smooth Riemannian metric. Let $\gamma \subset \Sigma$ be a C^4-smooth germ of curve with positive geodesic curvature that has evolution property. Then it has the string Poritsky property*[2]*. For every $p, q > 0$ the reflections from the string curves Γ_p and Γ_q commute as mappings acting on the space of those oriented geodesics that intersecting both of them and lie on the concave side $\mathcal{U}$ from the curve γ.*

Remark 4.4 In the Euclidean case the first part of Proposition 4.3 with a proof is contained in [20, 1]. Commutativity then follows by arguments from [23, chapter 3]. The proof of the first part of Proposition 4.3 given below is analogous to arguments from [20], [23, ch.3]. The analogue of evolution property for outer billiards was introduced and studied by E. Amiran [2].

Proof of Proposition 4.3. Billiard reflections acting on the manifold of oriented geodesics preserve a canonical symplectic form ω. See [20, section 3], [23, chapter 3] in the planar case. In the general case the form ω is given by Melrose construction, see [22, section 1.5], [17, 18, 3, 4] and Subsection 7.1 below. The string curves Γ_p form a foliation of $\overline{\mathcal{U}}$ by level curves of a function ϕ that is C^3-smooth on $\mathcal{U}$, C^1-smooth on $\overline{\mathcal{U}}$ and has no critical points. This follows from the fact that they are tangent to the line field Λ of the same regularity (Theorem 1.3). Consider the mapping R of the set $\overline{\mathcal{U}}$ to the space of oriented geodesics sending each point $Q \in \overline{\mathcal{U}}$ to the geodesic tangent to $\Lambda(Q)$. The orientations of the lines $\Lambda(Q)$ are chosen to converge to the orientation of the curve γ, as $Q \to \gamma$. This is a diffeomorphism onto $\overline{\mathcal{U}^*} := R(\overline{\mathcal{U}})$ of the same regularity, as Λ, by construction and since γ has positive geodesic curvature. The image $\Gamma_p^* := R(\Gamma_p)$ of each curve Γ_p is the family or geodesics tangent to Γ_p and oriented as Γ_p. The curves Γ_p^* form a foliation by level curves of the function $\psi := \phi \circ R^{-1}$, which has the above regularity and no critical points. For every $q < p$ the curve Γ_q^* is invariant under the reflection T_p from the curve Γ_p (evolution property). Therefore, the restriction of the function ψ to the strip between the curves Γ_0^* and Γ_p^* is also T_p-invariant. Hence, its Hamiltonian vector field H_ψ is also invariant and tangent to the curves Γ_q^*. Thus, for every $q < p$ the

[2] Very recently it was shown in a joint paper of the author with Sergei Tabachnikov and Ivan Izmestiev [9] that for a C^∞-smooth curve γ the *evolution property is* **equivalent** *to the Poritsky property.* And that it is also equivalent to the statement that the foliation by the curves Γ_p and its orthogonal foliation form a *Liouville net* on the concave side $\mathcal{U}$ from the curve γ.

reflection $T_p : \Gamma_q^* \to \Gamma_q^*$ acts by translation in the time coordinate t_q of the field H_ψ on Γ_q^*, and this also holds for $q = 0$. The time coordinate t_0 on Γ_0^* induces a parameter, also denoted by t_0, on the curve $\Gamma_0 = \gamma$. Therefore, γ has the Poritsky property with Poritsky–Lazutkin parameter t_0, by the above discussion. Any two reflections T_p and T_q commute while acting on the union of the curves Γ_r^* with $r \leq \min\{p, q\}$: the curves Γ_r^* are T_p- and T_q-invariant, and T_p, T_q act as translations there. Proposition 4.3 is proved. □

Proposition 1.6 follows from Proposition 4.3 and Example 4.2.

4.2 Preparatory coboundary property of length ratio

Let Σ be an oriented surface of constant curvature $K \in \{0, \pm 1\}$: either Euclidean plane, or unit sphere in $\mathbb{R}^3$, or hyperbolic plane. Let $O \in \Sigma$, and let $\gamma \subset \Sigma$ be a regular germ of curve through O with positive geodesic curvature. We consider that γ is oriented clockwise with respect to the orientation of the surface Σ. For every point $X \in \gamma$ by G_X we denote the geodesic tangent to γ at X. Let $A, B \in \gamma$ be two distinct points close to O such that the curve γ is oriented from B to A. Let $C = C_{AB}$ denote the unique intersection point of the geodesics G_A and G_B that is close to O. (Then CA is the right geodesic tangent to γ through C.) Set

$$L_A := |CA|; \; L_B := |CB|;$$

here $|CX|$ is the length of the geodesic arc CX. Recall that we denote

$$\psi(x) = \begin{cases} x, & \text{if } \Sigma \text{ is Euclidean plane,} \\ \sin x, & \text{if } \Sigma \text{ is unit sphere,} \\ \sinh x, & \text{if } \Sigma \text{ is hyperbolic plane.} \end{cases} \tag{4.1}$$

Proposition 4.5 *Let Σ be as above, $\gamma \subset \Sigma$ be a germ of C^2-smooth curve at a point $O \in \Sigma$ with the string Poritsky property. There exists a positive continuous function $u(X)$, $X \in \gamma$, such that for every $A, B \in \gamma$ close enough to O one has*

$$\frac{\psi(L_A)}{\psi(L_B)} = \frac{u(A)}{u(B)}. \tag{4.2}$$

The above statement holds for

$$u = \frac{1}{\kappa}\frac{dt}{ds}; \quad t \text{ is the Poritsky parameter.}$$

Proof For every $p > 0$ small enough and every $C \in \Gamma_p$ close enough to O there are two geodesics issued from the point C that are tangent to γ (Proposition 2.3). The corresponding tangency points $A = A(C)$ and $B = B(C)$ in γ depend smoothly on the point $C \in \Gamma_p$. Let s_p denote the natural length parameter of the curve Γ_p. We set $s = s_0$: the natural length parameter of the curve γ. We write $C = C(s_p)$, and consider the natural parameters $s_A(s_p)$, $s_B(s_p)$ of the points $A(C)$ and $B(C)$ as functions of s_p. Let $\alpha(C)$ denote the oriented angle between a vector $v \in T_C\Gamma_p$ orienting the curve Γ_p and a vector $\zeta_A \in T_C G_A$ directing the geodesic G_A from C to A. It is equal (but with opposite sign) to the oriented angle between the vector $-v$ and a vector $\zeta_B \in T_C G_B$ directing the geodesic G_B from C to B, since the tangent line to Γ_p at C is the exterior bisector of the angle between the geodesics G_A and G_B (Theorem 1.3). One has

$$\frac{ds_A}{ds_p} = \frac{\sin\alpha(C)}{\kappa(A(C))\psi(|AC|)}, \ \frac{ds_B}{ds_p} = \frac{\sin\alpha(C)}{\kappa(B(C))\psi(|BC|)}, \tag{4.3}$$

by (2.8), (2.7) and the above angle equality.

Let now t be the Poritsky parameter of the curve γ. Let $t_A(s_p)$ and $t_B(s_p)$ denote its values at the points $A(C)$ and $B(C)$ respectively as functions of s_p. Their difference is constant, by the Poritsky property. Therefore,

$$\frac{dt_A}{ds_p} = \frac{dt}{ds}(A)\frac{ds_A}{ds_p} = \frac{dt_B}{ds_p} = \frac{dt}{ds}(B)\frac{ds_B}{ds_p}.$$

Substituting (4.3) to the latter formula and cancelling out $\sin\alpha(C)$ yields (4.2) with $u = \frac{1}{\kappa}\frac{dt}{ds}$. □

4.3 Conics and Ceva's Theorem on surfaces of constant curvature. Proof of Theorem 1.7

Definition 4.6 Let Σ be a surface with Riemannian metric. We say that a germ of curve $\gamma \subset \Sigma$ at a point O with positive geodesic curvature has *tangent incidence property,* if the following statement holds. Let $A', B', C' \in \gamma$ be arbitrary three distinct points close enough to O. Let a, b, c denote the geodesics tangent to γ at A', B', C' respectively. Let A, B, C denote the points of intersection $b \cap c$, $c \cap a$, $a \cap b$ respectively. Then the geodesics AA', BB', CC' intersect at one point. See [20, p.462, fig.5] and Fig. 4 below.

Proposition 4.7 *Every germ of C^2-smooth curve with the string Poritsky property on a surface of constant curvature has tangent incidence property.*

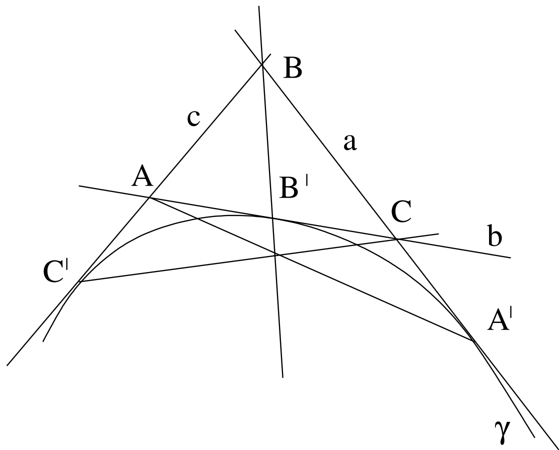

Figure 4: A curve γ with tangent incidence property.

As it is shown below, Proposition 4.7 follows from Proposition 4.5 and the next theorem.

Theorem 4.8 *[16, pp. 3201–3203] (Ceva's Theorem on surfaces of constant curvature.) Let Σ be a simply connected complete surface of constant curvature. Let $\psi(x)$ be the corresponding function in (4.1): the length of circle of radius x divided by 2π. Let $A, B, C \in \Sigma$ be three distinct points. Let A', B', C' be respectively some points on the* **sides** *BC, CA, AB of the geodesic triangle ABC. Then the geodesics AA', BB', CC' intersect at one point, if and only if*

$$\frac{\psi(|AB'|)}{\psi(|B'C|)}\frac{\psi(|CA'|)}{\psi(|A'B|)}\frac{\psi(|BC'|)}{\psi(|C'A|)} = 1. \tag{4.4}$$

Addendum to Theorem 4.8. *Let now in the conditions of Theorem 4.8 A', B', C' be points on the* **geodesics** *BC, CA, AB respectively so that some two of them, say A', C' do not lie on the corresponding* **sides** *and the remaining third point B' lies on the corresponding side AC, see Fig. 4.*

1) In the Euclidean and spherical cases the geodesics AA', BB', CC' intersect at the same point, if and only if (4.4) holds.

2) In the hyperbolic case (when Σ is of negative curvature) the geodesics

AA', BB', CC' intersect at the same point, if and only if some two of them intersect and (4.4) holds.

3) Consider the standard model of the hyperbolic plane Σ in the Minkowski space $\mathbb{R}^3$, see Subsection 1.1. Consider the 2-subspaces defining the geodesics AA', BB', CC', and let us denote the corresponding projective lines (i.e., their tautological projections to $\mathbb{RP}^2$) by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ respectively. The projective lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ intersect at one point (which may be not the projection of a point in Σ), if and only if (4.4) holds.

Proof Statements 1) and 2) of the addendum follow from Theorem 4.8 by analytic extension, when some two points A' and C' go out of the corressponding sides BC, BA while remaining on the same (complexified) geodesics BC, BA. Statement 3) is proved analogously. □

Proof of Proposition 4.7. Let O be the base point of the germ γ, and let A', B', C' be its three subsequent points close enough to O. Let a, b, c be respectively the geodesics tangent to γ at them. Then each pair of the latter geodesics intersect at one point close to O. Let A, B, C be the points of intersections $b\cap c$, $c\cap a$, $a\cap b$ respectively. The point B' lies on the geodesic arc $AC\subset b$. This follows from the assumption that the point B' lies between A' and C' on the curve γ and the inequality $\kappa\neq 0$. In a similar way we get that the points A' and C' lie on the corresponding geodesics a and c but outside the sides BC and AB of the geodesic triangle ABC so that A lies between C' and B, and C lies between A' and B. The geodesics BB' and AA' intersect, by the two latter arrangement statements. Let $u:\gamma\to\mathbb{R}$ be the function from Proposition 4.5. One has $\frac{\psi(|BA'|)}{\psi(|BC'|)}=\frac{u(A')}{u(C')}$, by (4.2), and similar equalities hold with B replaced by A and C. Multiplying the three latter equalities we get (4.4), since the right-hand side cancels out. Hence the geodesics AA', BB' and CC' intersect at one point, by statements 1), 2) of the addendum to Theorem 4.8. Proposition 4.7 is proved. □

Theorem 4.9 *Each conic on a surface of constant curvature has tangent incidence property. Vice versa, each C^2-smooth curve on a surface of constant curvature that has tangent incidence property is a conic.*

Proof The first, easy statement of the theorem follows from Propositions 1.6 and 4.7. The proof of its second statement repeats the arguments from [20, p.462], which are given in the Euclidean case but remain valid in the other cases of constant curvature without change. Let us repeat them briefly in full generality for completeness of presentation. Let γ be a germ of curve with tangent incidence property on a surface Σ of constant curvature. Let

A', B', C' denote three distinct subsequent points of the curve γ, and let a, b, c be respectively the geodesics tangent to γ at these points. Let A, B, C denote respectively the points of intersections $b \cap c$, $c \cap a$, $a \cap b$. Fix the points A' and C'. Consider the pencil $\mathcal{C}$ of conics through A' and C' that are tangent to $T_{A'}\gamma$ and $T_{C'}\gamma$. Then each point of the surface Σ lies in a unique conic in $\mathcal{C}$ (including two degenerate conics: the double geodesic $A'C'$; the union of the geodesics $G_{A'}$ and $G_{C'}$). Let $\phi \in \mathcal{C}$ denote the conic passing through the point B'.

Claim. *The tangent line $l = T_{B'}\phi$ coincides with $T_{B'}\gamma$.*

Proof Let L denote the geodesic through B' tangent to l. Let C_1 and A_1 denote respectively the points of intersections $L \cap a$ and $L \cap c$. Both curves γ and ϕ have tangent incidence property. Therefore, the three geodesics AA', BB', CC' intersect at the same point denoted X, and the three geodesics $A'A_1$, BB', $C'C_1$ intersect at the same point Y; both X and Y lie on the geodesic BB'. We claim that this is impossible, if $l \neq T_{B'}\gamma$ (or equivalently, if $L \neq b$). Indeed, let to the contrary, $L \neq b$. Let us turn the geodesic b continuously towards L in the family of geodesics b_t through B', $t \in [0, 1]$: $b_0 = b$, $b_1 = L$, the azimuth of the line $T_{B'}b_t$ turns monotonously (clockwise or counterclockwise), as t increases. Let A_t, C_t denote respectively the points of the intersections $b_t \cap c$ and $b_t \cap a$: $A_0 = A$, $C_0 = C$. Let X_t denote the point of the intersection of the geodesics $A'A_t$ and $C'C_t$: $X_0 = X$, $X_1 = Y$. At the initial position, when $t = 0$, the point X_t lies on the fixed geodesic BB'. As t increases from 0 to 1, the points A and C remain fixed, while the points C_t and A_t move monotonously, so that as C_t moves towards (out from) B along the geodesic a, the point A_t moves out from (towards) B along the geodesic c, see Fig. 5. In the first case, when C_t moves towards B and A_t moves out from B, the point X_t moves out of the geodesic BB', to the half-plane bounded by BB' that contains A, and its distance to BB' increases. Hence, $Y = X_1$ does not lie on BB'. The second case is treated analogously. The contradiction thus obtained proves the claim. □

For every point $Q \in \Sigma$ such that the conic $\phi_Q \in \mathcal{C}$ passing through Q is regular, set $l_Q := T_Q\phi_Q$. The lines l_Q form an analytic line field outside the union of three geodesics: $G_{A'}$, $G_{C'}$, $A'C'$. Its phase curves are the conics from the pencil $\mathcal{C}$. The curve γ is also tangent to the latter line field, by the above claim. Hence, γ is a conic. This proves Theorem 4.9. □

Proof of Theorem 1.7. Let γ be a germ of C^2-smooth curve with the string Poritsky property on a surface of constant curvature. Then it has tangent incidence property, by Proposition 4.7. Therefore, it is a conic, by Theorem 4.9. Theorem 1.7 is proved. □

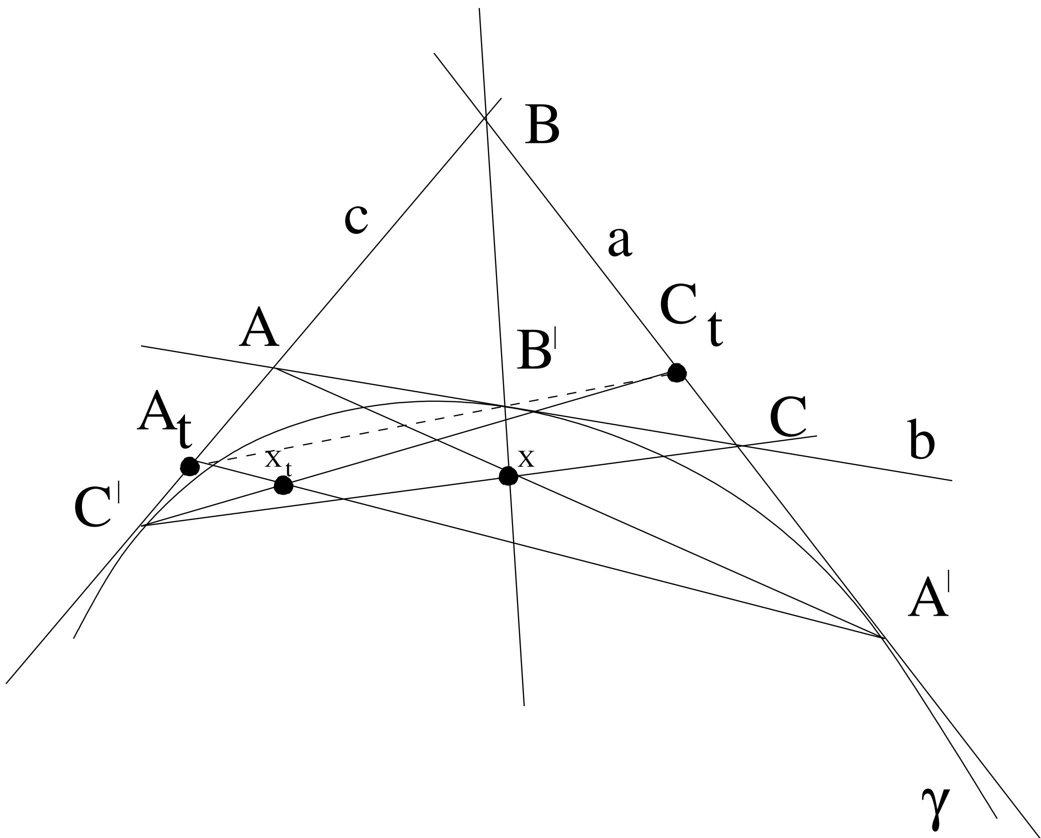

Figure 5: The intersection point X_t moves away from the geodesic BB'.

5 Case of outer billiards: proof of Theorem 1.13

Everywhere below in the present section Σ is a simply connected complete Riemannian surface of constant curvature, and $\gamma \subset \Sigma$ is a germ of C^2-smooth curve at a point $O \in \Sigma$ with positive geodesic curvature.

Proposition 5.1 *Let Σ, O, γ be as above, and let γ have the area Poritsky property. Then there exists a continuous function $u : \gamma \to \mathbb{R}_+$ such that for every $A, B \in \gamma$ close enough to O the following statement holds. Let α, β denote the angles between the chord AB and the curve γ at the points A and B respectively. Then*

$$\frac{\sin\alpha}{\sin\beta} = \frac{u(A)}{u(B)}. \tag{5.1}$$

Let t, s denote respectively the area Poritsky and length parameters of the curve γ. The above statement holds for the function

$$u := t'_s = \frac{dt}{ds}.$$

Proof Recall that for every $C, D \in \gamma$ by $\lambda(C, D)$ we denote the length of the arc CD of the curve γ. Fix A and B as above. Set $A(0) = A$, $B(0) = B$. For every small $\tau > 0$ let $A(\tau)$ denote the point of the curve γ such that $\lambda(A(\tau), A(0)) = \tau$ and the curve γ is oriented by the natural parameter from $A(0)$ to $A(\tau)$. Let $B(\tau) \in \gamma$ denote the family of points such that the area of the domain bounded by the chord $A(\tau)B(\tau)$ and the arc $A(\tau)B(\tau)$ of the curve γ remains constant, independent on τ. For every τ small enough the chord $A(\tau)B(\tau)$ intersects the chord $A(0)B(0)$ at a point $X(\tau)$ tending to the middle of the chord $A(0)B(0)$, see Fig. 6. This follows from constance of area and homogeneity (constance of curvature) of the surface Σ. One has

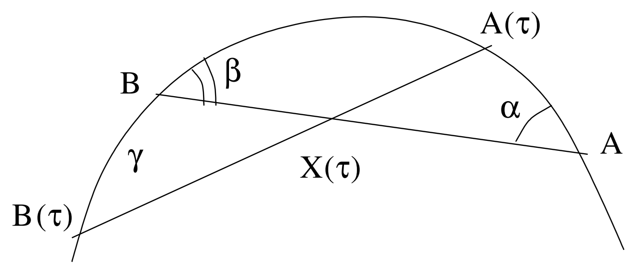

Figure 6: Curve γ with the area Poritsky property. The chords AB, $A(\tau)B(\tau)$.

$$t(A(\tau)) - t(A(0)) = t(B(\tau)) - t(B(0)) \text{ for every } \tau \text{ small enough,}$$

by the area Poritsky property. The above left- and right-hand sides are asymptotic to $u(A)\lambda(A(0), A(\tau))$ and $u(B)\lambda(B(0), B(\tau))$ respectively, as $\tau \to 0$, with $u = \frac{dt}{ds}$. Therefore,

$$\frac{\lambda(B(0), B(\tau))}{\lambda(A(0), A(\tau))} \to \frac{u(A)}{u(B)}, \text{ as } \tau \to 0. \tag{5.2}$$

The length $\lambda(A(0), A(\tau))$ is asymptotic to $\frac{1}{\sin\alpha}$ times $\text{dist}(A(\tau), B(0)A(0))$: the distance of the point $A(\tau)$ to the geodesic $X(\tau)A(0) = B(0)A(0)$. Similarly, $\lambda(B(0), B(\tau)) \simeq \frac{1}{\sin\beta}\,\text{dist}(B(\tau), B(0)A(0))$, as $\tau \to 0$. The above distances of the points $A(\tau)$ and $B(\tau)$ to the geodesic $A(0)B(0)$ are asymptotic to each other, since the intersection point $X(\tau)$ of the chords $A(\tau)B(\tau)$ and $A(0)B(0)$ tends to the middle of the chord $A(0)B(0)$ and by homogeneity. This implies that the left-hand side in (5.2) tends to the ratio $\frac{\sin\alpha}{\sin\beta}$, as $\tau \to 0$. This together with (5.2) proves (5.1). □

Proposition 5.2 *Let Σ, O and γ be as at the beginning of the section. Let there exist a function u on γ that satisfies (5.1) for every $A, B \in \gamma$ close to O. Then γ has tangent incidence property, see Definition 4.6.*

Proof Let A', B', C' be three subsequent points of the curve γ. Let a, b, c denote respectively the geodesics tangent to γ at these points. Let A, B, C denote respectively the points of intersections $b \cap c$, $c \cap a$, $a \cap b$ (all the points A', B', C', and hence A, B, C are close enough to the base point O), as at Fig. 4. Let ψ be the same, as in (4.1). One has

$$\frac{\sin \angle CA'B'}{\sin \angle CB'A'} = \frac{\psi(|CB'|)}{\psi(|CA'|)} = \frac{u(A')}{u(B')}, \tag{5.3}$$

by (5.1) and Sine Theorem on the Euclidean plane and its analogues for unit sphere and hyperbolic plane applied to the geodesic triangle $CA'B'$, see [13, p.215], [21, theorem 10.4.1]. Similar equalities hold for other pairs of points (B', C'), (C', A'). Multiplying all of them yields relation (4.4): the ratios of values of the function u at A', B', C' cancel out. This together with Theorem 4.8 and its addendum implies that γ has tangent indicence property and proves Proposition 5.2. □

Proof of Theorem 1.13. A curve with the area Poritsky property on a surface of constant curvature has tangent incidence property, by Propositions 5.1 and 5.2. Hence, it is a conic, by Theorem 4.9. Theorem 1.13 is proved. □

6 The function $L(A, B)$ and the Poritsky–Lazutkin parameter. Proofs of Theorems 1.16, 1.15 and Corollaries 1.17, 1.18

Proof of Theorem 1.16. Let g denote the metric. Let $C = C_{AB}$ denote the point of intersection of the geodesics G_A and G_B tangent to γ at the points A and B respectively. We will work in normal coordinates (x, y) centered at C and the corresponding polar coordinates (r, ϕ). The next two claims concern asymptotics of different quantities, as $\mathrm{dist}(A,B) \to 0$ so that A and B lie in a compact subarc in γ.

Claim 1. *The length $s_A - s_B$ of the arc AB of the curve γ differs from its Euclidean length in the coordinates (x, y) by a quantity $o((s_A - s_B)^3)$. The same statement also holds for the quantity $L(A,B)$. These asymptotics are uniform in the metric running through a closed bounded subset in the space of C^3-smooth Riemannian metrics.*

Proof It is known that the metric g is $O(r^2)$-close to the Euclidean metric, and the polar coordinates are g-orthogonal. In the polar coordinates g has the same radial part dr^2, as the Euclidean metric $dr^2 + r^2 d\phi^2$, and their angular parts differ by a quantity $\Delta = O(r^2)r^2 d\phi^2 = O(r^4 d\phi^2)$. The g-length of the arc AB is the integral of the g-norm of the Euclidean-unit tangent vector field to γ. The integration parameter is the Euclidean natural parameter. The contribution of the above difference Δ to the latter integral is bounded from above by the integral I of a quantity $O(r^2\alpha)$, where α is the angle of a tangent vector $\dot{\gamma}(Q)$ with the radial line CQ. Set $\delta := |s_A - s_B|$. The arc AB lies in a $O(\delta)$-neighborhood of the point C. The distance of the arc AB to C is of order $O(\delta^2)$. Those points in the arc AB where α is bounded away from zero are on distance $O(\delta^2)$ from the origin C. Therefore, $\alpha = o(1)$, as $\delta \to 0$, uniformly on the complement of the arc AB to the disk $D_{\delta^{\frac{3}{2}}}$ of radius $\delta^{\frac{3}{2}}$ centered at C. Hence, the above integral of $O(r^2\alpha)$ over the complement to the disk $D_{\delta^{\frac{3}{2}}}$ is $o(\delta^3)$. The integral inside this disk is also $o(\delta^3)$, since its intersection with γ has length of order $O(\delta^{\frac{3}{2}})$, while the subintegral expression is $O(\delta^2)$. Finally, the upper bound I for the contribution of the non-Euclidean angular part Δ is $o(\delta^3)$. This implies the statement of the claim for the g-length $s_A - s_B$, and hence, for the expression $L(A,B)$: the g-lengths of the segments AC, BC coincide with their Euclidean lengths by the definition of normal coordinates. The asymptotics of Claim 1 are uniform in the metric, as are the intermediate asymptotics used in the proof. □

Claim 2. *Let $\gamma \subset \mathbb{R}^2$ be a C^3-smooth curve with positive geodesic curvature. For every point $A \in \gamma$ consider the osculating circle S_A at A of the curve γ. For every $B \in \gamma$ close to A let us consider the point $B' \in S_A$ closest to B ($BB' \perp S_A$) and the corresponding expressions $\lambda(A, B')$, $L(A,B') = L_{S_A}(A,B')$ written for the circle S_A. One has*

$$\lambda(A,B') - \lambda(A,B) = o((s_A - s_B)^3), \quad L(A,B') - L(A,B) = o((s_A - s_B)^3).$$

Proof Recall that we denote $\delta = |s_A - s_B|$. The lengths of the arcs $AB \subset \gamma$ and $AB' \subset S_A$ differ by a quantity $o(\delta^3)$. Indeed, the projection of the arc

AB to the arc AB' along the radii of the circle S_A has norm of derivative of order $1+o(\delta^2)$. This is implied by the two following statements: 1) the distance between the source and the image is of order $o(\delta^2)$ (the circle is osculating); 2) the slopes of the corresponding tangent lines differ by a quantity $o(\delta)$. The asymptotics $1+o(\delta^2)$ for the norm of projection implies that $\lambda(A,B)-\lambda(A,B')=o(\delta^3)$. Let us now show that the straightline parts of the expressions $L(A,B)$ and $L(A,B')$ also differ by a quantity $o(\delta^3)$. The tangent lines $T_B\gamma$ and $T_{B'}S_A$ pass through $o(\delta^2)$-close points B and B', and their slopes differ by a quantity $o(\delta)$, see the above statements 1) and 2). Note that $BB'\perp T_{B'}S_A$. This implies that the distance between their points C and C' of intersection with the line $T_A\gamma$ is $o(\delta)$. Consider the line through C orthogonal to the line $T_{B'}S_A$. Let H denote their intersection point. The difference of the straighline parts of the expressions $L(A,B)$ and $L(A,B')$ is equal to $(|BC|-|B'H|)\pm(|CC'|-|C'H|)$. The second bracket is the difference of a cathet and a hypothenuse, both of order $o(\delta)$, in a right triangle with angle $O(\delta)$ between them. Hence, the latter difference is $o(\delta^3)$, since the cosine of the angle is $1+O(\delta^2)$. The first bracket is equal to the similar difference in another right triangle, with cathet $B'H$ and hypothenuse being the segment BC shifted by the vector $\overrightarrow{BB'}$; both are of order $O(\delta)$, and the angle between them is $o(\delta)$. Hence, the first bracket is $o(\delta^3)$ (the cosine being now $1+o(\delta^2)$). Finally, the difference of the straightline parts of the expressions $L(A,B)$ and $L(A,B')$ is $o(\delta^3)$. The claim is proved. □

Claims 1 and 2 reduce Theorem 1.16 to the case, when the metric is Euclidean and γ is a circle in $\mathbb{R}^2$. Let R denote its radius. Let AB be its arc cut by a sector of small angle ϕ. Then

$$L(A,B)=R(2\tan(\frac{\phi}{2})-\phi)\simeq\frac{R}{12}\phi^3=\frac{\kappa^2}{12}|s_A-s_B|^3,\ \kappa=R^{-1}.$$

This proves Theorem 1.16. □

Proof of Corollary 1.17. Let $C\in\Gamma_p$. Let $A=A(C)$, $B=B(C)\in\gamma$ denote the points such that the geodesics AC and BC are tangent to γ at A and B respectively. We order them so that $A(C)=\mathcal{T}_p(B(C))$. One has $L(A(C),B(C))=p$ for all $C\in\Gamma_p$, by definition. On the other hand, $L(A(C),B(C))\simeq\frac{1}{12}\kappa^2(A(C))|s(A(C))-s(B(C))|^3$, as C tends to a compact subarc $\widehat{\gamma}\Subset\gamma$, by Theorem 1.16. Therefore, all the quantities $\kappa^{\frac{2}{3}}(A(C))|s(A(C))-s(B(C))|$ are uniformly asymptotically equivalent. Substituting $A(C)=\mathcal{T}_p(B(C))$, we get (1.6). Corollary 1.17 is proved. □

Corollary 1.18 follows immediately from Corollary 1.17.

Proof of Theorem 1.15. Let the curve γ have the string Poritsky property. Let t denote its Poritsky parameter. Set $f := \frac{dt}{ds}$. For the proof of Theorem 1.15 it suffices to show that $f \equiv \kappa^{\frac{2}{3}}$ up to constant factor. Or equivalently,

$$\frac{f(Q)}{f(B)} = \frac{\kappa^{\frac{2}{3}}(Q)}{\kappa^{\frac{2}{3}}(B)} \text{ for every } B, Q \in \gamma. \tag{6.1}$$

Fix a small $p > 0$. Set $A := \mathcal{T}_p(B)$, $R = \mathcal{T}_p(Q)$. One has

$$t(A) - t(B) = t(R) - t(Q), \tag{6.2}$$

by the Poritsky property. On the other hand, the latter left- and right-hand sides are asymptotically equivalent respectively to $f(B)\lambda(A, B)$ and $f(Q)\lambda(R, Q)$. But

$$\kappa^{\frac{2}{3}}(B)\lambda(A, B) \simeq \kappa^{\frac{2}{3}}(Q)\lambda(R, Q), \text{ as } p \to 0,$$

by Corollary 1.17. Substituting the two latter asymptotics to (6.2) yields (6.1). Theorem 1.15 is proved. □

7 Symplectic generalization of Theorem 1.15

In Subsection 7.1 we give a background material on symplectic properties of the billiard ball reflection map. In Subsection 7.2 we introduce weakly billiard-like maps. We consider the so-called string type families of weakly billiard-like maps, which generalize the family of billiard reflections from string construction curves defined by a curve with the string Poritsky property. We state Theorem 7.10, which is a symplectic generalization of Theorem 1.15 (C^6-smooth case) to the string type billiard-like map families. Theorem 7.10 will be proved in Subsection 7.4. For its proof, in Subsection 7.3 we introduce an analogue of Lazutkin coordinates, the so-called modified Lazutkin coordinates, for weakly billiard-like maps (Theorem 7.11) and prove Lemma 7.13 on asymptotics of orbits in these coordinates.

In Subsection 7.5 we show how to retrieve Theorem 1.15 for C^6-smooth curves from Theorem 7.10.

The idea to extend Theorem 1.15 to a more general symplectic context was suggested by Sergei Tabachnikov.

7.1 Symplectic properties of billiard ball map

The background material recalled here can be found in [3, 4, 15, 17, 18, 22].

Let Σ be a surface with Riemannian metric. Let $\Pi : T\Sigma \to \Sigma$ denote the tautological projection. Let us recall that the *tautological 1-form* α on $T\Sigma$ (also called the *Liouville form*) is defined as follows: for every $(Q, P) \in T\Sigma$ with $Q \in \Sigma$ and $P \in T_Q\Sigma$ for every $v \in T_{(Q,P)}(T\Sigma)$ set

$$\alpha(v) :=< P, \Pi_* v > . \tag{7.1}$$

The differential

$$\omega = d\alpha$$

of the 1-form α is the *canonical symplectic form* on $T\Sigma$.

Let $O \in \Sigma$, and $\gamma \subset \Sigma$ be a germ of regular oriented curve at O. We parametrize it by its natural length parameter s. The corresponding function $s \circ \Pi$ on $T\gamma$ will be also denoted by s. For every $Q \in \gamma$ and $P \in T_Q\gamma$ set

$$\dot{\gamma}(Q) = \frac{d\gamma}{ds}(Q) := \text{ the directing unit tangent vector to } \gamma \text{ at } Q,$$

$$\sigma(Q, P) :=< P, \dot{\gamma}(Q) >, \quad y(Q, P) := 1 - \sigma(Q, P). \tag{7.2}$$

The restriction to $T\gamma$ of the form ω is a symplectic form, which will be denoted by the same symbol ω.

Proposition 7.1 *(see [15, formula (3.1)] in the Euclidean case). The coordinates (s, y) on $T\gamma$ are symplectic: $\omega = ds \wedge dy$ on $T\gamma$.*

Proof The proposition follows from the definition of the symplectic structure $\omega = d\alpha$, α is the same, as in (7.1): in local coordinates (s, σ) one has $\alpha = \sigma ds$, thus, $\omega = d\sigma \wedge ds = ds \wedge dy$. □

Let V denote the Hamiltonian vector field on $T\Sigma$ with the Hamiltonian $||P||^2$. It generates the geodesic flow. Consider the unit circle bundle:

$$S = \mathcal{T}_1\Sigma := \{||P||^2 = 1\} \subset T\Sigma.$$

It is known that for every point $x \in S$ the kernel of the restriction $\omega|_{T_xS}$ is the one-dimensional linear subspace spanned by the vector $V(x)$ of the field V. Each cross-section $W \subset S$ to the field V is identified with the (local) space of geodesics. The symplectic structure ω induces a well-defined symplectic structure on W called the *symplectic reduction.*

Remark 7.2 The symplectic reduction is invariant under holonomy along orbits of the geodesic flow. Namely, for every arc AB of its orbit and two germs of cross-sections W_1 and W_2 through A and B respectively the holonomy mapping $W_1 \to W_2$, $A \mapsto B$ along the arc AB is a symplectomorphism.

Consider the local hypersurface

$$\Gamma = \Pi^{-1}(\gamma) \cap S = (\mathcal{T}_1\Sigma)|_\gamma \subset S.$$

At those points $(Q, P) \in \Gamma$, for which the vector P is transverse to γ the hypersurface Γ is locally a cross-section for the restriction to S of the geodesic flow. Thus, near the latter points the hypersurface Γ carries a canonical symplectic structure given by the symplectic reduction. Set

$$\mathcal{O}_\pm := (O, \pm\dot{\gamma}(O)) \in \Gamma.$$

Let γ have positive geodesic curvature. For every $(Q, P) \in \Gamma$ close enough to $\mathcal{O}_\pm$ with Q lying on the convex side from γ the oriented geodesic through Q issued in the direction P intersects γ at two points Q and Q' (which coincide if P is tangent to γ). Let P' denote the orienting unit tangent vector of the latter geodesic at Q'. This defines the germ at $\mathcal{O}_\pm$ of involution

$$\beta : (\Gamma, \mathcal{O}_\pm) \to (\Gamma, \mathcal{O}_\pm), \quad \beta(Q, P) = (Q', P'), \quad \beta^2 = Id, \tag{7.3}$$

which will be called the *billiard ball geodesic correspondence.*

Consider the following open subset in $T\gamma$: the unit ball bundle

$$\mathcal{T}_{\leq 1}\gamma := \{(Q, P) \in T\gamma \mid ||P||^2 \leq 1\}.$$

Let $\pi : (T\Sigma)|_\gamma \to T\gamma$ denote the mapping acting by orthogonal projections

$$\pi : T_Q\Sigma \to T_Q\gamma, \ Q \in \gamma.$$

It induces the following projection also denoted by π:

$$\pi : \Gamma \to \mathcal{T}_{\leq 1}\gamma. \tag{7.4}$$

Let $\mathcal{V}$ denote a convex domain with boundary containing γ. Every point $(Q, P) \in \mathcal{T}_{\leq 1}\gamma$ has two π-preimages $(Q, w_\pm)$ in Γ: the vector w_+ (w_-) is directed inside (respectively, outside) the domain $\mathcal{V}$. The vectors $w_\pm$ coincide, if and only if $||P|| = 1$, and in this case they lie in $T_Q\gamma$. Thus, the mapping $\pi : \Gamma \to \mathcal{T}_{\leq 1}\gamma$ has two continuous inverse branches. Let $\mu_+ :=$

$\pi^{-1} : \mathcal{T}_{\leq 1}\gamma \to \Gamma$ denote the inverse branch sending P to w_+, cf. [15, section 2]. The above mappings define the germ of mapping

$$\delta_+ := \pi \circ \beta \circ \mu_+ : (\mathcal{T}_{\leq 1}\gamma, \mathcal{O}_\pm) \to (\mathcal{T}_{\leq 1}\gamma, \mathcal{O}_\pm). \tag{7.5}$$

Recall that Γ carries a canonical symplectic structure given by the above-mentioned symplectic reduction (as a cross-section), and $T\gamma$ carries the standard symplectic structure: the restriction to $T\gamma$ of the form $\omega = ds \wedge dy$.

Theorem 7.3 *[22, subsection 1.5], [17, 18, 3, 4] The mappings β, π, and hence, δ_+ given by (7.3), (7.4) and (7.5) respectively are symplectic.*

Proof Symplecticity of the mapping β follows from the definition of symplectic reduction and its holonomy invariance (Remark 7.2). Symplecticity of the projection π follows from definition and the fact that the π-pullback of the tautological 1-form α on $T\gamma$ is the restriction to Γ of the tautological 1-form on $T\Sigma$. This implies symplecticity of $\mu_+ = \pi^{-1}$, and hence, δ_+. $\square$

Let $I : \Gamma \to \Gamma$ denote the reflection involution

$$I : (Q, P) \mapsto (Q, P^*),$$

$$Q \in \gamma, \ P^* := \text{ the vector symmetric to } P \text{ with respect to the line } T_Q\gamma.$$

Let $\Gamma_+ \subset \Gamma$ denote the subset of those points (Q, P) for which P either is directed inside the convex domain $\mathcal{V}$, or coincides with $\dot{\gamma}(Q)$.

Proposition 7.4 *The involution I preserves the tautological 1-form α, and hence, is symplectic. The involutions I and β are C^r-smooth germs of mappings $(\Gamma, \mathcal{O}_\pm) \to (\Gamma, \mathcal{O}_\pm)$, if the metric and the curve γ are C^{r+1}-smooth. The mapping δ_+ is conjugated to their product acting on Γ_+:*

$$\widetilde{\delta}_+ := I \circ \beta = \mu_+ \circ \delta_+ \circ \mu_+^{-1}. \tag{7.6}$$

The proposition follows immediately from definitions.

The billiard transformation T of reflection from the curve γ acts on the space of oriented geodesics that intersect γ and are close enough to the geodesic tangent to γ at O. Each of them intersects γ at two points. To each oriented geodesic G we put into correspondence a point $(Q, P) \in \Gamma_+$, where Q is its first intersection point with γ (in the sense of the orientation of the geodesic G) and P is the orienting unit vector tangent to G at Q. This is a locally bijective correspondence.

Proposition 7.5 *Let the metric and the curve γ be C^3-smooth. The billiard mapping T written as a mapping $\Gamma_+ \to \Gamma_+$ via the above correspondence coincides with $\widetilde{\delta}_+$. Consider the coordinates (s,ϕ) on Γ: $s = s(Q)$ is the natural length parameter of a point $Q \in \gamma$; $\phi = \phi(Q,P)$ is the oriented angle of the vector $\dot{\gamma}(Q)$ with a vector $P \in T_Q\Sigma$. In the coordinates (s,ϕ) the mappings I, β and $T = \widetilde{\delta}_+$ are C^2-smooth and take the form*

$$I(s,\phi) = (s,-\phi),\ \beta(s,\phi) = (s + 2\kappa^{-1}(s)\phi + O(\phi^2), -\phi + O(\phi^2)), \quad (7.7)$$

$$\widetilde{\delta}_+(s,\phi) = (s + 2\kappa^{-1}(s)\phi + O(\phi^2), \phi + O(\phi^2)). \quad (7.8)$$

The asymptotics are uniform in s, as $\phi \to 0$. In the coordinates

$$(s,y), \quad y = 1 - \cos\phi, \quad (7.9)$$

see (7.2), the billiard mapping T coincides with δ_+ and takes the form

$$\delta_+(s,y) = (s + 2\sqrt{2}\kappa^{-1}(s)\sqrt{y} + O(y), y + O(y^{\frac{3}{2}})). \quad (7.10)$$

Proof All the statements of the proposition except for the formulas follow from definition. Formula (7.7) follows from the definitions of the mappings I and β: a geodesic issued from a point $Q \in \gamma$ at a small angle ϕ with the tangent vector $\dot{\gamma}(Q)$ intersects γ at a point Q' such that $\lambda(Q,Q') = 2\kappa^{-1}(Q)\phi + O(\phi^2)$. The latter formula follows from its Euclidean analogue (applied to the curve γ represented in normal coordinates centered at Q), Proposition 2.2 and smoothness. Formulas (7.7) and (7.6) imply (7.8), which in its turn implies (7.10), since $y = \frac{\phi^2}{2} + O(\phi^4)$. □

7.2 Families of billiard-like maps with invariant curves. A symplectic version of Theorem 1.15

In this and the next subsections we study the following class of area-preserving mappings generalizing the billiard mappings (7.10).

Definition 7.6 A *weakly billiard-like map* is a germ of mapping preserving the standard area form $dx \wedge dy$,

$$F : (\mathbb{R} \times \mathbb{R}_{\geq 0}, (0,0)) \to (\mathbb{R} \times \mathbb{R}_{\geq 0}, (0,0)),$$

$$F = (f_1, f_2) : (x,y) \mapsto (x + w(x)\sqrt{y} + O(y), y + O(y^{\frac{3}{2}})),\ w(x) > 0, \quad (7.11)$$

for which the x-axis is a line of fixed points and such that the variable change

$$(x,y) \mapsto (x,\phi),\ y = \phi^2$$

conjugates F to a C^2-smooth germ $\widetilde{F}(x,\phi)$. The above asymptotics are uniform in x, as $y \to 0$. If, in addition to the above assumptions, the latter mapping $\widetilde{F}$ is a product of two involutions:

$$\widetilde{F} = I \circ \beta, \ I(x,\phi) = (x, -\phi),$$

$$\beta(x,\phi) = (x + w(x)\phi + O(\phi^2), -\phi + O(\phi^2)), \ \beta^2 = Id, \tag{7.12}$$

then F will be called *a (strongly) billiard-like map.*

Example 7.7 The mapping δ_+ from (7.10) is strongly billiard-like in the coordinates (s,y) with $w(s) = 2\sqrt{2}\kappa^{-1}(s)$, see (7.6), (7.7), (7.8) and (7.10).

The next definition generalizes the notion of string curve family to weakly billiard-like maps.

Definition 7.8 A family $F_\varepsilon(x,y)$ of weakly billiard-like maps (7.11) depending on a parameter $\varepsilon \in [0,\varepsilon_0]$ is of *string type,* if the derivatives up to order 2 of the corresponding mappings $\widetilde{F}_\varepsilon(x,\phi)$ are continuous in (x,ϕ,ε) on a product $\{|x| \leq \delta_0\} \times [0,\phi_0] \times [0,\varepsilon_0]$ and there exist a $\delta \in (0,\delta_0]$ and a family γ_ε of F_ε-invariant graphs of continuous functions $h_\varepsilon : [-\delta,\delta] \to \mathbb{R}_{\geq 0}$,

$$\gamma_\varepsilon = \{y = h_\varepsilon(x)\}, \tag{7.13}$$

such that γ_ε converge to the x-axis: $h_\varepsilon(x) \to 0$ uniformly on $[-\delta,\delta]$.

Example 7.9 Let $\gamma \subset \Sigma$ be a germ of curve with positive geodesic curvature such that the corresponding string construction curves Γ_p are C^3-smooth and their 3-jets depend continuously on the base points. (For example, this holds automatically in the case, when the curve γ and the metric are C^6-smooth, see Theorem 1.3.) Then the family of billiard reflection maps from the curves Γ_p is a string type family. The invariant curves γ_p from (7.13) are identified with one and the same curve in the space of oriented geodesics: the family of geodesics tangent to the curve γ and oriented by its tangent vectors $\dot{\gamma}$. See Subsection 7.5 for more details.

The next theorem deals with string type families of weakly billiard-like maps satisfying an analogue of the Poritsky property. It extends Theorem 1.15 on coincidence of Poritsky and Lazutkin parameters.

Theorem 7.10 *Let $F_\varepsilon(x,y)$ be a string type family of weakly billiard maps. Let for every ε small enough there exist a continuous strictly increasing parameter t_ε on γ_ε in which $F|_{\gamma_\varepsilon}$ is a translation by ε-dependent constant,*

$$t_\varepsilon \circ F|_{\gamma_\varepsilon} = t_\varepsilon + c(\varepsilon), \tag{7.14}$$

such that the parameter $t_\varepsilon = t_\varepsilon(x)$ considered as a function of x converges to a strictly increasing function $t_0(x)$ uniformly on $[-\delta, \delta]$, as $\varepsilon \to 0$. Then

$$t_0 = aX + b, \ X := \int_0^x w^{-\frac{2}{3}}(z)dz, \ a, b \equiv const. \tag{7.15}$$

Here $w = w_0(x)$ is the function from (7.11) corresponding to the mapping F_ε with $\varepsilon = 0$.

Theorem 7.10 is proved in Subsection 7.4.

7.3 Modified Lazutkin coordinates and asymptotics

In the proof of Theorem 7.10 we use the following well-known theorem.

Theorem 7.11 *Let F be a weakly billiard-like map F, and let $w(x)$ be the corresponding function in (7.11). The transformation*

$$\mathcal{L} : (x, y) \mapsto (X, Y), \ \begin{cases} X(x) = \int_0^x w^{-\frac{2}{3}}(z)dz \\ Y(x, y) := w^{\frac{2}{3}}(x)y \end{cases} \tag{7.16}$$

is symplectic. Its post-composition with the variable change $(X, Y) \mapsto (X, \Phi)$, $\Phi := \sqrt{Y}$, conjugates F to a mapping with the asymptotics

$$F : (X, \Phi) \mapsto (X + \Phi + o(\Phi), \Phi(1 + o(\Phi))), \ \text{as } \Phi \to 0, \tag{7.17}$$

uniform in X. The coordinates (X, Φ) will be called the **modified Lazutkin coordinates**.

A version of Theorem 7.11 is implicitly contained in [14, 15]. For completeness of presentation, we present its proof using the following proposition.

Proposition 7.12 *The y-component of a weakly billiard-like map (7.11) admits the following more precise formula:*

$$f_2(x, y) = y - \frac{2}{3}w'(x)y^{\frac{3}{2}} + o(y^{\frac{3}{2}}). \tag{7.18}$$

Proof Recall that $\widetilde{F} = (\widetilde{f}_1, \widetilde{f}_2)$ is a C^2-smooth mapping, $\phi = \sqrt{y}$, $\widetilde{f}_2(x, \phi) = \sqrt{f_2(x, \phi^2)}$. Consider the Taylor expansion of the function $\widetilde{f}_2$ in ϕ:

$$\widetilde{f}_2(x, \phi) = \phi + c(x)\phi^2 + o(\phi^2),$$

$$f_2(x, y) = \widetilde{f}_2^2(x, \phi) = y(1 + c(x)\sqrt{y} + o(\sqrt{y}))^2 = y + 2c(x)y^{\frac{3}{2}} + o(y^{\frac{3}{2}}), \tag{7.19}$$

$$\frac{\partial f_2}{\partial y}(x,y) = \frac{1}{\sqrt{y}}\widetilde{f}_2\frac{\partial \widetilde{f}_2}{\partial \phi}(s,\phi) = 1 + 3c(x)\sqrt{y} + o(\sqrt{y}).$$

This together with analogous calculations of the other partial derivatives,

$$\frac{\partial f_1}{\partial x} = 1 + w'(x)\sqrt{y} + o(\sqrt{y}),$$

$$\frac{\partial f_1}{\partial y} = O(y^{-\frac{1}{2}}), \frac{\partial f_2}{\partial x} = 2\widetilde{f}_2(x,\phi)\frac{\partial \widetilde{f}_2}{\partial x}(x,\phi) = o(\phi^2) = o(y),$$

shows that the Jacobian of the mapping $F(x,y)$ equals $1+(w'(x)+3c(x))\sqrt{y}+o(\sqrt{y})$. But it should be equal to 1, by symplecticity. Therefore, $c(x) = -\frac{1}{3}w'(x)$. This together with (7.19) proves the proposition. □

Proof of Theorem 7.11. Symplecticity of the transformation $\mathcal{L}$ follows from definition. Let us show that the coordinate change $(x,y) \mapsto (X,\Phi)$ conjugates F to a mapping with asymptotics (7.17). One has

$$X \circ F(x,y) = X + \int_x^{x+w(x)\sqrt{y}+O(y)} w^{-\frac{2}{3}}(z)dz$$

$$= X + w(x)w^{-\frac{2}{3}}(x)\sqrt{y} + O(y) = X + \Phi + O(\Phi^2), \tag{7.20}$$

$$\Phi \circ F(x,y) = \sqrt{w^{\frac{2}{3}}(f_1(x,y))f_2(x,y)}$$

$$= w^{\frac{1}{3}}(x + w(x)\sqrt{y} + o(\sqrt{y}))\sqrt{y(1 - \frac{2}{3}w'(x)y^{\frac{1}{2}} + o(y^{\frac{1}{2}}))}.$$

Substituting the expressions $\sqrt{y} = \Phi w^{-\frac{1}{3}}(x)$ and

$$w^{\frac{1}{3}}(x + w(x)\sqrt{y} + o(\sqrt{y})) = w^{\frac{1}{3}}(x) + \frac{1}{3}w^{-\frac{2}{3}}(x)w'(x)w(x)\sqrt{y} + o(\sqrt{y})$$

$$= w^{\frac{1}{3}}(x)(1 + \frac{1}{3}w'(x)\sqrt{y} + o(\sqrt{y}))$$

to the above formula yields

$$\Phi \circ F(x,y) = \Phi(1 + \frac{1}{3}w'(x)y^{\frac{1}{2}} + o(y^{\frac{1}{2}}))(1 - \frac{1}{3}w'(x)y^{\frac{1}{2}} + o(y^{\frac{1}{2}})) = \Phi + o(\Phi^2).$$

This together with (7.20) proves (7.17). □

We use the following lemma on asymptotics of orbits of a mapping (7.17).

Lemma 7.13 *Let $V_{\Delta,\sigma} := [-\Delta,\Delta]\times[0,\sigma] \subset \mathbb{R}^2_{(X,\Phi)}$, $F : V_{\Delta,\sigma} \to F(V_{\Delta,\sigma})$ be a homeomorphism with asymptotics (7.17) uniform in $X \in [-\Delta,\Delta]$. There exist functions $\alpha(z), \beta(z) > 0$, $\alpha(z), \beta(z) \to 0$, as $z \to 0$ such that for every $\eta \in (0, \frac{\sigma}{4})$ small enough the following statements hold. Fix an arbitrary $\delta \in (0,\Delta)$. For every $q_0 \in V_{\delta,\eta}$ its two-sided orbit in $V_{\delta,\sigma}$ is a finite sequence:*

$$\mathcal{O} := (q_{j_{\min}}, \dots, q_{-1}, q_0, q_1, \dots, q_{j_{\max}}),\ q_j = (X_j, \Phi_j),\ q_{j+1} = F(q_j), \quad (7.21)$$

$$X_{j_{\min}-1} = X \circ F^{-1}(q_{j_{\min}}) < -\delta, \quad X_{j_{\max}+1} = X \circ F(q_{j_{\max}}) > \delta. \quad (7.22)$$

The following inequalities hold for every $j = j_{\min} - 1, \dots, j_{\max} + 1$:

$$|\ln \frac{\Phi_j}{\Phi_0}| \leq \alpha(\eta); \quad (7.23)$$

$$e^{-\beta(\eta)}\Phi_0 \leq X_{j+1} - X_j \leq e^{\beta(\eta)}\Phi_0. \quad (7.24)$$

Addendum to Lemma 7.13. *Let F_ε be a family of homeomorphisms defined on $V_{\Delta,\eta}$ and depending on a parameter $\varepsilon \in [0, \varepsilon_0]$ with asymptotics (7.17) being uniform in $(X, \varepsilon) \in [-\Delta,\Delta] \times [0,\varepsilon_0]$. Then all the statements of the lemma hold with functions α and β independent on ε.*

Proof of Lemma 7.13. The second component of asymptotics (7.17) is equivalent to the uniform asymptotics $\ln \frac{\Phi\circ F(X,\Phi)}{\Phi} = o(\Phi)$: to the existence of a non-decreasing function $u(\Phi) > 0$, $u(\Phi) \to 0$, as $\Phi \to 0$, such that

$$|\ln \frac{\Phi \circ F^{\pm 1}(X,\Phi)}{\Phi}| \leq \Phi u(\Phi). \quad (7.25)$$

The first component of asymptotics (7.17) is equivalent to the existence of a non-decreasing function $v(\Phi) > 0$, $v(\Phi) \to 0$, as $\Phi \to 0$, for which

$$\Phi(1 - v(\Phi)) \leq X \circ F(X,\Phi) - X \leq \Phi(1 + v(\Phi)). \quad (7.26)$$

Consider the maximal connected piece $\mathcal{O}_4$ of the orbit $\mathcal{O}$ containing q_0 whose points have Φ-coordinates satisfying the inequality $\frac{\Phi_0}{4} \leq \Phi_j \leq 4\Phi_0$:

$$\mathcal{O}_4 := (q_{j_{\min,4}}, \dots, q_0, \dots, q_{j_{\max,4}}),\ j_{\min} \leq j_{\min,4} \leq 0 \leq j_{\max,4} \leq j_{\max},$$

$$\frac{1}{4}\Phi_0 \leq \Phi_j \leq 4\Phi_0 \text{ for every } j \in [j_{\min,4}, j_{\max,4}]. \quad (7.27)$$

By definition, if $j_{\min,4} > j_{\min}$, then (7.27) does not hold for $j = j_{\min,4} - 1$. Analogous statement holds for $j_{\max,4}$. Let us choose an $\eta > 0$ small enough so that $u(4\eta), v(4\eta) < \frac{1}{4}$. Then for every $j = j_{\min,4}, \dots, j_{\max,4}$ one has

$$\frac{1}{8}\Phi_0 \leq X_{j+1} - X_j \leq 8\Phi_0, \quad (7.28)$$

by (7.27) amd (7.26). Set $N := j_{\max,4} - j_{\min,4} + 1 = |\mathcal{O}_4|$. One has

$$N \leq \frac{16\Delta}{\Phi_0}, \tag{7.29}$$

whenever $\Phi_0 < \frac{1}{8}(\Delta - \delta)$, by (7.27). For every $i \in [j_{\min,4} - 1, j_{\max,4}]$ one has

$$|\ln \frac{\Phi_{i+1}}{\Phi_i}| \leq 4\Phi_0 u(4\eta), \tag{7.30}$$

by (7.25) and (7.27). Summing up the latter inequality and using (7.29), we get the following inequality for $j \in [j_{\min,4} - 1, j_{\max,4} + 1]$:

$$|\ln \frac{\Phi_j}{\Phi_0}| \leq N u(4\eta)\Phi_0 \leq \alpha(\eta) := 16\Delta u(4\eta). \tag{7.31}$$

One has obviously $\alpha(\eta) \to 0$, as $\eta \to 0$. This proves (7.23) for $j \in [j_{\min,4} - 1, j_{\max,4} + 1]$. Inequality (7.24) for the same j with

$$\beta(\eta) = -\ln(1 - v(4\eta)) + \alpha(\eta)$$

follows from (7.23) and (7.26).

Claim. *For every $\eta > 0$ small enough (such that $\alpha(\eta), 4\eta u(4\eta) < \frac{1}{8}$) and every $q_0 \in V_{\delta,\eta}$ one has $\mathcal{O}_4 = \mathcal{O}$: that is, $j_{\min,4} = j_{\min}$, $j_{\max,4} = j_{\max}$.*

Proof Suppose the contrary, for some η as above and some $q_0 = (X_0, \Phi_0) \in V_{\delta,\eta}$ one has, say, $j_{\max,4} < j_{\max}$. Set $j_0 := j_{\max,4}$. Then

$$|\ln \frac{\Phi_{j_0+1}}{\Phi_{j_0}}|, |\ln \frac{\Phi_{j_0}}{\Phi_0}| < \frac{1}{8},$$

by (7.30) and (7.23) for $j = j_0$. Adding the latter inequalities we get $|\ln \frac{\Phi_{j_0+1}}{\Phi_0}| < \frac{1}{4}$, thus, $\frac{1}{4}\Phi_0 < \Phi_{j_0+1} < 4\Phi_0$. The contradiction thus obtained with the definition of the number $j_{\max,4}$ (maximality) proves the claim. □

Let η be small, as in the claim. One has $q_{j_{\max}+1} = F(q_{j_{\max}}) \notin V_{\delta,\sigma}$, by definition. But $\Phi(q_{j_{\max}+1}) \leq e^{\alpha(\eta)}\Phi_0 < 4\eta < \sigma$, by (7.23). Therefore, $X_{j_{\max}+1} > \delta$, by definition and (7.24). This together with a similar argument for the point $q_{j_{\min}-1}$ implies (7.22). Lemma 7.13 is proved. □

Proof of the Addendum to Lemma 7.13. The addendum follows from uniformity of asymptotics (7.17) in (X, ε) and from the above proof. □

7.4 Proof of Theorem 7.10

Everywhere below we write the mappings F_ε in the coordinates $(X_\varepsilon, \Phi_\varepsilon)$ given by (7.17). We consider that F_ε are well-defined on one and the same set $V_{\Delta,\eta} = [-\Delta, \Delta] \times [0, \eta] \subset \mathbb{R}^2_{(X_\varepsilon,\Phi_\varepsilon)}$ for all $\varepsilon \in [0, \varepsilon_0]$. Thus, we identify the above coordinates for all ε and denote them by (X, Φ). To show that the limit parameter t_0 is equal to the Lazutkin coordinate X up to multiplicative and additive constants, we have to show that for every four distinct points in the X-axis with X-coordinates $\mathcal{X}_j$,

$$-\Delta < \mathcal{X}_1 < \mathcal{X}_2 < \mathcal{X}_3 < \mathcal{X}_4 < \Delta,$$

the ratios of lengths of the segments

$$I_1 := [\mathcal{X}_1, \mathcal{X}_2], \quad I_3 := [\mathcal{X}_3, \mathcal{X}_4]$$

in the parameters X and t_0 are equal:

$$\frac{t_0(\mathcal{X}_2) - t_0(\mathcal{X}_1)}{t_0(\mathcal{X}_4) - t_0(\mathcal{X}_3)} = \frac{\mathcal{X}_2 - \mathcal{X}_1}{\mathcal{X}_4 - \mathcal{X}_3}. \tag{7.32}$$

Take a $\varepsilon > 0$ small enough, and consider the corresponding F_ε-invariant curve γ_ε. It can be represented as the graph $\{\Phi = H_\varepsilon(X)\}$ of a continuous function. The parameter t_ε on γ_ε in which F_ε is a translation induces a parameter on the X-axis via projection; the induced parameter will be also denoted by t_ε. Fix a $\delta \in (0, \Delta)$ such that $-\delta < \mathcal{X}_1 < \mathcal{X}_4 < \delta$. Consider the corresponding orbit $\mathcal{O}$ of the point $q_{0,\varepsilon} = (\mathcal{X}_1, H_\varepsilon(\mathcal{X}_1)) \in \gamma_\varepsilon$, see (7.21), and let us denote its points by $q_{j,\varepsilon} := F^j(q_{0,\varepsilon})$. Set

$$\nu(\varepsilon) := \Phi(q_{0,\varepsilon}) = H_\varepsilon(\mathcal{X}_1); \ \nu(\varepsilon) \to 0, \text{ as } \varepsilon \to 0.$$

The sequence $X_j := X(q_{j,\varepsilon})$ is strictly increasing with steps having uniform asymptotics $\nu(\varepsilon)(1+o(1))$, as $\varepsilon \to 0$, by Lemma 7.13 and its addendum. For every $i = 1, 2, 3, 4$ let $j_i = j_{i,\varepsilon}$ denote the maximal number j for which $X_j \leq \mathcal{X}_i$. By definition, $j_1 = 0$. For every $i = 2, 3, 4$ and every ε small enough one has $\mathcal{X}_i - X_{j_i} < 2\nu(\varepsilon)$, by the above asymptotics. The sequence $t_\varepsilon(X_j)$ is an arithmetic progression, since $F_\varepsilon|_{\gamma_\varepsilon}$ acts as a translation in the parameter t_ε. Its step tends to zero, as $\varepsilon \to 0$, since t_ε limits to a strictly increasing continuous parameter t_0 and the X-lengths of steps tend to zero uniformly. This implies that the ratio of the t_ε-lengths of the segments I_1 and I_3 has the same finite asymptotics, as the ratio

$$R_{1,3}(\varepsilon) := \frac{j_{2,\varepsilon} - j_{1,\varepsilon}}{j_{4,\varepsilon} - j_{3,\varepsilon}}.$$

But the ratio of their X-lengths has also the same asymptotics, as $R_{1,3}(\varepsilon)$, since all the steps of the sequence $X(q_{j,\varepsilon})$ are asymptotically equivalent to one and the same quantity $\nu(\varepsilon)$. This proves (7.32) and Theorem 7.10.

7.5 Deduction of Theorem 1.15 (case C^6) from Theorem 7.10

Let the metric on Σ be C^6-smooth. Let $\gamma \subset \Sigma$ be a germ of C^6-smooth curve with the Poritsky property. Let Γ_ε be the corresponding family of string curves. Let $\widetilde{F}_\varepsilon := \widetilde{\delta}_{+,\varepsilon}$ be the billiard ball maps (7.6) defined by reflections from the curves Γ_ε; see also (7.8). We write them in the coordinates $(s_\varepsilon, \phi_\varepsilon)$ associated to Γ_ε on the space of oriented geodesics, see Proposition 7.5. The curves Γ_ε form a foliation tangent to a C^2-smooth line field on the closure of the concave domain adjacent to γ. Their the 3-jets depend continuously on points. Both statements follow from Theorem 1.3. This implies that the mappings $\widetilde{F}_\varepsilon = \widetilde{\delta}_{+,\varepsilon}(s_\varepsilon, \phi_\varepsilon)$ have derivatives of order up to 2 that are continuous in $(s_\varepsilon, \phi_\varepsilon, \varepsilon)$. Therefore, the corresponding maps $F_\varepsilon := \delta_{+,\varepsilon} = \delta_\varepsilon(s_\varepsilon, y_\varepsilon)$, $y_\varepsilon = 1 - \cos\phi_\varepsilon$, see (7.9), (7.5), (7.10), are strongly billiard-like.

The maps F_ε have invariant curves γ_ε, which are identified with the family of geodesics tangent to the curve γ and oriented as γ. In the coordinates $(s_\varepsilon, y_\varepsilon)$ the curves γ_ε are graphs of continuous functions converging to zero uniformly, as $\varepsilon \to 0$, by construction.

Let now γ have the string Poritsky property. Then the Poritsky parameter t induces a parameter denoted by t_ε on each invariant curve γ_ε: by definition, the value of the parameter t_ε at a geodesic tangent to γ is the value of the Poritsky parameter t at the tangency point. The maps $\delta_\varepsilon : \gamma_\varepsilon \to \gamma_\varepsilon$ act by translations in the parameters t_ε. The parameters t_ε obviously converge uniformly to the Poritsky parameter $t = t_0$ of the curve $\gamma = \Gamma_0$, as $\varepsilon \to 0$. Therefore, the billiard ball maps F_ε satisfy the conditions of Theorem 7.10 with $w = 2\sqrt{2}\kappa^{-1}$, see Example 7.7. This together with Theorem 7.10 implies that $t_0 = at_L + b$, $a, b \equiv const$, and proves Theorem 1.15 in the case, when the metric and the curve γ are C^6-smooth.

8 Osculating curves with the string Poritsky property. Proof of Theorem 1.19

Here we prove Theorem 1.19, which states that a germ of curve with the string Poritsky property is uniquely determined by its 4-jet.

8.1 Cartan distribution, a generalized version of Theorem 1.19 and plan of the section

Everywhere below for a curve (function) γ by $j_p^r\gamma$ we denote its r-jet at the point p. Set

$$\mathcal{F}^r := \text{ the space of } r\text{-jets of functions of one variable } x \in \mathbb{R}.$$

Let Σ be a C^m-smooth two-dimensional manifold. For every $r \in \mathbb{Z}_{\geq 0}$, $r \leq m$, set

$$\mathcal{J}^r = \mathcal{J}^r(\Sigma) := \text{ the space of } r\text{-jets of regular curves in } \Sigma.$$

In more detail, a *germ of regular curve* is the graph of a germ of function $\{y = h(x)\}$ in appropriate local chart (x, y). Locally a neighborhood in $\mathcal{J}^r$ of the jet of a given C^r-germ of regular curve is thus identified with a neighborhood of a jet in $\mathcal{F}^r$. One has $\dim \mathcal{F}^r = \dim \mathcal{J}^r = r + 2$. There are local coordinates $(x, b_0, \dots, b_r)$ on $\mathcal{F}^r$ defined by the condition that for every jet $j_p^r h \in \mathcal{F}^r$ one has

$$x(j_p^r h) = p,\ b_0(j_p^r h) = h(p),\ b_1(j_p^r h) = h'(p), \dots, b_r(j_p^r h) = h^{(r)}(p). \quad (8.1)$$

Recall that the *r-jet extension* of a function (curve) is the curve in the jet space $\mathcal{F}^r$ (respectively, $\mathcal{J}^r$) consisting of its r-jets at all points.

Definition 8.1 (see an equivalent definition in [19, pp.122–123]). Consider the space $\mathcal{F}^r$ equipped with the above coordinates $(x, b_0, \dots, b_r)$. The *Cartan (or contact) distribution* $\mathcal{D}_r$ on $\mathcal{F}^r$ is the field of two-dimensional subspaces in its tangent spaces defined by the system of Pfaffian equations

$$db_0 = b_1 dx,\ db_1 = b_2 dx,\ \dots,\ db_{r-1} = b_r dx. \quad (8.2)$$

For every C^m-smooth surface Σ and every $r \leq m$ the *Cartan (or contact) distribution (plane field) on* $\mathcal{J}^r$, which is also denoted by $\mathcal{D}_r$, is defined by (8.2) locally on its domains identified with open subsets in $\mathcal{F}^r$; the distributions (8.2) defined on intersecting domains V_i, V_j with respect to different charts (x_i, y_i) and (x_j, y_j) coincide and yield a global plane field on $\mathcal{J}^r$.

Remark 8.2 Recall that the r-jet extension of each function (curve) is tangent to the Cartan distribution.

The main result of the present section is the following theorem, which immediately implies Theorem 1.19. Proofs of both theorems will be given in Subsection 8.7.

Theorem 8.3 *Let Σ be a two-dimensional surface with a C^6-smooth Riemannian metric. There exists a C^1-smooth line field $\mathcal{P}$ on $\mathcal{J}^4 = \mathcal{J}^4(\Sigma)$ lying in the Cartan plane field $\mathcal{D}_4$ such that the 4-jet extension of every C^5-smooth curve on Σ with positive geodesic curvature and the string Poritsky property (if any) is a phase curve of the field $\mathcal{P}$.*

Let γ be a germ of curve with the string Poritsky property at a point $O \in \Sigma$. The Poritsky–Lazutkin parameter t on γ is given by already known formula (1.4). We normalize it by additive and multiplicative constants so that $t(O) = 0$ and $\frac{dt}{ds}(O) = \kappa(O)$, see (8.4). We identify points of the curve γ with the corresponding values of the parameter t. Consider the function $L(A, B)$ defined in (1.2). Let $t(A) = a$, $t(B) = a + \tau$. The Poritsky property implies that the function $L(a, a + \tau) = L(0, \tau)$ is independent on a. In particular, the function

$$\Lambda(t) := L(0, t) - L(-t, 0) \tag{8.3}$$

vanishes. For the proof of Theorem 8.3 we show (in the Main Lemma stated in Subsection 8.2) that for every odd $n > 3$ the "differential equation" $\Lambda^{(n+1)}(0) = 0$ is equivalent to an equation saying that the coordinate $b_n = \frac{db_{n-1}}{dx}$ of the n-jet of the curve γ is equal to a function of the other coordinates $(x, b_0, \dots, b_{n-1})$. For $n = 5$ this yields an ordinary differential equation on $\mathcal{J}^4$ satisfied by the 4-jet extension of the curve γ. It will be represented by a line field contained in $\mathcal{D}_4$.

The proof of the Main Lemma takes the most of the section. For its proof we study (in Subsection 8.3) two germs of curves γ and $\gamma_{n,b}$ at a point O having contact of order $n \geq 3$. More precisely, they are graphs of functions $y = h(x)$ and $y = h_{n,b}(x)$ such that $h_{n,b}(x) - h(x) = bx^n + o(x^n)$. We show that the corresponding functions $\Lambda(t)$ and $\Lambda_{n,b}(t)$ differ by $c_n b t^{n+1} + o(t^{n+1})$, with c_n being a known constant depending on the second jet of the curve γ; $c_n \neq 0$ for odd $n > 3$. To this end, we consider a local normal chart (x, y) centered at O with x-axis being tangent to γ at O. We compare different quantities related to both curves, all of them being considered as functions of x: the natural parameters, the curvature etc. In Subsection 8.4 we show that the asymptotic Taylor coefficients of order $(n + 1)$ of the functions $L(0, t)$ and $\Lambda(t)$ depend only on the n-jet of the metric at O. We show in Subsection 8.5 that the above Taylor coefficients are analytic functions of the n-jets of metric and the curve (using results of Subsections 8.3 and 8.4). In Subsection 8.6 we show that the degree $n + 1$ coefficient of the function $\Lambda(t)$ is a linear non-homogeneous function in $b_n = b_n(\gamma)$ with coefficients depending on b_j, $j < n$; the coefficient at b_n being expressed via c_n (using results of Subsection 8.3). This will prove the Main Lemma.

8.2 Differential equations in jet spaces and the Main Lemma

Let s denote the natural orienting length parameter of the curve γ, $s(O) = 0$. Let κ be its geodesic curvature considered as a function $\kappa(s)$, and let $\kappa > 0$. We already know that if the curve γ has the string Poritsky property, then its Poritsky–Lazutkin parameter t is expressed as a function of a point $Q \in \gamma$ in terms of the parameter s via formula (1.1), up to constant factor and additive constant, which can be chosen arbitrarily. We normalize it as follows:

$$t(Q) := \kappa^{\frac{1}{3}}(0) \int_0^{s(Q)} \kappa^{\frac{2}{3}}(s) ds \tag{8.4}$$

We can define the parameter t given by (8.4) on any curve γ, not necessarily having the Poritsky property. We identify the points of the curve γ with the corresponding values of the parameter t; thus, $t(O) = 0$.

Remark 8.4 The parameter t on a curve γ given by (8.4) is invariant under rescaling of the metric by constant factor. This follows from the fact that if the norm induced by the metric is multiplied by a constant factor C, then the Levi-Civita connexion remains unchanged, the unit tangent vectors $\dot{\gamma}$ are divided by C, and the geodesic curvature $||\nabla_{\dot{\gamma}}\dot{\gamma}||$ of the curve γ considered as a function of a point in γ is divided by C.

Let $G = G(0)$ denote the geodesic tangent to γ at its base point O. We will work in normal coordinates (x, y) centered at O, in which G coincides with the x-axis. For every t let $G(t)$ denote the geodesic tangent to γ at the point t, and let $C(t)$ denote the point of the intersection $G \cap G(t)$.

Let $L(A, B)$ the function of $A, B \in \gamma$ defined in (1.2). We consider $L(A, B)$ as a function of the corresponding parameters $t(A)$ and $t(B)$, thus,

$$L(0, t) = L(O, \gamma(t)) = |OC(t)| + |C(t)\gamma(t)| - \lambda(0, t), \tag{8.5}$$

where $\lambda(0, t) = \lambda(O, \gamma(t))$ is the length of the arc $O\gamma(t)$ of the curve γ.

The main part of the proof of Theorem 8.3 is the following lemma.

Lemma 8.5 (The Main Lemma). *Let $n \in \mathbb{N}$, $n \geq 3$. Let Σ be a surface equipped with a C^{n+1}-smooth Riemannian metric. Let $V \subset \Sigma$ be a domain equipped with a C^{n+1}-smooth chart (x, y). Let $\mathcal{J}_y^n(V)$ denote the space of n-jets of curves in V that are graphs of C^n-smooth functions $\{y = h(x)\}$; thus,* *it is naturally identified with an open subset $\mathcal{F}^n_y(V) \subset \mathcal{F}^n$. Let $(x, b_0, \dots, b_n)$ denote the corresponding coordinates on $\mathcal{F}^n_y(V) \simeq \mathcal{J}^n_y(V)$ given by (8.1). Set*

$$J_2 := (x, b_0, b_1, b_2).$$

There exist C^1-smooth functions $\sigma_n(J_2)$ and $P_n(J_2; b_3, \dots, b_n)$,

$$\sigma_n \neq 0 \text{ for odd } n > 3; \ \sigma_n \equiv 0 \text{ for } n = 3 \text{ and for every even } n > 3, \quad (8.6)$$

such that every jet $J_n = (x, b_0, \dots, b_n) \in \mathcal{J}^n_y(V)$ extending J_2 satisfies the following statement. Let γ be a C^n-smooth germ of curve representing the jet J_n, and let t be the parameter on γ defined by (8.4). Let $t > 0$, $L(0, t)$ be the same, as in (8.5). The corresponding function $\Lambda(t)$ from (8.3) admits an asymptotic Taylor formula of degree $n+1$ at 0 of the following type:

$$\Lambda(t) = \sum_{k=3}^{n+1} \widehat{\Lambda}_k t^k + o(t^{n+1}), \quad (8.7)$$

$$\widehat{\Lambda}_{n+1} = \sigma_n(J_2) b_n - P_n(J_2; b_3, \dots, b_{n-1}). \quad (8.8)$$

Definition 8.6 A *pure* n-jet of curve γ in $\mathbb{R}^2$ is a class of n-jets of curves modulo translations. If $\gamma = \{y = h(x)\}$, then it is identified with the collection of Taylor coefficients of the function $h(x)$ at monomials of degrees from 1 to n. A *pure* n-jet of metric on a planar domain is a class of n-jets of metrics modulo translations. It is identified with the collection of Taylor coefficients of the metric tensor at monomials of degrees from 0 to n.

Addendum to Lemma 8.5. *The function σ_n depends analytically on the pure 1-jet of the metric and the pure 2-jet of the curve. The function P_n depends analytically on the pure n-jet of the metric and the pure $(n-1)$-jet of the curve. The function σ_n is defined by the following formula. Set $u = u(J_2) := (1, b_1) \in T_{(x,b_0)}\Sigma$. Let $w \in T_{(x,b_0)}\Sigma$ denote the image of the vector $\frac{\partial}{\partial y} \in T_{(x,b_0)}\Sigma$ under the Riemannian-orthogonal projection to the line $\mathbb{R}u^{\perp}$. Let $\kappa = \kappa(J_2)$ denote the geodesic curvature of a curve γ representing the jet J_2 (it depends on the pure 2-jet of the curve and the pure 1-jet of the metric). Then for every odd $n \geq 3$*

$$\sigma_n(J_2) = \frac{(n-2)(n-3)}{6(n+1)!} ||w|| (||u|| \kappa(J_2))^{-n}. \quad (8.9)$$

Lemma 8.5 and its addendum will be proved in Subsection 8.6.

8.3 Comparison of functions $L(0,t)$ and $\Lambda(t)$ for osculating curves

Let $n \geq 3$. Let Σ be a surface equipped with a Riemannian metric, $O \in \Sigma$. Let us consider normal coordinates (x, y) centered at O. We consider that the metric under question is C^3-smooth in the normal coordinates. Let $b \in \mathbb{R}$, and let $\gamma, \gamma_{n,b} \subset \Sigma$ be two germs of C^n-smooth curves at O with the same $(n-1)$-jet that are tangent to the x-axis at O,

$$\gamma = \{y = h(x)\}, \ \gamma_{n,b} = \{y = h_{n,b}(x)\}, \ h_{n,b}(x) = h(x) + bx^n + o(x^n),$$

$h, h_{n,b} \in C^n$. Here $o(x^n)$ is a function tending to zero together with its derivatives up to order n, as $x \to 0$. Their geodesic curvatures at O are equal to the same number $\kappa(O) = h''(0) = h''_{n,b}(0)$, by (2.1). Without loss of generality we consider that $\kappa(O) = 1$. One can achieve this by rescaling the norm of the metric by constant factor $\kappa(O)$, see Remark 8.4.

The main result of the present subsection is the following lemma.

Lemma 8.7 *In the above conditions let t be the parameter on γ given by (8.4). Let $L(0,t)$, $L_{n,b}(0,t)$ and $\Lambda(t)$, $\Lambda_{n,b}(t)$ be the functions from (8.3) defined for the curves γ and $\gamma_{n,b}$ respectively. For every $t > 0$ one has*

$$L_{n,b}(0,t) - L(0,t) = \frac{(n-2)(n-3)}{12(n+1)} bt^{n+1} + o(t^{n+1}), \ \textit{as } t \to 0, \qquad (8.10)$$

$$\Lambda_{n,b}(t) - \Lambda(t) = \begin{cases} \frac{(n-2)(n-3)}{6(n+1)} bt^{n+1} + o(t^{n+1}), \ \textit{if } n \textit{ is odd}, \\ o(t^{n+1}), \ \textit{if } n \textit{ is even}. \end{cases} \qquad (8.11)$$

For the proof of Lemma 8.7 we first compare the natural parameters $s(x)$, $s_{n,b}(x)$ centered at O and the parameters $t(x)$, $t_{n,b}(x)$ given by (8.4) for the curves γ and $\gamma_{n,b}$ as functions of x. We also compare the corresponding inverse functions $x = x(t)$ and $x = x_{n,b}(t)$ as functions of t, see Proposition 8.8 below. Afterwards we prove formula (8.10) using the above-mentioned comparison results and the results of Section 2. Then we deduce (8.11).

Proposition 8.8 *As $x \to 0$ (or equivalently, $t \to 0$), one has*

$$t(x) \simeq t_{n,b}(x) \simeq x, \quad x(t) \simeq t \simeq h'(x(t)), \qquad (8.12)$$

$$s_{n,b}(x) - s(x) = \frac{n}{n+1} bx^{n+1} + o(x^{n+1}), \qquad (8.13)$$

$$\kappa_{n,b}(x) - \kappa(x) = n(n-1)bx^{n-2} + o(x^{n-2}), \qquad (8.14)$$

$$t_{n,b}(x) - t(x) = \frac{2n}{3} b x^{n-1} + o(x^{n-1}), \tag{8.15}$$

$$x_{n,b}(t) - x(t) = -\frac{2n}{3} b t^{n-1} + o(t^{n-1}). \tag{8.16}$$

Here $o(x^k)$, $o(t^k)$ are functions that tend to zero together with their derivatives up to order k, as $x \to 0$ ($t \to 0$).

Proof Formulas (8.12) follow from (8.4), since $\kappa(O) = 1$. In the parametrizations $\gamma = \gamma(x)$, $\gamma_{n,b} = \gamma_{n,b}(x)$ one has

$$s(x) = \int_0^x ||\dot\gamma(u)||du, \ s_{n,b}(x) = \int_0^x ||\dot\gamma_{n,b}(u)||du. \tag{8.17}$$

We claim that

$$||\dot\gamma_{n,b}(x)|| - ||\dot\gamma(x)|| = nbx^n + o(x^n). \tag{8.18}$$

Indeed, let us identify the tangent spaces $T_{(x,y)}\Sigma$ at different points (x, y) by translations. One has $\dot\gamma(x), \dot\gamma_{n,b}(x) = (1, x + o(x))$,

$$v(x) := \dot\gamma_{n,b}(x) - \dot\gamma(x) = (0, nbx^{n-1} + o(x^{n-1})) : \tag{8.19}$$

$h'(x) \simeq x$, since $h''(0) = \kappa(O) = 1$, by assumption. The metric has trivial 1-jet at the base point O. Therefore, the difference of metric tensors at the $O(x^n)$-close points $\gamma(x)$, $\gamma_{n,b}(x)$, which are $O(x)$-close to O, is $O(x^{n+1})$. Hence, it suffices to prove (8.18) for the vector $\dot\gamma_{n,b}(x)$ being translated to the point $\gamma(x)$. The Euclidean angle α between the vectors $v(x)$ and $\dot\gamma(x)$ is $\frac{\pi}{2} - x + o(x)$, by (8.19). Therefore, the angle between them in the metric of the tangent plane $T_{\gamma(x)}\Sigma$ has the same asymptotics. Hence,

$$||\dot\gamma_{n,b}(x)||^2 = ||v(x) + \dot\gamma(x)||^2 = ||\dot\gamma(x)||^2 + 2nbx^n + o(x^n),$$

by Cosine Theorem and since $||v(x)||^2 = O(x^{2n-2}) = O(x^{n+1})$ ($n \geq 3$). The latter formula together with the obvious formula $||\dot\gamma(x)|| = 1 + O(x)$ imply (8.18), which together with (8.17) implies (8.13).

Let us prove (8.14). The Christoffel symbols at the $O(x^n)$-close points $\gamma(x)$ and $\gamma_{n,b}(x)$ are $O(x^n)$-close, as in the above discussion. Therefore, the difference $\kappa_{n,b}(x) - \kappa(x)$ is equal up to $O(x^n)$ to the same difference, where each curvature is calculated in the metric (Christoffel symbols) of the point $\gamma(x)$. The difference of the Christoffel parts of the curvatures, which are quadratic in the vectors $\frac{1}{||\dot\gamma(x)||}\dot\gamma(x)$, $\frac{1}{||\dot\gamma_{n,b}(x)||}\dot\gamma_{n,b}(x)$, is $O(||v(x)||) = O(x^{n-1})$, by (8.18). The difference of their second derivative terms is equal to $h''_{n,b}(x) - h''(x) + O(x^n) = n(n-1)bx^{n-2} + o(x^{n-2})$, by definition and (8.18). This together with the above discussion implies (8.14).

Let us prove (8.15). One has

$$t_{n,b}(x) - t(x) = \int_0^x (\kappa_{n,b}^{\frac{2}{3}}(u)||\dot{\gamma}_{n,b}(u)|| - \kappa^{\frac{2}{3}}(u)||\dot{\gamma}(u)||)du$$

$$\simeq \int_0^x (\kappa_{n,b}^{\frac{2}{3}}(u) - \kappa^{\frac{2}{3}}(u))||\dot{\gamma}(u)||du + O(x^n),$$

by definition and (8.18). The latter right-hand side is asymptotic to $\frac{2}{3}\int_0^x n(n-1)bu^{n-2}du = \frac{2n}{3}bx^{n-1}$, by (8.14) and since $\kappa(0) = 1$. This proves (8.15).

Formula (8.16) follows from (8.15). Proposition 8.8 is proved. □

In the proof of formula (8.10) we use the following notations:

$$P = P(t) := \gamma(t),\ Q = Q(t) := (x_{n,b}(t), h(x_{n,b}(t))) \in \gamma,\ A = A(t) := \gamma_{n,b}(t),$$

$$G(t) := \text{ the geodesic tangent to } \gamma \text{ at } P,\ G(0) = \text{ the } x-\text{axis},$$

$$C = C(t) := G(t) \cap G(0),\ V = V(t) := \{x = x_{n,b}(t)\},\ B = B(t) := G(t) \cap V,$$

$$G_{n,b}(t) := \text{ the geodesic tangent to } \gamma_{n,b} \text{ at } A,\ D = D(t) := G_{n,b}(t) \cap G(0),$$

see Fig. 7. By definition, $Q = Q(t) = \gamma \cap V$. In what follows for any two points $E, F \in \Sigma$ close to O the length of the geodesic segment connecting F to E will be denoted by $|EF|$. By definition,

$$L(0,t) = |OC| + |CP| - \lambda(O,P),\ L_{n,b}(0,t) = |OD| + |DA| - \lambda(O,A). \quad (8.20)$$

Recall that $\lambda(O,A)$, $\lambda(O,P)$ are lengths of arcs OA and OP of the curves $\gamma_{n,b}$ and γ respectively. Set

$$L_1 = L_1(t) := |OC| + |CB| - \lambda(O,Q),\ L_2 = L_2(t) := |OC| + |CA| - \lambda(O,A),$$

$$\Delta_1 = \Delta_1(t) := L_1(t) - L(0,t) = \lambda(Q,P) - |BP|,$$

$$\Delta_2 = \Delta_2(t) := L_2(t) - L_1(t),$$

$$\Delta_3 = \Delta_3(t) := L_{n,b}(0,t) - L_2(t) :$$

$$L_{n,b}(0,t) - L(0,t) = \Delta_1(t) + \Delta_2(t) + \Delta_3(t). \quad (8.21)$$

In what follows we find asymptotics of each Δ_j.

Proposition 8.9 *One has*

$$\Delta_1(t) = O(t^{2n-1}) = O(t^{n+2}) \text{ whenever } n \geq 3. \quad (8.22)$$

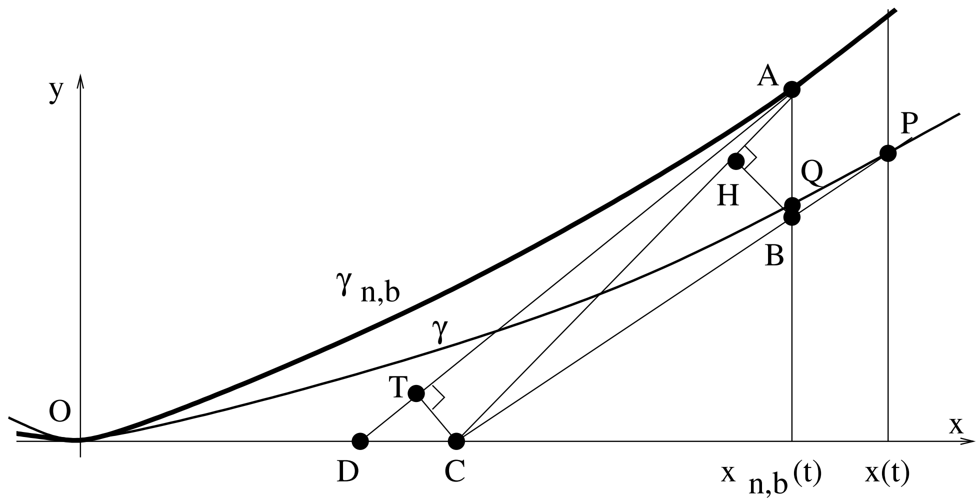

Figure 7: Auxiliary geodesics for calculation of the asymptotic of the difference $L_{n,b}(0,t) - L(0,t)$.

Proof In the curvilinear triangle QPB with $QP \subset \gamma$, PB being geodesic and QB vertical segment one has $|PB| = O(x_{n,b}(t) - x(t)) = O(t^{n-1})$, by (8.16). Its angle at Q is $\frac{\pi}{2} + O(t)$. Therefore, by (2.14),

$$\Delta_1 = \lambda(Q,P) - |PB| = O(|PB|^3) + O(t|PB|^2) = O(t^{3n-3}) + O(t^{2n-1}).$$

The latter right-hand side is $O(t^{2n-1}) = O(t^{n+2})$, since $n \geq 3$. □

Proposition 8.10 *One has*

$$\Delta_2(t) = \frac{b}{n+1}t^{n+1} + o(t^{n+1}). \tag{8.23}$$

Proof By definition,

$$\begin{aligned}\Delta_2(t) &= |OC| + |CA| - \lambda(O,A) - (|OC| + |CB| - \lambda(O,Q)) \\ &= (|CA| - |CB|) - (\lambda(O,A) - \lambda(O,Q)),\end{aligned} \tag{8.24}$$

$$\lambda(O,A) - \lambda(O,Q) = s_{n,b}(x_{n,b}(t)) - s(x_{n,b}(t)) = \frac{nbt^{n+1}}{n+1} + o(t^{n+1}), \tag{8.25}$$

by (8.13) and (8.12). To find the asymptotics of the difference $|CA| - |CB|$, let us consider the height denoted by BH of the geodesic triangle ABC, which splits it into two triangles ABH and CBH, see Fig. 7. We use the following asymptotic formula for lengths of their sides:

$$|AB| \simeq bt^n + o(t^n) \simeq |BH|, \tag{8.26}$$

$$|CB| \simeq |CP| \simeq |CA| \simeq \frac{t}{2} \tag{8.27}$$

$$|AH| \simeq bt^{n+1} + o(t^{n+1}) \simeq |AC| - |BC|. \tag{8.28}$$

Proof of (8.26). The Euclidean distance in the coordinates (x, y) between the points A and Q is $bx^n_{n,b}(t) + o(x^n_{n,b}(t)) = bt^n + o(t^n)$, by construction. Therefore, the distance between them in the metric g is asymptotic to the same quantity, since g is Euclidean on $T_O\Sigma$. The Euclidean distance between the points Q and B is of order $O((x(P)-x(B))^2) \simeq O(t^{2(n-1)}) = O(t^{n+1})$, by (8.16) and since $n \geq 3$: $2(n-1) \geq n+1$ for $n \geq 3$. The two latter statements together imply that $|AB| = bt^n + o(t^n)$; this is the first asymptotics in (8.26).

In the proof of the second asymptotics in (8.26) and in what follows we use the two next claims.

Claim 1. *The azimuths of the tangent vectors of the geodesic arcs CA, CP, DA at all their points are uniformly asymptotically equivalent to $t = t(P)$, as $t \to 0$.*

Proof Let us prove the above statement for the geodesic arc CP; the proof for the arcs CA and DA is analogous. The slope of the tangent vector to the curve γ at the point P is asymptotic to $x(P) = x(t) \simeq t$, and it is equal to the slope of the tangent vector of the geodesic CP at P. On the other hand, let us apply formula (2.5) to the geodesic arc $\alpha = CP$: its right-hand side is a quantity of order $O(t)$. The length of the arc CP is $O(t)$. Hence, the difference between the azimuths of tangent vectors at any two points of the geodesic arc CP is of order $O(t^2)$. This proves the claim. □

Claim 2. *The angle A of the geodesic triangle ABH is asymptotic to $\frac{\pi}{2} - t + O(t^2)$. Its angle B is asymptotic to $t + O(t^2)$, and $|AH| \simeq t|AB|$.*

Proof The first statement of the claim follows from Claim 1 applied to CA and the fact that the slopes of the tangent vectors to the geodesic arc BA are uniformly $O(|BA|) = O(t^n)$-close to $\frac{\pi}{2}$. This follows from the second formula in (2.13) and formula (2.5) applied to the geodesic arc BA. The second statement of the claim follows from the first one and (2.12). □

The first statement of Claim 2 implies that $|AB| \simeq |HB|$, which yields the second asymptotics in (8.26). Formula (8.26) is proved. □

Proof of (8.27). The asymptotics $|CP| \simeq \frac{x(P)}{2} \simeq \frac{t}{2}$ follows from Claim 1 and the fact that the height of the point P over the x-axis is asymptotic to $\frac{x^2(P)}{2} \simeq \frac{t^2}{2}$. The other asymptotics in (8.27) follow from the above one, formula (8.26) and the fact that $|BP| = O(t^{n-1})$ (follows from (8.16)). $\square$

Proof of (8.28). The geodesic triangle ABH has right angle at H. This together with Claim 2 and (8.26) implies the first asymptotic formula in (8.28). In the proof of the second formula in (8.28) we use the following claim.

Claim 3. *The angle* $\phi := \angle BCH$ *equals* $2bt^{n-1} + o(t^{n-1})$.

Proof The triangle BCH has right angle at H, $|BH| = bt^n + o(t^n)$, $|BC| \simeq \frac{t}{2}$, by (8.26) and (8.27). Hence, $\phi \simeq |BH|/\frac{t}{2} = 2bt^{n-1} + o(t^{n-1})$. $\square$

Now let us prove the second asymptotic formula in (8.28). One has

$$|BC| - |HC| \simeq \frac{1}{2}|BC|\phi^2,$$

by formula (2.12) applied to the family of triangles BCH. The right-hand side in the latter formula is $b^2t^{2n-1} + o(t^{2n-1}) = O(t^{n+2})$, by (8.27) and Claim 3 and since $2n - 1 \geq n + 2$ for $n \geq 3$. Thus,

$$|BC| - |HC| = O(t^{n+2}), \tag{8.29}$$

$$|AC| - |BC| = (|HC| - |BC|) + |AH| = |AH| + O(t^{n+2}) = bt^{n+1} + o(t^{n+1}),$$

by the first formula in (8.28) proved above. Formula (8.28) is proved. $\square$

Substituting formulas (8.25) and (8.28) to (8.24) yields

$$\Delta_2(t) = bt^{n+1} - \frac{n}{n+1}bt^{n+1} + o(t^{n+1}) = \frac{b}{n+1}t^{n+1} + o(t^{n+1}).$$

Proposition 8.10 is proved. $\square$

Proposition 8.11 *One has*

$$\Delta_3(t) = \frac{n-6}{12}bt^{n+1} + o(t^{n+1}). \tag{8.30}$$

Proof Recall that

$$\Delta_3(t) = L_{n,b}(0,t) - L_2(t) = |OD| + |DA| - \lambda(O,A) - (|OC| + |CA| - \lambda(O,A))$$

$$= |DA| - (DC + |CA|). \tag{8.31}$$

Here DC is the "oriented length" $DC := |OC| - |OD|$.

Let CT denote the height of the geodesic triangle DCA. To find an asymptotic formula for the right-hand side in (8.31), we first find asymptotics of the length of the height CT and the angle $\angle DAC$.

Claim 4. *Let $\alpha := \angle DAC$ denote the oriented angle between the geodesics AD and AC: it is said to be positive, if D lies between O and C, as at Fig.7. One has $\alpha = \frac{6-n}{3}bt^{n-1} + o(t^{n-1})$.*

Proof Consider the following tangent lines of the geodesic arcs AD, AC, BC, CP and the curve γ:

$$\ell_1 := T_A AD = T_A\gamma_{n,b}, \ \ell_2 := T_A AC, \ \ell_3 := T_B BC,$$

$$\ell_4 := T_Q\gamma, \ \ell_5 := T_P CP = T_P\gamma.$$

We orient all these lines "to the right". One has

$$\alpha \simeq \mathrm{az}(\ell_2) - \mathrm{az}(\ell_1), \tag{8.32}$$

by definition and since the Riemannian metric at the point A written in the normal coordinates (x, y) tends to the Euclidean one, as $t \to 0$. Let us find asymptotic formula for the above difference of azimuths by comparing azimuths of appropriate pairs of lines $\ell_1, \dots, \ell_5$. One has

$$\mathrm{az}(\ell_4) - \mathrm{az}(\ell_1) = -nbt^{n-1} + o(t^{n-1}),$$

since the above azimuth difference is asymptotically equivalent to the difference of the derivatives of the functions $h(x)$ and $h_{n,b}(x) = h(x)+bx^n+o(x^n)$ at the same point $x = x(B) \simeq t$: hence, to $-nbx^{n-1} + o(x^{n-1})$. One has

$$\mathrm{az}(\ell_5) - \mathrm{az}(\ell_4) \simeq h'(x(t)) - h'(x_{n,b}(t)) \simeq x(t) - x_{n,b}(t) = \frac{2n}{3}bt^{n-1} + o(t^{n-1}),$$

by (8.16) and since the function $h'(x) \simeq x$ has unit derivative at 0,

$$\mathrm{az}(\ell_3) - \mathrm{az}(\ell_5) = O(t(x(B) - x(P))) = O(t(x_{n,b}(t) - x(t))) = O(t^n),$$

by (2.5) and (8.16),

$$\mathrm{az}(\ell_2) - \mathrm{az}(\ell_3) \simeq \angle BCA = 2bt^{n-1} + o(t^{n-1}),$$

by (2.11), (2.5) and Claim 3. The right-hand sides of the above asymptotic formulas for azimuth differences are all of order t^{n-1}, except for one, which is $O(t^n)$. Summing up all of them yields the statement of Claim 4:

$$\alpha \simeq \mathrm{az}(\ell_2) - \mathrm{az}(\ell_1) = \frac{6-n}{3}bt^{n-1} + o(t^{n-1}).$$

□

Claim 5. *In the right triangle*[3] CDT $\angle TDC \simeq t$, $CT = \frac{6-n}{6} bt^n + o(t^n)$,

$$CD \simeq DT = \frac{6-n}{6} bt^{n-1} + o(t^{n-1}), \quad CD - DT = \frac{6-n}{12} bt^{n+1} + o(t^{n+1}). \tag{8.33}$$

Proof The angle asymptotics follows from Claim 1. The length asymptotics for the side CT is found via the adjacent right triangle ACT, from the formula $CT \simeq AC\angle CAT$ after substituting $\angle CAT = \frac{6-n}{3} bt^{n-1} + o(t^{n-1})$ (Claim 4) and $AC \simeq \frac{t}{2}$, see (8.27). This together with formula (2.12) applied to the right triangle CDT implies (8.33). □

Now let us prove formula (8.30). Recall that

$$\Delta_3(t) = |DA| - (DC + |CA|) = (DT - DC) + (|AT| - |AC|), \tag{8.34}$$

see (8.31). One has $DT - DC = \frac{n-6}{12} bt^{n+1} + o(t^{n+1})$, by (8.33), and $|AT| - |AC| = O(t^{n+2})$, analogously to formula (8.29). Substituting the two latter formulas to (8.34) yields to (8.30). Proposition 8.11 is proved. □

Proof of Lemma 8.7. Let us prove formula (8.10). Summing up formulas (8.22), (8.23), (8.30) and substituting their sum to (8.21) yields to (8.10):

$$L_{b,n}(t) - L(0,t) = \Delta_1(t) + \Delta_2(t) + \Delta_3(t) = \frac{b}{n+1} t^{n+1} + \frac{n-6}{12} bt^{n+1} + o(t^{n+1})$$

$$= (\frac{1}{n+1} + \frac{n-6}{12}) bt^{n+1} + o(t^{n+1}) = \frac{(n-2)(n-3)}{12(n+1)} bt^{n+1} + o(t^{n+1}).$$

Let us prove formula (8.11). Consider the points of the curves γ and $\gamma_{n,b}$ with $x < 0$. Taking them in the coordinates $(\hat{x}, y)$, $\hat{x} := -x$ results in multiplying the coefficient b by $(-1)^n$. This implies that for every $t > 0$

$$L_{n,b}(-t,0) - L(-t,0) = (-1)^n \frac{(n-2)(n-3)}{12(n+1)} bt^{n+1} + o(t^{n+1}). \tag{8.35}$$

Thus, for odd (even) n the main asymptotic terms in (8.35) and (8.10) are opposite (respectively, coincide). Hence, in the expression

$$\Lambda_{n,b}(t) - \Lambda(t) = (L_{n,b}(0,t) - L(0,t)) - (L_{n,b}(-t,0) - L(-t,0))$$

they are added (cancel out), and we get (8.11). Lemma 8.7 is proved. □

[3] We treat the lengths of sides of the triangle CDT as oriented lengths (without module sign): we take them with the sign equal to $\text{sign}(\alpha)$, where α is the same, as in Claim 4.

8.4 Dependence of functions $L(0,t)$ and $\Lambda(t)$ on the metric

Here we prove the following lemma, which shows that the $(n+1)$-jets of the quantities $L(0,t)$ and $\Lambda(t)$ depend only on the n-jet of the metric.

Lemma 8.12 *Let $n \geq 3$, Σ be a two-dimensional surface. Let $O \in \Sigma$, $\gamma \subset \Sigma$ be a germ of C^n-smooth curve at O. Let g and $\widetilde{g}$ be two C^n-smooth Riemannian metrics on Σ having the same n-jet at O: $\widetilde{g}(q) - g(q) = o(\text{dist}^n(q, O))$, as $q \to O$. Then the differences $L_{\widetilde{g}}(0,t) - L_g(0,t)$, $\Lambda_{\widetilde{g}}(t) - \Lambda_g(t)$ of quantities $L(0,t)$ and $\Lambda(t)$ defined by the metrics $\widetilde{g}$ and g are $o(t^{n+1})$.*

Proof Let s, $\widetilde{s}$, t, $\widetilde{t}$, κ, $\widetilde{\kappa}$ denote the natural and Lazutkin parameters centered at O, see (8.4), and the geodesic curvature of the curve γ defined by the metrics g and $\widetilde{g}$ respectively. One has $\kappa(O) = \widetilde{\kappa}(O)$, since $n \geq 3$. Let us rescale the metrics by the same constant factor so that $\kappa(O) = 1$. Fix coordinates (x, y) centered at O so that the x-axis is tangent to the curve γ and $||\frac{\partial}{\partial x}|| = 1$ at O. Consider x as a local parameter on γ. We consider the above quantities as functions of x; $s(0) = \widetilde{s}(0) = t(0) = \widetilde{t}(0) = 0$.

Let $x(t)$, $\widetilde{x}(t)$ denote the functions inverse to $t(x)$ and $\widetilde{t}(x)$ respectively. Let $\gamma(t)$, $\widetilde{\gamma}(t)$ denote the points of the curve γ with x-coordinates $x(t)$ and $\widetilde{x}(t)$ respectively. Let now $s(t)$ and $\widetilde{s}(t)$ denote the natural length parameters of the metrics g and $\widetilde{g}$, now considered as functions of the parameter t defined by the metric under question (g or $\widetilde{g}$).

Proposition 8.13 *One has $t \simeq x \simeq \widetilde{t} \simeq s \simeq \widetilde{s}$,*

$$\widetilde{s}(x) - s(x) = o(x^{n+1}), \ \widetilde{\kappa}(x) - \kappa(x) = o(x^{n-1}), \ \widetilde{t}(x) - t(x) = o(x^n), \tag{8.36}$$

$$\widetilde{x}(t) - x(t) = o(t^n), \ \text{dist}(\gamma(t), \widetilde{\gamma}(t)) = o(t^n), \tag{8.37}$$

$$\widetilde{s}(t) - s(t) = o(t^n), \ \widetilde{s}'(t) - s'(t) = o(t^{n-1}). \tag{8.38}$$

Proof The asymptotic equivalences follow from (8.4). The first formula in (8.36) is obvious. The second one holds by definition and since the Christoffel symbols of the two metrics differ by a quantity $o(x^{n-1})$. The third formula follows from the second one. Formula (8.37) follows from the third formula in (8.36). Formula (8.38) follows from (8.36) and (8.37). □

Fix a small value $t \in \mathbb{R}$, say, $t > 0$. Set

$$P = \gamma(t), \ A = \widetilde{\gamma}(t).$$

Let C ($\widetilde{C}$) be the point of intersection of the g-(respectively, $\widetilde{g}$-) geodesics $G(P)$, $G(O)$ tangent to γ at P and O. Let D ($\widetilde{D}$) be the analogous points of intersection of the geodesics tangent to γ at A and O. See Fig. 8a). The distance (arc length) between points E and F in a metric h will be denoted by $|EF|_h$ (respectively, $\lambda_h(E,F)$). One has

$$L_g(0,t) = |OC|_g + |CP|_g - \lambda_g(O,P), \ L_{\widetilde{g}}(0,t) = |O\widetilde{D}|_{\widetilde{g}} + |\widetilde{D}A|_{\widetilde{g}} - \lambda_{\widetilde{g}}(O,A),$$

by definition. Set

$$\Delta_1(t) := |OC|_g + |CP|_g - |OD|_g - |DA|_g - (\lambda_g(O,P) - \lambda_g(O,A)); \quad (8.39)$$

$$\Delta_2(t) := (|OD|_g - |OD|_{\widetilde{g}}) + (|DA|_g - |DA|_{\widetilde{g}}) - (\lambda_g(O,A) - \lambda_{\widetilde{g}}(O,A)); \quad (8.40)$$

$$\Delta_3(t) := (|OD|_{\widetilde{g}} - |O\widetilde{D}|_{\widetilde{g}}) + (|DA|_{\widetilde{g}} - |\widetilde{D}A|_{\widetilde{g}}). \quad (8.41)$$

One has

$$L_g(0,t) - L_{\widetilde{g}}(0,t) = \Delta_1 + \Delta_2 + \Delta_3. \quad (8.42)$$

Claim 1. *One has* $\Delta_1(t) = o(t^{n+1})$.

Proof Let us introduce the point B of intersection of the g-geodesic PC with the vertical line through A, see Fig. 8a): $x(B) = \widetilde{x}(t)$. One has

$$\Delta_1 = (|OC|_g + |CB|_g - |OD|_g - |DA|_g) + (BP_g - \widehat{\lambda}_g(A,P)). \quad (8.43)$$

Here BP_g and $\widehat{\lambda}_g(A,P)$ are the corresponding oriented lengths, which are positive if and only if A lies between O and P on the curve γ. Consider the curvilinear triangle APB formed by the arc AP of the curve γ, the g-geodesic PB and the vertical segment BA. Its sides AP and BA have g-length $o(t^n)$, by definition and (8.37). Its angle B is $\frac{\pi}{2} + O(\widetilde{x}(t)) = \frac{\pi}{2} + O(t)$, as in Claim 2 in Subsection 8.3. This together with (2.14) implies that the second bracket in (8.43) is $o(t^{n+1})$. Let us prove the same statement for the first bracket. It is equal to

$$DC_g + |CA|_g - |DA|_g + (|CB|_g - |CA|_g) = DC_g + |CA|_g - |DA|_g + o(t^{n+1}), \quad (8.44)$$

since $||CB|_g - |CA|_g| \leq |BA| = O((x(P) - x(B))^2) = o(t^{n+1})$. Here DC_g is the oriented length $|OC|_g - |OD|_g$. One has

$$DC_g + |CA|_g - |DA|_g = o(t^{n+1}). \quad (8.45)$$

Indeed, consider the height CT of the triangle ADC, which splits it into two triangles. One has $\angle CAD = O(x(A) - x(P)) = o(t^n)$, as in the proof of Claim 4 in the previous subsection. This together with right triangle arguments using (2.12) analogous to those from the proof of Claim 5 (Subsection

8.3) implies (8.45). Substituting (8.45) to (8.44) and then substituting everything to (8.43) yields $\Delta_1(t) = o(t^{n+1})$. Claim 1 is proved. □

Claim 2. *One has* $\Delta_2(t) = o(t^{n+1})$.

Proof All the points in (8.40) are $O(t)$-close to O. The g- and $\widetilde{g}$-distances between any two points (which will be denoted by E and F) differ by a quantity $o(t^{n+1})$. Indeed, the $\widetilde{g}$-length of the g-geodesic segment EF differs from its g-length by $o(t^{n+1})$, since the metrics differ by $o(t^n)$. The distance $|EF|_{\widetilde{g}}$ is no greater than the latter $\widetilde{g}$-length, and hence, no greater than $|EF|_g + o(t^{n+1})$. Applying the same arguments to interchanged metrics yields that the above distances differ by $o(t^{n+1})$. Similarly, $\lambda_g(O,A) - \lambda_{\widetilde{g}}(O,A) = o(t^{n+1})$. This proves the claim. □

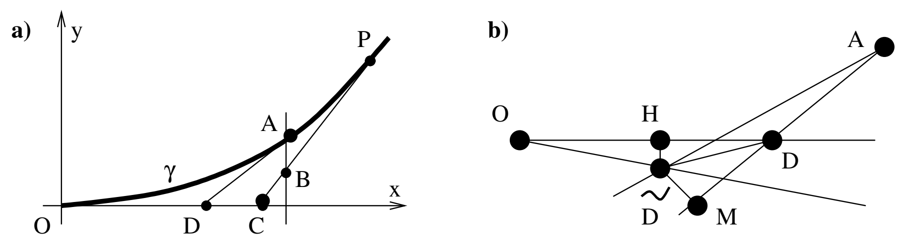

Figure 8: The curve γ, points P, A, C, D, B (fig. a)). The points $\widetilde{D}$, H, M; case 2) (fig. b)).

Let H and M denote the points in the $\widetilde{g}$-geodesics OD and DA respectively that are $\widetilde{g}$-closest to $\widetilde{D}$: $\widetilde{D}H \perp OD$; $\widetilde{D}M \perp DA$; see Fig. 8b).

Claim 3. *One has* $|\widetilde{D}H|_{\widetilde{g}} = o(t^{n+1})$, $|\widetilde{D}M|_{\widetilde{g}} = o(t^{n+1})$.

Proof Let EF_h denote the geodesic EF in the metric h. The $\widetilde{g}$-geodesic $O\widetilde{D}_{\widetilde{g}}$ is tangent to the g-geodesic OD_g at O, and the metrics g and $\widetilde{g}$ have the same n-jet at O. Therefore, their Christoffel symbols have the same $(n-1)$-jet, and hence, their difference is asymptotically dominated by the g-distance to O in power $n-1$. This together with the equation of geodesics implies that the azimuths of the unit vectors tangent to both latter geodesics (as functions of the natural parameter based at O) differ by a quantity asymptotically dominated by n-th power to the g-distance to O. Therefore, the distance (in any metric) between points of the geodesics corresponding to the same natural parameter value is asymptotically dominated by the above distance in power $n+1$. Hence, $\text{dist}(D, O\widetilde{D}_{\widetilde{g}}) = o(t^{n+1})$. Therefore, the $\widetilde{g}$-geodesic $O\widetilde{D}$ should be turned at O by an angle of order $o(t^n)$ in order to hit the point D, by the above statement and since $|OD|_{\widetilde{g}} \simeq \frac{t}{2}$, as in (8.27).

This implies that the points in $O\widetilde{D}_{\widetilde{g}}$ lying on a distance of order $O(t)$ from O are $o(t^{n+1})$-close to the geodesic $OD_{\widetilde{g}}$. This proves the statement of the claim for the distance $|\widetilde{D}H|_{\widetilde{g}}$. The proof for $|\widetilde{D}M|_{\widetilde{g}}$ is analogous. □

Claim 4. *One has* $\Delta_3(t) = o(t^{n+1})$.

Proof All the distances below are measured in the metric $\widetilde{g}$. One has

$$|O\widetilde{D}| - |OH| = O(\frac{|\widetilde{D}H|^2}{|O\widetilde{D}|}) = o(t^{2n+1}) = o(t^{n+1}), \tag{8.46}$$

$$|A\widetilde{D}| - |AM| = O(\frac{|\widetilde{D}M|^2}{|A\widetilde{D}|}) = o(t^{2n+1}) = o(t^{n+1}), \tag{8.47}$$

by (2.12) (applied to the right $\widetilde{g}$-triangles $O\widetilde{D}H$ and $A\widetilde{D}M$) and Claim 3,

$$|OD| - |OH| = \pm|DH|, \ |AD| - |AM| = \pm|DM|, \tag{8.48}$$

see the cases of signs (which do not necessarily coincide) below. Taking sum of equalities (8.48) and its difference with (8.46), (8.47) yields

$$\Delta_3(t) = (\pm)|DH| \pm |DM| + o(t^{n+1}). \tag{8.49}$$

Case 1). In the right triangle $D\widetilde{D}H$ the angle D is bounded from below (along some sequence of parameter values t converging to 0). Then the same statement holds in the right triangle $\widetilde{D}MD$, since the angle between the geodesics DA and OD tends to 0 as $O(t)$. This implies that $|DH| = O(|\widetilde{D}H|) = o(t^{n+1})$, and $|DM| = O(|\widetilde{D}M|) = o(t^{n+1})$, by Claim 3. This together with (8.49) implies Claim 4 (along the above sequence)

Case 2). In the right tringle $D\widetilde{D}H$ the angle D tends to zero along some sequence of parameter values t converging to 0, see Fig. 8b). Then the same holds in $\widetilde{D}MD$. In this case the signs in (8.49) are different. For example, if H lies between O and D, then the angle $\angle\widetilde{D}DA$ is obtuse and D lies between M and A. The opposite case is treated analogously. Let us denote

$$\alpha(t) := \angle\widetilde{D}DH, \ \beta(t) := \angle\widetilde{D}DM; \ \alpha(t), \beta(t) \to 0 \text{ as } t \to 0.$$

Applying (2.12) to the above right triangles together with Claim 3 yields

$$|\widetilde{D}D| - |DH| = O(\alpha(t)|\widetilde{D}H|) = o(t^{n+1}),$$

$$|\widetilde{D}D| - |DM| = O(\beta(t)|\widetilde{D}M|) = o(t^{n+1}).$$

Hence, $|DH| - |DM| = o(t^{n+1})$. This together with (8.49) implies the asymptotics of Claim 4 (along the above sequence). Claim 4 is proved. □

Claims 1, 2 and 4 together with (8.42) imply the statement of Lemma 8.12 on the function L. In its turn, it implies the same statement on Λ. □

8.5 Taylor coefficients of $\Lambda(t)$: analytic dependence on jets

Lemma 8.14 *Let (x, y) be coordinates on a neighborhood of a point $O \in \Sigma$. Let a metric on Σ be C^n-smooth, $n \geq 3$, and let γ be a germ of C^n-smooth curve on Σ at O. Then the corresponding functions $L(0,t)$, $\Lambda(t)$ are $O(t^3)$. They admit asymptotic Taylor expansions up to t^{n+1}. Their coefficients at t^{n+1} are analytic functions of the pure n-jets of the metric and the curve γ.*

Proof The asymptotics $L(0,t), \Lambda(t) = O(t^3)$ follows from Theorem 1.16.

Case 1): the curve γ and the metric are analytic. Consider the metric and the curve with variable Taylor coefficients of orders up to n; the other, higher Taylor coefficients are fixed. Consider $L(0,t)$ and $\Lambda(t)$ as functions in t and in the latter variable Taylor coefficients. They are analytic on the product of a small disk centered at 0 with coordinate t and a domain in the space of collections of the above Taylor coefficients. In more detail, complexifying everything we get that $L(0,t)$ has a well-defined holomorphic extension to complex domain. (The complexified lengths of segments in the definition of the function $L(0,t)$ become integrals of appropriate holomorphic forms along paths.) Well-definedness follows from the fact that through each point C in a complex neighborhood of the real curve γ there are two complex geodesics tangent to its complexification. This follows by quadraticity of tangencies (non-vanishing of geodesic curvature) and Implicit Function Theorem. Analytic extendability to the locus $\{t = 0\}$ follows from the Erasing Singularity Theorem on bounded functions holomorphic on complement to a hypersurface. Therefore, both functions admit a Taylor series in t with coefficients being analytic functions in the above Taylor coefficients.

Case 2) of general C^n-smooth metric g and curve γ. Consider other, analytic metric $\widetilde{g}$ and curve $\widetilde{\gamma}$ representing their n-jets. The functions $\widetilde{L}(0,t)$ and $\widetilde{\Lambda}(t)$ defined by them are analytic and coincide with the functions $L(0,t)$ and $\Lambda(t)$ corresponding to g and γ up to $o(t^{n+1})$. Indeed, if the C^n-smooth function $y = f(x)$ representing γ as a graph changes in the same n-jet, i.e., by a quantity $o(x^n)$, then $L(0,t)$, $\Lambda(t)$ change by a quantity of order $o(t^{n+1})$. This follows from Lemma 8.7 applied to $b = 0$. A similar statement holds for change of metric, by Lemma 8.12. This together with the discussion in Case 1) implies that $L(0,t)$ and $\Lambda(t)$ have asymptotic Taylor expansions of order up to t^{n+1} coinciding with those of $\widetilde{L}(0,t)$ and $\widetilde{\Lambda}(t)$, and hence, having coefficients being analytic functions of the n-jets of g and γ. They depend only on pure n-jets, since applying a translation of both the curve and the metric leaves $L(0,t)$ and $\Lambda(t)$ invariant. Lemma 8.14 is proved. □

8.6 Proof of Lemma 8.5

Let Σ be a two-dimensional surface equipped with a C^{n+1}-smooth Riemannian metric g. Let $V \subset \Sigma$ be a domain equipped with a C^{n+1}-smooth chart (x, y) (not necessarily normal). Consider a C^n-smooth germ of curve γ at a point $O \in V$ with positive geodesic curvature that is a graph of C^n-smooth function $\{y = h(x)\}$; the tangent line $T_O\gamma$ is not necessarily horizontal. The corresponding function $\Lambda(t)$ admits an asymptotic Taylor expansion

$$\Lambda(t) = \sum_{k=3}^{n+1} \widehat{\Lambda}_k t^k + o(t^{n+1}).$$

Its coefficients are analytic functions of the pure n-jets of the metric and γ at O (Lemma 8.14). Therefore, without loss of generality we consider that O is the origin in the coordinates (x, y), applying a translation. Then

$$\gamma = \{y = h(x)\}, \quad h(x) = b_1 x + \frac{b_2}{2}x^2 + \frac{1}{3!}b_3 x^3 + \cdots + \frac{1}{n!}b_n x^n + o(x^n).$$

By definition, the coordinates of the pure jet $j_O^n\gamma$ are $(b_1, \dots, b_n)$.

We already know that $\widehat{\Lambda}_{n+1}$ is an affine function in b_n, which follows from Lemma 8.7, see (8.11). To obtain a precise formula for its coefficient at b_n, we use the following proposition.

Proposition 8.15 *Let $n \geq 3$, Σ, O, (x, y), $h(x)$ be as above. Consider a family of tangent germs of C^n-smooth curves $\gamma_{n,b} = \{y = h_{n,b}(x)\}$ at O, $h_{n,b}(x) = h(x) + bx^n + o(x^n)$; $h_{n,0} := h$, $\gamma_{n,0} := \gamma$. Let $w \in T_O\Sigma$ denote the orthogonal projection of the vector $\frac{\partial}{\partial y}$ to $(T_O\gamma)^\perp$. Let $u = (1, b_1) \in T_O\gamma$: the tangent vector to γ with unit x-component. Let $\kappa(O)$ denote the geodesic curvature of the curve γ at O, which coincides with that of $\gamma_{n,b}$. Let $(\widetilde{x}, \widetilde{y})$ be normal coordinates centered at O such that the $\widetilde{x}$-axis is tangent to γ. Set*

$$\hat{x} := \kappa(O)\widetilde{x}, \ \hat{y} := \kappa(O)\widetilde{y}.$$

In the coordinates $(\hat{x}, \hat{y})$ the family of curves $\gamma_{n,b}$ is the family of graphs of C^n-functions $\{\hat{y} = \hat{h}_{n,b}(\hat{x})\}$, set $\hat{h}_{n,0} := \hat{h}$, such that $\hat{h}(\hat{x}) = \frac{\hat{x}^2}{2} + O(\hat{x}^3)$,

$$\hat{h}_{n,b}(\hat{x}) = \hat{h}(\hat{x}) + \mu_n b \hat{x}^n + o(\hat{x}^n), \ \mu_n = ||w|| ||u||^{-n} \kappa^{1-n}(O). \tag{8.50}$$

Proof Note that the normal coordinates $(\widetilde{x}, \widetilde{y})$ are C^n-smooth, and the metric g is C^n-smooth there, since $g \in C^{n+1}$. Hence, the curves under question are also C^n-smooth in these coordinates. Fix a point $A = (\widetilde{x}, 0)$ on the $\widetilde{x}$-axis. Let ℓ denote the geodesic through A orthogonal to the $\widetilde{x}$-axis. We have to calculate the gap (i.e., distance) $\widetilde{\Delta}(\widetilde{x})$ between the intersection points of the geodesic ℓ with the curves $\gamma_{n,b}$ and γ. Let $\Delta(\widetilde{x})$ denote the gap between the points of the intersection of the curves with the vertical line $\{x = x(A)\}$. Their ratio $\widetilde{\Delta}(\widetilde{x})/\Delta(\widetilde{x})$ tends to the cosine of the angle between the vector $\frac{\partial}{\partial y} \in T_O\Sigma$ and the line $(T_O\gamma)^{\perp}$, as $\widetilde{x} \to 0$. One has $\Delta(\widetilde{x}) = ||\frac{\partial}{\partial y}||bx^n + o(x^n)$. Hence, by definition,

$$\widetilde{\Delta}(\widetilde{x}) = ||w||bx^n + o(x^n). \tag{8.51}$$

One has $dx = \alpha d\widetilde{x} + \beta d\widetilde{y}$ on $T_O\Sigma$, $\alpha = dx(\frac{\partial}{\partial \widetilde{x}}) = ||u||^{-1}$, by definition; $x = \alpha\widetilde{x} + \beta\widetilde{y} + O(|\widetilde{x}|^2 + |\widetilde{y}|^2)$. One has $\widetilde{y} = \frac{\kappa(O)}{2}\widetilde{x}^2 = O(\widetilde{x}^2)$ along each curve $\gamma_{n,b}$, by (2.1). This together with (8.51) implies that

$$\widetilde{\Delta}(\widetilde{x}) = ||w||||u||^{-n}b\widetilde{x}^n + o(\widetilde{x}^n). \tag{8.52}$$

Hence, in the coordinates $(\widetilde{x}, \widetilde{y})$

$$\gamma_{n,b} = \{\widetilde{y} = \widetilde{h}_{n,b}(\widetilde{x})\}, \ \ \widetilde{h}_{n,b}(\widetilde{x}) = \widetilde{h}_{n,0}(\widetilde{x}) + ||w||||u||^{-n}b\widetilde{x}^n + o(\widetilde{x}^n).$$

Now rescaling to the coordinates $(\hat{x}, \hat{y})$ yields that $\gamma_{n,b}$ is a family of graphs of functions $\hat{h}_{n,b}(\hat{x})$ satisfying (8.50). The proposition is proved. □

Proposition 8.16 *Consider the above family of curves $\gamma_{n,b}$ and the corresponding functions $\Lambda^{n,b}(t)$, set $\Lambda^{n,0} := \Lambda$. For every $n \geq 3$ one has*

$$\widehat{\Lambda}^{n,b}_{n+1} = \widehat{\Lambda}_{n+1} + \nu_n b, \ \nu_n := \begin{cases} \frac{(n-2)(n-3)}{6(n+1)}||w||(||u||\kappa(O))^{-n}, \ \textit{for odd } n, \\ 0, \ \textit{for even } n. \end{cases} \tag{8.53}$$

Proof The coordinates $(\hat{x}, \hat{y})$ are normal coordinates for the rescaled metric $\hat{g} := \kappa(O)g$. The common geodesic curvature at O of the curves $\gamma_{n,b}$ in the metric $\hat{g}$ is equal to 1, by Remark 8.4. Therefore, for the metric $\hat{g}$ one has $\widehat{\Lambda}^{n,b}_{n+1} - \widehat{\Lambda}_{n+1} = \frac{(n-2)(n-3)}{6(n+1)}\mu_n b$ for odd n, and the latter difference vanishes for even n, by Lemma 8.7 and (8.50). Rescaling the metric back to g by the factor $\kappa^{-1}(O)$ rescales the functions $\Lambda_{n,b}$ and their Taylor coefficients by the same factor (Remark 8.4). This implies (8.53). □

Proposition 8.17 *Let $n \geq 3$, γ be a germ of C^n-smooth curve at a point $O \in \Sigma$ lying in a chart with coordinates (x, y). Let γ be a graph $\{y = h(x)\}$.*

Let $b_1, \dots, b_n$ denote the coordinates of the pure n-jet $j^n_O h$. Let $w, u \in T_O\Sigma$ be the vectors from Proposition 8.15. Then the Taylor coefficient $\widehat{\Lambda}_{n+1}$ of the corresponding function $\Lambda(t)$ is equal to

$$\widehat{\Lambda}_{n+1} = \sigma_n b_n - P_n, \tag{8.54}$$

$$\sigma_n = \frac{(n-2)(n-3)}{6(n+1)!}||w||\,(||u||\kappa(O))^{-n} \text{ for odd } n, \tag{8.55}$$

$\sigma_n = 0$ for even n, where P_n is an analytic function in $b_1, \dots, b_{n-1}$ and in the pure n-jet of the metric at O.

Proof The fact that $\widehat{\Lambda}_{n+1}$ depends on b_n as an affine function with factor σ_n at b_n follows from definition and Proposition 8.16; the b from Proposition 8.16 is $\frac{1}{n!}$ times the difference of the b_n-coordinates of jets of functions $h_{n,b}(x)$ and $h(x)$. The function P_n is thus independent on b_n and hence, has the required type, by Lemma 8.14. □

Proof of Lemma 8.5 and its addendum. All the statements of Lemma 8.5 and its addendum follow from the above proposition, except for the following points discussed below. Note that σ_n depends only on the pure 2-jet of the curve γ and the pure 1-jet of the metric, by definition. The function P_n is an analytic function of the pure n-jet of the metric and the pure $(n-1)$-jet of the curve γ. Let us treat it as a function of a point and a pure $(n-1)$-jet of curve. We have to prove its smoothness. To this end, we use the assumption that the metric is C^{n+1}-smooth. (This is the main place in the proof where we use this assumption.) Then its pure n-jet is a C^1-smooth function of a point. This together with the above analyticity statement proves C^1-smoothness and finishes the proof of Lemma 8.5. □

8.7 Proof of Theorems 8.3 and 1.19

Proof of Theorem 8.3. Let $O \in \Sigma$. Let (x, y) be local coordinates on a neighborhood $V = V(O) \subset \Sigma$. Let $\mathcal{J}^4_y(V)$ denote the space of 4-jets of curves, as in Lemma 8.5, which are graphs of functions $\{y = h(x)\}$. Let $J_2 = (x, b_0, b_1, b_2)$, $\sigma_5 = \sigma_5(J_2)$ and $h_5 := P_5(J_2; b_3, b_4)$ be the same, as in (8.8). Consider the field of kernels K_4 of the following 1-form ν_4 on $\mathcal{J}^4_y(V)$:

$$\nu_4 := db_4 - \sigma_5^{-1} h_5(x, b_0, b_1, b_2, b_3, b_4)dx;\; K_4 := \mathrm{Ker}(\nu_4).$$

Let $\mathcal{D}_4$ denote the canonical distribution on $\mathcal{J}^4_y(V) \simeq \mathcal{F}^4_y(V)$, see (8.2):

$$\mathcal{D}_4 = \mathrm{Ker}(db_0 - b_1dx, db_1 - b_2dx, db_2 - b_3dx, db_3 - b_4dx).$$

Set

$$\mathcal{P} := K_4 \cap \mathcal{D}_4. \tag{8.56}$$

This is a line field, since the above intersections are obviously transverse and $\dim(\mathcal{D}_4) = 2$. It is C^1-smooth, since so are σ_2 and h_5 (Lemma 8.5). Let γ be an arbitrary C^5-smooth germ of curve γ based at a point $A \in V$ such that the line $T_A\gamma$ is not parallel to the y-axis. Let γ have the string Poritsky property. Then $\Lambda(t) \equiv 0$, hence, $\widehat{\Lambda}_6 = 0$, thus,

$$\sigma_5(J_2)b_5 - h_5(J_2; b_3, b_4) = 0, \tag{8.57}$$

by (8.8). On the other hand, the 5-jet extension of the curve γ is tangent to the canonical distribution $\mathcal{D}_5$, and hence, to the hyperplane field $\{db_4 = b_5dx\}$. This together with (8.57) implies that its 4-jet extension is tangent to the hyperplane field $\{db_4 = \frac{h_5}{\sigma_5}dx\}$. Thus, it is tangent to the kernel field K_4, and hence, to $\mathcal{P} = K_4 \cap \mathcal{D}_4$. This proves Theorem 8.3. □

Proof of Theorem 1.19. Two germs of curves with the string Poritsky property and the same 4-jet correspond to the same point in $\mathcal{J}^4$. Therefore, their 4-jet extensions coincide with one and the same phase curve of the line field $\mathcal{P}$, by Theorem 8.3 and the Uniqueness Theorem for ordinary differential equations. Thus, the germs coincide. This proves Theorem 1.19. □

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