ALGEBRAS DERIVED EQUIVALENT TO BRAUER GRAPH ALGEBRAS AND DERIVED INVARIANTS OF BRAUER GRAPH ALGEBRAS REVISITED

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Abstract. In this paper the class of Brauer graph algebras is proved to be closed under derived equivalence. For that we use the rank of the maximal torus of the identity component of the group of outer automorphisms $Out^0(A)$ of a symmetric stably biserial algebra A.

1. Introduction

Brauer graph algebras or equivalently symmetric special biserial algebras, originating from modular representation theory, are studied quite extensively. They appear in classifications of various classes of algebras including blocks with cyclic or dihedral defect groups [14, 15], blocks of Hecke algebras [9, 10] and others. Brauer tree algebras, the subclass of Brauer graph algebras of finite representation type, contain all blocks with cyclic defect group.

In this paper we make a final step in the proof of the fact that Brauer graph algebras are closed under derived equivalence. This fact was believed to be true, based on the work of Pogorzały [27]. In [11], counterexamples to some of the statements of [27] were given. In [8], we revised the proof of the fact that the only algebras possibly stably (and thus derived) equivalent to self-injective special biserial algebras (a class containing Brauer graph algebras) are self-injective stably biserial (see Section 2). As a finite-dimensional algebra derived equivalent to a symmetric algebra is itself symmetric [31] we can restrict our attention to symmetric stably biserial algebras. It turns out that in odd characteristic the class of symmetric stably biserial algebras coincides with the class of Brauer graph algebras, whereas in characteristic 2 this is not the case [8].

The general strategy of the proof of the fact that Brauer graph algebras are closed under derived equivalence follows the classical proof for Brauer tree algebras. The fact that Brauer tree algebras are closed under stable equivalence was proved in [19]. Since by [30] derived equivalence for self-injective algebras implies stable equivalence, it follows that this class is closed under derived equivalence as well. It turns out that the proof for the whole class of Brauer graph algebras is much more involved and requires an extra step in characteristic 2, which is provided in this paper.

A symmetric stably biserial algebra can be given by the same combinatorial data as the Brauer graph algebra, that is a graph on a surface and a number attached to each vertex of this graph, called the multiplicity. Additionally, one needs to fix a distinguished class of loops in the quiver, satisfying certain conditions, which we call deformed loops (see Section 2). In case the number of deformed loops is 0 we recover the usual definition of a Brauer graph algebra. Since for local algebras derived equivalence implies Morita equivalence [34], we will sometimes assume that A has at least 2 simple modules. For further reference, let us denote by $V(\Gamma), E(\Gamma)$ and $F(\Gamma)$ the vertices, edges and faces of the Brauer graph Γ.

The main technique, used in this paper, is the computation of the rank of the maximal torus $D(A)$ of the identity component of the group of outer automorphisms $Out^0(A)$ for a symmetric stably biserial algebra A. The group $Out^0(A)$ is a derived invariant [23, 33] used quite seldom. The only previous systematic application we know of is the proof of the fact that the number of arrows in the quiver of a gentle algebra is a derived invariant [1].

Theorem 1.1. *Let $\mathbf{k}$ be an algebraically closed field. Let A be a symmetric stably biserial algebra over $\mathbf{k}$ ($char(\mathbf{k}) = 2$) or a symmetric special biserial algebra over $\mathbf{k}$ ($char(\mathbf{k}) \neq 2$) with at least two non-isomorphic simple modules, which is not a caterpillar (see Section 3). Let Γ be the Brauer graph of A and let d be the number of deformed loops in A ($d = 0$ for the symmetric special biserial case). The rank of $D(A)$ is $|E(\Gamma)| - |V(\Gamma)| - d + 2$.*

In section 3, we investigate basic properties of symmetric stably biserial algebras. In section 4, we revisit the known derived invariants for Brauer graph algebras [3, 4, 5, 6] in arbitrary characteristic, providing simpler proofs of their invariance for the larger class of symmetric stably biserial algebras and correcting some inaccuracies in the existing literature.

Theorem 1.2. *Let A be a symmetric stably biserial algebra with a Brauer graph Γ and with at least two simple modules. The following are invariants of A under a derived equivalence of symmetric stably biserial algebras: $|V(\Gamma)|$, $|E(\Gamma)|$, $|F(\Gamma)|$, the multiset of perimeters of faces, the multiset of multiplicities and bipartivity of Γ.*

As a corollary of Theorems 1.1 and 1.2 and the fact that Brauer graph algebras can be derived equivalent only to symmetric stably biserial algebras [8] we obtain the following:

Corollary 1.3. *The class of Brauer graph algebras is closed under derived equivalence. Namely, if A is an algebra Morita equivalent to a Brauer graph algebra and B is an algebra such that $D^b(A) \simeq D^b(B)$, then B is Morita equivalent to a Brauer graph algebra.*

In forthcoming work [21], among other results, W. Gnedin independently obtains Corollary 1.3 in characteristic 2 and for bipartite Brauer graphs by different methods. Note that the list of invariants from Theorem 1.2 is crucial to the forthcoming joint work [26] of S. Opper and the second named author, where a complete classification of Brauer graph algebras up to derived equivalence will be provided.

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2. Preliminaries

Throughout this paper, A is a basic, connected, finite dimensional algebra over an algebraically closed field $\mathbf{k}$ and mod-A is the category of right A-modules. The stable category of mod-A will be denoted by $\underline{\text{mod}}$-A and $\Omega : \underline{\text{mod}}\text{-}A \to \underline{\text{mod}}\text{-}A$ will denote the syzygy. The bounded derived category of the category mod-A will be denoted by $D^b(A)$. A quiver Q consists of a set of vertices Q_0 and a set of arrows Q_1. The map $s : Q_1 \to Q_0$ will denote the start of an arrow, the map $e : Q_1 \to Q_0$ will denote the end of an arrow. In the path algebra $\mathbf{k}Q$ the multiplication of arrows α and β is $\alpha\beta \neq 0$, provided $e(\alpha) = s(\beta)$, by $J(A)$ we will denote the Jacobson radical of the algebra A, which is the ideal generated by the arrows of the quiver Q in case $A \simeq \mathbf{k}Q/I$. By $K_0(\mathcal{C})$ we are going to denote the Grothendieck group of an Abelian or a triangulated category $\mathcal{C}$.

In this paper we are going to be interested in symmetric special biserial and symmetric stably biserial algebras.

Definition 2.1. *Let Q be a quiver, I an admissible ideal of $\mathbf{k}Q$. A self-injective algebra $A = \mathbf{k}Q/I$ is called* ***special biserial*** *if the following conditions are satisfied.*

(1) For each vertex $v \in Q$, the number of outgoing arrows and the number of incoming arrows are less than or equal to 2.
(2) For each arrow $\alpha \in Q$, there is at most one arrow $\beta \in Q$ such that $\alpha\beta \notin I$.
(3) For each arrow $\alpha \in Q$, there is at most one arrow $\beta \in Q$ such that $\beta\alpha \notin I$.

Definition 2.2. *Let Q be a quiver, I an admissible ideal of $\mathbf{k}Q$. A self-injective algebra $A = \mathbf{k}Q/I$ is called* ***stably biserial*** *if the following conditions are satisfied.*

(1) For each vertex $v \in Q$, the number of outgoing arrows and the number of incoming arrows are less than or equal to 2.
(2) For each arrow $\alpha \in Q$, there is at most one arrow $\beta \in Q$ such that $\alpha\beta \notin \alpha\mathrm{rad}(A)\beta + \mathrm{soc}(A)$.
(3) For each arrow $\alpha \in Q$, there is at most one arrow $\beta \in Q$ such that $\beta\alpha \notin \beta\mathrm{rad}(A)\alpha + \mathrm{soc}(A)$.

The following description of stably biserial algebras was provided in [11]:

Proposition 2.3 (Proposition 7.5 [11]). *If $A = \mathbf{k}Q/I$ is stably biserial then we can choose the presentation of A in such a way that the following conditions hold.*

(1) If $\alpha\beta \neq 0$, $\alpha\gamma \neq 0$, $\beta \neq \gamma$, for arrows α, β, γ then either $\alpha\beta \in \mathrm{soc}(A)$ or $\alpha\gamma \in \mathrm{soc}(A)$.
(2) If $\beta\alpha \neq 0$, $\gamma\alpha \neq 0$, $\beta \neq \gamma$, for arrows α, β, γ then either $\beta\alpha \in \mathrm{soc}(A)$ or $\gamma\alpha \in \mathrm{soc}(A)$.

Self-injective special biserial algebras are a subclass of stably biserial algebras. We will call an algebra **symmetric special biserial** (SSB for short) or **symmetric stably biserial**, if in addition to being special biserial or stably biserial it is symmetric.

Consider the following data:

(1) A quiver Q such that every vertex has two incoming and two outgoing arrows.
(2) A permutation π on Q_1 with $e(\alpha) = s(\pi(\alpha))$ for all $\alpha \in Q_1$.
(3) A function $m : C(\pi) \to \mathbb{N}$, where $C(\pi)$ is the set of cycles of π. We will denote by $C(\alpha) := \alpha\pi(\alpha)\pi^2(\alpha)\dots\pi^{|C(\alpha)|-1}$ the cycle, containing $\alpha \in Q_1$ and call $m(C(\alpha))$ the **multiplicity** of the cycle $C(\alpha)$.
(4) A set of loops $\mathcal{L} = \{\alpha_{i_1}, \dots, \alpha_{i_d}\}$, such that $\pi(\alpha_{i_j}) \neq \alpha_{i_j}$ and a set of elements $\{t_{\alpha_{i_1}}, \dots, t_{\alpha_{i_d}}\}$, with $t_{\alpha_{i_j}} \in \mathbf{k}^*$.

In [8] the following description of symmetric stably biserial algebras in terms of generators and relations was obtained:

Theorem 2.4. *Any symmetric stably biserial algebras can be given as $A = \mathbf{k}Q/I$, where the ideal of relations is generated by*

(1) $\alpha\beta$ for all $\alpha, \beta \in Q_1$, $\beta \neq \pi(\alpha)$, α does not belong to the set of loops $\mathcal{L}$,
(2) $\Big(\alpha\pi(\alpha)\pi^2(\alpha)\dots\pi^{|C(\alpha)|-1}(\alpha)\Big)^{m(C(\alpha))} - \Big(\beta\pi(\beta)\pi^2(\beta)\dots\pi^{|C(\beta)|-1}(\beta)\Big)^{m(C(\beta))}$ for all $\alpha, \beta \in Q_1$ with $s(\alpha) = s(\beta)$,
(3) $\alpha^2 - t_\alpha\Big(\alpha\pi(\alpha)\pi^2(\alpha)\dots\pi^{|C(\alpha)|-1}(\alpha)\Big)^{m(C(\alpha))}$ for each $\alpha \in \mathcal{L}$,

(4) $\left(\alpha\pi(\alpha)\pi^2(\alpha)\dots\pi^{|C(\alpha)|-1}(\alpha)\right)^{m(C(\alpha))}\beta$ *for all* $\alpha,\beta\in Q_1$.

Moreover, any symmetric stably biserial algebra over an algebraically closed field $\mathbf{k}$ *with* $char(\mathbf{k})\neq 2$ *is isomorphic to an algebra* $\mathbf{k}Q/I$ *as above with the empty set of loops* $\mathcal{L}$.

Remark 2.5. *In [8] we considered quivers* Q *such that every vertex has either two incoming and two outgoing arrows or one incoming and one outgoing arrow, and an admissible ideal of relations* I*. To pass to this equivalent description from the description in Theorem 2.4 one needs to delete loops* α *such that* $\pi(\alpha)=\alpha$, $m(\alpha)=1$ *and modify the ideal of relations accordingly. In the case where the algebra has only two loops* α_i *such that* $\pi(\alpha_i)=\alpha_i$, $m(\alpha_i)=1$*, one needs to delete only one loop to get the algebra isomorphic to* $\mathbf{k}[x]/(x^2)$.

The loops from the set $\mathcal{L}$ will be called **deformed loops**. It is well known that any SSB-algebra can be given in the above form with the empty set of deformed loops.

Note also that for this description of stably biserial algebras $|Q_1|=2|Q_0|$ is invariant under derived equivalence, since $|Q_0|$ is the rank of $K_0(D^b(A))$.

It is well known [32, 7, 35] that the class of SSB-algebras coincide with the class of Brauer graph algebras. **Brauer graph** is a graph with a cyclic ordering of (half-)edges around each vertex and a number assigned to each vertex. This graph Γ can be constructed using the data (Q,π,m) as follows: the vertices of Γ correspond to the cycles of π, the edges of Γ correspond to the vertices of Q, an edge connects two vertices of Γ if the corresponding π-cycles have the corresponding vertex of Q in common. The cyclic ordering of edges around a vertex comes from the order in which vertices of Q appear in the π-cycle, the multiplicities come from the function m. Along the same lines, to each Brauer graph one can assign the data (Q,π,m) and the corresponding SSB-algebra.

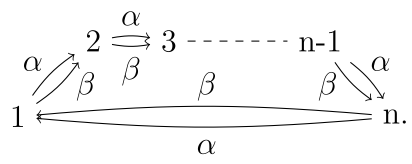
In [4] each Brauer graph was considered together with a minimal compact oriented surface $\mathcal{S}$, into which it is embedded, in such a way that its complement is a union of disks (see also [24]). The orientation of the edges around the vertices of Γ comes from the orientation of the surface, now it makes sense to consider not only vertices and edges of Γ but also **faces** of Γ. The set $\mathcal{L}$ corresponds to a subset of faces of perimeter 1 of Γ. We will use the terms SSB-algebra and Brauer graph algebra interchangeably.

3. Stably biserial algebras

In this section we are going to investigate basic properties of symmetric stably biserial algebras. As stated in Theorem 2.4 any symmetric stably biserial algebra A can be given as a certain deformation of a Brauer graph algebra with a Brauer graph Γ. We are going to show that with one exception the Brauer graph Γ does not depend on the presentation of A and that any deformation from Theorem 2.4 is indeed symmetric.

Let us introduce a special class of algebras, called caterpillar in this paper. This class of algebras behaves differently form other symmetric special biserial algebras and has to be excluded from some considerations.

The algebra $\mathbf{k}Q/I$ will be called a **caterpillar** of length $n>1$ if Q is of the form



In this case, the ideal of relations has the form I_1 or I_2. Where I_1 is generated by relations $\alpha e_i\beta=0=\beta e_i\alpha, i\neq 1$, $\alpha e_1\alpha=0=\beta e_1\beta$, $(\alpha^k e_1\beta^n\alpha^{n-k})^{m_\alpha}=(\beta^k e_1\alpha^n\beta^{n-k})^{m_\alpha}$, thus it has one π-cycle, the multiplicity of which is m_α. The ideal I_2 is generated by relations $\alpha\beta = 0 = \beta\alpha$, $\alpha^{nm_\alpha} = \beta^{nm_\beta}$, thus it has two π-cycles, the multiplicity of one π-cycle is m_α, the multiplicity of the other π-cycle is m_β. The Brauer graphs of these algebras are:

The following Lemma is most likely know for Brauer graph algebras, but we could not find the proof, so we include it for the larger class of symmetric stably biserial algebras.

Lemma 3.1. *Let A be a symmetric stably biserial algebra with a presentation $\mathbf{k}Q/I$ as in Theorem 2.4 and the associated Brauer graph Γ. If Γ is not a loop with 1 as the multiplicity of the unique vertex, or an edge with 2 as the multiplicity of both vertices, then Γ does not depend on the choice of the presentation $\mathbf{k}Q/I$.*

Proof. Assume that the algebra A has two presentations $\mathbf{k}Q/I \simeq \mathbf{k}Q'/I'$ as in Theorem 2.4. Let us delete loops α such that $\pi(\alpha) = \alpha$, $m(\alpha) = 1$ and modify the ideal of relations accordingly in both presentations. These deleted loops correspond to the leafs with multiplicity 1 in the Brauer graph and can always be reconstructed from the valency of the vertices in the quiver. Since the ideals of relations I and I' are admissible after the deletion of extra loops, we can assume that Q and Q' coincide, thus there is a bijection between primitive idempotents for these two presentations and between simple modules over $\mathbf{k}Q/I$ and $\mathbf{k}Q'/I'$, simple modules will be denoted by S_i. This extends to a bijection between the edges of the Brauer graphs Γ and Γ', constructed from these two presentations, since the edges of the Brauer graph correspond to simple modules. For the projective cover P_i of S_i we can consider the module $\mathrm{rad}P_i/\mathrm{soc}P_i$ which has either one or two indecomposable summands M_i and N_i. These modules are uniserial and each of them gives a unique sequence of simple modules, corresponding to it's radical series $(S_{i_1}, \cdots, S_{i_n})$, where S_{i_1} is the top of M_i or N_i respectively. Adding S_i to this sequence $(S_{i_0} = S_i, S_{i_1}, \cdots, S_{i_n})$ and numbering the sequence by the elements of $\mathbb{Z}/(n+1)\mathbb{Z}$ we get a collection of cycles of simple modules (coming from each P_i for all i's), which we identify up to a cyclic permutation of $\mathbb{Z}/(n+1)\mathbb{Z}$. If for some P_i the modules M_i and N_i are both zero, then $A \simeq \mathbf{k}[x]/(x^2)$. The radical series of the modules M_i, N_i do not depend on the presentation of the algebra, so in this case the Brauer graph is determined uniquely and is an edge with both vertices of multiplicity 1.

Note that by construction of the permutation π the cyclic ordering of the simples in the sequences constructed above coincides with the cyclic ordering of edges in the Brauer graph.

If the module S_i appears in two different cyclic sequences, then the edge, corresponding to S_i is not a loop and we can reconstruct the cyclic ordering around the ends of the edge, corresponding to S_i from the subsequence of the form $(S_i, S_{i_1}, \cdots, S_{i_l}, S_i)$, where $(S_{i_1}, \cdots, S_{i_l})$ does not contain S_i. The multiplicities of the vertices is the number of times the subsequences $(S_i, S_{i_1}, \cdots, S_{i_l})$ has to be repeated to get the whole sequences.

If S_i appears in only one cyclic sequence, but this cyclic sequence has a subsequence of the form $(S_i, S_{i_1}, \cdots, S_{i_l}, S_i, S_{i_{l+2}}, \cdots, S_{i_m}, S_i)$, where the subsequences $(S_{i_1}, \cdots, S_{i_l})$ and $(S_{i_{l+2}}, \cdots, S_{i_m})$ do not contain S_i, are different and at least one of them is not empty, then the edge corresponding to S_i is a loop and we can reconstruct the cyclic ordering of the edges around the vertex adjacent to this loop and the multiplicity is the number of times the subsequence $(S_i, S_{i_1}, \cdots, S_{i_l}, S_i, S_{i_{l+2}}, \cdots, S_{i_m})$ has to be repeated to get the whole sequence.

If S_i appears in only one cyclic sequence and this sequence does not have a subsequence as before, but the projective module P_i is uniserial then we can reconstruct the cyclic ordering of the edges around one vertex incident to the edge corresponding to S_i and its multiplicity as before, the other end of this edge has no other edges incident to it and has multiplicity 1.

The only case left to consider is when S_i appears in only one cyclic sequence and this sequence does not have a subsequence as before, but the projective module P_i is not uniserial. In this case the modules M_i and N_i have the same radical series but are both nonzero. If the cyclic sequence containing S_i is of the form $(S_i, S_{i_1}, \cdots, S_{i_l}, S_i)$, where $(S_{i_1}, \cdots, S_{i_l})$ does not contain S_i, then the edge, corresponding to S_i is not a loop and we can reconstruct the cyclic ordering around each end of this edge, the multiplicities of the ends are 1. Assume that $(S_i, S_{i_1}, \cdots, S_{i_l}, S_i)$, where $(S_{i_1}, \cdots, S_{i_l})$ does not contain S_i is a subsequence of the cyclic sequence and it has to be repeated $m > 1$ times to get the whole sequence. If $(S_{i_1}, \cdots, S_{i_l})$ is empty, then $|Q_0| = 1$, this situation will be considered later. If the edge corresponding to S_i is a loop, then all edges corresponding to $(S_{i_1}, \cdots, S_{i_l})$ are loops and we get a caterpillar with one vertex in the Brauer graph with multiplicity $m/2$ (this can happen only for even m). If the edge corresponding to S_i is not a loop, then all edges corresponding to $(S_{i_1}, \cdots, S_{i_l})$ are not loops and we get a caterpillar with two vertices in the Brauer graph, both with multiplicity m. The two algebras we get for even m are not isomorphic, since they are not even derived equivalent by Proposition 4.4. Note that the proof of Proposition 4.4 does not relay on the results of this section.

Let us consider the case $|Q_0| = 1$. The Brauer graph is either an edge or a loop. If it is an edge, there are no deformed loops and $A \simeq A_{k,l} = \mathbf{k}[x, y]/\langle xy, x^k - y^l\rangle$, $k, l \geq 1$, which is a commutative algebra. If it is a loop, then for multiplicity greater then one, A is non-commutative. So it is sufficient to consider algebra $B_{t_x,t_y} = \mathbf{k}[x, y]/\langle x^2y, y^2x, x^2 - t_xxy, y^2 - t_yxy\rangle$, which is 4-dimensional. If it is isomorphic to $A_{k,l}$, then either $k = 1, l = 3$, which is not possible, or $k = l = 2$. In the last case the algebras can be indeed isomorphic, even when $t_x = t_y = 0$, $char(\mathbf{k}) \neq 2$. $\square$

Remark 3.2. *The cyclic ordering of edges in the Brauer graph played an important role in the proof of Theorem 2.4. Namely, for a symmetric stably biserial algebra A with an arbitrary presentation as in Proposition 2.3, with an admissible ideal of relations, we first fixed the permutation π and then using the change of basis produced a presentation as in Theorem 2.4. We would like to note here that the change of basis from* [8, Lemma 10] *does not work for the algebras A_t and $B_{t,s}$ (see below), which was not noted in the proof of Lemma 10. This does not effect the result, since these algebras turn out not to be symmetric. For the algebra A_t the element $\alpha - t\beta$ belongs to the socle of A_t, for the algebra $B_{t,s}$ the element $\gamma_0 - s\gamma_1$ belongs to the socle of $B_{t,s}$, which is a contradiction. Here*

$$\begin{tikzcd} Q: 1 \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd} \qquad \begin{tikzcd} Q: 1 \arrow[r, shift right, "\gamma_0"'] \arrow[r, shift right=3, "\gamma_1"'] & 2 \arrow[l, shift right, "\beta_0"'] \arrow[l, shift right=3, "\beta_1"'] \end{tikzcd}$$

$$\begin{gathered} I = \langle J(\mathbf{k}Q)^3, \alpha^2 = \beta^2, \\ \alpha\beta = t\alpha^2, \beta\alpha = t\beta^2\rangle, t^2 = 1 \\ A_t = \mathbf{k}Q/I, \pi(\alpha) = \beta, \pi(\beta) = \alpha \end{gathered} \qquad \begin{gathered} I = \langle J(\mathbf{k}Q)^3, \gamma_0\beta_0 = \gamma_1\beta_1, \\ \beta_0\gamma_0 = \beta_1\gamma_1, \beta_0\gamma_1 = t\beta_0\gamma_0, \\ \gamma_1\beta_0 = t\gamma_1\beta_1, \beta_1\gamma_0 = s\beta_1\gamma_1, \\ \gamma_0\beta_1 = s\gamma_0\beta_0\rangle, st = 1 \\ B_{t,s} = \mathbf{k}Q/I, \pi(\gamma_i) = \beta_i, \pi(\beta_i) = \gamma_i \end{gathered}$$

Let us also denote by A_∞ an algebra isomorphic to any symmetric stably biserial algebras with one vertex, two loops and one π-cycle of multiplicity 1.

Proposition 3.3. *Let us consider any data of the form $(Q, \pi, m, \mathcal{L}, \{t_\alpha\}_{\alpha \in \mathcal{L}})$, and $A \simeq \mathbf{k}Q/I$, where I is the ideal of relations described in Theorem 2.4. If A is not isomorphic to A_∞ with both loops deformed, then the algebra A is symmetric.*

Proof. Recall that an algebra A is symmetric if and only if there exists a non-degenerate symmetric $\mathbf{k}$-bilinear form $\langle a, b\rangle : A \times A \to \mathbf{k}$ such that $\langle ab, c\rangle = \langle a, bc\rangle$ for all $a, b, c \in A$.

Let us define the standard bilinear form $\langle a, b\rangle := \phi(ab)$, where the value of ϕ on the path basis of A is defined as follows: $\phi(C(\alpha)^{m(C(\alpha))}) = 1$ for any arrow α, thus $\phi(\gamma^2) = l_\gamma$ for any deformed loop γ, and $\phi(p) = 0$ for any path $p \notin \mathrm{soc}A$. The values of ϕ on other elements of A is defined by linearity.

The defined form is bilinear, symmetric and satisfies the property $\langle ab, c\rangle = \langle a, bc\rangle$ for all $a, b, c \in A$. Let us check that it is non-degenerate.

Let us assume that ϕ is degenerate, that is $\phi((\sum c_i p_i)a) = 0$ for some $\sum c_i p_i \neq 0$ and for all $a \in A$, where $c_i \in \mathbf{k}^*$ and p_i are paths, by the symmetry of ϕ, for all $a \in A$ we have $\phi(a(\sum c_i p_i)) = 0$. We can assume that all p_i start at the same vertex i and end at the same vertex j (multiplying by two idempotents and keeping $\sum c_i p_i$ non-zero). All p_i's are subpaths of the standard socle paths of the form $C(\alpha)^{m(C(\alpha))}$. Since there are at most two such standard socle paths starting at i, all p_i's can be divided into two groups depending on the socle path. Let us chose the shortest path from one of these two groups p_1. Let $\bar{p_1}p_1$ be the standard socle path containing p_1 (that is not γ^2 for a deformed loop γ), then $\bar{p_1}(\sum c_i p_i) = c_1\bar{p_1}p_1 + c_2\bar{p_1}p_2$, where $\bar{p_1}p_2$ appears only in case when the shortest path from the second group is an arrow p_2 and $\bar{p_1}$ is also an arrow. In this case p_2 must be a deformed loop $p_2 = \bar{p_1}$. If $\bar{p_1}p_2 = 0$, then $\phi(\bar{p_1}(\sum c_i p_i)) = 0$ iff $c_1 = 0$ and we are done.

Let us do the same exchanging p_1 and p_2. Then $\bar{p_2}(\sum c_i p_i) = c_2\bar{p_2}p_2 + c_1\bar{p_2}p_1$, where $\bar{p_2}p_1$ appears only in case when the shortest path from the first group is an arrow p_1 and $\bar{p_2}$ is also an arrow. In this case p_1 must be a deformed loop $p_1 = \bar{p_2}$. And we get exactly the excluded case of 2 deformed loops at one vertex. $\square$

4. Combinatorial derived invariants

The aim of this section is to show that the following combinatorial data are invariant under derived equivalences of stably biserial algebras: number of vertices, edges and faces of the Brauer graph, multisets of perimeters of faces, multisets of multiplicities of vertices, bipartivity. Note that the corresponding results were shown to be true for Brauer graph algebras with some minor inaccuracies in [2, 3, 4, 6], the proofs are identical or relay on the corresponding results for Brauer graph algebras, except for some simplifications. From here on we are going to exclude the case $|Q_0| = 1$ from some considerations, since by [34] a local algebra can be derived equivalent only to itself.

4.1. **The centre of a symmetric stably biserial algebra.** In this subsection we compute the centre $Z(A)$ of a symmetric stably biserial algebra A, which is known to be invariant under derived equivalence, see [29]. We will use this to establish, that the number of π-cycles, or the vertices of the Brauer graph, is invariant under derived equivalence. This will also gives us an opportunity to correct the above-mentioned inaccuracies in the description of the centre of an SSB-algebra made in [4]. Let $\{C_1, C_2, \dots, C_r\}$ be the set of π-cycles. For each $i = 1, \dots, r$ consider a cyclic sequence $(\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,l_i})$ of arrows of the cycle C_i, where $\pi(\alpha_{i,j}) = \alpha_{i,j+1}$, l_i denotes the length of the cycle C_i. Let

$m(C_1), m(C_2), \dots, m(C_r)$ denote the multiplicities of the π-cycles and let $r' \le r$ be an integer such that $m(C_i) > 1, i = 0, \dots, r'$ and $m(C_i) = 1, i = r'+1, \dots, r$. For each loop γ such that $\pi(\gamma) \neq \gamma$ there are i and j such that $\gamma = \alpha_{i,j}$. For each such loop γ set $q_\gamma = q_{\alpha_{i,j}} = (\alpha_{i,j+1}\alpha_{i,j+2}\dots\alpha_{i,l_i}\alpha_{i,1}\dots\alpha_{i,j})^{m(C_i)-1}\alpha_{i,j+1}\alpha_{i,j+2}\dots\alpha_{i,l_i}\alpha_{i,1}\dots\alpha_{i,j-1}$.

Proposition 4.1. *Let A be a symmetric stably biserial algebra with the corresponding data $(Q, \pi, m, \mathcal{L})$. As a vector space over $\mathbf{k}$ the centre $Z(A)$ is generated by 1 and by the elements of the following form:*

a) Elements $m_{i,t} = (\alpha_{i,1}\alpha_{i,2}\dots\alpha_{i,l_i})^t + (\alpha_{i,2}\alpha_{i,3}\dots\alpha_{i,1})^t + \dots + (\alpha_{i,l_i}\alpha_{i,1}\dots\alpha_{i,l_i-1})^t$, for $i = 1, 2, \dots, r'$ and $t = 1, \dots, m(C_i)-1$.

b) Elements q_γ for each loop γ such that $\pi(\gamma) \neq \gamma$.

c) Elements s_v for each vertex $v \in Q_0$, where s_v is the socle element corresponding to v.

Moreover, if A is not isomorphic to some A_∞, then considered as an algebra, $Z(A)/(\mathrm{soc}(Z(A)) \simeq \mathbf{k}[x_1, x_2, \dots, x_{r'}]/\langle x_i^{m(C_i)}, (x_ix_j)_{i\neq j}\rangle$. So the multiset $\{m(C_1), m(C_2), \dots, m(C_{r'})\}$ is invariant under derived equivalence. The number of loops γ such that $\pi(\gamma) \neq \gamma$, or equivalently, the number of faces of Γ of perimeter 1 is a derived invariant as well.

Proof. It is clear that all the listed elements belong to the centre of A. Let us prove that any element of the centre is a linear combination of elements of the form (a)-(c).

Each $z \in Z(A)$ has a form $z = \sum_{i=1}^N a_ip_i + z'$, where p_i are the elements of the path basis of A which do not belong to the socle of A and $z' \in \mathrm{soc}(A)$. Without loss of generality, we can assume $z' = 0$. All elements p_i with $a_i \neq 0$ are necessarily closed paths, that is $p_i = e_vp_ie_v$ for some idempotent e_v, corresponding to a vertex v. Fix $p_i = \beta_1\beta_2\dots\beta_m$ for some $\beta_j \in Q_1$, let $\beta_{m+1} = \pi(\beta_m)$, then $p_i\beta_{m+1} \neq 0$. Assume that $p_i\beta_{m+1}$ does not belong to the socle of A, then $\beta_1\beta_2\dots\beta_m\beta_{m+1}$ has coefficient a_i in the sum $\beta_{m+1}z$, hence $\beta_{m+1} = \beta_1$ and the coefficient of $\beta_2\dots\beta_m\beta_{m+1}$ in z is a_i, so $p_i = (\alpha_{j,s}\alpha_{j,s+1}\dots\alpha_{j,s-1})^t$ for some π-cycle, and z contains $a_im_{j,t}$ as a summand, $z - a_im_{j,t}$ contains less summands, then z. If $\beta_1\beta_2\dots\beta_m\beta_{m+1}$ belongs to the socle of A, then β_{m+1} is a loop, since p_i is a closed path. Then p_i is either $m_{j,m(C_j)-1}$ for a cycle C_j, consisting of a single loop (if $\pi(\beta_{m+1}) = \beta_{m+1}$) or $q_{\beta_{m+1}}$ (if $\pi(\beta_{m+1}) \neq \beta_{m+1}$). Either $z - a_iq_{\beta_{m+1}}$ or $z - a_im_{j,m(C_j)-1}$ has less summands then z and we can proceed by induction on the number of nonzero coefficients a_i in the sum $z = \sum_{i=1}^N a_ip_i$. By induction we get that z is a linear combination of elements of the form (a)-(c).

In case $A \not\simeq A_\infty$, $\mathrm{soc}(Z(A))$ is clearly generated by the elements of type (b) and (c). Moreover, $m_{i,t_1}m_{j,t_2} = \delta_{i,j}m_{i,t_1+t_2}$ and $m_{i,1}^{m(C_i)} \in \mathrm{soc}Z(A)$. Hence $Z(A)/(\mathrm{soc}(Z(A))) \simeq \mathbf{k}[x_1, x_2, \dots, x_r]/\langle x_i^{m(C_i)}, (x_ix_j)_{i\neq j}\rangle$. Since $Z(A)$ is invariant under derived equivalence as an algebra, the multiset $\{m(C_1), m(C_2), ..., m(C_{r'})\}$ is invariant under derived equivalence. The socle of $Z(A)$ is spanned by the elements of the form s_v, $v \in Q_0$ and q_γ, γ is a loop, $\pi(\gamma) \neq \gamma$. Since the number of the elements s_v is a derived invariant, so is the number of loops γ such that $\pi(\gamma) \neq \gamma$. $\square$

4.2. **Number and perimeters of faces.** Let A be a symmetric stably biserial algebra with the corresponding Brauer graph Γ, let $p_1, p_2, \dots p_m$ be the perimeters of faces of Γ. Namely, using the graph Γ the surface $\mathcal{S}$ can be cut into polygons, by a **perimeter** of a face F we mean the number of edges in the corresponding polygon, thus, for example, the perimeter of a self-folded triangle is 3. Note that the perimeter of a face F coincides with the length of the corresponding Green walk (see [32, 16]). The aim of this section is to prove that the multiset $\{p_1, \dots, p_m\}$ (and, in particular, the number of faces m)

is an invariant of the derived category of A. For this we are going to use the structure of the Auslander-Reiten quiver of the stable category of mod-A. Note that by [30] for self-injective and in particular for symmetric algebras derived equivalence implies stable equivalence, so any invariant of stable equivalence is automatically a derived invariant.

Indecomposable modules over special biserial algebras are classified in terms of strings and bands, the description of the Auslander-Reiten sequences and of the Auslander-Reiten quiver for such algebras is well understood [20, 13, 36, 17, 18]. Let us consider the AR-quiver $\Gamma_{\underline{\mathrm{mod}}\text{-}A}$ of $\underline{\mathrm{mod}}$-A. If A is SSB, then each periodic component of $\Gamma_{\underline{\mathrm{mod}}\text{-}A}$ is a tube. Moreover, all tubes are either tubes of rank 1, consisting of band modules or tubes consisting of string modules, called exceptional tubes. Exceptional tubes correspond to faces of Γ: if a face has an even perimeter p, then it produces two tubes of rank $p/2$ such that they are permuted by Ω; if a face has an odd perimeter, then it produces one tube of rank p, which is stable under the action of Ω. For a detailed exposition see [16, Section 4].

In case A is symmetric stably biserial and not necessarily special biserial its AR-quiver $\Gamma_{\underline{\mathrm{mod}}\text{-}A}$ of $\underline{\mathrm{mod}}$-A coincides with the same quiver for the SSB-algebra A' constructed from the same data (Q, π, m). Indeed, $A/\mathrm{soc}(A) \simeq A'/\mathrm{soc}(A')$ is a string algebra, so the classification of indecomposable non-projective modules is the same. The AR-sequences not ending at the module $P/\mathrm{soc}(P)$ for a projective module P coincide for A and $A/\mathrm{soc}(A)$ by [12, Proposition 4.5], the fact that the sequences $0 \to \mathrm{rad}P \to \mathrm{rad}P/\mathrm{soc}P \oplus P \to P/\mathrm{soc}P \to 0$ give the same in $\Gamma_{\underline{\mathrm{mod}}\text{-}A}$ and $\Gamma_{\underline{\mathrm{mod}}\text{-}A'}$ can be checked by hand.

Consequently, we get that the number of faces of a given perimeter $p > 2$ is the number of tubes of rank p, stable under Ω, in case p is odd and the number of tubes of rank $p/2$, not stable under Ω, divided by 2, in case p is even. In both cases the perimeter can be also reconstructed from the stable category. By Proposition 4.1 the number of faces of perimeter 1 is a derived invariant. The number of faces of perimeter 2 can be reconstructed as follows: $(2|E(\Gamma)| - \sum_{p_i \neq 2} p_i)/2$. Thus, the following holds:

Proposition 4.2. *Let A_1, A_2 be two symmetric stably biserial algebras with Brauer graphs Γ_1 and Γ_2, such that neither Γ_1 nor Γ_2 is a loop with multiplicity 1 or an edge with multiplicity of both vertices 2. If $D^b(A_1) \simeq D^b(A_2)$, then the number of faces and the multisets of perimeters of faces of Γ_1 and Γ_2 coincide.*

Remark 4.3. *The proof of the fact that the number of faces and the multisets of perimeters of faces of Γ is invariant under an equivalence of stable categories of SSB-algebras was provided in* [3] *with a mistake, which was corrected in* [5]. *Note that the proof is much more involved, since one can not use the centre of the algebra $Z(A)$ (as in Proposition 4.1), so one has to deal with the tubes coming from the faces of perimeter 1 and 2 and with the tubes containing band modules.*

4.3. **Number of vertices and their multiplicities.** Let $\mathbb{Z}^{|Q_0|}$ be the Grothendieck group of a self-injective algebra A with the Cartan matrix $C(A)$. Then $C(A)$ defines a group homomorphism ϕ_A from $\mathbb{Z}^{|Q_0|}$ to itself and $K_0(\underline{\mathrm{mod}}\text{-}A) \simeq \mathbb{Z}^{|Q_0|}/\mathrm{Im}(\phi_A)$. To obtain the standard description of this Abelian group one can use Smith's normal form of $C(A)$, which can be obtained by computing the greatest common divisors of all $t \times t$ minors of $C(A)$, this was done for SSB-algebras in [2, 6]. We are going to use only the rank of $C(A)$, which is equal to $|Q_1/\pi| - 1$ if the Brauer graph Γ of A is bipartite and to $|Q_1/\pi|$, otherwise. Note that by construction $|Q_1/\pi|$ is the number of vertices of Γ.

Proposition 4.4. *Let A_1, A_2 be stably biserial algebras with Brauer graphs Γ_1 and Γ_2, such that neither Γ_1 nor Γ_2 is a loop with multiplicity 1 or an edge with multiplicity of* *both vertices 2. If $D^b(A_1) \simeq D^b(A_2)$, then $|V(\Gamma_1)| = |V(\Gamma_2)|$. Moreover, the multisets of multiplicities of the vertices and the bipartivity of Γ_1 and Γ_2 coincide.*

Proof. Let A_i' be the special biserial algebra corresponding to the data given by the Brauer graph Γ_i. As A_i and A_i' have the same Cartan matrices, we can use the description of the structure of the Grothendieck group of A_i' for A_i. Since derived equivalences of self-injective algebras imply stable equivalences, $K_0(\underline{\text{mod}}\text{-}A_1) \simeq K_0(\underline{\text{mod}}\text{-}A_2)$, thus the ranks of $C(A_1)$ and $C(A_2)$ coincide.

By [6], $\text{rk}(C(A_i))$ is equal to $|V(\Gamma_i)| - 1$ if the Brauer graph $\Gamma(A_i)$ of A_i is bipartite and to $|V(\Gamma_i)|$ otherwise. By Proposition 4.2 $|F(\Gamma_1)| = |F(\Gamma_2)|$, the same holds for $|E(\Gamma_1)| = |E(\Gamma_2)|$, since $|V(\Gamma_i)| - |E(\Gamma_i)| + |F(\Gamma_i)|$ is even as the Euler characteristic of the surface $\mathcal{S}_i$, we see that $|V(\Gamma_i)|$ can not differ by 1, hence, $|V(\Gamma_1)| = |V(\Gamma_2)|$. Since the ranks of the Cartan matrices of A_1 and A_2 coincide, Γ_i are either simultaneously bipartite or simultaneously not bipartite.

The multiplicities of the vertices > 1 can be detected by the centre of the algebra, see Proposition 4.1. The invariance of the number of the vertices with multiplicity 1 follows from the invariance of the number of all vertices. □

Lemma 4.5. *Let A_1, A_2 be derived equivalent symmetric stably biserial algebras, where A_1 is a caterpillar. Then A_2 is special biserial.*

Proof. The algebra A_1 has no loops γ such that $\pi(\gamma) \neq \gamma$, so by Proposition 4.1 A_2 has no such loops as well, hence, A_2 is symmetric special biserial. □

Proof of Theorem 1.2. Combining the results of Proposition 4.2 and 4.4 we get that the following are derived invariants of a symmetric stably biserial algebra A with at least two non-isomorphic simple modules and the corresponding Brauer graph Γ: $|V(\Gamma)|, |E(\Gamma)|, |F(\Gamma)|$, the multiset of perimeters of faces, the multiset of multiplicities of vertices, bipartivity of Γ. □

5. The group of outer automorphisms

Throughout this section we are going to assume that either $char(\mathbf{k}) = 2$ or that $char(\mathbf{k}) \neq 2$ and the number of deformed loops $d = 0$ (that is A is symmetric special biserial); that A is not a caterpillar and that A has at least two non-isomorphic simple modules. We are going to show that derived equivalent symmetric stably biserial algebras have the same number of deformed loops using the identity component of the group of outer automorphisms. By [23, 33] the identity component of the group of outer automorphisms $Out^0(A)$ of an algebra A is invariant under derived equivalence as an algebraic group. We are going to use the necessary notions and facts about algebraic groups freely, for more details see [25].

5.1. H' **is trigonalizable.** Let $A = \mathbf{k}Q/I$ be a stably biserial algebra in the standard form given in Theorem 2.4, i.e. the ideal of relations is not necessarily admissible. Let $\mathcal{L} \subset Q_1$ be the set of deformed loops. Let $A = B \oplus J(A)$ be a Wedderburn-Maltsev decomposition (i.e. B is semisimple subalgebra). Then it is known that $Out(A) = H/H \cap Inn(A)$, where $H = \{f \in Aut(A) | f(B) \subset B\}$ [22, 28]. If $\{e_v\}_{v \in Q_0}$ is a set of primitive idempotents and $B = \langle \{e_v\}_{v \in Q_0} \rangle$, then obviously for any $v \in Q_0$ and $f \in H$, $f(e_v) = e_{v'}$ for some $v' \in Q_0$. Therefore $H' = \{f \in H \mid f|_B = Id\}$ is a closed subgroup of finite index in H, i.e. it is a union of connected components, since $H \cap Inn(A) = H' \cap Inn(A) = Inn(A)$ acts on each component, to understand $Out^0(A)$ we can consider only H' without loss of generality.

Lemma 5.1. *If A is not a caterpillar and* $\mathrm{rk}K_0(A) \geq 2$*, then there is an embedding* $i : H' \to T(l, \mathbf{k})$ *of algebraic groups over* $\mathbf{k}$*, where* $T(l, \mathbf{k})$ *is the group of lower triangular matrices and* $l = \dim A$.

Proof. Let P be a set of paths in Q which forms a basis for $\mathbf{k}Q/I$, such that $\alpha^2 \notin P$ for $\alpha \in \mathcal{L}$ and all primitive idempotents $\{e_v\}_{v \in Q_0}$ and all arrows of Q are in P. For each $p \in P$ let $l_p = max\{k : p \in J(A)^k\}$, with the convention that $l_{e_v} = 0, v \in Q_0$. A pair $(\beta, \beta') \in Q_1 \times Q_1$ with $s(\beta) = s(\beta'), e(\beta) = e(\beta')$ (β, β' are parallel arrows) and $\pi^2(\beta) \neq \beta, \pi^2(\beta') = \beta'$ will be called an exceptional pair.

Let us consider some linear extension of the following partial order on P:

1)If $l_p < l_q$, then $p < q$.

2)If (β, β') is an exceptional pair, then $\beta < \beta'$. Note that since $|Q_0| > 1$, $l_\beta = l_{\beta'} = 1$.

We are going to express the matrix of an automorphisms of A in the basis P with respect to this linear order and show that, it is lower-triangular, i.e, for $f \in H'$ and $p \in P$ we have $f(p) = k_p p + \sum_{p' > p} k_{p,p'} p'$.

Let us consider $p = \beta \in Q_1$, such that $l_\beta = 1$. If β has no parallel arrows, then $f(\beta) = k_\beta \beta + r$ where $r \in J(A)^2$ and we are done. Now suppose that β has a parallel arrow β', in this case we can have $f(\beta) = k_\beta \beta + k_{\beta,\beta'} \beta' + r$ with $k_{\beta,\beta'} \neq 0$. Note that since $|Q_0| \neq 1$, β is not a loop. There are three possible cases:

1) $\pi(\beta), \pi(\beta')$ are not parallel. In this case $f(\pi(\beta')) = k_{\pi(\beta')} \pi(\beta') + r_1$, $r_1 \in J(A)^2$. Then $0 = f(\beta \pi(\beta')) = f(\beta) f(\pi(\beta')) = k_{\beta,\beta'} k_{\pi(\beta')} \beta' \pi(\beta') + r'$, $r' \in J(A)^3$. A path of length two $\beta' \pi(\beta')$ belongs to $J(A)^3$, hence $\beta' \pi(\beta') \in \mathrm{soc(A)}$. Since $\pi(\beta), \pi(\beta')$ are not parallel, (β, β') is an exceptional pair and $\beta < \beta'$. The same argument for (β', β) gives $k_{\beta',\beta} k_{\pi(\beta)} \beta \pi(\beta) \in \mathrm{soc(A)}$, hence $k_{\beta',\beta} k_{\pi(\beta)} = 0$, thus $k_{\beta',\beta} = 0$, so $f(\beta') = k_{\beta'} \beta' + r''$, $r'' \in J(A)^2$.

2) $\pi(\beta), \pi(\beta')$ are parallel arrows but $\pi^l(\beta), \pi^l(\beta')$ are not parallel for some l (we take the minimal l). In this case $(\pi^{l-1}(\beta), \pi^{l-1}(\beta'))$ is not an exceptional pair (otherwise $s(\pi^{l-1}(\beta))$ has 3 incoming arrows) and $f(\pi^l(\beta')) = k_{\pi^l(\beta')} \pi^l(\beta') + r'$, $r' \in J(A)^2$. So $0 = f(\pi^{l-1}(\beta)) f(\pi^l(\beta'))$ implies $f(\pi^{l-1}(\beta)) = k_{\pi^{l-1}(\beta)} \pi^{l-1}(\beta) + r$, $r \in J(A)^2$ and the same holds for β'. Then by decreasing induction on i we obtain in the same way that $f(\pi^i(\beta')) = k_{\pi^i(\beta')} \pi^i(\beta') + r'$, $r' \in J(A)^2$, $f(\pi^i(\beta)) = k_{\pi^i(\beta)} \pi^i(\beta) + r$, $r \in J(A)^2$ for all $0 \leq i \leq l$, in particular, for $i = 0$.

3) $\pi^l(\beta), \pi^l(\beta')$ are parallel for all l. In this case $A = \mathbf{k}Q/I$ is a caterpillar.

For an arbitrary $p_i \in P$, chose a presentation $p_i = \beta_1 \ldots \beta_n$ with n maximal. Then $f(p_i) = f(\beta_1) \ldots f(\beta_n) = \prod_i k_{\beta_i} \beta_1 \ldots \beta_n + \sum k_j p_j$, where $p_j > \beta_1 \ldots \beta_n$. Indeed, $\beta_1 \ldots \beta_n$ is of the form $\beta_1 \pi(\beta_1) \ldots \pi^n(\beta_1)$ and since for an exceptional pair (β, β') we have $\beta' \pi(\beta), \pi^{-1}(\beta) \beta' \in J(A)^3$ the sum $\sum k_j p_j$ belongs to $J(A)^{n+1}$. □

5.2. **Decomposition with the unipotent subgroup.** We have seen in the previous subsection that H' is a subgroup of the group of lower-triangular matrices. In order not to compute the groups $Out^0(A)$ for all symmetric stably biserial algebras, which might turn out to be quite technical, we want to deduce some easier invariant of $Out^0(A)$ preserved by isomorphisms of algebraic groups.

Let us consider maximal unipotent subgroups in H' and $Inn(A)$, denoted respectively by $U_{H'}$ and U_I. These groups are given by the intersection of H' (respectively $Inn(A)$) with $U(l, \mathbf{k})$ the group of (lower) unitriangular matrices. We can consider the following diagram of algebraic groups:

$$\begin{tikzcd}
1 \arrow[r] & U_I \arrow[r] \arrow[d] & Inn(A) \arrow[r] \arrow[d] & D_I \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & U_{H'} \arrow[r] \arrow[d] & H' \arrow[r] \arrow[d] & D_{H'} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & U_{H'}/U_I \arrow[r] & Out(A) \arrow[r] & D_{H'}/D_I \arrow[r] & 1
\end{tikzcd}$$

An easy diagram chasing shows that the map $D_I \to D_{H'}$ is an embedding. As a quotient of a trigonalizable group $Out(A)$ is trigonalizable, $D_{H'}/D_I$ is diagonalizable and $U_{H'}/U_I$ is the maximal unipotent subgroup of $Out(A)$, it contains all unipotent subgroups of $Out(A)$ [25, Theorem 16.6]. Thus we can consider another diagram:

$$\begin{tikzcd}
 & 1 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & X \arrow[d] & \\
1 \arrow[r] & (U_{H'}/U_I)^0 \arrow[r] \arrow[d] & Out^0(A) \arrow[r] \arrow[d] & D(A) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & U_{H'}/U_I \arrow[r] \arrow[d] & Out(A) \arrow[r] \arrow[d] & D_{H'}/D_I \arrow[r] \arrow[d] & 1 \\
 & Y \arrow[r] & Z \arrow[r] & W \arrow[r] & 1
\end{tikzcd}$$

Since $Out^0(A)$ is connected and solvable its maximal unipotent subgroup is connected and thus coincides with $(U_{H'}/U_I)^0$. Note also that all maximal tori of $Out^0(A)$ are conjugate. The groups $D_I, D_{H'}, D_{H'}/D_I$ are diagonalizable. We also get the following exact sequence $1 \to X \to Y \to Z \to W \to 1$, where Y, Z are finite, then X, W are finite as well. Hence the rank of $D(A)$ and $D_{H'}/D_I$ coincide. We have proved the following lemma.

Lemma 5.2. *In the notation of the previous construction, the rank of $D_{H'}/D_I$ coincides with the rank of the maximal torus $D(A)$ of $Out^0(A)$ and so is a derived invariant of A.*

5.3. **Computation of the rank of $D_{H'}/D_I$.** Both $D_{H'}$ and D_I are induced by the projection map from the group of lower triangular matrices to the group of diagonal matrices. So we are going to find out what elements can appear on the diagonal of the matrices from H' and $Inn(A)$. Clearly the diagonal entries corresponding to the arrows of the quiver determine all other diagonal entries, so we are going to restrict our attention to them.

Lemma 5.3. *There is an isomorphism of affine algebraic groups $D_I \simeq (\mathbf{k}^*)^{|Q_0|-1}$.*

Proof. Recall that $Inn(A) = \{f_a | a \in A^*\}$, where $f_a(x) = axa^{-1}$. Each $a \in A^*$ can be uniquely written as $a = \sum_{i \in Q_0} a_i e_i + r$, where $a_i \in \mathbf{k}^*$, $r \in J(A)$. Then $a^{-1} = \sum_{i \in Q_0} a_i^{-1} e_i + r'$, $r' \in J(A)$ and the action of f_a on $J(A)/J(A)^2$ depends only on a_i's. For $c \in \mathbf{k}^*$, f_a clearly coincides with f_{ca}.

Let $\overline{f_a} = f_a \pmod{U_I}$. Consider any spanning tree of Q (ignoring the orientation of Q), let $\{\alpha_i\}_{1 \le i \le n-1}$ be the corresponding set of arrows. For each arrow α_i we have $f_a(\alpha_i) = a\alpha_i a^{-1} = a_{s(\alpha_i)} a^{-1}_{e(\alpha_i)} \alpha_i \pmod{J(A)^2}$. Let us define the map $D_I \xrightarrow{\eta} (\mathbf{k}^*)^{|Q_0|-1}$ as $\overline{f_a} \to (a_{s(\alpha_1)} a^{-1}_{e(\alpha_1)}, \ldots, a_{s(\alpha_{|Q_0|-1})} a^{-1}_{e(\alpha_{|Q_0|-1})})$. The equality $f_a(\alpha_i) = \alpha_i \pmod{J(A)^2}$ for the elements of U_I guarantees that the map is well defined.

Since α_i form a spanning tree, for any element $(k_1, \ldots, k_{|Q_0|-1}) \in (\mathbf{k}^*)^{|Q_0|-1}$ one can uniquely determine $\{a_i\}_i \in Q_0$ up to multiplication of all a_i by a common constant $c \in \mathbf{k}^*$. Setting $a = \sum_{i \in Q_0} a_i e_i$ one can define a map $D_I \overset{\theta}{\leftarrow} (\mathbf{k}^*)^{|Q_0|-1}$, clearly $\theta\eta((k_1, \ldots, k_{|Q_0|-1})) = (k_1, \ldots, k_{|Q_0|-1})$.

For $a - a' \in J(A)$ and any path $p = \beta_1\beta_2 \ldots \beta_k$ we have $f_a(p) - f_{a'}(p) \in J(A)^{k+1}$. As the order on the basis P agrees with the path length $\overline{f_a} = \overline{f_{a'}}$. Since $f_a = f_{ca}$ for $c \in \mathbf{k}^*$, $\eta\theta(\overline{f_a}) = \overline{f_a}$ and we get the desired bijection. □

Let us consider the following algebraic group D_Γ, which can be constructed from the data $(Q, \pi, m, \mathcal{L})$ or equivalently from the data of a Brauer graph and a fixed number of 1-perimeter faces of Γ (the set $\mathcal{L}$ is assumed to be empty in case $char(\mathbf{k}) \neq 2$). The group D_Γ is a subgroup of $(\mathbf{k}^*)^{2|E(\Gamma)|+1}$. The first $2|E(\Gamma)|$ entries k_α are labelled by the arrows of Q, the last entry is denoted $\underline{k}$. The subgroup is given by the following equations: $(k_\alpha)^2 = t_\alpha \underline{k}$ for each deformed loop α and $\prod_{\alpha \in C} k_\alpha^{m(C)} = \underline{k}$ for each $C \in Q_1/\pi$ with multiplicity $m(C)$.

Proposition 5.4. *Let A be a symmetric stably biserial algebra corresponding to the data $(Q, \pi, m, \mathcal{L})$, then $D_{H'} \simeq D_\Gamma$.*

Proof. As before any $f \in H'$ has the form $f(\alpha) = k_\alpha \alpha + \sum_{p > \alpha} k_{\alpha,p} p$. We need to check that the set of elements of $(\mathbf{k}^*)^{2|E(\Gamma)|}$ appears as the set (k_α) for some $f \in H'$ if and only if there exists $\underline{k} \in \mathbf{k}^*$ such that the equations from the description of the group D_Γ are satisfied.

If for a set of elements (k_α) in $(\mathbf{k}^*)^{2|E(\Gamma)|}$ there exists $\underline{k} \in \mathbf{k}^*$ such that $(k_\alpha, \underline{k}) \in D_\Gamma$, then we can define $f \in H'$ by $f(\alpha) = k_\alpha \alpha$, which clearly gives an automorphism of A.

Let us prove that for any $f \in H'$ the set (k_α) is an element of D_Γ with some $\underline{k} \in \mathbf{k}^*$. For that we need to better understand which coefficients $k_{\alpha,p}$ can be non-zero. For an arrow α, let us denote by $\bar{C}(\alpha) := C(\alpha)^{m(C(\alpha))} = (\alpha\pi(\alpha) \cdots \pi^{|\pi(\alpha)|-1}(\alpha))^{m(\langle\pi\rangle\alpha)}$ the maximal power of the cycle passing through α. Let us show that if $k_{\alpha,p} \neq 0$ and p is not a subpath of $\bar{C}(\alpha)$, then $p = \beta^{-1}\bar{C}(\beta)$ for some arrow β with $s(\beta) = e(\alpha)$ and $e(\beta) = s(\alpha)$. Assume this is not the case and let us take $p = \beta_1 \cdots \beta_t$ with t minimal, since $s(p) = s(\alpha)$, $e(p) = e(\alpha)$ and p is not a subpath of $\bar{C}(\alpha)$, $p \notin \mathrm{soc}(A)$, then p is a subpath of some $\bar{C}(\delta)$ and such p is unique. Let β be the arrow such that $\beta p \in \bar{C}(\beta)$, such β exists and $e(\beta) = s(\alpha)$ but $\alpha \neq \pi(\beta)$. Note that β is not a loop with $\pi(\beta) \neq \beta$, otherwise $\alpha = \beta$ and p is a subpath of $\bar{C}(\alpha)$. The relation $f(\beta)f(\alpha) = 0$ implies that the coefficient before βp, which contains $k_\beta k_{\alpha,p}$ should be 0. Assume $\beta p \notin \mathrm{soc} A$, then by the minimality of the length of p, $k_{\alpha,p} = 0$. So $\beta p \in \mathrm{soc} A$ as desired.

Let us now check that for any $f \in H'$ the set (k_α) satisfies the equations $\prod_{\alpha \in C} k_\alpha^{m(C)} = \underline{k}$ for some $\underline{k} \in \mathbf{k}$. Let us compute $f(\bar{C}(\alpha))$ for some $\alpha \in Q_1$. It has a summand $\prod_{\alpha' \in C(\alpha)} k_{\alpha'}^{m(C(\alpha))} \bar{C}(\alpha)$, if it has any other summand, then this summand can only appear in one of the following 3 situations:

1) as a product of the elements of the form $k_{\pi^i(\alpha),p}$, where p is not a subpath of $\bar{C}(\pi^i(\alpha))$. This situation is possible only in the case of a caterpillar with two simple modules, which we do not consider.

2) as a product of subpaths of $\bar{C}(\alpha)$ and paths, which are not subpaths of $\bar{C}(\alpha)$, this is possible only in the situation $|Q_0| = 1$ and A has a deformed loop, which we also do not consider.

3) as a product of subpaths of $\bar{C}(\alpha)$, at least one of which comes from $f(\pi^i(\alpha))$ and is not $\pi^i(\alpha)$. Note that all these subpaths are arrows, otherwise the product is zero. Since at least one of the subpaths comes from $f(\pi^i(\alpha))$ and is not $\pi^i(\alpha)$ all of them come from

$f(\pi^j(\alpha))$ but are not $\pi^j(\alpha)$, otherwise we are in the situation $|Q_0| = 1$ and A has a deformed loop again. Hence every $\pi^i(\alpha)$ has a parallel arrow and A is a caterpillar, which we do not consider.

So $f(\bar{C}(\alpha)) = \prod_{\alpha' \in C(\alpha)} k_{\alpha'}^{m(C(\alpha))} \bar{C}(\alpha)$. Since the relation $f(\bar{C}(\alpha)) = f(\bar{C}(\beta))$ holds for any $\alpha, \beta \in Q_1$ with $s(\alpha) = s(\beta)$ and the graph Γ is connected, we can denote by $\underline{k}$ the product $\prod_{\alpha' \in C(\alpha)} k_{\alpha'}^{m(C(\alpha))}$ for some fixed α and get that $\prod_{\alpha \in C} k_{\alpha}^{m(C)} = \underline{k}$ for any π-cycle C.

From here on we assume $char(\mathbf{k}) = 2$. Let us deal with the equations $(k_\alpha)^2 = t_\alpha \underline{k}$, for each deformed loop α. For a deformed loop α let w_α be the path that makes αw_α into a π-cycle. Let m be the multiplicity of this cycle. Then $f(\alpha) = k_\alpha \alpha + \sum_p k_p p$ (we will use this simplification of the notation for the rest of the proof), where p can have the following form: $(w_\alpha \alpha)^i, i = 1, \dots m-1$, $\alpha(w_\alpha \alpha)^i, i = 0, \dots m-1$, $(w_\alpha \alpha)^i w_\alpha, i = 0, \dots m-1$, $\alpha(w_\alpha \alpha)^i w_\alpha, i = 0, \dots m-1$. Since α is a deformed loop it appears only in socle relations and there are no restrictions on k_p so far.

Let us consider $f(\alpha)^2$: the coefficient before $\alpha w_\alpha \alpha$ should be zero, this gives $k_\alpha k_{w_\alpha \alpha} + k_\alpha k_{\alpha w_\alpha} = 0$, so since $k_\alpha \in \mathbf{k}^*$, $k_{w_\alpha \alpha} + k_{\alpha w_\alpha} = 0$. Let us assume $k_{(w_\alpha \alpha)^i} + k_{(\alpha w_\alpha)^i} = 0$, for $i < j < m$ and prove the same for j. Let us consider the coefficient of $\alpha(w_\alpha \alpha)^j$ in $f(\alpha)^2$. For any entry of the form $k_{(\alpha w_\alpha)^i} k_{(\alpha w_\alpha)^{j-i}\alpha}$ there is an entry of the form $k_{(\alpha w_\alpha)^{j-i}\alpha} k_{(w_\alpha \alpha)^i}$ and by induction hypothesis they cancel out. The only entries left are $k_\alpha k_{(w_\alpha \alpha)^j} + k_{(\alpha w_\alpha)^j} k_\alpha = 0$, so we are done.

Let us now consider the coefficient of $\alpha^2 = (\alpha w_\alpha)^m = (w_\alpha \alpha)^m$, which should coincide with $t_\alpha \underline{k}$. For any entry of the form $k_{(w_\alpha \alpha)^i w_\alpha} k_{(\alpha w_\alpha)^{m-i-1}\alpha}$ there is an entry $k_{(\alpha w_\alpha)^{m-i-1}\alpha} k_{(w_\alpha \alpha)^i w_\alpha}$, they cancel out since $char(\mathbf{k}) = 2$, the entries $k_{(w_\alpha \alpha)^i} k_{(w_\alpha \alpha)^{m-i}}$ and $k_{(\alpha w_\alpha)^i} k_{(\alpha w_\alpha)^{m-i}}$ cancel out because of the previous paragraph (which we need only for m even, otherwise they cancel anyway). So we are left with $k_\alpha^2 = t_\alpha \underline{k}$, as desired. $\square$

Lemma 5.5. *The rank of $D_{H'}$ is $|Q_1| - |Q_1/\pi| - d + 1$, where d is the number of deformed loops in A.*

Proof. Let us construct an epimorphism $j : D_{H'} \to (\mathbf{k}^*)^{|Q_1|-|Q_1/\pi|-d+1}$ such that the kernel $ker(j)$ is finite.

Each cycle of π contains an arrow, which is not a deformed loop. Let us fix one such arrow in each cycle and denote the collection of these arrows by $\mathcal{F}$. Let us label the elements of $(\mathbf{k}^*)^{|Q_1|-|Q_1/\pi|-d+1}$ by x_α, $\alpha \in Q_1, \alpha \notin \mathcal{F} \cup \mathcal{L}$ and by an additional indeterminant x. Define the map j as follows: $j((k_\alpha, \underline{k})) := ((x_\alpha, x))$, where $x_\alpha = k_\alpha, \alpha \notin \mathcal{F} \cup \mathcal{L}$, $x = \underline{k}$. The map j is surjective, since for any (x_α, x) we can define $k_\alpha = \sqrt{t_\alpha x}$ for $\alpha \in \mathcal{L}$ and $k_\alpha = \sqrt[m(C(\alpha))]{x / \prod_{\alpha' \in C(\alpha), \alpha' \neq \alpha, \alpha' \notin \mathcal{L}} x_{\alpha'}^{m(C(\alpha))} \prod_{\alpha' \in C(\alpha), \alpha' \in \mathcal{L}} (\sqrt{t_{\alpha'} x})^{m(C(\alpha))}}$ for $\alpha \in \mathcal{F}$.

Let us compute the kernel of j. $(k_\alpha, \underline{k}) \in ker(j)$ if and only if $\underline{k} = 1$, $k_\alpha = 1$ for $\alpha \notin \mathcal{F} \cup \mathcal{L}$, $k_\alpha^2 = t_\alpha$ for $\alpha \in \mathcal{L}$, $k_\alpha^{2m(C(\alpha))} = 1 / \prod_{\alpha' \in C(\alpha), \alpha' \in \mathcal{L}} t_{\alpha'}^{m(C(\alpha))}$ for $\alpha \in \mathcal{F}$. This clearly defines a finite group.

Passing to the groups of characters, if necessary, and using the equivalence between the category of diagonalizable groups and finitely generated commutative groups [25, Theorem 12.9], we see that the rank of $D_{H'}$ is $|Q_1| - |Q_1/\pi| - d + 1$. $\square$

Proof of Theorem 1.1. Since D_I is connected, its image belongs to the maximal torus in $D_{H'}$ and passing to the groups of characters again, the exact sequence $1 \to D_I \to D_{H'} \to D_{H'}/D_I \to 1$, gives that the rank of $D_{H'}/D_I$ is $|Q_1| - |Q_1/\pi| - d + 1 - |Q_0| + 1 = |Q_0| - |Q_1/\pi| - d + 2 = |E(\Gamma)| - |V(\Gamma)| - d + 2$. $\square$

Using the fact that Brauer graph algebras can be stably (and hence derived) equivalent only to symmetric stably biserial algebras [8, Theorem 1 and 3], Corollary 1.3 can be deduced from Theorems 1.1 and 1.2 and Lemma 4.5, as well as from the fact that for local algebras derived equivalence implies Morita equivalence.

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