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KZ equations and Bethe subalgebras in generalized Yangians related to compatible R-matrices

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The notion of compatible braidings was introduced in Isaev *et al.* (1999, *J. Phys. A*, **32**, L115–L121). On the base of this notion, the authors of Isaev *et al.* (1999, *J. Phys. A*, **32**, L115–L121) defined certain quantum matrix algebras generalizing the RTT algebras and Reflection Equation ones. They also defined analogues of some symmetric polynomials in these algebras and showed that these polynomials generate commutative subalgebras, called Bethe. By using a similar approach, we introduce certain new algebras called generalized Yangians and define analogues of some symmetric polynomials in these algebras. We claim that they commute with each other and thus generate a commutative Bethe subalgebra in each generalized Yangian. Besides, we define some analogues (also arising from couples of compatible braidings) of the Knizhnik–Zamolodchikov equation—classical and quantum.

Keywords: compatible braidings; braided Yangians; Bethe subalgebra; braided r-matrix; braided (quantum) KZ connections.

1. Introduction

The notion of compatible R-matrices (we call them *braidings*) was introduced in [1]. By a braiding, we mean a linear operator $R : V^{\otimes 2} \to V^{\otimes 2}$ subject to the braid relation

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R), \tag{1.1}$$

where V, $\dim V = N < \infty$, is a vector space over the ground field $\mathbb{C}$ and I is the identity operator or its matrix.

According to [1], two braidings R and F are called *compatible* if they are subject to the system

$$R_{12}\,F_{23}\,F_{12} = F_{23}\,F_{12}\,R_{23}, \quad R_{23}\,F_{12}\,F_{23} = F_{12}\,F_{23}\,R_{12}. \tag{1.2}$$

As usual, the low indexes indicate the positions, where a matrix (or an operator) is located. Observe that the matrices A_k, $k \geq 2$ are obtained from that A_1 by means of the usual flip P:

$$A_2 = P_{12}\,A_1\,P_{12}, \quad A_3 = P_{23}\,A_2\,P_{23} = P_{23}\,P_{12}\,A_1\,P_{12}\,P_{23}, \quad \text{and so on.}$$

Hereafter, A is an $N \times N$ matrix (may be with non-commutative entries), whereas $A_1 = A \otimes I_{2\ldots p}$, $A_2 = I_1 \otimes A \otimes I_{3\ldots p}$ and so on are $N^p \times N^p$ matrices.

Following [1], we introduce the following notations

$$A_{\overline{2}} = F_{12}\,A_1\,F_{12}^{-1}, \quad A_{\overline{3}} = F_{23}\,A_{\overline{2}}\,F_{23}^{-1} = F_{23}\,F_{12}\,A_1\,F_{12}^{-1}\,F_{23}^{-1},$$

and so on, where the overlined indexes mean that the matrix A_1 is pushed forward to higher positions by means of the second braiding F. For the sake of the uniformity we also put $A_{\overline{1}} = A_1$.

Below, we fix a basis in the space V and the corresponding bases in $V^{\otimes p}$ and identify operators and their matrices.

Following [1] introduce an algebra $\mathcal{A}(R, F)$ defined by the following system of relations

$$R_{12}\,T_{\overline{1}}\,T_{\overline{2}} = T_{\overline{1}}\,T_{\overline{2}}\,R_{12}, \tag{1.3}$$

where $T = \|t_i^j\|_{1 \leq i,j \leq N}$ is a matrix with entries t_i^j. The matrix T is called *generating matrix* of the algebra $\mathcal{A}(R, F)$.

As was shown in [1], in these algebras (under some conditions on the braidings) it is possible to define analogues of some symmetric polynomials and to establish analogues of the Cayley–Hamilton and Newton identities. (In [1], these identities are combined in the so-called Cayley–Hamilton–Newton ones.) Also, it is possible to show that quantum elementary symmetric polynomials commute with each other and consequently generate a subalgebra of $\mathcal{A}(R, F)$ called *the Bethe subalgebra*.

The first purpose of the present article is to introduce some algebras similar to those $\mathcal{A}(R, F)$ but with infinite number of generators and to generalize the mentioned results to them. Each of these algebras is defined via the system

$$R(u, v)\,T_{\overline{1}}(u)\,T_{\overline{2}}(v) = T_{\overline{1}}(v)\,T_{\overline{2}}(u)\,R(u, v), \tag{1.4}$$

where, $T(u) = \sum_{k \geq 0} T[k]u^{-k}$ is a matrix expanded in a Laurent series and the current (i.e. depending on parameters) quantum R-matrices $R(u, v)$ arises from a braiding R via the Baxterization procedure. We denote this algebra $\mathbf{Y}(R, F)$ and call it the *generalized Yangian.*[1]

[1] Note that if

$$R(u, v) = P - \frac{I}{u - v} \quad \text{and} \quad F = P,$$

we get the famous Drinfeld's Yangian $Y(gl(N))$. (Usually, one also imposes the condition $T[0] = I$.) Its generators are entries of the matrices

$$T[k] = \|t_i^j[k]\|_{1 \leq i,j \leq N}, \quad k = 0, 1, 2...$$

So far, our generalized Yangians are the most general quantum matrix algebras associated with rational and trigonometric R-matrices for which analogues of some symmetric polynomials, namely, elementary ones and power sums, are constructed and their commutativity is claimed. Note that a detailed proof of this claim in a particular case $F = R$ was given in [2]. The general case can be treated in a similar way.

The second purpose of the article is to introduce braided analogues of the Knizhnik–Zamolodchikov (KZ) equation—classical and quantum—and to establish their compatibility. Observe that the former ones are based on a braided version of the first Sklyanin bracket. In its turn, this version is based on braided current r-matrices. In the rational case, such a braided current r-matrix can be easily defined with the help of an involutive symmetry F as follows $r(u,v) = \frac{F}{u-v}$. (So, in this case any second braiding is not needed.)

In the trigonometric case, we are looking for a braided current r-matrix under the form $r(u,v) = \frac{F\,u}{u-v} + r$. In order to find a (constant) summand r, we need a Hecke symmetry $R = R(q)$ analytically depending on q in a vicinity of $q = 1$ and deforming an involutive symmetry F (i.e. $R(1) = F$). Then by expanding $\mathcal{R} = \mathcal{R}(q) = R(q)\,F$ at the point $q = 1$, we get r.

The article is organized as follows. In the next section, we define compatible braidings and exhibit some examples. In Section 3, we introduce the aforementioned quantum symmetric polynomials in the generalized Yangians. In Section 4, we describe braided versions of current r-matrices and the first Sklyanin bracket. In Section 5, we introduce braided analogues of the classical and quantum KZ equation in the spirit of [3] and [4], respectively.

2. Compatible braidings

Let (R,F) be a couple of compatible braidings. As noticed in Section 1, the braiding F is used for transferring the generating matrix T (depending on parameters or not) to the higher positions and the symmetry R comes in the defining relations of the algebras $\mathcal{A}(R,F)$ or that $\mathbf{Y}(R,F)$.

We impose the following conditions on these braidings. We assume R to be an involutive or Hecke symmetry. Remind that a braiding R is called an *involutive symmetry* (respectively, a *Hecke symmetry*) if it is subject to the condition

$$R^2 = I \quad (\text{resp., } (R - q\,I)(R + q^{-1}\,I) = 0,\ q \in \mathbb{C},\ q \neq \pm 1).$$

For such symmetries, we construct current R-matrices according to the following *Baxterization procedure*.

PROPOSITION 1 [5] Let R be an involutive or a Hecke symmetry. Define the operators

$$R(u,v) = R - \frac{I}{u-v} \tag{2.1}$$

for an involutive symmetry R and

$$R(u,v) = R - \frac{(q-q^{-1})\,u\,I}{u-v} \tag{2.2}$$

A similar treatment is valid for our generalized Yangians. However, below we do not use this treatment and deal with the *generating* matrix $T(u)$ in whole. Also, note that even in a particular case $F = P$ the corresponding Yangians of RTT type represent a far-reaching generalization of the Drinfeld's Yangian, since R entering their construction could be any skew-invertible involutive or Hecke symmetry, including those which are deformations of the usual or super-flips.

for a Hecke symmetry. The operators $R(u,v)$ are current R-matrices, i.e. they meet the Quantum Yang–Baxter equation (YBE) with parameters

$$R_{12}(u,v)R_{23}(u,w)R_{12}(v,w) = R_{23}(v,w)R_{12}(u,w)R_{23}(u,v).$$

Namely, these current R-matrices are used in defining generalized Yangians. An R-matrix (2.1) (respectively (2.2)) and the corresponding algebra $\mathbf{Y}(R,F)$ is called *rational* (respectively *trigonometric*).

As for the braidings F, we assume them to be *skew-invertible*. This means that there exists an operator $\Psi^F \in \mathrm{End}(V^{\otimes 2})$ such that

$$\mathrm{Tr}_{(2)} F_{12}\Psi^F_{23} = \mathrm{Tr}_{(2)}\Psi^F_{12}F_{23} = P_{13}.$$

If it is so, we can introduce the so-called F-trace of any square $N \times N$ matrix X by setting

$$\mathrm{Tr}_F X = \mathrm{Tr}(C^F \cdot X), \qquad C^F = \mathrm{Tr}_{(2)}\Psi^F_{12}.$$

Also, we put

$$\mathrm{Tr}_{F(1\dots k)} X_{1\dots k} = \mathrm{Tr}_{(1\dots k)}(C^F_1 \dots C^F_k X_{1\dots k})$$

for any matrix $X_{1\dots k}$ of the appropriate size.

Now, in analogy with the notation $A_{\overline{k}}$ we introduce a similar notation for $N^2 \times N^2$ matrices. Let A_{12} be such a matrix located at positions number 1 and 2, we put

$$A_{\overline{12}} = A_{12}, \quad A_{\overline{13}} = F_{23}A_{\overline{12}}F^{-1}_{23}, \quad A_{\overline{23}} = F_{12}A_{\overline{13}}F^{-1}_{12} = F_{12}F_{23}A_{\overline{12}}F^{-1}_{23}F^{-1}_{12},$$

and so on. In general, the notation $A_{\overline{kl}},\ k < l$ means that transferring of the matrix $A_{12} = A_{\overline{12}}$ to the positions number k and l is performed by means of the symmetry F as follows

$$A_{\overline{kl}} = (F_{k-1\,k}\,F_{k-2\,k-1}...F_{12})(F_{l-1\,l}F_{l-2\,l-1}...F_{23})A_{\overline{12}}(F^{-1}_{23}...F^{-1}_{l-2\,l-1}\,F^{-1}_{l-1\,l})(F^{-1}_{12}...\,F^{-1}_{k-2\,k-1}\,F^{-1}_{k-1\,k}).$$

Thus, if (R,F) is a couple of compatible braidings, the following holds

$$R_{23} = F_{12}\,F_{23}\,R_{12}\,F^{-1}_{23}\,F^{-1}_{12} = R_{\overline{23}}. \tag{2.3}$$

This means that transferring of the braiding R_{12} to the positions 2 and 3 performed either by means of the usual flip P or by means of the braiding F leads to the same result: $R_{\overline{23}} = R_{23}$. This entails that $R_{\overline{i\,i+1}} = R_{i\,i+1}$ for any i.

It is clear that the relation (1.1) for the operator R is equivalent to the quantum YBE

$$\mathcal{R}_{12}\,\mathcal{R}_{13}\,\mathcal{R}_{23} = \mathcal{R}_{23}\,\mathcal{R}_{13}\,\mathcal{R}_{12} \tag{2.4}$$

for the operator $\mathcal{R} = R\,P$.

However, if (R,F) is a couple of compatible braidings, it is easy to see that the operator $\mathcal{R} = R\,F$ is subject to the following 'braided version' of the quantum YBE

$$\mathcal{R}_{\overline{12}}\,\mathcal{R}_{\overline{13}}\,\mathcal{R}_{\overline{23}} = \mathcal{R}_{\overline{23}}\,\mathcal{R}_{\overline{13}}\,\mathcal{R}_{\overline{12}}. \tag{2.5}$$

Moreover, the following generalization of (2.5) is valid

$$\mathcal{R}_{\overline{ij}}\,\mathcal{R}_{\overline{ik}}\,\mathcal{R}_{\overline{jk}} = \mathcal{R}_{\overline{jk}}\,\mathcal{R}_{\overline{ik}}\,\mathcal{R}_{\overline{ij}}, \tag{2.6}$$

provided i, j, k are positive integers such that $i < j < k$.

The operators subject to (2.5) are called *braided R-matrices*.

Observe that if F is an involutive symmetry, the relation (2.6) becomes valid for any positive pairwise distinct integers i, j, k. Besides, the notation $A_{\overline{ij}}$ is well defined for any matrix A_{12} and any distinct positive integers i and j. For instance, $A_{\overline{21}} = F\,A_{12}\,F$.

Consider a few examples of compatible braidings. The braidings R and $F = P$ are compatible for any R. The corresponding algebra $\mathbf{Y}(R,P)$ is called the (generalized) Yangian of RTT type. The Drinfeld's Yangian $\mathbf{Y}(gl(N))$ is a particular case, respective to $R = P$. Another example of such an algebra is the so-called q-Yangian, as it is defined in [6]. The Hecke symmetry R entering its definition is that coming from the quantum group $U_q(sl(N))$.

It is also evident that if $F = R$, then the braidings R and F are compatible. We say that the corresponding braided Yangian $\mathbf{Y}(R,R)$ is of Reflection Equation (RE) type.

If $F = P_{(m|n)}$ is a super-flip, and R is the Hecke symmetry coming from the Quantum super-group $U_q(sl(m|n))$, the braidings R and F are compatible. In the case $m = n = 1$, the Hecke symmetry R is represented in a basis by the following matrix

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}.$$

Note that this Hecke symmetry $R = R(q)$ is a deformation of the involutive symmetry $F = P_{(m|n)}$ and it depends analytically on q. Thus, we are in the frameworks of setting discussed at the end of Section 1.

3. Symmetric polynomials in generalized Yangians

In this section, we define (quantum) symmetric polynomials in generalized Yangians. Namely, we are dealing with elementary symmetric polynomials and power sums. However, first we consider an $N \times N$ numerical matrix M. In this case, the elementary symmetric polynomials $e_k(M)$ and quantum power sums $p_k(M)$ for this matrix are respectively defined as follows:

$$\det(M - t\,I) = \sum_{0}^{N} (-t)^{N-k} e_k(M), \quad p_k(M) = \mathrm{Tr}\, M^k. \tag{3.1}$$

Note that if M is triangular matrix, these polynomials are respectively equal to the elementary symmetric polynomials and power sums in eigenvalues of M. This motivates the terminology.

If M is a matrix with entries belonging to a non-commutative algebra, it is not in general possible to define analogues of these polynomials with interesting properties. Fortunately, it is possible to do in the algebras $\mathcal{A}(R,F)$ (see [1]) and generalized Yangians $\mathbf{Y}(R,F)$. First, define elementary symmetric polynomials in the trigonometric generalized Yangians

If R is a Hecke symmetry, we put

$$e_0(u) = 1, \quad e_k(u) = \mathrm{Tr}_{F(1\dots k)} A^{(k)}_{1\dots k}(R)\, T_{\overline{1}}(u) T_{\overline{2}}(q^{-2}u) \dots T_{\overline{k}}(q^{-2(k-1)}u), \quad k = 1, 2 \dots \tag{3.2}$$

Here, $A^{(k)}_{1\dots k}(R)$ is the skew-symmetrizer acting in the space $V^{\otimes k}$ and arising from the Hecke symmetry R. It can be defined by the following recurrent relations

$$A^{(1)} = I, \qquad A^{(k+1)}_{1\dots k+1} = \frac{k_q}{(k+1)_q} A^{(k)}_{1\dots k}\left(\frac{q^k}{k_q} I - R_k\right) A^{(k)}_{1\dots k}, \quad k \ge 1. \tag{3.3}$$

Hereafter, we sometimes use the following simplified notation R_k instead of that R_{kk+1}.

In the rational case, the corresponding elementary symmetric polynomials are defined as follows

$$e_0(u) = 1, \quad e_k(u) = \mathrm{Tr}_{F(1\dots k)} A^{(k)}_{1\dots k}(R)\, T_{\overline{1}}(u) T_{\overline{2}}(u-1) \dots T_{\overline{k}}(u-k+1), \quad k = 1, 2\dots \tag{3.4}$$

Here, $A^{(k)}_{1\dots k}(R)$ is the skew-symmetrizer respective to the involutive symmetry R. Its explicit formula can be obtained from that (3.3) at $q = 1$.

As for the power sums, we define them respectively as follows

$$p_0(u) = 1, \quad p_k(u) = \mathrm{Tr}_{F(1\dots k)} T_{\overline{1}}(q^{-2(k-1)}u) T_{\overline{2}}(q^{-2(k-2)}u) \dots T_{\overline{k}}(u) R_{k-1} \dots R_2 R_1, \quad k \ge 1,$$
$$p_0(u) = 1, \quad p_k(u) = \mathrm{Tr}_{F(1\dots k)} T_{\overline{1}}(u-k+1) T_{\overline{2}}(u-k+2) \dots T_{\overline{k}}(u) R_{k-1} \dots R_2 R_1, \quad k \ge 1.$$

It should be emphasized that in the braided Yangians of RE type these formulae could be reduced to the following forms respectively

$$p_k(u) = \mathrm{Tr}_F T(q^{-2(k-1)}u) T(q^{-2(k-2)}u) \dots T(u), \qquad p_k(u) = \mathrm{Tr}_F T(u-k+1) T(u-k+2) \dots T(u).$$

Observe that these formulae are in a sense similar to the second formula from (3.1) but they contain shifts of the arguments, multiplicative and additive, respectively.

Also, note that the elementary symmetric polynomials and power sums are related via a quantum version of the Newton identities. If R is a Hecke symmetry these identities are

$$k_q e_k(u) - q^{k-1} p_1(q^{-2(k-1)}u) e_{k-1}(u) + q^{k-2} p_2(q^{-2(k-2)}u) e_{k-2}(u) + \dots + (-1)^k p_k(u) e_0(u) = 0.$$

If R is involutive, then we have

$$k e_k(u) - p_1(u-k+1) e_{k-1}(u) + p_2(u-k+2) e_{k-2}(u) + \dots + (-1)^k p_k(u) = 0.$$

A proof of these identities is given in [5] for the braided Yangians of RE type. For braided Yangians of general form, these identities can be proved in a similar way.

Also, note that if the bi-rank of R is $(m|0)$, then $e_k(u) \equiv 0$ for $k > m$. The subalgebra generated in $\mathbf{Y}(R, F)$ by the elements $e_k(u)$ is called the *Bethe* one.

PROPOSITION 2 Let $\mathbf{Y}(R, F)$ be a rational or trigonometric generalized Yangian. Then the elements $e_k(u), \quad k = 1, 2\dots$ and consequently these $p_k(u), \quad k = 1, 2\dots$ commute with each other:

$$e_k(u)\, e_p(v) = e_p(v)\, e_k(u), \qquad \forall\, k, p, \quad \forall\, u, v,$$

and consequently the Bethe subalgebra is commutative.

A detailed proof of this claim is given in [2] for the braided Yangians of RE type. Generalized Yangians corresponding to all couples (R, F) under consideration can be treated in a similar way.

REMARK 3 In [7], a notion of half-quantum algebras (HQA) was introduced. Each of these algebras is also defined via a couple of compatible braidings, namely, by the system

$$S^{(2)}_{12}(R)\, T_{\overline{1}}\, T_{\overline{2}}\, A^{(2)}_{12}(R) = 0, \quad \text{where} \quad A^{(2)}_{12}(R) = \frac{q\,I - R}{q + q^{-1}}, \quad S^{(2)}_{12}(R) = \frac{q^{-1}\,I + R}{q + q^{-1}} \tag{3.5}$$

are the skew-symmetrizer and symmetrizer respectively, provided R is a Hecke symmetry. As usual, the braiding F is employed for defining the overlined indexes.

Let us exhibit an equivalent form of (3.5), which is useful in the study of the generalized Yangians,

$$A^{(2)}_{12}(R)\, T_{\overline{1}}\, T_{\overline{2}}\, A^{(2)}_{12}(R) = T_{\overline{1}}\, T_{\overline{2}}\, A^{(2)}_{12}(R).$$

In the HQA, there exist analogues of the elementary symmetric polynomials and power sums and those of the Newton and Cayley–Hamilton identities (see [7]).

However, in general in the HQA the commutativity of the these symmetric polynomials is not valid.

The HQA are related to the braided Yangians as follows. In the trigonometric R-matrix $R(u, v)$ we put $v = q^{-2}\,u$. Then, we have

$$R(u, q^{-2}u) = R(q) - \frac{(q - q^{-1})q^2\, I}{(q^2 - 1)} = R(q) - q\,I.$$

This operator coincides up to a factor $-(q + q^{-1})$ with the skew-symmetrizer $A^{(2)}(R)$.

After multiplying the defining system of the corresponding braided Yangian evaluated at $v = q^{-2}\,u$ by $S^{(2)}_R$ from the right-hand side, we get the relation

$$A^{(2)}_{12}(R)\, T_{\overline{1}}(u)\, T_{\overline{2}}(q^{-2}u)\, S^{(2)}_{12}(R) = 0,$$

which can be written as follows

$$A^{(2)}_{12}(R)\, (q^{-2u\partial_u}\, T_{\overline{1}}(u))\, (q^{-2u\partial_u}\, T_{\overline{2}}(u))\, S^{(2)}_{12}(R) = 0, \quad \partial_u = \frac{d}{d\,u}.$$

This relation looks like that in a HQA but the role of the matrices $T(u)$ is played by the operator $q^{-2u\partial_u}\, T(u)$.

A similar treatment is possible in the rational braided Yangians but in them the shifts are additive, since the parameters u and v are related as follows $u - v = 1$.

4. Braided r-matrices and braided Sklyanin brackets

Let F be an involutive symmetry. Let us consider the following operator

$$r(u, v) = \frac{F}{u - v}, \tag{4.1}$$

which is a braided generalization of the rational r-matrix $r(u, v) = \frac{P}{u-v}$.

It is easy to see that it meets the following relations

$$1.\ r_{\overline{21}}(v,u) + r_{\overline{12}}(u,v) = 0, \tag{4.2}$$

$$2.\ [r_{\overline{12}}(u,v)\, r_{\overline{13}}(u,w)] + [r_{\overline{12}}(u,v)\, r_{\overline{23}}(v,w)] + [r_{\overline{13}}(u,w)\, r_{\overline{23}}(v,w)] = 0. \tag{4.3}$$

In order to define a braided analogue of *trigonometric* r-matrix, we need two compatible braidings. Again, let (R, F) be a couple of compatible braidings. Also, suppose that F is an involutive symmetry and $R = R(q)$ is a Hecke symmetry deforming F as described in Section 1. Let us expand the operator $\mathcal{R} = R\,F$ at the point $q = 1$:

$$\mathcal{R} = I + h\,r + O(h^2), \quad q = e^h. \tag{4.4}$$

Definition 4 The element $r \in End(V^{\otimes 2})$ entering this expansion is called a (constant) braided r-matrix.

Proposition 5 This braided r-matrix r has the following properties

$$1.\ r_{\overline{12}} + r_{\overline{21}} = 2F, \tag{4.5}$$

$$2.\ [r_{\overline{12}}\, r_{\overline{13}}] + [r_{\overline{12}}\, r_{\overline{23}}] + [r_{\overline{13}}\, r_{\overline{23}}] = 0. \tag{4.6}$$

Proof. Similarly to the classical case, the relation (4.6) immediately follows from (2.5). The relation (4.6) follows from that

$$(\mathcal{R}\,F)^2 = (q - q^{-1})(\mathcal{R}\,F) + I.$$

Now, consider the following operator

$$r(u,v) = \frac{F\,u}{u - v} - \frac{r}{2}. \tag{4.7}$$

We call this operator *braided trigonometric r-matrix*. As usual, the term 'trigonometric' is justified by another form of this operator obtained by the change $u \to q^u$, $v \to q^v$. □

Proposition 6 The operator (4.7) meets the relations (4.2) and (4.3).

Proof. The first relation follows immediately from (4.5). In order to show (4.3), we have to compute the braided Schouten bracket of the operator $r(u,v)$ with itself. Let us precise that by the *braided Schouten bracket* of two such operators $A(u,v)$ and $B(u,v)$ we mean the following expression

$$\begin{aligned}[][[A,B]](u,v,w) = {} & [A_{\overline{12}}(u,v), B_{\overline{13}}(u,w)] + [A_{\overline{12}}(u,v), B_{\overline{23}}(v,w)] + [A_{\overline{13}}(u,w), B_{\overline{23}}(v,w)] \\ & + [B_{\overline{12}}(u,v), A_{\overline{13}}(u,w)] + [B_{\overline{12}}(u,v), A_{\overline{23}}(v,w)] + [B_{\overline{13}}(u,w), A_{\overline{23}}(v,w)].\end{aligned}$$

If A and/or B are constant, the corresponding parameters should be omitted.

By direct computations, we have that the Schouten bracket of the summand $A = \frac{F\,u}{u-v}$ with itself is equal to

$$[[A, A]](u, v, w) = \frac{2\,u}{u-w}\,[F_{23}, F_{12}].$$

Also, the following holds

$$\begin{aligned}[][[A, r]](u, v, w) &= \left[\frac{F_{\overline{12}}\,u}{u-v}, r_{\overline{13}}\right] + \left[r_{\overline{12}}, \frac{F_{\overline{13}}\,u}{u-w}\right] + \left[\frac{F_{\overline{12}}\,u}{u-v}, r_{\overline{23}}\right] + \left[r_{\overline{12}}, \frac{F_{\overline{23}}\,v}{v-w}\right] \\ &\quad + \left[\frac{F_{\overline{13}}\,u}{u-w}, r_{\overline{23}}\right] + \left[r_{\overline{13}}, \frac{F_{\overline{23}}\,v}{v-w}\right] \\ &= [F_{\overline{13}}, r_{\overline{23}} - r_{\overline{12}}]\frac{u}{u-w} = (-r_{\overline{23}} + r_{\overline{12}} + r_{\overline{21}} - r_{\overline{32}})\,\frac{F_{\overline{13}}u}{u-w} = 2[F_{23}, F_{12}]\frac{u}{u-w}.\end{aligned}$$

Besides, $[[r, r]] = 0$ in virtue of (4.6). This completes the proof. □

Let us remark that due to the property (4.2) of the braided r-matrix $r(u, v)$ defined by (4.7) it can be cast under the following form

$$r(u, v) = \frac{1}{2}\left(\frac{F(u+v)}{u-v} - \frac{r - r_{\overline{21}}}{2}\right).$$

It should be emphasized that we do not use any concrete form of the symmetries R and F. Below, we also need following properties of braided R and r-matrices.

PROPOSITION 7 If (R, F) is a couple of compatible braidings, then the following holds

$$\mathcal{R}_{12}\,A_{\overline{3}} = A_{\overline{3}}\,\mathcal{R}_{12}$$

for any $N \times N$ matrix A.

Proof. By using the compatibility of the braidings R and F, we get

$$\begin{aligned}\mathcal{R}_{12}\,F_{23}\,F_{12}\,A_1\,F_{12}^{-1}\,F_{23}^{-1} &= R_{12}\,F_{12}\,F_{23}\,F_{12}\,A_1\,F_{12}^{-1}\,F_{23}^{-1} = R_{12}\,F_{23}\,F_{12}\,F_{23}\,A_1\,F_{12}^{-1}\,F_{23}^{-1} \\ &= F_{23}\,F_{12}\,R_{23}\,F_{23}\,A_1\,F_{12}^{-1}\,F_{23}^{-1} = F_{23}\,F_{12}\,A_1\,R_{23}F_{23}\,F_{12}^{-1}\,F_{23}^{-1} = F_{23}\,F_{12}\,A_1\,F_{12}^{-1}\,F_{23}^{-1}\,\mathcal{R}_{12}.\end{aligned}$$

COROLLARY 8 Additionally, suppose F to be an involutive symmetry. Then for any pairwise distinct positive integers i, j, k we have

$$\mathcal{R}_{\overline{ij}}\,A_{\overline{k}} = A_{\overline{k}}\,\mathcal{R}_{\overline{ij}}. \tag{4.8}$$

PROPOSITION 9 Under the same hypothesis, if A is an $N \times N$ matrix and $r(u, v)$ is a braided current (rational or trigonometric) r-matrix, then the following holds

$$r_{\overline{ij}}(u, v)\,A_{\overline{k}} = A_{\overline{k}}\,r_{\overline{ij}}(u, v). \tag{4.9}$$

Proof. It suffices to consider the case $i = 1, j = 2, k = 3$. Then this relation is clear for any braided rational r-matrix. It is also so for the first summand of any braided trigonometric r-matrix. For the second summand it follows from the previous claim. □

Now, we pass to describing a braided analogue of the first Sklyanin bracket. Let F be an involutive symmetry and $r(u,v)$ be a braided rational or trigonometric r-matrix defined correspondingly by (4.1) or (4.7). Define an involutive symmetry on the space of the currents as follows

$$\mathcal{F}(T_{\overline{1}}(u) \otimes T_{\overline{2}}(v)) = F\,(T_{\overline{1}}(v) \otimes T_{\overline{2}}(u))\,F = T_{\overline{2}}(v) \otimes T_{\overline{1}}(u).$$

Now, we introduce a Lie type operator, which is a braided analogue of the first Sklyanin bracket:

$$[T_{\overline{1}}(u), T_{\overline{2}}(v)]^{\mathcal{F}} = [T_{\overline{1}}(u) + T_{\overline{2}}(v), r(u,v)]. \tag{4.10}$$

PROPOSITION 10 The operator (4.10) meets the following conditions

1. $[T_{\overline{1}}(u), T_{\overline{2}}(v)]^{\mathcal{F}} = -[\ ,\ ]^{\mathcal{F}}\mathcal{F}(T_{\overline{1}}(u), T_{\overline{2}}(v)) = -[\ ,\ ]^{\mathcal{F}}(T_{\overline{2}}(v), T_{\overline{1}}(u))$,
2. $[\ ,\ ]^{\mathcal{F}}\,[\ ,\ ]^{\mathcal{F}}_{23}(I + \mathcal{F}_{12}\,\mathcal{F}_{23} + \mathcal{F}_{23}\,\mathcal{F}_{12})(T_{\overline{1}}(u) \otimes T_{\overline{2}}(v) \otimes T_{\overline{3}}(w)) = 0.$

Proof. The first claim follows immediately from the skew-symmetry of the braided r-matrix $r(u,v)$. In order to prove the second one, we use the relations (4.9) with $A = T$. Then we have

$$\begin{aligned}
[T_{\overline{1}}(u), [T_{\overline{2}}(v), T_{\overline{3}}(w)]^{\mathcal{F}}]^{\mathcal{F}} &= [T_{\overline{1}}(u), [T_{\overline{2}}(v) + T_{\overline{3}}(w), r_{\overline{23}}(v,w)]]^{\mathcal{F}} \\
&= [T_{\overline{1}}(u), (T_{\overline{2}}(v) + T_{\overline{3}}(w))\, r_{\overline{23}}(v,w) - r_{\overline{23}}(v,w)\,(T_{\overline{2}}(v) + T_{\overline{3}}(w))]^{\mathcal{F}} \\
&= ([T_{\overline{1}}(u), T_{\overline{2}}(v)]^{\mathcal{F}} + [T_{\overline{1}}(u), T_{\overline{3}}(w)]^{\mathcal{F}})\, r_{\overline{23}}(v,w) - r_{\overline{23}}(v,w)\,([T_{\overline{1}}(u), T_{\overline{2}}(v)]^{\mathcal{F}} + [T_{\overline{1}}(u), T_{\overline{3}}(w)]^{\mathcal{F}}).
\end{aligned}$$

By using the relation (4.9) once more, we can reduce this expression to the following form

$$[\mathfrak{X}, r_{\overline{12}}(u,v)]\, r_{\overline{23}}(v,w) + [\mathfrak{X}, r_{\overline{13}}(u,w)]\, r_{\overline{23}}(v,w) - r_{\overline{23}}(v,w)\,[\mathfrak{X}, r_{\overline{12}}(u,v)] + r_{\overline{23}}(v,w)[\mathfrak{X}, r_{\overline{13}}(u,w)],$$

where $\mathfrak{X} = T_{\overline{1}}(u) + T_{\overline{2}}(v) + T_{\overline{3}}(w)$.

Now, by applying the operators $\mathcal{F}_{\overline{12}}\,\mathcal{F}_{\overline{23}}$ and $\mathcal{F}_{\overline{23}}\,\mathcal{F}_{\overline{12}}$ to this expression and by adding all results we arrive to the conclusion. □

5. Braided KZ equations—classical and quantum

Let us pass to constructing a family of commuting differential operators looking like the famous KZ ones. To this end, we consider two families of matrices

$$M_i = g_{\overline{i}} + \kappa \sum_{k \neq i}^{n} \frac{F_{\overline{ik}}}{u_i - u_k}, \quad i = 1, \ldots, n$$

and

$$N_i = g_{\overline{i}} + \kappa \sum_{k \neq i}^{n} \left(\frac{F_{\overline{ik}}\, u_i}{u_i - u_k} - \frac{r_{\overline{ik}}}{2} \right), \quad i = 1, \ldots, n,$$

associated with braided rational and trigonometric r-matrices correspondingly. Here $n \geq 2$ is an integer, $\kappa \in \mathbb{C}$ is an arbitrary parameter, $u_i \in \mathbb{C}, i = 1, ..., n$ are pairwise distinct numbers and g is a numerical $N \times N$ matrix. Also, throughout this section F is assumed to be a skew-invertible involutive symmetry.

Also, constitute two families of differential operators

$$\partial_i - M_i = \partial_i - \left(g_{\overline{i}} + \kappa \sum_{k \neq i}^{n} \frac{F_{\overline{i}\,\overline{k}}}{u_i - u_k} \right), \qquad i = 1, 2, ..., n, \tag{5.1}$$

and

$$u_i\, \partial_i - N_i = u_i\, \partial_i - \left(g_{\overline{i}} + \kappa \sum_{k \neq i}^{n} \left(\frac{F_{\overline{i}\,\overline{k}}\, u_i}{u_i - u_k} - \frac{r_{\overline{i}\,\overline{k}}}{2} \right) \right), \quad i = 1, 2, ..., n. \tag{5.2}$$

where $\partial_i = \partial_{u_i}$. We call the operators (5.1) and (5.2) *the braided KZ connections*—rational and trigonometric, respectively.

PROPOSITION 11 Assume that the matrix g is subject to the relation $g_{\overline{1}}\, g_{\overline{2}} = g_{\overline{2}}\, g_{\overline{1}}$. Then the following equations hold

$$\partial_i\, M_j - \partial_j\, M_i - [M_i, M_j] = 0, \quad u_i \partial_i\, N_j - u_j \partial_j\, N_i - [N_i, N_j] = 0. \tag{5.3}$$

Proof. First, note that the condition on the matrix g can be cast under the following form

$$g_1\, F_{12}\, g_1\, F_{12} = F_{12}\, g_1\, F_{12}\, g_1, \tag{5.4}$$

which means that this matrix realizes a one-dimensional representation of the RE algebra associated with the involutive symmetry F.

The proof of the first relation in (5.3) results from the fact that the operator $\frac{F}{u-v}$ is a braided r-matrix and the following relations take place:

$$\partial_i \left(\frac{F_{\overline{j}\,\overline{i}}}{u_j - u_i} \right) = \partial_j \left(\frac{F_{\overline{i}\,\overline{j}}}{u_i - u_j} \right), \tag{5.5}$$

$$[g_{\overline{i}}, F_{\overline{j}\,\overline{k}}] = 0, \quad [g_{\overline{i}}, F_{\overline{j}\,\overline{i}}] + [g_{\overline{j}}, F_{\overline{i}\,\overline{j}}] = 0, \quad [g_{\overline{i}}, g_{\overline{j}}] = 0,$$

where i, j and k are pairwise distinct integers. Note that $F_{\overline{i}\,\overline{j}} = F_{\overline{j}\,\overline{i}}$.

The second relation in (5.3) can be proved in the same way, with using the relation (4.9) and the following one

$$u_i \partial_i \left(\frac{F_{\overline{j}\,\overline{i}}\, u_j}{u_j - u_i} \right) = u_j \partial_j \left(\frac{F_{\overline{i}\,\overline{j}}\, u_i}{u_i - u_j} \right)$$

instead of (5.5). ∎

COROLLARY 12 The corresponding systems of differential equations (called the KZ ones)

$$\partial_i \,\Psi = M_i \,\Psi, \quad i = 1, 2...n,$$
$$u_i \,\partial_i \,\Psi = N_i \,\Psi, \quad i = 1, 2...n,$$

where Ψ is a vector-function of the length N, are compatible.

Our next purpose is to introduce quantum counterparts of these systems. In the case related to the affine Quantum Groups these counterparts were introduced in [4]. Similarly to [4] (also, see [8]), we get systems of difference equations. However, we deal without any Quantum Group. Instead, we use the rational and trigonometric braidings $R(u, v)$ defined by (4.1) or (4.7) correspondingly.

Consider the following operators

$$\mathcal{R}(u, v) = R(u, v)\, F f(u, v)^{-1} \tag{5.6}$$

where

$$f(u, v) = 1 - \frac{1}{u - v} \quad \text{or} \quad f(u, v) = q - \frac{(q - q^{-1})\, u}{u - v} \tag{5.7}$$

The former (respectively, latter) $f(u, v)$ corresponds to a rational (respectively, trigonometric) case.

Observe that due to the factor $f(u, v)^{-1}$ in (5.6) the following relation

$$\mathcal{R}_{\overline{12}}(u, v)\, \mathcal{R}_{\overline{21}}(v, u) = I$$

holds.

Similarly to the classical braided KZ operators above, below we use a matrix g subject to the condition $g_{\bar{i}}\, g_{\bar{j}} = g_{\bar{j}}\, g_{\bar{i}}$. Observe that in virtue of (4.8), we have

$$[R_{\overline{ij}}, g_{\bar{k}}] = 0.$$

Here i, j and k are pairwise distinct positive integers. Let us additionally demand

$$[R_{\overline{ij}}, g_{\bar{i}} \otimes g_{\bar{j}}] = 0.$$

Besides, introduce operators

$$T_i^{\text{add}} f(u_1...u_i...u_n) = f(u_1...u_i + p...u_n), \quad T_i^{\text{mult}} f(u_1...u_i...u_n) = f(u_1...p\, u_i...u_n) \tag{5.8}$$

which perform the shifts of the variables, additive and multiplicative respectively. Here, $p \in \mathbb{C}$ is an arbitrary non-trivial number.

Observe that any rational (respectively, trigonometric) R-matrix $R(u,v)$ is invariant with respect to the operator T_i^{add} (respectively T_i^{mult}) applied to the both parameters u and v. Namely, we have

$$(T_i \otimes T_i)(R(u,v)) = R(u,v).$$

Hereafter, T_i for $i = 1,2$ stands for the operator T_i^{add} or $T_i = T_i^{\text{mult}}$ depending on the type of $R(u,v)$.

Introduce the following notation

$$\mathcal{R}_{i\downarrow} = \mathcal{R}_{\overline{i\,i-1}}(u_i,u_{i-1})...\mathcal{R}_{\overline{i\,1}}(u_i,u_1),$$
$$\mathcal{R}_{i\uparrow} = \mathcal{R}_{\overline{i\,n}}(u_i,u_n)...\mathcal{R}_{\overline{i\,i+1}}(u_i,u_{i+1}).$$

Also, we consider the operators

$$\Delta_i = \mathcal{R}_{i\downarrow}\, T_i^{-1}\, g_{\bar{i}}\, \mathcal{R}_{i\uparrow} = \mathcal{R}_{i\downarrow}\, \Theta_i\, \mathcal{R}_{i\uparrow}, \tag{5.9}$$

where the notation $\Theta_i = T_i^{-1}\, g_{\bar{i}}$ is used.

We are interested in the compatibility condition of the system

$$T_i\, \Psi(u_1...u_n) = \kappa\, T_i\, \mathcal{R}_{i\downarrow}\, \Theta_i\, \mathcal{R}_{i\uparrow}\, \Psi(u_1...u_n), \quad i = 1,...,n, \tag{5.10}$$

where κ is an arbitrary nontrivial number.

More explicitly, in the rational case the ith equation of this system reads

$$\Psi(u_1...u_i + p...u_n) = \kappa \mathcal{M}_i \Psi(u_1...u_i...u_n), \tag{5.11}$$

where,

$$\mathcal{M}_i = T_i\, \mathcal{R}_{i\downarrow}\, \Theta_i\, \mathcal{R}_{i\uparrow} = \mathcal{R}_{\overline{i\,i-1}}(u_i+p,u_{i-1})...\mathcal{R}_{\overline{i\,1}}(u_i+p,u_1) g_{\bar{i}} \mathcal{R}_{\overline{i\,n}}(u_i,u_n)...\mathcal{R}_{\overline{i\,i+1}}(u_i,u_{i+1}).$$

The compatibility condition of the system (5.11) is called the holonomy condition (see [8, Section 10.5]). It takes the form

$$T_i \mathcal{M}_j T_i^{-1} \mathcal{M}_i = T_j \mathcal{M}_i T_j^{-1} \mathcal{M}_j. \tag{5.12}$$

The commutativity of T_i and T_j provides an equivalent form of (5.12)

$$[T_j^{-1}\mathcal{M}_j, T_i^{-1}\mathcal{M}_i] = 0.$$

PROPOSITION 13 The system (5.10) satisfies the holonomy condition in both rational and trigonometric cases.

Proof. The demonstration is similar to the non-braided case. We exhibit it to make our paper self-contained.

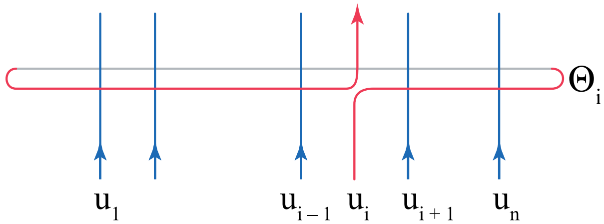

FIG. 1. Δ_i.

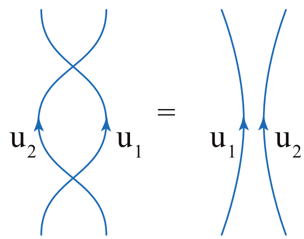

FIG. 2. Second Reidemeister move.

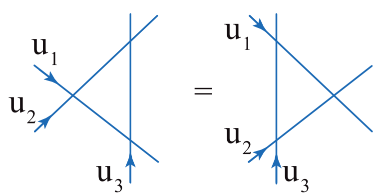

FIG. 3. The Yang–Baxter equation.

Below, we systematically use the following relations

$$\mathcal{R}_{\overline{ij}}\mathcal{R}_{\overline{kl}} = \mathcal{R}_{\overline{kl}}\mathcal{R}_{\overline{ij}},$$

provided i, j, k, l are positive pairwise distinct integers.

Let us illustrate the expression Δ_i by the Fig. 1. Here, we use the standard pictorial representation for the basic structure equations involving $\mathcal{R}$-operator. These resemble the second and third Reidemeister moves illustrated in Figs 2 and 3.

All pictures imply the application of $\mathcal{R}$ at the crossing points to the vector spaces with corresponding numbers and with appropriate spectral parameters marked on the diagram. Here, the symbol $\Theta_i = T_i^{-1} g_{\bar{i}}$ means the application of this operator to the corresponding space.

On Fig. 4, we draw the diagram, corresponding to the expression $\Delta_i \Delta_j$. Without loss of generality, we assume that $j < i$. We perform a series of mutations using the YBE and once the commutation relation with the shift operator. The green arrow on the Fig. 5 illustrates the following algebraic transformation.

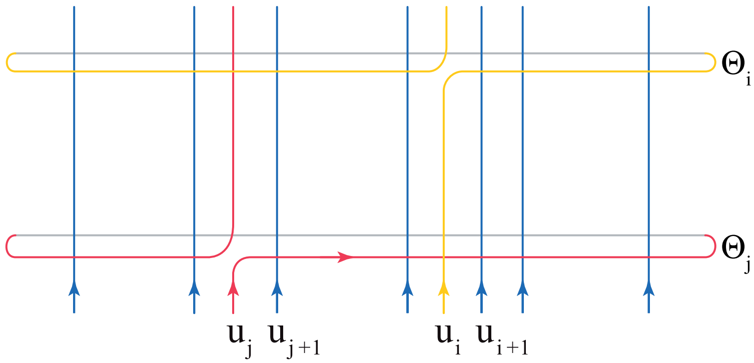

FIG. 4. $\Delta_i\Delta_j$.

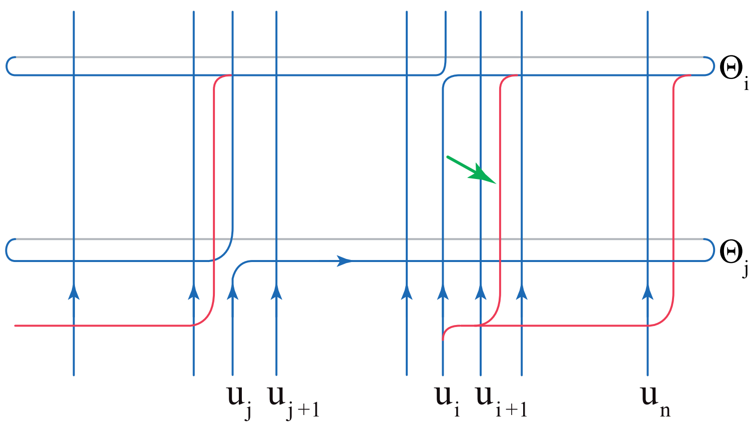

FIG. 5. Elementary YB transformation.

First, we push forward the element $\mathcal{R}_{\overline{i\,i+1}}$ till $\mathcal{R}_{\overline{j\,i+1}}\mathcal{R}_{\overline{j\,i}}$.

$$\begin{aligned}\Delta_i\Delta_j &= \mathcal{R}_{\overline{i\,i-1}}\dots\mathcal{R}_{\overline{i\,1}}\Theta_i\mathcal{R}_{\overline{i\,n}}\dots\mathcal{R}_{\overline{i\,i+1}}\mathcal{R}_{\overline{j\,j-1}}\dots\mathcal{R}_{\overline{j\,1}}\Theta_j\mathcal{R}_{\overline{j\,n}}\dots\mathcal{R}_{\overline{j\,j+1}}\\ &= \mathcal{R}_{\overline{i\,i-1}}\dots\mathcal{R}_{\overline{i\,1}}\Theta_i\mathcal{R}_{\overline{i\,n}}\dots\mathcal{R}_{\overline{i\,i+2}}\mathcal{R}_{\overline{j\,j-1}}\dots\mathcal{R}_{\overline{j\,1}}\Theta_j\mathcal{R}_{\overline{j\,n}}\dots\mathcal{R}_{\overline{j\,i+2}}\\ &\times \mathcal{R}_{\overline{i\,i+1}}\mathcal{R}_{\overline{j\,i+1}}\mathcal{R}_{\overline{j\,i}}\dots\mathcal{R}_{\overline{j\,j+1}}.\end{aligned}$$

Now, by using the YBE we arrive to the following formula

$$\begin{aligned}\Delta_i\Delta_j &= \mathcal{R}_{\overline{i\,i-1}}\dots\mathcal{R}_{\overline{i\,1}}\Theta_i\mathcal{R}_{\overline{i\,n}}\dots\mathcal{R}_{\overline{i\,i+2}}\mathcal{R}_{\overline{j\,j-1}}\dots\mathcal{R}_{\overline{j\,1}}\Theta_j\mathcal{R}_{\overline{j\,n}}\dots\mathcal{R}_{\overline{j\,i+2}}\\ &\times \mathcal{R}_{\overline{j\,i}}\mathcal{R}_{\overline{j\,i+1}}\mathcal{R}_{\overline{j\,i-1}}\dots\mathcal{R}_{\overline{j\,j+1}}\mathcal{R}_{\overline{i\,i+1}}.\end{aligned}$$

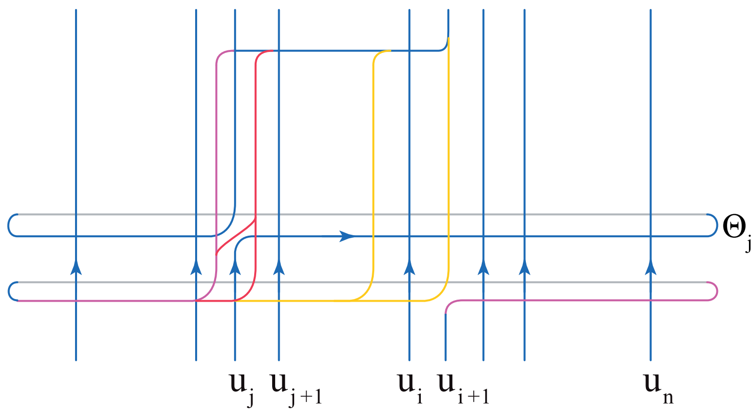

FIG. 6. Involutivity transformation.

Then, by the similar transformation we transpose the groups of magenta elements and the group of blue ones.

$$\begin{aligned}\Delta_i\Delta_j &= \mathcal{R}_{\overline{i\,i-1}}\dots\mathcal{R}_{\overline{i\,1}}\Theta_i\mathcal{R}_{\overline{i\,n}}\dots\mathcal{R}_{\overline{i\,i+2}}\mathcal{R}_{\overline{j\,j-1}}\dots\mathcal{R}_{\overline{j\,1}}\Theta_j\mathcal{R}_{\overline{j\,n}}\dots\mathcal{R}_{\overline{j\,i+2}}\\ &\times \mathcal{R}_{\overline{j\,i}}\mathcal{R}_{\overline{j\,i+1}}\mathcal{R}_{\overline{j\,i-1}}\dots\mathcal{R}_{\overline{j\,j+1}}\mathcal{R}_{\overline{i\,i+1}}\\ &= \mathcal{R}_{\overline{i\,i-1}}\dots\mathcal{R}_{\overline{i\,1}}\Theta_i\mathcal{R}_{\overline{j\,j-1}}\dots\mathcal{R}_{\overline{j\,1}}\Theta_j\mathcal{R}_{\overline{j\,i}}\mathcal{R}_{\overline{j\,n}}\dots\mathcal{R}_{\overline{j\,i+2}}\mathcal{R}_{\overline{j\,i+1}}\\ &\times \mathcal{R}_{\overline{j\,i-1}}\dots\mathcal{R}_{\overline{j\,j+1}}\mathcal{R}_{\overline{i\,n}}\dots\mathcal{R}_{\overline{i\,i+2}}\mathcal{R}_{\overline{i\,i+1}}.\end{aligned}$$

In this procedure, we used the YBE involving the green operator $\mathcal{R}_{\overline{j\,i}}$ several times. Then, we use the formula

$$\Theta_i\Theta_j\mathcal{R}_{\overline{j\,i}} = \mathcal{R}_{\overline{j\,i}}\Theta_i\Theta_j.$$

Hence, we have the following expression

$$\begin{aligned}\Delta_i\Delta_j &= \mathcal{R}_{\overline{i\,i-1}}\dots\mathcal{R}_{\overline{i,j+1}}\mathcal{R}_{\overline{i,j}}\mathcal{R}_{\overline{i,j-1}}\dots\mathcal{R}_{\overline{i\,1}}\mathcal{R}_{\overline{j\,j-1}}\dots\mathcal{R}_{\overline{j\,1}}\mathcal{R}_{\overline{j\,i}}\\ &\times \Theta_i\Theta_j\mathcal{R}_{\overline{j\,n}}\dots\mathcal{R}_{\overline{j\,i+1}}\mathcal{R}_{\overline{j\,i-1}}\dots\mathcal{R}_{\overline{j\,j+1}}\mathcal{R}_{\overline{i\,n}}\dots\mathcal{R}_{\overline{i\,i+1}}.\end{aligned}$$

Now, we perform the transposition of blue and magenta elements with the help of the YBE involving the red operator:

$$\begin{aligned}\Delta_i\Delta_j &= \mathcal{R}_{\overline{j\,j-1}}\dots\mathcal{R}_{\overline{j\,1}}\mathcal{R}_{\overline{i\,i-1}}\dots\mathcal{R}_{\overline{i,j+1}}\mathcal{R}_{\overline{i,j-1}}\dots\mathcal{R}_{\overline{i\,1}}\mathcal{R}_{\overline{i,j}}\mathcal{R}_{\overline{j\,i}}\\ &\times \Theta_i\Theta_j\mathcal{R}_{\overline{j\,n}}\dots\mathcal{R}_{\overline{j\,i+1}}\mathcal{R}_{\overline{j\,i-1}}\dots\mathcal{R}_{\overline{j\,j+1}}\mathcal{R}_{\overline{i\,n}}\dots\mathcal{R}_{\overline{i\,i+1}}.\end{aligned}$$

By using the fact that

$$\mathcal{R}_{\overline{i,j}}\mathcal{R}_{\overline{j\,i}} = \mathcal{R}_{\overline{j\,i}}\mathcal{R}_{\overline{i,j}} = 1,$$

we arrive to the formula

$$\Delta_i\Delta_j = \mathcal{R}_{\overline{j\,j-1}}\dots\mathcal{R}_{\overline{j\,1}}\Theta_j\mathcal{R}_{\overline{i\,i-1}}\dots\mathcal{R}_{\overline{i,j+1}}\mathcal{R}_{\overline{i,j-1}}\dots\mathcal{R}_{\overline{i\,1}}\mathcal{R}_{\overline{j\,n}}\dots\mathcal{R}_{\overline{j\,i+1}} \\ \times\, \mathcal{R}_{\overline{j\,i-1}}\dots\mathcal{R}_{\overline{j\,j+1}}\Theta_i\mathcal{R}_{\overline{i\,n}}\dots\mathcal{R}_{\overline{i\,i+1}}.$$

Then transposing the commuting elements (blue ones and magenta ones), we finalize by the expression:

$$\Delta_i\Delta_j = \mathcal{R}_{\overline{j\,j-1}}\dots\mathcal{R}_{\overline{j\,1}}\Theta_j\mathcal{R}_{\overline{j\,n}}\dots\mathcal{R}_{\overline{j\,i+1}}\mathcal{R}_{\overline{i\,i-1}}\dots\mathcal{R}_{\overline{i,j+1}}\mathcal{R}_{\overline{j\,i-1}}\dots\mathcal{R}_{\overline{j\,j+1}} \\ \times\, \mathcal{R}_{\overline{i,j-1}}\dots\mathcal{R}_{\overline{i\,1}}\Theta_i\mathcal{R}_{\overline{i\,n}}\dots\mathcal{R}_{\overline{i\,i+1}}.$$

Now, we insert the product

$$\mathcal{R}_{\overline{j\,i}}\mathcal{R}_{\overline{i,j}}$$

and by using the YBE several times in the same fashion as below, we arrive to the following expression

$$\begin{aligned}\Delta_i\Delta_j &= \mathcal{R}_{\overline{j\,j-1}}\dots\mathcal{R}_{\overline{j\,1}}\Theta_j\mathcal{R}_{\overline{j\,n}}\dots\mathcal{R}_{\overline{j\,i+1}}\mathcal{R}_{\overline{j\,i}}\mathcal{R}_{\overline{i,j}}\mathcal{R}_{\overline{i\,i-1}}\dots\mathcal{R}_{\overline{i,j+1}}\mathcal{R}_{\overline{j\,i-1}}\dots\mathcal{R}_{\overline{j\,j+1}} \\ &\times \mathcal{R}_{\overline{i,j-1}}\dots\mathcal{R}_{\overline{i\,1}}\Theta_i\mathcal{R}_{\overline{i\,n}}\dots\mathcal{R}_{\overline{i\,i+1}} \\ &= \mathcal{R}_{\overline{j\,j-1}}\dots\mathcal{R}_{\overline{j\,1}}\Theta_j\mathcal{R}_{\overline{j\,n}}\dots\mathcal{R}_{\overline{j\,i+1}}\mathcal{R}_{\overline{j\,i}}\mathcal{R}_{\overline{j\,i-1}}\dots\mathcal{R}_{\overline{j\,j+1}}\mathcal{R}_{\overline{i\,i-1}}\dots\mathcal{R}_{\overline{i,j+1}}\mathcal{R}_{\overline{i,j}} \\ &\times \mathcal{R}_{\overline{i,j-1}}\dots\mathcal{R}_{\overline{i\,1}}\Theta_i\mathcal{R}_{\overline{i\,n}}\dots\mathcal{R}_{\overline{i\,i+1}} = \Delta_j\Delta_i.\end{aligned}$$

The last few transformations are represented in Fig. 6. This completes the proof. □

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