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Nonlinear random dynamical model for the stock market

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Abstract. A new model of a stock market as a nonlinear random dynamical system with additive noise in three-dimensional phase space is offered. Implementations of this model in the adiabatic approximation possess all the key signs of fractal market, making the model a reasonable evolutionary model for a stock market. The use of adiabatic approximation allows us to model a stock market as a nonlinear random dynamical system with multiplicative noise with the power-law in one-dimensional phase space. This model shows fractality, long memory and $1/f$ noise in a stock market.

1. Introduction

An interdisciplinary research field, known as econophysics, was formed in the middle 1990s as an approach to solve various problems in economics, such as uncertainty or stochastic processes and nonlinear dynamics, by applying theories and methods originally developed by physicists. The term “econophysics” was coined by H E Stanley in order to describe the large number of papers written by physicists in the problems of (stock and other) markets (for econophysics reviews see refs. [1-4]).

Current state of theoretical economics allows one to effectively use advanced methods of physical and mathematical modelling for economical system. A remarkable example is applying nonlinear dynamics to analysis of financial time series [5, 6]. Moreover, in 1963 B B Mandelbrot [7] during his research of cotton prices found out that the prices follows a scaled distribution in time. These probability distributions are considered fractal. In 1994 was formed fractal market hypothesis (FMH) as an alternative investment theory to efficient market hypothesis (EMH). The FMH is a model of investor behaviour that unlike the EMH assumes investors have multiple time horizons and interpret information based upon their horizon.

According to the FMH, the financial time series (FTS) have the following key features: power law distribution, $1/f$ noise, and long memory.

The most comprehensive survey of mathematical models of financial markets can be found in the book of R J Elliott and P E Kopp [8]. Although the book and other relevant publications contain numerous conceptual models, we have not found any econophysical model of a stock market that can explain its fundamental functioning mechanisms and the key features. Thus, the purpose of this work is building of econophysical model of a stock market as a nonlinear random dynamical system which is explaining the key features of FTS.

2. Nonlinear random dynamical model for the stock market

2.1. Deterministic dynamical system

In detail, the construction of the nonlinear dynamical model of the stock is presented in works [9-11]. In this section, we only indicate the basic principles.



The evolution of a stock market can be described by the well-known Lorenz system of equations:

$$\begin{cases} \tau_\eta \dot{\eta}_t = -\eta_t + a_\eta h_t \\ \tau_h \dot{h}_t = -h_t + a_h \eta_t S_t \\ \tau_s \dot{S}_t = (S_0 - S_t) - a_s \eta_t h_t \end{cases}, \quad (1)$$

where $\eta_t \equiv \eta_t - \eta_{eq}$ is the variation of “ask” price (η_t) relative to equilibrium value (η_{eq} is the “ask” price in equilibrium state); $h_t \equiv h_t - h_{eq}$ is the variation of “bid” price (h_t) relative to equilibrium value (h_{eq} is the “bid” price in equilibrium state); $S_t \equiv N_{|a\rangle_t} - N_{|p\rangle_t}$ is an instantaneous difference between numbers of agents in $|a\rangle$ -state and $|p\rangle$ -state.

A particular market agent being in $|a\rangle$ -state has maximum amount of valuable information about financial asset ($I_{|a\rangle}$) and has minimum information ($I_{|p\rangle}$) being in $|p\rangle$ -state. The agent being in $|a\rangle$ -state is able to generate local demand on deal with the asset and send an “ask-quantum” to other agents. If the agent is in $|p\rangle$ -state (he/she does not have enough valuable information about the asset), then the agent's rational decision is do not generate demand on deal (“bid-quantum”). Moreover, for the agent in $|p\rangle$ -state generating of a deal offer depends on the agent's reaction on received “ask-quantum” or can be his or her own decision. General pattern in stock markets is that local “ask” waves (“quanta”) induce local “bid” waves (“quanta”).

Indeed, stock market is an open system that continuously exchanges information and money flows with the external world. Sources of external information include corporate financial reports, financial news feeds, stock-ticker data and others. This information flow, in some sense, “pump up” the stock market, making inverse population of market agents: $N_{|a\rangle} \gg N_{|p\rangle}$, where $N_{|a\rangle}$ is the number of agents being in $|a\rangle$ -state, $N_{|p\rangle}$ is the number of agents being in $|p\rangle$ -state.

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The system of self-consistent equations (1) is a well-known method to describe a self-organizing system. The Lorenz synergetic model was first developed as a simplification of hydrodynamic equations describing the Rayleigh-Bénard heat convection in the atmosphere; it is now a classical model of chaotic dynamics. Further research on the Lorenz system presented in a series of publications proved that the system provides an appropriate kinetic picture of the cooperative behavior of particles in any macroscopic dynamical system where the actualization of potential order is possible. Processes in such self-organizing complex systems in nonequilibrium state lead to the selection of a small number of parameters from the complete set of variables that describe the system; all other degrees of freedom adjust to correspond to these selected parameters. Following the terminology used in the synergy theory, these parameters are the order parameter (η_t), conjugated field (h_t), and control parameter (S_t). According to the Ruelle-Takens theorem, we can observe a nontrivial self-organization with strange attractors if the number of selected degrees of freedom is three or more.

In the system of equations (1) a_η is a coefficient, and positive constants a_h , a_s are measures of feedback in a stock market. Functions η_t/τ_η , h_t/τ_h , $(S_0 - S_t)/\tau_s$ describe the autonomous relaxation of the variation of “ask” price, variation of “bid” price and instantaneous difference between

numbers of agents in $|a\rangle$ -state and $|p\rangle$ -state to the stationary values $\eta_t = 0$, $h_t = 0$, $S_t = S_0$ with relaxation times τ_η , τ_h , τ_S .

Financial interpretation of the results is as follows [9-11]:

- In case of relatively small intensity of external information pumping ($S_0 \rightarrow 0$), the stock market tends to stable equilibrium. However, practically this stable equilibrium state cannot be reached, since the market is an open system with permanent external information pumping.
- If $S_0 \rightarrow 28$, then the stock market functions as an open nonequilibrium system with deterministic chaos. Such behavior is typical for a financial market with considerably intense external information.
- 3-dimensional dynamical model (1) explains some properties of the stock market functioning such as fractality (fractal dimension equals 1.497) and chaotic nature (correlation dimension equals 1.896) of FTS [12].

2.2. Random dynamical system

The nonlinear dynamic model (1) explains the fractality and chaotic nature of empirical as well as the dissipative nature of the system. On the other hand, Eq. (1) cannot explain some other phenomena found in empirical data [13-17], and first of all, the key signs of fractal market signals: a power law of PDF, 1/f noise, and long memory. Let us consider different ways of improving (generalizing) Eq. (1) in order to adequately describe a stock market.

Taking into account stochastic terms and the fractionality of the order parameter, Eq. (1) takes the following form:

$$\begin{cases} \tau_\eta \dot{\eta}_t = -\eta_t^\alpha + a_\eta h_t + \sqrt{I_\eta} \xi_t \\ \tau_h \dot{h}_t = -h_t + a_h \eta_t^\alpha S_t + \sqrt{I_h} \xi_t \\ \tau_S \dot{S}_t = (S_0 - S_t) - a_S \eta_t^\alpha h_t + \sqrt{I_S} \xi_t \end{cases} \quad (2)$$

In Eq. (2), I are noise intensities for each phase variable; ξ_t is white noise, where $\langle \xi_t \rangle = 0$, $\langle \xi_t, \xi_{t'} \rangle = \delta(t - t')$; $\alpha \in (0, 1]$. The random dynamic system (RDS) (2) is a generalization of the deterministic dynamic system (1) where stochastic sources are added, the feedback is weakened, and the order parameter is relaxed. The replacement of the order parameter η_t by a smaller value (η_t^α) means that the process of ordering influences the self-consistent behavior of the system to a lesser extent than it does in the ideal case of $\alpha = 1$.

Let us analyze RDS (2) in adiabatic approximation when the characteristic relaxation time of the variation of “ask” price considerably exceeds the corresponding relaxation times of the variation of “bid” price and the instantaneous difference between numbers of agents in $|a\rangle$ -state and $|p\rangle$ -state:

$$\tau_\eta \gg \tau_h, \tau_S.$$

For a stock market functioning as an open nonequilibrium system, the adiabatic approximation means that when the external information feed tends to zero, the stream of “ask-quanta” slowly decreases and at the same time the “bid-quanta” and the number of agents being in $|a\rangle$ -state decrease as well.

An adiabatic approximation is a necessary condition for the transformation of the three-dimensional RDS with additive noise (2) into a one-dimensional RDS with multiplicative noise of the following form:

$$\tau_\eta \dot{\eta} = f(\eta) + \sqrt{I(\eta)} \xi. \quad (3)$$

The terms of Eq. (3) corresponding to the drift and diffusion (intensity of the chaotic source) have the following form:

$$f(\eta) \equiv -\eta^\alpha + S_0 \eta^\alpha (1 + \eta^{2\alpha}), \quad (4)$$

$$I(\eta) \equiv I_\eta + (I_h + I_s \eta^{2\alpha})(1 + \eta^{2\alpha})^2. \quad (5)$$

The stationary probability distribution density of the deviation of the variation of “ask” price from the corresponding equilibrium value has the following form:

$$p(\eta) = Z^{-1} (1 + \eta^{2\alpha})^{-2} \exp \left[\int \frac{(1 + \eta^{2\alpha})^{-2}}{\eta^\alpha} d\eta \right], \quad (6)$$

where Z is a normalization constant.

Before we draw any conclusion about probability density function (PDF) (6), let's direct our attention to one significant fact that distinguishes theory from practice. Distributions of real systems and processes regardless of their nature cannot have an infinite expected value or variance. Therefore, power-law PDFs like $p(x) \propto x^{-2\alpha}$ (2α is chosen for the purposes of analysis of expression (6)) are approximate and not valid for large x . The exponential decrease of PDF corresponds to the intermediate asymptotics, and in practice instead of heavy tails we should have semi-heavy tails:

$$p(\eta) \propto \eta^{-2\alpha} P\left(\frac{\eta}{\eta_c}\right), \quad (7)$$

where the scaling function $P(\eta/\eta_c)$ is approximately constant at $\eta \cong \eta_c$ and quickly decreases when $\eta \rightarrow \infty$. Here the “heaviness of the tail” is shifted toward the intermediate range of η values. Thus, the dimensionless variation of “ask” price η scaled for η_c serves as a scaling variable η/η_c in (7). Since the integral in Fokker-Planck equation is regular at $\eta \rightarrow 0$, the PDF obtained has a power-law form.

The power law for PDF of the deviation of the variation of “ask” price, which is equivalent to for large times, was obtained and justified analytically. However, we could not obtain analytical expressions for power spectral density ($S(f)$), autocorrelation function (ACF), or the fractal dimensions (D_F). Therefore, we present below the results of numerical calculations for a family of realizations of RDS (3): $p(x) \propto x^{-1.18}$, $S(f) \propto f^{-1.36}$, $ACF(\tau) \propto \tau^{-0.04}$, $D_F = 1.235$.

3. Conclusions

The generalized Lorentz system (2) adequately models the evolution of a stock market as a complex system. The characteristics of η -realizations of RDS (3) are quantitatively close to the corresponding characteristics of empirical FTS.

For a description of the evolution of a stock market, the nonlinear dynamical system model (1) is a rough, not very accurate approximation. First, the model does not predict the occurrence of catastrophic values in a time series of the variation of “ask” price which would signify the complexity of a stock market, or the existence of long memory or the time series' tendency to follow trends. Despite this deficiency, Eq. (1) allows one to study stock market far from equilibrium, and it also explains the existence of dynamical chaos in a time series as well as their fractality.

The nonlinear random dynamical system (2) is a generalization of the model (1) that accounts for external stochastic sources and the fractionality of the order parameter (weakening of feedback and relaxation of the order parameter). This model adequately describes the evolution of a stock market.

Quantitative characteristics of the model (2) in adiabatic approximation are close to the corresponding characteristics of the empirical FTS. An adiabatic approximation allows us to reduce a three-dimensional random adiabatic system with additive noise (2) to a one-dimensional random dynamical system with exponential multiplicative noise (3).

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