# MODELING OF GRADIENT-LIKE FLOWS ON $n$-SPHERE

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A general idea of the qualitative study of dynamical systems, going back to the works by A. Andronov, E. Leontovich, A. Mayer, is a possibility to describe dynamics of a system using combinatorial invariants. So M. Peixoto proved that the structurally stable flows on surfaces are uniquely determined, up to topological equivalence, by the isomorphic class of a directed graph. Multidimensional structurally stable flows does not allow entering their classification into the framework of a general combinatorial invariant. However, for some subclasses of such systems it is possible to achieve the complet combinatorial description of their dynamics.

In the present paper, based on classification results by S. Pilyugin, A. Prishlyak, V. Grines, E. Gurevich, O. Pochinka, any connected bi-color tree implemented as gradient-like flow of $n$-sphere, $n > 2$ without heteroclinic intersections. This problem is solved using the appropriate gluing operations of the so-called Cherry boxes to the flow-shift. This result not only completes the topological classification for such flows, but also allows to model systems with a regular behavior. For such flows, the implementation is especially important because they model, for example, the reconnection processes in the solar corona.

## 1. Introduction and statement of results

A general idea of a qualitative study of dynamical systems, going back to the works by A. Andronov [**1**], E. Leontovich, A. Mayer [**2**], [**3**], is a possibility to describe dynamics of a system using combinatorial invariants. A brilliant example of the implementation of such approach is the topological classification of Morse-Smale flows on surfaces obtained by M. Peixoto [**4**]. He proved that structurally stable flows on surfaces are uniquely determined, up to topological equivalence, by the isomorphic class of a directed graph.

Multidimensional structurally stable flows do not allow to classify their into the framework of a general combinatorial invariant. However, for some subclasses of such systems it is possible to achieve a completely combinatorial description of their dynamics. Thus, according to the results by S. Pilyugin [**5**], A. Prishlyak [**6**], V. Grines, E. Gurevich, O. Pochinka [**7**], the topological equivalence class of a *gradient-like flow* (Morse-Smale flow without periodic orbits) on $n$-sphere, $n > 2$, without heteroclinic intersections is completely determined by a bi-color tree corresponding to a skeleton of co-dimensional one saddle invariant manifolds. A problem of the realization of an abstract invariant is an integral part of the topological classification. For such flows, it is especially important because they model regular processes in various natural sciences (see, for example, [**8**]). In particular, such flows model reconnection processes in the solar corona (see, for example, [**9**], [**10**]).

In such processes the corona of the Sun is divided on domains by fans and spines (2- and 1-dimensional invariant manifolds) of null points of the magnetic field (the points, at which the strength of magnetic field is null). Restructuring of this domains underlie such effects as the flares and prominences. This energy processes are very important for explanation of many nature laws. The topological structure of a magnetic field is defined by null points, spines, fans and separators, the union of those forming the so-called skeleton of the magnetic field. Experiments and observations show that the evolution of the structure of the magnetic field is similar to relaxation processes. At first plasma evolves slowly for some considerable time but at some point there occurs a topological restructuring of the magnetic configuration (reconnection) [**11**].

Indeed below we model different stable states of the magnetic field.

Let us recall that *$n$-ball* or *$n$-disk* is a manifold with a boundary homeomorphic to a *standard $n$-ball*

$$\mathbb{B}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1^2 + ... + x_n^2 \leq 1\}.$$

An *open $n$-ball* or *$n$-disk* we call a manifold homeomorphic to an interior of $\mathbb{B}^n$. We call by *$(n-1)$-sphere* a manifold $S^{n-1}$ homeomorphic to a *standard $(n-1)$-sphere*

$$\mathbb{S}^{n-1} = \partial\mathbb{B}^n.$$

Let us consider a class $G$ of gradient-like flows without heteroclinic intersections on $n$-dimensional sphere $S^n, n \geq 3$, that is, flows whose the non-wandering set consists of a finite many hyperbolic fixed points such that the invariant manifolds of saddle points have no intersections.

Let $f^t \in G$. According to the result by S. Pilyugin [**5**, Lemma 2.2], the dimensions of the invariant saddle manifolds of $f^t$ have to be only $(n-1)$ and 1. Let us denote by $\Omega_{f^t}$ the non-wandering set of $f^t$, and let

$$\Omega^i_{f^t} = \{p \in \Omega_{f^t} \mid \dim W^u_p = i\}.$$

By [**12**, Theorem 2.3],

$$S^n = \bigcup_{p\in\Omega_{f^t}} W^u_p = \bigcup_{p\in\Omega_{f^t}} W^s_p.$$

It follows from [**13**, Proposition 2.3] that for any saddle point $\sigma$ of a flow $f^t$ the closure of its invariant manifold $W^\delta_\sigma$ with dimension $(n-1)$ contains, except the manifold itself, exactly one fixed point. That point is a sink if $\delta = u$ and a source if $\delta = s$. Then the set $cl\,W^\delta_\sigma$ is a sphere with dimension $(n-1)$. By [**14**] and [**15**] this sphere is cylindrically embedded[1]. Denote by $m_{f^t}$ the number of saddle points of a flow $f^t$. Then the union

$$\mathcal{W}_{f^t} = \bigcup_{p\in\Omega^1_{f^t}} cl\,W^s_p \;\cup \bigcup_{q\in\Omega^{n-1}_{f^t}} cl\,W^u_q$$

of closures of all invariant manifolds of dimension $(n-1)$ divides a sphere $S^n$ into $k_{f^t} = m_{f^t} + 1$ connected components. Denote such components by $D_1,\dots,D_{k_{f^t}}$, and let

$$\mathcal{D}_{f^t} = \bigcup_{i=1}^{k_{f^t}} D_i.$$

*A bi-colour graph of a flow* $f^t \in G$ is a graph $\Gamma_{f^t}$, such that:

1) the set $\Gamma^0_{f^t}$ of vertices of $\Gamma_{f^t}$ bijectively corresponds to $\mathcal{D}_{f^t}$ by a bijection

$$\xi_0\colon \Gamma^0_{f^t} \to \mathcal{D}_{f^t};$$

2) two vertices $v_i$, $v_j$ are connected by an edge $e_{i,j}$ iff domains $D_i = \xi_0(v_i)$, $D_j = \xi_0(v_j)$ have a common boundary;

3) an edge $e_{i,j}$ has a colour $u$ (resp. $s$) if the common boundary of $D_i$ and $D_j$ is the closure of an unstable (resp. stable) saddle manifold (see Fig. 1).

Two graphs $\Gamma_{f^t}$ and $\Gamma_{f'^t}$ of some flows $f^t$, $f'^t$ are called *isomorphic* if there exists an isomorphism $\eta\colon \Gamma_{f^t} \to \Gamma_{f'^t}$ mapping vertices of $\Gamma_{f^t}$ into vertices of $\Gamma_{f'^t}$ preserving adjacency and coloring.

It follows from [**7**], that the flows $f^t, f'^t \in G$ are topologically equivalent iff their graphs $\Gamma_{f^t}$ and $\Gamma_{f'^t}$ are isomorphic. Indeed, for any flow $f^t \in G$ its bi-color graph is a tree, i.e. connected graph without cycles.

The main result of the present paper is the following theorem.

**Theorem 1.** *For every bi-color tree* $\Gamma$ *there is a flow* $f^t \in G$ *whose graph* $\Gamma_{f^t}$ *is isomorphic to graph* $\Gamma$.

Notice that flows of the considered class, under the assumption that they have a unique sink, were classified and realized in [**16**] by means of a directed graph.

[1] A sphere $S^{n-1} \subset M^n$ is called *cylindrically embedded in* $M^n$, if there exists a topological embedding $h : \mathbb{S}^{n-1} \times [-1;+1] \to M^n$, such that $h(\mathbb{S}^{n-1} \times \{0\}) = S^{n-1}$.

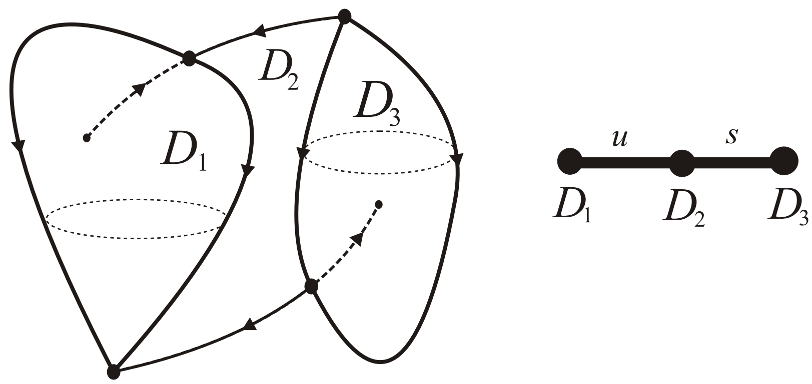


Figure 1: Example of a flow and its bi-color graph

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## 2. Realization of a flow by a bi-color tree

### 2.1 Description of bi-color tree

Recall some definitions from the graph theory (see, for example, [**17**] for details).

A *graph* is a pair $(V, E)$, where $V$ is a set of *vertices* and $E$ is a set of *pairs of vertices*, which are called *edges*. If $E$ contains ordered pairs, then the graph is called a *directed* one. A *k-edge-colouring* of a graph is an assignment of $k$ colours to its edges.

Two vertices are called *adjacent* if they are *connected* by an edge (i.e. they constitute the edge), and the edge is *incident to* each of the vertices. A *loop* is an edge, whose end-vertices coincide. A *simple graph* is an undirected graph without loops.

The number of edges incident to a vertex is called *degree* of the vertex.

A set $\{v_1, (v_1, v_2), v_2, \dots, v_{k-1}, (v_{k-1}, v_k), v_k\}$ is called a *path* of the length $k$. A path is called a *cycle* if $v_1 = v_k$. A graph is called *connected* if every two its vertices are joined by a path.

*A tree* is a connected acyclic graph. It means that any two its vertices are connected by exactly one path.

Any tree with at least 2 vertices has at least two *pendant vertex*, that is a vertex of the degree 1.

Any tree becomes a *out-tree* if arbitrary its vertex $r$ is selected, as *a root.* In the other words a planted tree is a tree in which one vertex $r$ has been designated as the root and every edge is directed away from the root.

If $v$ is a vertex in a planted tree other than the root, *the parent of* $v$ is the unique vertex $w$ such that there is a directed edge $(w, v)$. If $w$ is the parent of $v$, then $v$ is called *a child of* $w$.

The rooted vertex $r$ by definition has a level 0. *The level* $d$ *of any other vertex* $v$ in a such planted tree is the number of edges in the unique path between the vertex $v$ and the root $r$. *The depth of a tree* $D$ is the maximum level of any vertex there.

*An ordered out-tree* is an out-tree where the children of each vertex are ordered.

**2.2 Construction of the flow by the graph**

To construct a required flow on the $n$-sphere $\mathbb{S}^n$ for the given bi-color tree $\Gamma$ choose a pendant vertex $r$ of $\Gamma$ as a root and an order all children to get from the tree $\Gamma$ an ordered out-tree. Denote by $N$ the number of all vertices of $\Gamma$.

To realize of a flow from the bi-color tree $\Gamma$, we will use the idea of embedding of $N-1$ pairwise disjoint *Cherry boxes* $B_v$ in a *flow-shift* $g_0^t : \mathbb{R}^n \to \mathbb{R}^n$, given by the formula

$$g_0^t(x_1, \ldots, x_n) = (x_1 + t, \ldots, x_n)$$

and the cherry box $B_v$ has a form

$$B_v = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : |x_1 - \alpha_v| \leq \delta_v, (x_2 - \beta_v)^2 + x_3^2 + \ldots x_n^2 \leq \delta_v^2\}$$

for some $\alpha_v, \beta_v \in \mathbb{R}, \delta_v > 0$ which depends on parameters of $v$. The dynamics in $B_v$ coincides with the flow-shift dynamics on the boundary of $B_v$ and differs from one inside the box due to the appearance of a saddle and a node. We will say that the dynamics in $B_v$ has a type $u$ ($s$), if the saddle point has $(n-1)$-dimensional unstable (stable) manifold and the node point is a source (sink) (see Fig. 2).

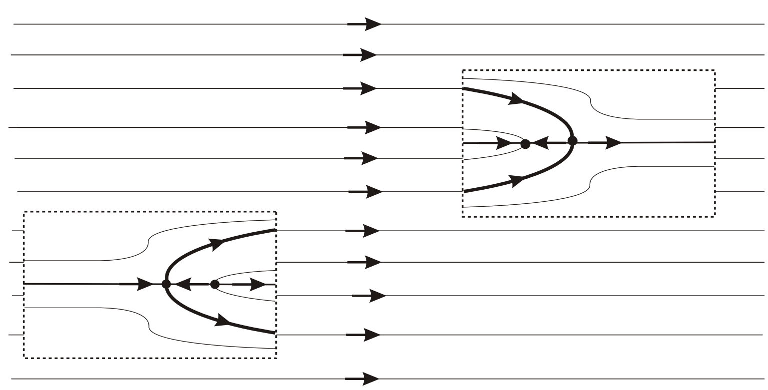

Figure 2: An embedding of Cherry-boxes of the types $u$ and $s$ to the flow-shift

Below we give formulas for the following things:

1. The calculation of a position and a size of the Cherry-box $B_v$;
2. The definition of a flow $g_v^t$ in $B_v$;
3. The embedding of a resulting dynamics in $\mathbb{S}^n$.

**1. The calculation of a position and a size of the Cherry-box $B_v$.** For the vertex $y$ which is a unique child of the root $r$ we put

$$\alpha_y = 2\rho_y\left(\frac{1}{2}+\frac{1}{2N-4}+\dots+\frac{1}{(2N-4)^{D-2}}\right),\ \ \beta_y=0,\ \ \delta_y=1,$$

where $\rho_y$ equals 1 ($-1$) if the edge $(r,y)$ has a colour $s$ ($u$) and $D$ is the depth of the tree $\Gamma$. For any other vertex $v$ with the level $d_v \geq 2$ the parameters of the box $B_v$ are determined through the parameters $\alpha_w, \beta_w, \delta_w$ of its parent's box $B_w$, the order $k_v$ of $v$ as a child and a number $\rho_v$ which equals 1 ($-1$) if the edge $(w,v)$ has a colour $s$ ($u$) by the following way:

$$\delta_v=\frac{\delta_w}{2N-4},\ \ \alpha_v=\rho_v\left(|\alpha_w|-\delta_w-\delta_v\right),\ \ \beta_v=\beta_w+\frac{\delta_w}{2}-(2k_v-1)\delta_v.$$

So, the size and position are defined for each Cherry-box corresponding to every vertex of $\Gamma$ except the root.

**2. The definition of a flow $g_v^t$ in $B_v$.** Let

$$\Sigma_v=(x_1-\alpha_v)^2+(x_2-\beta_v)^2+x_3^2+\dots+x_n^2.$$

Define the flow $g_v^t:\mathbb{R}^n\to\mathbb{R}^n$ by the formulas:

$$\begin{cases}
\dot{x}_1=\begin{cases}1-\dfrac{16\delta_v^2}{9}\left(\Sigma_v-\delta_v^2\right)^2, & \Sigma_v\leq\delta_v^2\\ 1, & \text{otherwise}\end{cases}\\
\dot{x}_2=\begin{cases}\dfrac{x_2-\beta_v}{2}\left(\sin\left(\dfrac{\pi}{2}\left(\frac{4\Sigma_v}{\delta_v^2}-3\right)\right)-1\right), & \dfrac{\delta_v^2}{2}<\Sigma_v\leq\delta_v^2\\ -(x_2-\beta_v), & \Sigma_v\leq\dfrac{\delta_v^2}{2}\\ 0, & \text{otherwise}\end{cases}\\
\dots\dots\dots\dots\dots\dots\\
\dot{x}_n=\begin{cases}\dfrac{x_n}{2}\left(\sin\left(\dfrac{\pi}{2}\left(\frac{4\Sigma_v}{\delta_v^2}-3\right)\right)-1\right), & \dfrac{\delta_v^2}{2}<\Sigma_v\leq\delta_v^2\\ -x_n, & \Sigma_v\leq\dfrac{\delta_v^2}{2}\\ 0, & \text{otherwise}\end{cases}
\end{cases}$$

By construction flow $g_v^t$ has exactly two hyperbolic fixed points: the saddle (source) point $P_v(\alpha_v+\rho_v\delta_v/2,\beta_v,0,\dots,0)$ and the sink (saddle) point $Q_v(\alpha_v-\rho_v\delta_v/2,\beta_v,0,\dots,0)$ for $\rho_v=1$ ($\rho_v=-1$). Define $g_\Gamma^t:\mathbb{R}^n\to\mathbb{R}^n$ in such a way that it coincides with $g_v^t$ in $B_v$ and is $g_0^t$ outside all Cherry-boxes (see figure 3).

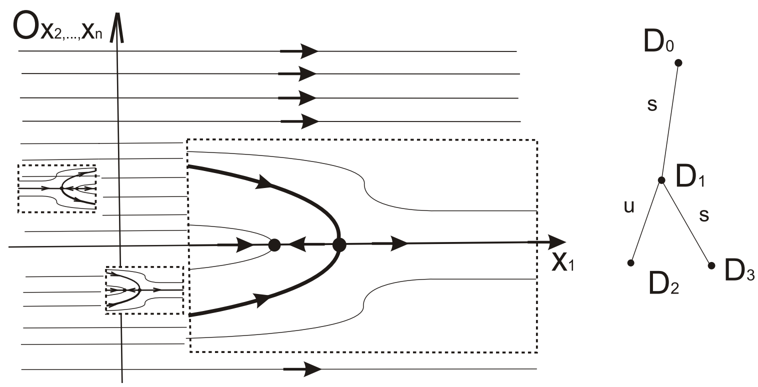


Figure 3: An example of a tree $\Gamma$ and the flow $g^t_\Gamma$

Let us notice that the flow $g^t_\Gamma$ has no heteroclinic intersections. Indeed, by the construction the interiors of the Cherry-boxes are pairwise disjoint. Moreover, in the hyperplane $x_1 = \alpha_v$ we have

$$\dot{x}_1 < 0 \ \ if \ \ (x_2 - \beta_v)^2 + x_3^2 + \ldots x_n^2 < (\delta_v/2)^2,$$

$$\dot{x}_1 > 0 \ \ if \ \ (\delta_v/2)^2 < (x_2 - \beta_v)^2 + x_3^2 + \ldots x_n^2 < \delta_v^2.$$

Also in $B_v$ we have

$$\dot{x}_2 \leq 0 \ \ if \ \ x_2 \geq \beta_v, \ \ \dot{x}_2 \geq 0 \ \ if \ \ x_2 \leq \beta_v,$$

$$\dot{x}_i \leq 0 \ \ if \ \ x_i \geq 0, \ \ \dot{x}_i \geq 0 \ \ if \ \ x_i \leq 0, i = 3, \ldots, n.$$

Thus, the invariant $(n-1)$-manifold of saddle point from $B_v$ outside $B_v$ coincides with a cylinder

$$C_v = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : (x_2 - \beta_v)^2 + x_3^2 + \ldots x_n^2 \leq \nu_v^2\},$$

where $\delta_v/2 < \nu_v < \delta_v$. By the construction these cylinders are pairwise disjoint, that proves the fact.

**3. The embedding of a resulting dynamics in $\mathbb{S}^n$.**

Let us define a flow $h^t : \mathbb{R}^n \to \mathbb{R}^n$ by the formula:

$$h^t(x_1, x_2, \ldots, x_n) = \left(2^t x_1, 2^t x_2, \ldots, 2^t x_n\right).$$

Let $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}$ and $C = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_2^2 + \cdots + x_n^2 \leq 1\}$. It is easy to verify that a diffeomorphism $\zeta : \mathbb{R}^n_+ \setminus O \to C$ given by the formula

$$\zeta(x_1, \ldots, x_n) = \left(\log_2 \varrho, \frac{x_2}{\varrho}, \ldots, \frac{x_n}{\varrho}\right), \ \varrho = \sqrt{x_1^2 + \cdots + x_n^2}$$

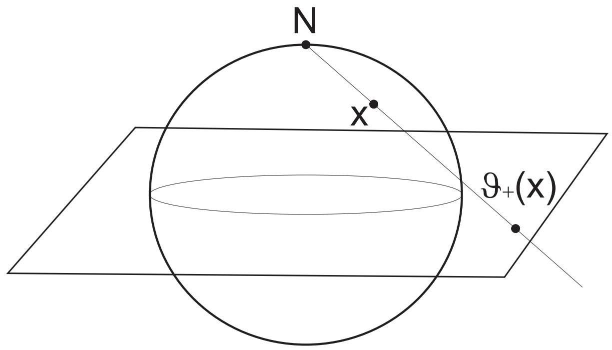


Figure 4: The stereographic projection

conjugates the diffeomorphisms $h^t|_{\partial\mathbb{R}^n_+\setminus O}$ and $g^t|_{\partial C}$. It allows to define a flow $\varphi^t\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ in such a way that $\varphi^t$ coincides with $h^t$ outside $int\,\mathbb{R}^n_+$ and coincides with $\zeta^{-1}g^t_\Gamma\zeta$ on $\mathbb{R}^n_+$.

Let us project the flow $\varphi^t$ to the $n$-sphere by means the stereographic projection.

Denote by $\mathcal{N}(\underbrace{0,\dots,0}_{n},1)$ the North Pole of the sphere $\mathbb{S}^n$. For every point $x=(x_1,\dots,x_{n+1})$ in $\mathbb{S}^n\subset\mathbb{R}^{n+1}$ there is the unique line passing through the points $\mathcal{N}$ and $x$. This line intersects $\mathbb{R}^n = Ox_1\dots x_n$ at exactly one point $\vartheta(x)$ (see figure 4), which is called *the stereographic projection of the point* $x$. One can easily to check that $\vartheta:\mathbb{S}^n\setminus\{\mathcal{N}\}\to\mathbb{R}^n$ is a diffeomorphism, given by the formula

$$\vartheta(x_1,\dots,x_{n+1}) = \left(\frac{x_1}{1-x_{n+1}},\dots,\frac{x_{n-1}}{1-x_{n+1}},\frac{x_n}{1-x_{n+1}}\right).$$

As flow $\varphi^t$ coincides with $h^t$ in some neighborhoods of the origin $O$ and of the infinity point, hence, it induces on $\mathbb{S}^n$ the required flow

$$f^t(x) = \begin{cases}\vartheta^{-1}(\varphi^t(\vartheta(x))),\ x\neq\mathcal{N};\\ \mathcal{N},\ x=\mathcal{N}\end{cases}.$$

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