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# Scenario of a Simple Transition from a Structurally Stable 3-Diffeomorphism with a Two-Dimensional Expanding Attractor to a DA Diffeomorphism

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**Abstract**—The Smale surgery on the three-dimensional torus allows one to obtain a so-called DA diffeomorphism from the Anosov automorphism. The nonwandering set of a DA diffeomorphism consists of a single two-dimensional expanding attractor and a finite number of source periodic orbits. As shown by V. Z. Grines, E. V. Zhuzhoma, and V. S. Medvedev, the dynamics of an arbitrary structurally stable 3-diffeomorphism with a two-dimensional expanding attractor generalizes the dynamics of a DA diffeomorphism: such a 3-diffeomorphism exists only on the three-dimensional torus, and the two-dimensional attractor is its unique nontrivial basic set, but its nonwandering set may contain isolated saddle periodic orbits together with source periodic orbits. In the present study, we describe a scenario of a simple transition (through elementary bifurcations) from a structurally stable diffeomorphism of the three-dimensional torus with a two-dimensional expanding attractor to a DA diffeomorphism. A key moment in the construction of the arc is the proof that the closure of the separatrices of boundary periodic points of a nontrivial attractor and of isolated saddle periodic points are tamely embedded. This result demonstrates the fundamental difference of the dynamics of such diffeomorphisms from the dynamics of three-dimensional Morse–Smale diffeomorphisms, in which the closure of the separatrices of saddle periodic points may be wildly embedded.

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## 1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let $f \in \mathrm{Diff}^1(M^n)$ be a $C^1$-smooth diffeomorphism of a closed $n$-dimensional ($n \geq 2$) manifold $M^n$ equipped with a Riemannian metric $d$. An $f$-invariant set $\Lambda \subset M^n$ is said to be *hyperbolic* if the restriction $T_\Lambda M^n$ of the tangent bundle $TM^n$ of $M^n$ to $\Lambda$ can be represented as the Whitney sum $E^{\mathrm{s}}_\Lambda \oplus E^{\mathrm{u}}_\Lambda$ of $df$-invariant subbundles $E^{\mathrm{s}}_\Lambda$ and $E^{\mathrm{u}}_\Lambda$ ($\dim E^{\mathrm{s}}_x + \dim E^{\mathrm{u}}_x = n$, $x \in \Lambda$) and there exist constants $C_{\mathrm{s}} > 0$, $C_{\mathrm{u}} > 0$, and $0 < \lambda < 1$ such that

$$\|df^m(v)\| \leq C_{\mathrm{s}}\lambda^m\|v\| \quad \text{for } v \in E^{\mathrm{s}}_\Lambda, \qquad \|df^{-m}(v)\| \leq C_{\mathrm{u}}\lambda^m\|v\| \quad \text{for } v \in E^{\mathrm{u}}_\Lambda, \qquad m > 0.$$

The hyperbolic structure leads to the existence of so-called *stable* and *unstable* manifolds, which comprise points with identical asymptotic behavior under positive and negative iterations, respectively. For every point $x \in \Lambda$, there exists an injective immersion $J^{\mathrm{s}}_x\colon \mathbb{R}^s \to M^n$ (whose image $W^{\mathrm{s}}(x) = J^{\mathrm{s}}_x(\mathbb{R}^s)$ is called the *stable manifold of $x$*) such that the following properties hold:

(1) $T_x W^{\mathrm{s}}(x) = E^{\mathrm{s}}_\Lambda$;

(2) points $x, y \in M^n$ belong to the same manifold $W^{\mathrm{s}}(x)$ if and only if $d(f^m(x), f^m(y)) \to 0$ as $m \to \infty$;

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(3) $f(W^{\mathrm{s}}(x)) = W^{\mathrm{s}}(f(x))$;
(4) if $x, y \in \Lambda$, then either $W^{\mathrm{s}}(x) = W^{\mathrm{s}}(y)$ or $W^{\mathrm{s}}(x) \cap W^{\mathrm{s}}(y) = \varnothing$;
(5) if points $x, y \in \Lambda$ are close on $M^n$, then $W^{\mathrm{s}}(x)$ and $W^{\mathrm{s}}(y)$ are $C^1$-close on compact sets. This property is usually referred to as the *theorem on the continuous dependence of stable manifolds on the initial conditions.*

The *unstable manifold* $W^{\mathrm{u}}(x)$ of a point $x \in \Lambda$ is defined as the stable manifold with respect to the diffeomorphism $f^{-1}$. Unstable manifolds have similar properties. In view of property (3), stable and unstable manifolds are called *invariant manifolds.*

A point $x \in M^n$ is said to be *nonwandering* if for any neighborhood $U(x)$ of $x$ and any positive integer $N$ there exists an $n_0 \in \mathbb{Z}$, $|n_0| \geq N$, such that $f^{n_0}(x) \in U(x)$. We will denote the set of nonwandering points of a diffeomorphism $f$ by $\mathrm{NW}(f)$. A diffeomorphism $f$ is called an *axiom* A *diffeomorphism* (or, which is the same, an A-*diffeomorphism*) if the set $\mathrm{NW}(f)$ is hyperbolic and the periodic points are everywhere dense in $\mathrm{NW}(f)$.

Smale [20] proved the following statement, which is known as the *spectral decomposition theorem.* Let $f \in \mathrm{Diff}^1(M^n)$ be an axiom A diffeomorphism. Then the set $\mathrm{NW}(f)$ can be represented as a finite union of pairwise disjoint closed invariant sets $\Lambda_1, \dots, \Lambda_k$, called *basic sets*, each of which contains an everywhere dense orbit. In this case (see [15]), the manifold $M^n$ can be represented as

$$M^n = \bigcup_{i=1}^{k} W^{\mathrm{s}}(\Lambda_i) = \bigcup_{i=1}^{k} W^{\mathrm{u}}(\Lambda_i), \qquad \text{where} \quad W^{\mathrm{s}}(\Lambda_i) = \bigcup_{x \in \Lambda_i} W^{\mathrm{s}}(x), \quad W^{\mathrm{u}}(\Lambda_i) = \bigcup_{x \in \Lambda_i} W^{\mathrm{u}}(x).$$

A basic set is said to be *nontrivial* if it is not a periodic orbit (in particular, is not a fixed point).

According to [6], a basic set $\Lambda$ is said to be *orientable* if for any point $x \in \Lambda$ and any fixed numbers $\alpha > 0$ and $\beta > 0$ the intersection index of $W^{\mathrm{s}}_{\alpha}(x) \cap W^{\mathrm{u}}_{\beta}(x)$ is the same at all intersection points (either $+1$ or $-1$), where $W^{\mathrm{s}}_{\alpha}(x)$ and $W^{\mathrm{u}}_{\beta}(x)$ are $\alpha$- and $\beta$-neighborhoods of $x$ in the intrinsic metric of invariant manifolds. Otherwise a basic set $\Lambda$ is said to be *nonorientable.*

A compact $f$-invariant set $A \subset M^n$ is called an *attractor* of a diffeomorphism $f$ if $A$ has a compact neighborhood $U_A$ such that $f(U_A) \subset \operatorname{int} U_A$ and $A = \bigcap_{k \geq 0} f^k(U_A)$. A *repeller* is defined as an attractor for $f^{-1}$.

For an axiom A diffeomorphism $f$, an attractor $\Lambda$ is called an *expanding attractor* if the topological dimension $\dim \Lambda$ is equal to the dimension of the unstable manifold $W^{\mathrm{u}}(x)$, $x \in \Lambda$. A repeller of $f$ is said to be *contracting* if it is an expanding attractor for $f^{-1}$.

Two diffeomorphisms $f, g \in \mathrm{Diff}^1(M^n)$ are said to be *topologically conjugate* if there exists a homeomorphism $\varphi\colon M^n \to M^n$ such that $\varphi \circ f = g \circ \varphi$. A diffeomorphism $f \in \mathrm{Diff}^1(M^n)$ is said to be *structurally stable* if there exists a neighborhood $U(f) \subset \mathrm{Diff}^1(M^n)$ of $f$ such that any diffeomorphism $g \in U(f)$ is conjugate to $f$.

A significant role in the formulation of conditions of structural stability is played by the so-called strong transversality condition. Let $W_1, W_2 \subset M^n$ be two immersed manifolds that have a nonempty intersection. By definition, $W_1$ and $W_2$ *intersect transversally* if for every point $x \in W_1 \cap W_2$ the tangent space $T_x M^n$ is generated by the tangent subspaces $T_x W_1$ and $T_x W_2$. In particular, if $W_1$ and $W_2$ intersect transversally, then $\dim T_x W_1 + \dim T_x W_2 \geq \dim T_x M^n$.

An axiom A diffeomorphism is said to *satisfy the strong transversality condition* if for any points $x, y \in \mathrm{NW}(f)$ the manifolds $W^{\mathrm{s}}(x)$ and $W^{\mathrm{u}}(y)$ have only transversal intersections. It is known [16, 19] that a diffeomorphism is structurally stable if and only if it is an axiom A diffeomorphism satisfying the strong transversality condition.

Anosov diffeomorphisms underlie the construction of expanding attractors of codimension 1. Following [20], one can construct a structurally stable diffeomorphism of the torus $\mathbb{T}^n$ whose nonwandering set consists of a finite number of periodic sources and a codimension 1 expanding attractor. To this end, one takes a codimension 1 Anosov diffeomorphism of the $n$-torus $\mathbb{T}^n$ and applies the so-called *Smale surgery*. The resulting diffeomorphism, as well as its inverse, is called a *DA diffeomorphism*.

In the present paper, we consider structurally stable diffeomorphisms with an expanding attractor (contracting repeller) of codimension 1 on closed $n$-manifolds $M^n$, $n \geq 3$. A topological classification of such systems under the assumption that the expanding attractor is orientable was obtained for $n \geq 3$ in [10–13]. In particular, Grines and Zhuzhoma proved [12] that for the diffeomorphisms under study the manifold $M^n$ is homotopy equivalent to the torus $\mathbb{T}^n$, and if $n \neq 4$, then $M^n$ is homeomorphic to the torus $\mathbb{T}^n$. In [21], it was proved that for $n = 3$ there exist no structurally stable diffeomorphisms with nonorientable expanding attractors of codimension 1.

In particular, it follows from the results of the above papers that the dynamics of an arbitrary structurally stable diffeomorphism $f$ with an orientable expanding attractor (contracting repeller) of codimension 1 is a generalization of the dynamics of a DA diffeomorphism: such a diffeomorphism $f$ may exist only on the $n$-dimensional torus, and the nontrivial basic set is unique; however, the nonwandering set of $f$ may contain not only nodal but also saddle trivial basic sets.

An *arc of diffeomorphisms* is a one-parameter family of diffeomorphisms $\varphi_t\colon M^n \to M^n$, $t \in [0,1]$, that constitute a smooth mapping $\Phi\colon M^n \times [0,1] \to M^n$ such that $\Phi|_{M^n \times \{t\}} = \varphi_t$. An arc $\varphi_t$ is said to be *simple* if all its points are structurally stable diffeomorphisms except for a finite number of bifurcation points such that the transition through each of them is a generically unfolded saddle–node or period-doubling bifurcation.

The presence of a simple arc connecting a structurally stable diffeomorphism with an orientable basic set of codimension 1 with a DA diffeomorphism was announced in [22] for $n \geq 4$. It was pointed out there that if the dimension of the ambient manifold is 3, an obstacle to the proof of a similar conclusion is the possibility of wild embedding of the closures of one-dimensional separatrices of isolated saddle periodic points.

The following theorem is the main result of the present study.

**Theorem 1.1.** *For any structurally stable diffeomorphism of a closed* 3*-manifold* $M^3$ *whose nonwandering set contains a two-dimensional expanding attractor, there exists a simple arc that connects this diffeomorphism with a DA diffeomorphism.*

## 2. NECESSARY PRELIMINARY INFORMATION

**2.1. Embedding of frames of arcs and circles in 3-manifolds.** In the three-dimensional Euclidean space $\mathbb{R}^3$, consider the unit sphere $\mathbb{S}^2 = \{(x_1,x_2,x_3) \in \mathbb{R}^3\colon x_1^2 + x_2^2 + x_3^2 = 1\}$ centered at the origin $O(0,0,0) \in \mathbb{R}^3$. We can represent the manifold $\mathbb{R}^3 \setminus O$ as $\mathbb{S}^2 \times \mathbb{R}$ by establishing a correspondence between the points $x = (x_1,x_2,x_3) \in \mathbb{R}^3 \setminus O$ and $p(x) = (q(x), r(x)) \in \mathbb{S}^2 \times \mathbb{R}$ with

$$q(x) = \left(\frac{x_1}{\|x\|}, \frac{x_2}{\|x\|}, \frac{x_3}{\|x\|}\right), \qquad r(x) = \log_2\|x\|, \qquad \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

We say that $\mathbb{L} \subset \mathbb{R}^3 \setminus O$ is an *open ray emanating from* $O$ if $p(\mathbb{L}) = \{s\} \times \mathbb{R}$ for a point $s \in \mathbb{S}^2$. A one-dimensional manifold $L \subset \mathbb{R}^3 \setminus O$ homeomorphic to $\mathbb{R}$ is called an *infinite arc emanating from* $O$ if $r(L) = \mathbb{R}$.

Let $\nu \in \mathbb{N}$ and $L_1, \dots, L_\nu$ be a set of pairwise disjoint infinite arcs emanating from $O$. The set

$$F_\nu = \bigcup_{j=1}^{\nu} L_j \cup O$$

is called a *frame* of $\nu$ arcs. We assume that each arc of the frame $F_\nu$ is invariant with respect to the homothety $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by the formula

$$A(x_1,x_2,x_3) = (2x_1, 2x_2, 2x_3).$$

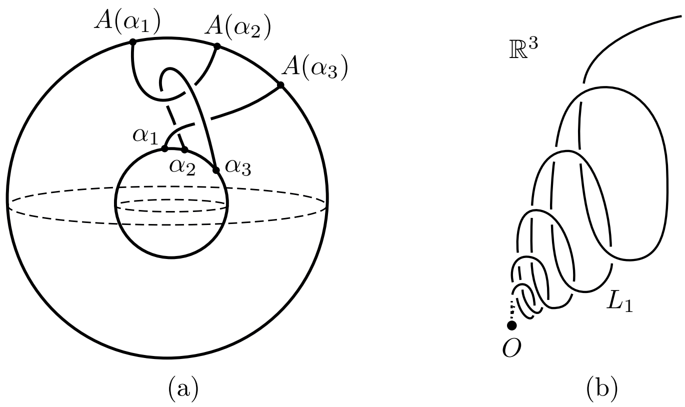


**Fig. 1.** Construction of a wild arc in $\mathbb{R}^3$.

A frame

$$\mathbb{F}_\nu = \bigcup_{j=1}^{\nu} \mathbb{L}_j \cup O$$

of pairwise different open rays $\mathbb{L}_1, \dots, \mathbb{L}_\nu$ emanating from $O$ is said to be *standard.* A frame $F_\nu$ is said to be *tame* if there exists a homeomorphism $H\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $F_\nu = H(\mathbb{F}_\nu)$; otherwise the frame $F_\nu$ is *wild.*

If $\nu = 1$, then the frame consists of a single arc. In 1948, Artin and Fox [4] constructed an example of a wild arc (Fig. 1b). The intersection of the arc $L_1$ and the three-dimensional annulus $\{(x_1, x_2, x_3) \in \mathbb{R}^3\colon 1 \le x_1^2 + x_2^2 + x_3^2 \le 4\}$ is shown in Fig. 1a. Here the arc $L_1$ passing through the points $\alpha_i$ and $A(\alpha_i)$, $i = 1, 2, 3$, is the union of all positive and negative iterations of all arcs in this annulus under the homothety $A$.

The following tameness criterion is a corollary to [9, Lemmas 4.2 and 4.3].

**Proposition 2.1.** *An arc $F_1$ is tame if and only if there exists a* 3*-ball $B(O)$, $O \in B(O)$, such that $B(O)$ is a topological submanifold of $\mathbb{R}^3$ and $\partial B(O) \cap L_1$ consists of a single point.*

Note that even if each arc of a frame $F_\nu \subset \mathbb{R}^3$, $\nu > 1$, is tame, the whole frame need not be tame. In 1960, Debrunner and Fox [3] constructed an example of a wild frame with an arbitrary number $\nu > 1$ of arcs each of which is tame. Moreover, for this frame $F_\nu = \bigcup_{j=1}^{\nu} L_j \cup O$, the sphere $\mathbb{S}^2$ intersects the arc $L_j$ for each $j \in \{1, \dots, \nu\}$ at exactly one point. Figure 2 demonstrates a wild frame of six tame arcs $L_i$ passing through the points $\alpha_i$ and $A(\alpha_i)$, $i = 1, \dots, 6$, respectively.

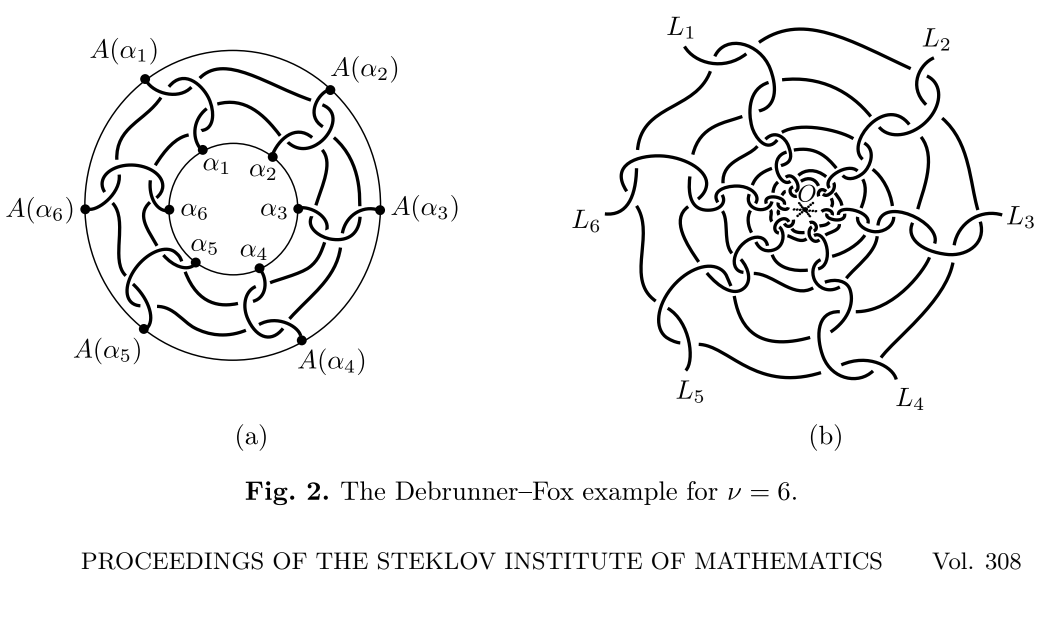


**Fig. 2.** The Debrunner–Fox example for $\nu = 6$.

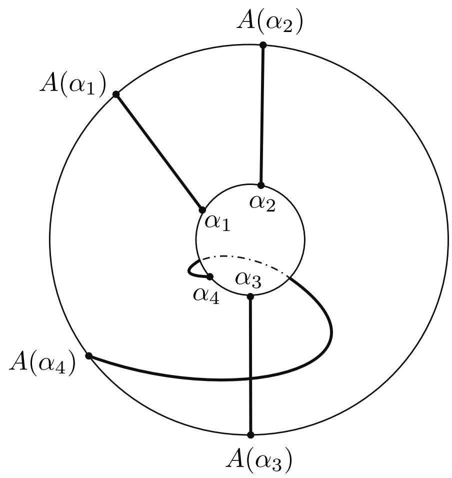


**Fig. 3.** Nontrivial frame of circles generating a tame frame of arcs for $\nu = 4$.

Thus, Proposition 2.1 cannot be generalized to the case of $\nu > 1$. On the other hand, a necessary condition for a frame $F_\nu$ with any $\nu$ to be tame is the existence of a foliation of the set $\mathbb{R}^3 \setminus O$ by 2-spheres each of which intersects each curve $L_j$ of the frame at a single point. Indeed, it follows from the definition of a tame frame $F_\nu = H(\mathbb{F}_\nu)$ that such a foliation is provided by the $H$-images of the concentric spheres

$$\mathbb{P}_t = \{(x_1, x_2, x_3) \in \mathbb{R}^3\colon\ x_1^2 + x_2^2 + x_3^2 = t^2\}, \qquad t > 0.$$

In [8], the following tameness criterion for a frame of arcs $F_\nu$ with any $\nu \in \mathbb{N}$ was proved.

**Proposition 2.2.** *A frame $F_\nu = \bigcup_{j=1}^{\nu} L_j \cup O$ is tame if and only if there exists a homeomorphism $W\colon \mathbb{R}^3 \setminus O \to \mathbb{R}^3 \setminus O$ such that every sphere $W(\mathbb{P}_t)$, $t > 0$, intersects each arc $L_j$, $j \in \{1, \dots, \nu\}$, at a single point.*

Every standard frame of arcs $\mathbb{F}_\nu$ in $\mathbb{R}^3$ is directly related to a frame of circles in the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ through the cover $\pi\colon \mathbb{R}^3 \setminus O \to \mathbb{S}^2 \times \mathbb{S}^1$ defined by the formula

$$\pi(x) = (q(x), r(x) \bmod 1).$$

It can be verified directly that the standard frame $\mathbb{F}_\nu$ and the space $\mathbb{R}^3$ are mapped by $\pi$ to the frame of circles

$$\mathbb{J}_\nu = \bigcup_{j=1}^{\nu} \mathbb{C}_j, \qquad \mathbb{C}_j = \{q_j\} \times \mathbb{S}^1.$$

We will call $\mathbb{J}_\nu$ the *standard* frame of circles. Any set $C_1, \dots, C_\nu$ of pairwise disjoint topologically embedded circles each of which is a generator of the fundamental group $\pi_1(\mathbb{S}^2 \times \mathbb{S}^1)$ forms a *frame of $\nu$ circles*

$$J_\nu = \bigcup_{j=1}^{\nu} C_j.$$

We call a frame $J_\nu$ a *trivial* frame if there exists a homeomorphism $\Phi\colon \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ such that $J_\nu = \Phi(\mathbb{J}_\nu)$. Note that any frame of circles $J_\nu$ lifts to the frame of arcs $F_{J_\nu} = \pi^{-1}(J_\nu) \cup O$, and if the frame of circles $J_\nu$ is trivial, then by Proposition 2.2 the corresponding frame of arcs is tame. The converse does not hold in general. For instance, in [7, Lemma 5.1], a nontrivial frame of circles $J_4$ is constructed for which the corresponding frame of arcs $F_{J_4}$ (passing through the points $\alpha_i$ and $A(\alpha_i)$, $i = 1, 2, 3, 4$) is tame (Fig. 3).

This example can be generalized to the case of any $\nu \geq 3$. On the other hand, in [3, Lemmas 4.3 and 4.4], Debrunner and Fox proved that for $\nu = 1$ the triviality of a circle is equivalent to the tameness of the corresponding arc. In [8], a similar fact was proved for $\nu = 2$.

**Proposition 2.3.** *A frame of circles $J_2$ is trivial if and only if the corresponding frame of arcs $F_{J_2}$ is tame.*

**2.2. Simple arcs in the space of diffeomorphisms.** Consider a one-parameter family of diffeomorphisms (*arc*) $\varphi_t\colon M^n \to M^n$, $t \in [0,1]$. Recall that an arc $\{\varphi_t\}$ is said to be *simple* if all its points are structurally stable diffeomorphisms except for a finite number of bifurcation points $b_i \in (0,1)$ such that the transition through each of them is a generically unfolded saddle–node or period-doubling bifurcation.

An arc $\{\varphi_t\}$ is said to *unfold generically through a saddle–node bifurcation* $\varphi_{b_i}$ if, in some neighborhood of a nonhyperbolic point $(p, b_i)$, the arc $\varphi_t$ is conjugate to the arc

$$\widetilde{\varphi}_{\widetilde{t}}(x_1, x_2, \dots, x_{1+n_u}, x_{2+n_u}, \dots, x_n) = \left(x_1 + \frac{1}{2}x_1^2 + \widetilde{t}, \pm 2x_2, \dots, \pm 2x_{1+n_u}, \frac{\pm x_{2+n_u}}{2}, \dots, \frac{\pm x_n}{2}\right),$$

where $(x_1, \dots, x_n) \in \mathbb{R}^n$, $|x_i| < 1$, $|\widetilde{t}| < 1/10$.

An arc $\{\varphi_t\}$ is said to *unfold generically through a period-doubling* (*flip*) *bifurcation* $\varphi_{b_i}$ if, in some neighborhood of a nonhyperbolic point $(p, b_i)$, the arc $\varphi_t$ is conjugate to the arc

$$\widetilde{\varphi}_{\widetilde{t}}(x_1, x_2, \dots, x_{1+n_u}, x_{2+n_u}, \dots, x_n) = \left(-x_1(1 \pm \widetilde{t}) + x_1^3, \pm 2x_2, \dots, \pm 2x_{1+n_u}, \frac{\pm x_{2+n_u}}{2}, \dots, \frac{\pm x_n}{2}\right),$$

where $(x_1, \dots, x_n) \in \mathbb{R}^n$, $|x_i| < 1/2$, and $|\widetilde{t}| < 1/10$.

## 3. EMBEDDING OF SADDLE SEPARATRICES OF A STRUCTURALLY STABLE DIFFEOMORPHISM OF THE 3-TORUS WITH A TWO-DIMENSIONAL EXPANDING ATTRACTOR

A key moment in the construction of a simple arc connecting a structurally stable diffeomorphism of the 3-torus with a two-dimensional expanding attractor with a DA diffeomorphism is the proof that the embedding of saddle separatrices is tame. This result demonstrates the fundamental difference of the dynamics of such diffeomorphisms from the dynamics of three-dimensional Morse–Smale diffeomorphisms, which are characterized by wild behavior of saddle separatrices. Let us first describe the dynamics of a structurally stable diffeomorphism of the 3-torus with a two-dimensional expanding attractor.

Let $f\colon M^3 \to M^3$ be a structurally stable diffeomorphism of the 3-torus with a two-dimensional expanding attractor. Without loss of generality, we assume that the manifold $M^3$ is the torus $\mathbb{T}^3$ and $f\colon \mathbb{T}^3 \to \mathbb{T}^3$ is a structurally stable diffeomorphism whose nonwandering set contains an expanding attractor $\Lambda$ of topological dimension 2, which is orientable according to the results of [14]. The proofs of the facts given below can be found, for example, in [12].

Since $\dim W^{\mathrm{s}}(x) = 1$ for every point $x \in \Lambda$, we can introduce the notation $(y,z)^{\mathrm{s}}$ and $[y,z]^{\mathrm{s}}$ for open and closed arcs of $W^{\mathrm{s}}(x)$ bounded by points $y, z \in W^{\mathrm{s}}(x)$, respectively. The set $W^{\mathrm{s}}(x) \setminus x$ consists of two connected components. At least one of these components has a nonempty intersection with the set $\Lambda$. A point $x \in \Lambda$ is called a *boundary* point if one of the connected components of $W^{\mathrm{s}}(x) \setminus x$ does not intersect $\Lambda$. Denote this component by $W^{\mathrm{s}\varnothing}(x)$.

The set $\Gamma_\Lambda$ of all boundary points of $\Lambda$ is nonempty and consists of a finite number of periodic points, which are divided into *associated pairs* $(p, q)$ of points of the same period such

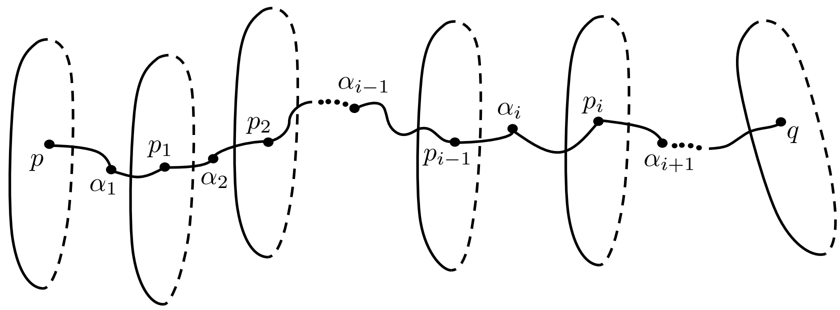


**Fig. 4.** Arc $l_{pq}$.

that the 2-bunch $B_{pq} = W^{\mathrm{u}}(p) \cup W^{\mathrm{u}}(q)$ is a $\mathcal{V}$-accessible boundary of a connected component $\mathcal{V}$ of $\mathbb{T}^3 \setminus \Lambda$.[1]

For every pair $(p, q)$ of associated boundary points of $\Lambda$, we construct a so-called characteristic sphere. Let $B_{pq}$ be a 2-bunch of $\Lambda$ consisting of the two unstable manifolds $W^{\mathrm{u}}(p)$ and $W^{\mathrm{u}}(q)$ of associated boundary points $p$ and $q$, respectively, and let $m_{pq}$ be the period of the points $p$ and $q$. Then, for every point $x \in W^{\mathrm{u}}(p) \setminus p$, there exists a unique point $y \in W^{\mathrm{u}}(q) \cap W^{\mathrm{s}}(x)$ such that the arc $(x, y)^{\mathrm{s}}$ does not intersect $\Lambda$. Define a mapping

$$\xi_{pq}\colon\ B_{pq} \setminus \{p, q\} \to B_{pq} \setminus \{p, q\}$$

by setting $\xi_{pq}(x) = y$ and $\xi_{pq}(y) = x$. Then $\xi_{pq}(W^{\mathrm{u}}(p) \setminus p) = W^{\mathrm{u}}(q) \setminus q$ and $\xi_{pq}(W^{\mathrm{u}}(q) \setminus q) = W^{\mathrm{u}}(p) \setminus p$; i.e., the mapping $\xi_{pq}$ sends the punctured unstable manifolds of the 2-bunch to each other and is an involution ($\xi_{pq}^2(x) = \mathrm{id}$). By the theorem on the continuous dependence of invariant manifolds on initial conditions in compact sets, the mapping $\xi_{pq}$ is a homeomorphism.

The restriction $f^{m_{pq}}|_{W^{\mathrm{u}}(p)}$ has exactly one hyperbolic repelling fixed point $p$; therefore, there exists a smooth closed 2-disk $D_p \subset W^{\mathrm{u}}(p)$ such that $p \in D_p \subset \operatorname{int} f^{m_{pq}}(D_p)$. Then the set $C_{pq} = \bigcup_{x \in \partial D_p} [x, \xi_{pq}(x)]^{\mathrm{s}}$ is homeomorphic to the closed cylinder $\mathbb{S}^1 \times [0, 1]$. The set $C_{pq}$ is called a *connecting cylinder*. The circle $\xi_{pq}(\partial D_p)$ bounds a 2-disk $D_q$ in $W^{\mathrm{u}}(q)$ such that $q \in D_q \subset \operatorname{int} f^{m_{pq}}(D_q)$. The set $S_{pq} = D_p \cup C_{pq} \cup D_q$ is homeomorphic to a 2-sphere, which is called a *characteristic sphere* corresponding to the bunch $B_{pq}$.

Denote by $T(f)$ the set of nonwandering points of $f$ that do not belong to $\Lambda$. Then

(1) each characteristic sphere $S_{pq}$ bounds a 3-ball $Q_{pq}$ such that $T(f) \in \bigcup_{(p,q) \subset \Gamma_\Lambda} Q_{pq}$;

(2) for every associated pair $(p, q)$ of boundary points, there exists a positive integer $k_{pq}$ such that $T(f) \cap Q_{pq}$ consists of $k_{pq}$ periodic sources $\alpha_1, \dots, \alpha_{k_{pq}}$ and $k_{pq} - 1$ periodic saddle points $p_1, \dots, p_{k_{pq}-1}$ that alternate on the arc

$$l_{pq} = W^{\mathrm{s}\varnothing}(p) \cup \bigcup_{i=1}^{k_{pq}-1} W^{\mathrm{s}}(p_i) \cup \bigcup_{i=1}^{k_{pq}} \alpha_i \cup W^{\mathrm{s}\varnothing}(q)$$

(see Fig. 4).

Note that since the diffeomorphism $f$ is structurally stable, every arc $(x, y)^{\mathrm{s}}$ with $x \in W^{\mathrm{u}}(p)$ and $y \in W^{\mathrm{u}}(q)$ that are not boundary points intersects $W^{\mathrm{u}}(p_i)$ at exactly one point for all $i = 1, \dots, k_{pq} - 1$. Indeed, the converse would imply that there exists a point $z \in \Lambda$ such that $W^{\mathrm{s}}(z)$ is tangent to $W^{\mathrm{u}}(p_i)$, which contradicts the strong transversality condition. Then the intersection of

[1] Let $V \subset M$ be an open set with boundary $\partial V$ ($\partial V = \operatorname{cl} V \setminus \operatorname{int} V$). Then a subset $\delta V \subset \partial V$ is called a *$V$-accessible boundary* (or *boundary accessible from the interior of* $V$) if for every point $x \in \delta V$ there exists an open arc that lies completely in $V$ and has $x$ as one of its endpoints.

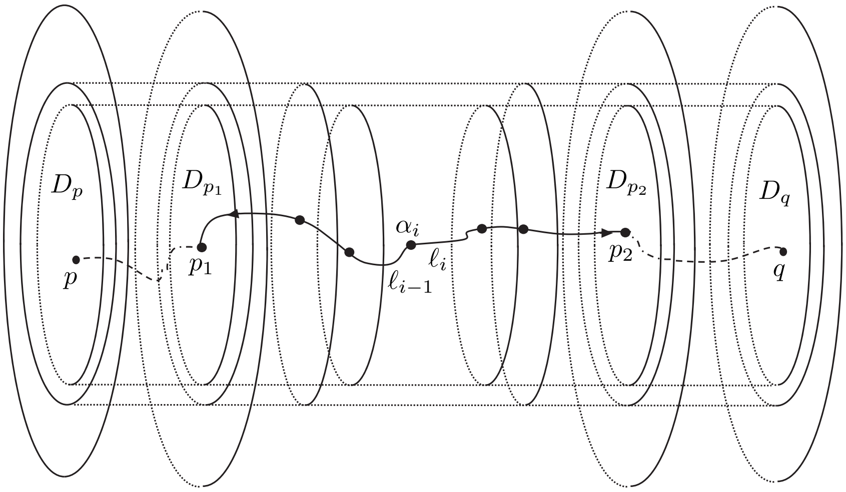


**Fig. 5.** Illustration to the proof of Theorem 3.1.

the 3-ball $Q_{pq}$ and the two-dimensional unstable manifold of the saddle point $p_i$, $i = 1, \dots, k_{pq} - 1$, is precisely the 2-disk $D_{p_i}$.

Thus, the disks $D_{p_i}$ divide the ball $Q_{pq}$ into $k_{pq}$ three-dimensional balls $Q_{\alpha_i}$, $i = 1, \dots, k_{pq}$, such that $\alpha_i \in Q_{\alpha_i}$.

Set $p = p_0$ and $q = p_{k_{pq}}$. Then the basin $W^{\mathrm{u}}(\alpha_i)$ of each periodic source $\alpha_i \in l_{pq}$, $i = 1, 2, \dots, k_{pq}$, contains exactly two stable separatrices $\ell_{i-1} = W^{\mathrm{s}}(p_{i-1}) \cap W^{\mathrm{u}}(\alpha_i)$ and $\ell_i = W^{\mathrm{s}}(p_i) \cap W^{\mathrm{u}}(\alpha_i)$ of the saddles $p_{i-1}$ and $p_i$, respectively. The separatrices $\ell_{i-1}$ and $\ell_i$ are said to be associated with the source $\alpha_i$. Note that the period of these separatrices is equal to that of the source $\alpha_i$ and is $m_{pq}$. Since $\alpha_i$ is a hyperbolic point, the diffeomorphism $f^{m_{pq}}|_{W^{\mathrm{u}}(\alpha_i)}$ is topologically conjugate to the homothety $A$ by a homeomorphism $h_{\alpha_i}\colon W^{\mathrm{u}}(\alpha_i) \to \mathbb{R}^3$. Let $L_{i-1} = h_{\alpha_i}(\ell_{i-1})$ and $L_i = h_{\alpha_i}(\ell_i)$. Then the union

$$F_2^{\alpha_i} = L_{i-1} \cup L_i \cup O$$

is a frame for two arcs *associated with the source* $\alpha_i$. Let

$$J_2^{\alpha_i} = \pi(F_2^{\alpha_i}).$$

Then $J_2^{\alpha_i}$ is a frame of two circles in $\mathbb{S}^2 \times \mathbb{S}^1$ *associated with the source* $\alpha_i$.

**Lemma 3.1.** *The frame of circles* $J_2^{\alpha_i}$ *is trivial.*

**Proof.** By Propositions 2.2 and 2.3, to prove that the frame of circles $J_2^{\alpha_i}$ associated with $\alpha_i$ is trivial, it suffices to construct a foliation of $W^{\mathrm{u}}(\alpha_i) \setminus \alpha_i$ such that each of its leaves is a 2-sphere that encloses $\alpha_i$ and intersects each of the separatrices $\ell_i$ and $\ell_{i-1}$ at a single point.

Consider a point $x_j \in \ell_j$, $j \in \{i-1, i\}$, and a fundamental domain $I_j \subset \ell_j$ with boundary points $x_j$ and $f^{m_{pq}}(x_j)$. Take a tubular neighborhood $V(I_j)$ diffeomorphic to $\mathbb{D}^2 \times [0,1]$ under a diffeomorphism $h_j\colon \mathbb{D}^2 \times [0,1] \to V(I_j)$ such that $f^{m_{pq}}(h_j(\mathbb{D}^2 \times \{0\})) \subset h_j(\mathbb{D}^2 \times \{1\})$ (Fig. 5). Since each arc $(x,y)^{\mathrm{s}}$ with nonboundary $x \in W^{\mathrm{u}}(p)$ and $y \in W^{\mathrm{u}}(q)$ intersects $W^{\mathrm{u}}(p_j)$ at exactly one point, by the $\lambda$-lemma we can assume (up to an iteration of the neighborhood $V(I_j)$) that each disk $h_j(\mathbb{D}^2 \times \{t\})$ intersects the arc $(x,y)^{\mathrm{s}}$ with $x \in D_p \setminus p$ at a single point.

The set $K_p = D_p \setminus \operatorname{int} f^{-m_{pq}}(D_p)$ is diffeomorphic to the set $\mathbb{S}^1 \times [0,1]$ under a diffeomorphism $h\colon \mathbb{S}^1 \times [0,1] \to K_p$ such that $h(\mathbb{S}^1 \times \{1\}) = \partial D_p$. Denote by $S_t$, $t \in [0,1]$, the 2-sphere bounded by the disks $h_{i-1}(\mathbb{D}^2 \times \{t\})$ and $h_i(\mathbb{D}^2 \times \{t\})$ and the cylinder $C_t = \bigcup_{x \in h(\mathbb{S}^1 \times \{t\})} [x, \xi_{pq}(x)]^{\mathrm{s}}$. The iterations of these spheres under $f^{m_{pq}}$ give the sought foliation of the set $W^{\mathrm{u}}(\alpha_i) \setminus \alpha_i$. $\square$

In fact, we can prove a stronger statement on the embedding of the frame $J_2^{\alpha_i}$ in $\mathbb{S}^2 \times \mathbb{S}^1$.

**Lemma 3.2.** *For any standard frame of circles $\mathbb{J}_2$, there exists a diffeomorphism $\widehat{g}\colon \mathbb{S}^2\times\mathbb{S}^1\to\mathbb{S}^2\times\mathbb{S}^1$ such that $\widehat{g}$ is isotopic to the identity and $\widehat{g}(\mathbb{J}_2)=J_2^{\alpha_i}$.*

**Proof.** By Lemma 3.1, there exists a homeomorphism $\widehat{G}\colon \mathbb{S}^2\times\mathbb{S}^1\to\mathbb{S}^2\times\mathbb{S}^1$ with $\widehat{G}(\mathbb{J}_2)=J_2^{\alpha_i}$. By the results of [18], the homeomorphism $\widehat{G}$ can be taken to be a diffeomorphism. Without loss of generality, we will also assume that $\mathbb{J}_2=\pi(OX_1)$. Since $\widehat{G}$ acts identically in the fundamental group, there are two possibilities: either $\widehat{G}$ is isotopic to the identity mapping or $\widehat{G}$ is isotopic to the involution $\widehat{v}$ obtained as follows. For any $\Theta\in\mathbb{R}$, denote by $R_\Theta$ the map that rotates the 2-sphere $S_r=\{(x,y,z)\in\mathbb{R}^3\colon x^2+y^2+z^2=r^2\}$ through an angle of $\Theta$ about the axis $OX_1$. Let $\widetilde{v}\colon\mathbb{R}^3\to\mathbb{R}^3$ be a diffeomorphism such that $\widetilde{v}|_{S_r}=R_{2\pi(r-1)}$ for $1\le r\le 2$ and $\widetilde{v}$ coincides with the identity mapping outside the ring $K_1=\{(x,y,z)\in\mathbb{R}^3\colon 1\le x^2+y^2+z^2\le 4\}$. Then

$$\widehat{v}=\pi\widetilde{v}(\pi|_{K_1})^{-1}\colon \mathbb{S}^2\times\mathbb{S}^1\to\mathbb{S}^2\times\mathbb{S}^1$$

is a sought diffeomorphism. Notice that $\widehat{v}(\mathbb{J}_2)=\mathbb{J}_2$.

If $\widehat{G}$ is isotopic to the identity mapping, then $\widehat{g}=\widehat{G}$, and if $\widehat{G}$ is isotopic to the involution $\widehat{v}$, then $\widehat{g}=\widehat{v}\widehat{G}$. $\square$

## 4. CONSTRUCTION OF A SIMPLE ARC

For $\mu\ge 0$, denote by $G_\mu$ the set of all structurally stable diffeomorphisms of the 3-torus with a two-dimensional expanding attractor and with exactly $\mu$ isolated saddle orbits. Thus, to prove Theorem 1.1, it suffices to construct a simple arc $\Gamma_{f_\mu,f_{\mu-1}}$ that connects a diffeomorphism $f_\mu\in G_\mu$, $\mu>0$, with a diffeomorphism $f_{\mu-1}\in G_{\mu-1}$ (this is done in Lemma 4.2 below). Indeed, then the simple arc

$$\Gamma_{f_\mu,f_0}=\Gamma_{f_\mu,f_{\mu-1}}*\ldots*\Gamma_{f_1,f_0}$$

connects[2] the structurally stable diffeomorphism $f_\mu$ whose nonwandering set contains a two-dimensional expanding attractor with the DA diffeomorphism $f_0$.

We decrease the number of isolated saddle points of a diffeomorphism $f$ by constructing an arc that unfolds generically through a saddle–node or period-doubling bifurcation. To implement such a scenario, we should reduce the confluence objects to a canonical form. To this end, we reduce the dynamics in the neighborhood of a source to a canonical expansion, and in Lemma 4.1 we lay an unstable saddle separatrix in the basin of a canonical sink on a smooth arc. An important technical tool of all the constructions is provided by the following classical facts.

**Proposition 4.1** (Thom's isotopy extension theorem, see [17, Theorem 5.8])**.** *Let $Y$ be a smooth manifold without boundary, $X$ a smooth compact submanifold in $Y$, and $\{f_t\colon X\to Y,\ t\in[0,1]\}$ a smooth isotopy such that $f_0$ is an inclusion mapping of $X$ into $Y$. Then, for an arbitrary compact set $A\subset Y$ containing the support* $\operatorname{supp}\{f_t\}$ *of the isotopy,*[3] *there exists a smooth isotopy $\{g_t\in\operatorname{Diff}(Y),\ t\in[0,1]\}$ such that $g_0=\mathrm{id}$, $g_t|_X=f_t|_X$ for all $t\in[0,1]$, and* $\operatorname{supp}\{g_t\}$ *belongs to $A$.*

**Proposition 4.2** (Franks's lemma [5, Lemma 1.1])**.** *Let $\theta$ be a finite set of points of a manifold $M^n$, $\varphi\colon M^n\to M^n$ a diffeomorphism, $T=\bigcup_{x\in\theta}TM^n_x$, and $T'=\bigcup_{x\in\theta}TM^n_{\varphi(x)}$. Then there exists a neighborhood $U(\theta)\supset\theta$ and a number $\varepsilon>0$ such that for any isomorphism $G\colon T\to T'$ satisfying the condition $\|G-D\varphi\|<\varepsilon/10$ there exists a diffeomorphism $\psi\colon M^n\to M^n$ such that $\psi$ is $\varepsilon$-close to $\varphi$ in the $C^1$ topology, $D\psi=G$ on $T$, and $\psi=f$ outside $U(\theta)$.*

---

[2] If $c_1$ and $c_2$ are paths in a topological space $X$ with $c_1(1)=c_2(0)$, then by the *product* of the paths $c_1$ and $c_2$ we mean the path $c_1*c_2$ defined as $(c_1*c_2)(t)=c_1(2t)$ for $0\le t\le 1/2$ and as $(c_1*c_2)(t)=c_2(2t-1)$ for $1/2\le t\le 1$.

[3] The *support* $\operatorname{supp}\{f_t\}$ of an isotopy $\{f_t\}$ is the closure of the set $\bigcup_{t\in[0,1]}\{x\in X\colon f_t(x)\ne f_0(x)\}$.

Since between any hyperbolic automorphisms of the same index (the number of eigenvalues greater than 1 in absolute value) there exists a path of hyperbolic automorphisms, Franks's lemma can be generalized as follows.

**Proposition 4.3.** *Suppose that a diffeomorphism $\varphi_0\colon M^n \to M^n$ has a hyperbolic point $r_0$ of period $m_0$, and let $(U_0, h)$ be a local chart of the manifold $M^n$ such that $r_0 \in U_0$ and $h(r_0) = O$. Then, for any hyperbolic automorphism $G$ with the same index as the automorphism $(D\varphi_0^{m_0})_{r_0}$, there exist neighborhoods $U_1$ and $U_2$ of $r_0$, $U_2 \subset U_1 \subset U_0$, and an arc $\varphi_t\colon M^n \to M^n$, $t \in [0,1]$, without bifurcations, such that*

(1) *the diffeomorphism $\varphi_t$, $t \in [0,1]$, coincides with the diffeomorphism $\varphi_0$ outside the set $\bigcup_{k=0}^{m-1} \varphi_0^k(U_1)$, and $\bigcup_{k=0}^{m-1} \varphi_0^k(r_0)$ is a hyperbolic orbit of period $m_0$ for every $\varphi_t$;*
(2) *the diffeomorphism $h\varphi_1^m h^{-1}$ coincides with the diffeomorphism $G$ on the set $h(U_2)$.*

Let us proceed to the details of the construction.

Let $B_r = \{(x_1, x_2, x_3) \in \mathbb{R}^3\colon x_1^2 + x_2^2 + x_3^2 < r^2\}$, $r > 0$, be an open ball centered at the origin $O(0,0,0)$ in the three-dimensional space $\mathbb{R}^3$. Consider the homothety $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined as $A(x_1, x_2, x_3) = (2x_1, 2x_2, 2x_3)$.

**Lemma 4.1.** *Suppose that a diffeomorphism $\varphi_0\colon M^3 \to M^3$ has a hyperbolic source $\alpha$, and let stable one-dimensional saddle separatrices $\gamma^1_{\varphi_0}$ and $\gamma^2_{\varphi_0}$ of saddles $P_1$ and $P_2$ lie in the basin of the source $W^{\mathrm{u}}(\alpha)$, have period $m$ equal to the period of $\alpha$, and form a tame frame $F_2^{\varphi_0} = \gamma^1_{\varphi_0} \cup \gamma^2_{\varphi_0} \cup \alpha$. Let $(U_0, h)$ be a local chart of the manifold $M^3$ such that $\alpha \in U_0$, $h(\alpha) = O$, and $\varphi_0^m(U_0) \supset U_0$. Then there exist neighborhoods $V_1$ and $V_2$ of $\alpha$ such that $V_2 \subset V_1 \subset U_0$ and an arc $\varphi_t\colon M^3 \to M^3$, $t \in [0,1]$, without bifurcations that have the following properties*:

(1) *the diffeomorphism $\varphi_t$, $t \in [0,1]$, coincides with the diffeomorphism $\varphi_0$ outside the set $\bigcup_{k=0}^{m-1} \varphi_0^k(V_1)$, and $\bigcup_{k=0}^{m-1} \varphi_0^k(\alpha)$ is a hyperbolic source orbit of period $m$ for every $\varphi_t$;*
(2) *for every $t \in [0,1]$, the one-dimensional separatrices $\gamma^1_{\varphi_t}$ and $\gamma^2_{\varphi_t}$ of the saddles $P_1$ and $P_2$ form a tame frame $F_2^{\varphi_t} = \gamma^1_{\varphi_t} \cup \gamma^2_{\varphi_t} \cup \alpha$, and $h(F_2^{\varphi_1} \cap V_2) \subset OX_1$.*

**Proof.** Introduce the notation $\phi_0 = \varphi_0^m$ and $\overline{\phi}_0 = h\phi_0 h^{-1}\colon \mathbb{R}^3 \to \mathbb{R}^3$. By Proposition 4.3, we can assume without loss of generality that $\overline{\phi}_0 = A$ on the disk $B_{2r_0}$ for some $r_0 > 0$. Set $\gamma^i_{\overline{\phi}_0} = h(\gamma^i_{\varphi_0})$, $i = 1, 2$, and $K_r = B_{2r} \setminus B_r$ for any $r > 0$.

Denote by $E$ the set of expansions $\overline{\phi}\colon \mathbb{R}^3 \to \mathbb{R}^3$ that coincide with $\overline{\phi}_0$ outside $B_{r_0/2}$ and with $A$ on $B_{2r_{\overline{\phi}}}$ for some $r_{\overline{\phi}} > 0$. For any $\overline{\phi} \in E$, set

$$\gamma^i_{\overline{\phi}} = \bigcup_{k\in\mathbb{Z}} \overline{\phi}^k\bigl(\gamma^i_{\overline{\phi}_0} \cap K_{r_0}\bigr), \quad i = 1, 2, \qquad F_2^{\overline{\phi}} = \gamma^1_{\overline{\phi}} \cup \gamma^2_{\overline{\phi}} \cup O.$$

By construction, the $\overline{\phi}$-invariant curve $\gamma^i_{\overline{\phi}}$ coincides with the $\overline{\phi}_0$-invariant curve $\gamma^i_{\overline{\phi}_0}$ outside $B_{r_0}$. To prove the lemma, it suffices to construct an arc of expansions $\overline{\phi}_t\colon \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0,1]$, such that

(1′) the diffeomorphism $\overline{\phi}_t$, $t \in [0,1]$, coincides with the diffeomorphism $\overline{\phi}_0$ outside the set $B_{r_0}$;
(2′) $F_2^{\overline{\phi}_1} \cap B_{r_{\overline{\phi}_1}} \subset OX_1$.

Then the arc $\varphi_t\colon M^3 \to M^3$ can be obtained from the arc $\overline{\phi}_t$ as follows. Let $V_1 = h^{-1}(B_{r_0})$, $V_2 = h^{-1}\bigl(B_{r_{\overline{\phi}_1}}\bigr)$, and $\phi_t = h^{-1}\overline{\phi}_t h$ on $V_1$. Then, for every $t \in [0,1]$, $\varphi_t$ coincides with $\varphi_0$ outside the set $\bigcup_{k=0}^{m-1} \varphi_0^k(V_1)$, $\varphi_t(z) = \varphi_0(z)$ for $z \in \varphi_0^k(V_2)$, $k \in \{0, \dots, m-2\}$, and $\varphi_t(z) = \phi_t(\varphi_0^{1-m}(z))$ for $z \in \varphi_0^{m-1}(V_2)$.

To construct the arc $\overline{\phi}_t$, we fix the knots

$$\widehat{\gamma}_1 = \pi(OX_1^-), \qquad \widehat{\gamma}_2 = \pi(OX_1^+)$$

on the manifold $\mathbb{S}^2\times\mathbb{S}^1$. For $\overline{\phi}\in E$, set

$$\widehat{\gamma}^i_{\overline{\phi}}=\pi\big(\gamma^i_{\overline{\phi}}\cap K_{r_{\overline{\phi}}}\big),\qquad i=1,2.$$

Then the curve $\widehat{\gamma}^i_{\overline{\phi}}$ is a knot on $\mathbb{S}^2\times\mathbb{S}^1$ whose homotopy class is 1 (see, for example, [2]).

By Lemma 3.2, there exists a diffeomorphism $\widehat{g}\colon\mathbb{S}^2\times\mathbb{S}^1\to\mathbb{S}^2\times\mathbb{S}^1$ such that $\widehat{g}$ is isotopic to the identity and $\widehat{g}\big(\widehat{\gamma}^i_{\overline{\phi}_0}\big)=\widehat{\gamma}_i$, $i=1,2$. Take an open covering $D=\{D_1,\dots,D_q\}$ of $\mathbb{S}^2\times\mathbb{S}^1$ such that a connected component $\mathcal{D}_i$ of $p^{-1}(D_i)$ is a subset of $K_{r_i}$ for some $r_i<r_{i-1}/2$ and $r_1\le r_0/4$, $i=1,\dots,q$. According to [1, Fragmentation lemma], there exist diffeomorphisms $\widehat{w}_1,\dots,\widehat{w}_q\colon\mathbb{S}^2\times\mathbb{S}^1\to\mathbb{S}^2\times\mathbb{S}^1$ that are smoothly isotopic to the identity and have the following properties:

(i) for each $i\in\{1,\dots,q\}$, there exists a smooth isotopy $\{\widehat{w}_{i,t}\}$ that is the identity outside $D_i$ and connects the identity mapping and $\widehat{w}_i$;

(ii) $\widehat{g}=\widehat{w}_1\dots\widehat{w}_q$.

Let $\overline{w}_{i,t}\colon\mathbb{R}^3\to\mathbb{R}^3$ be a diffeomorphism that coincides with $(p|_{K_{r_i}})^{-1}\widehat{w}_{i,t}p$ on $K_{r_i}$ and with the identity mapping outside $K_{r_i}$. Then the sought arc is defined by the formula

$$\overline{\phi}_t=\overline{w}_{1,t}\dots\overline{w}_{q,t}.\qquad\Box$$

**Lemma 4.2.** *There exists a simple arc $\Gamma_{f_\mu,f_{\mu-1},t}$ that connects a diffeomorphism $f_\mu\in G_\mu$, $\mu>0$, with a diffeomorphism $f_{\mu-1}\in G_{\mu-1}$.*

**Proof.** According to the results of Section 3, all trivial basic sets of $f_\mu$ lie on the arcs $l_{pq}=W^{\mathrm{s}\varnothing}(p)\cup\bigcup_{i=1}^{k_{pq}-1}W^{\mathrm{s}}(p_i)\cup\bigcup_{i=1}^{k_{pq}}\alpha_i\cup W^{\mathrm{s}\varnothing}(q)$, which connect boundary points $p$ and $q$. In the basin $W^{\mathrm{u}}(\alpha_i)$ of every periodic source $\alpha_i\in l_{pq}$, $i=1,2,\dots,k_{pq}$, there are exactly two stable separatrices $\ell_{i-1}=W^{\mathrm{s}}(p_i)\cap W^{\mathrm{u}}(\alpha_i)$ and $\ell_i=W^{\mathrm{s}}(p_{i+1})\cap W^{\mathrm{u}}(\alpha_i)$ of the saddles $p_{i-1}$ and $p_i$, which form a frame of arcs $F_2^{\alpha_i}$. In this case, Lemma 4.1 allows us to assume that there exists a local chart $(U,\psi)$ on $\mathbb{T}^3$ such that $\alpha_i\in U$, $\psi(\alpha_i)=O$, $f^{m_{pq}}(U)\supset U$, and $\psi(F_2^{\alpha_i}\cap U)\subset OX_1$.

Consider an arc $l_{pq}$ passing through at least one saddle point $p_i$. Then two situations are possible:

(1) the point $p_i$ has a positive type of orientation;

(2) the point $p_i$ has a negative type of orientation.

In case (1), the source $\alpha_i$ has the same period $m_{pq}$ as the point $p_i$. Set $l_i=W^{\mathrm{s}}_{p_i}\cup\psi^{-1}(OX_1)$. Then $l_i$ is a smooth curve containing $\ell_i$, and the points $\alpha_i$ and $p_i$ are interior points for $l_i$.

Set $\widetilde{\Pi}_1=\{(x_1,x_2,x_3)\in\mathbb{R}^3\colon|x_i|<1/2\}$. On the interval $[-1/2,1/2]\subset OX_1$, we define a family of mappings $\varphi_t$, $t\in[0,1]$, by the formula

$$\varphi_t(x_1)=x_1+\frac{1}{2}x_1^2+\frac{1}{10}(2t-1).$$

Then we define a diffeomorphism $\widetilde{\varphi}_t\colon\widetilde{\Pi}_1\to\mathbb{R}^3$ as

$$\widetilde{\varphi}_t(x_1,x_2,x_3)=(\varphi_t(x_1),2x_2,2x_3).$$

By construction, the diffeomorphism $\widetilde{\varphi}_0$ has a saddle point $P_1(-1/\sqrt{5},0,0)$ and a source point $P_2(1/\sqrt{5},0,0)$. For $\varepsilon>0$ and $\delta>0$, set

$$I_\varepsilon=\bigg[-\frac{1}{\sqrt{5}}-\varepsilon,\frac{1}{\sqrt{5}}+\varepsilon\bigg]\subset OX_1,\qquad V_{\varepsilon,\delta}=\bigg\{(x_1,x_2,x_3)\in\mathbb{R}^3\colon\ |x_1|<\frac{1}{\sqrt{5}}+\varepsilon,\ x_2^2+x_3^2<\delta^2\bigg\}.$$

Take a neighborhood $\Pi_1$ of $\ell_i$ and a diffeomorphism $\beta\colon\Pi_1\to\widetilde{\Pi}_1$ such that $\beta(p_i)=P_1$, $\beta(\alpha_i)=P_2$, and $\beta(l_i\cap\Pi_1)=Ox_1\cap\widetilde{\Pi}_1$. Then the following diffeomorphism is well defined in some neighborhood $V_{\varepsilon_1,\delta_1}$:

$$\widetilde{f}=\beta f^{m_{pq}}\beta^{-1}.$$

By Proposition 4.3, we can assume that in the neighborhoods

$$V_{P_1} = \left\{(x_1,x_2,x_3) \in \mathbb{R}^3\colon \ \left|x_1 + \frac{1}{\sqrt{5}}\right| < \varepsilon_1,\ x_2^2 + x_3^2 < \delta_1^2\right\},$$
$$V_{P_2} = \left\{(x_1,x_2,x_3) \in \mathbb{R}^3\colon \ \left|x_1 - \frac{1}{\sqrt{5}}\right| < \varepsilon_1,\ x_2^2 + x_3^2 < \delta_1^2\right\}$$

of the points $P_1$ and $P_2$, the diffeomorphism $\widetilde{f}$ coincides with $(D\widetilde{\varphi}_0)_{P_1}$ and $(D\widetilde{\varphi}_0)_{P_2}$, respectively. Denote the restriction of $\widetilde{f}$ to the interval $I_{\varepsilon_1}$ by $\phi$. On the cylinder $V_{\varepsilon_1,\delta_1}$, define a diffeomorphism $\widetilde{\phi}$ by setting

$$\widetilde{\phi}(x_1,x_2,x_3) = (\phi(x_1), 2x_2, 2x_3).$$

For $\delta > 0$, define a cutoff function $\rho_\delta(r)$, $r \ge 0$, equal to 1 for $r \in [0,\delta]$ and to 0 for $r \ge 2\delta$. For $\delta_2 = \delta_1/2$, define a family of diffeomorphisms $\widetilde{a}_t$, $t \in [0,1]$, on the cylinder $V_{\varepsilon_1,\delta_1}$ as

$$\widetilde{a}_t(x_1,x_2,x_3) = t\rho_{\delta_2}\big(\sqrt{x_2^2+x_3^2}\,\big)\widetilde{\phi}(x_1,x_2,x_3) + \big(1 - t\rho_{\delta_2}\big(\sqrt{x_2^2+x_3^2}\,\big)\big)\widetilde{f}(x_1,x_2,x_3).$$

By construction, $\widetilde{a}_t$, $t \in [0,1]$, coincides with $\widetilde{f}$ on $\partial V_{\varepsilon_1,\delta_1}$, $\widetilde{a}_0 = \widetilde{f}$ on $V_{\varepsilon_1,\delta_1}$, and $\widetilde{a}_1 = \widetilde{\phi}$ on $V_{\varepsilon_1,\delta_2}$. Since the diffeomorphisms $\widetilde{f}$ and $\widetilde{\phi}$ coincide on the interval $I_{\varepsilon_1}$ and cylinders $V_{P_1}$ and $V_{P_2}$, we can assume without loss of generality that the value of $\delta_2$ is chosen so that the diffeomorphism $\widetilde{a}_t$ has no nonwandering points different from $P_1$ and $P_2$.

For $\varepsilon_2 = \varepsilon_1/4$ and $t \in [0,1]$, set

$$v_t(x_1) = \rho_{\varepsilon_2}(|x_1|)\varphi_t(x_1) + (1 - \rho_{\varepsilon_2}(|x_1|))\phi(x_1), \qquad x_1 \in I_{\varepsilon_1}.$$

By construction, the diffeomorphism $v_t$ coincides with $\varphi_t$ on $I_{\varepsilon_2}$ and with $\phi$ on $I_{\varepsilon_1} \setminus I_{2\varepsilon_2}$. For $t \in [0,1]$, set $\nu_t(x_1) = tv_0(x_1) + (1-t)\phi(x_1)$. By construction, $\nu_0$ coincides with $\phi$ and $\nu_1$ coincides with $v_0$. Set $w_t = \nu_t * v_t$ and $\widetilde{w}_t(x_1,x_2,x_3) = (w_t(x_1), 2x_2, 2x_3)$ for $(x_1,x_2,x_3) \in V_{\varepsilon_1,\delta_1}$.

For $\delta_3 = \delta_2/2$, define a family of diffeomorphisms $\widetilde{b}_t$, $t \in [0,1]$, on the cylinder $V_{\varepsilon_1,\delta_1}$ as follows:

$$\widetilde{b}_t(x_1,x_2,x_3) = \rho_{\delta_3}\big(\sqrt{x_2^2+x_3^2}\,\big)\widetilde{w}_t(x_1,x_2,x_3) + \big(1 - \rho_{\delta_3}\big(\sqrt{x_2^2+x_3^2}\,\big)\big)\widetilde{a}_1(x_1,x_2,x_3).$$

By construction, $\widetilde{b}_t$, $t \in [0,1]$, coincides with $\widetilde{f}$ on $\partial V_{\varepsilon_1,\delta_1}$, $\widetilde{b}_0 = \widetilde{a}_1$ on $V_{\varepsilon_1,\delta_1}$, $\widetilde{b}_t = \widetilde{w}_t$ on $V_{\varepsilon_1,\delta_3}$, and $\widetilde{b}_1 = \widetilde{v}_1$ on $V_{\varepsilon_1,\delta_3}$.

Set $\widetilde{c}_t = \widetilde{a}_t * \widetilde{b}_t$ and $U_2 = \beta^{-1}(V_{\varepsilon_1,\delta_1})$. Then the sought arc $\Gamma_{f_\mu,f_{\mu-1}}$ coincides with $f$ outside $\bigcup_{k=0}^{m-1} f^k(U_2)$; $f_t(z) = f(z)$ for $z \in f^k(U_2)$, $k = 0,\dots,m_{pq}-2$; and $f_t(z) = \beta^{-1}(\widetilde{c}_t(\beta(f^{1-m_{pq}}(z))))$ for $z \in f^{m_{pq}-1}(U_2)$.

In case (2), the saddle point $p_i$ has period $m_{pq}/2$, and the points $\alpha_i$ and $\alpha_{i+1}$ have period $m_{pq}$; in this case, $f^{m_{pq}/2}(\alpha_i) = \alpha_{i+1}$. Set $l = W^{\mathrm{s}}_{p_i} \cup \psi^{-1}(OX_1) \cup f(\psi^{-1}(OX_1))$. Then $l$ is a smooth curve for which the points $\alpha_i$, $p_i$, and $\alpha_{i+1}$ are interior points. Let $\Pi_1 = \{(x_1,x_2,x_3) \in \mathbb{R}^2\colon |x_i| < 1/2\}$. On the interval $[-1/2,1/2] \subset OX_1$, we define a family of diffeomorphisms $\varphi_t$ by setting

$$\varphi_t(x_1) = -x_1\left(1 + \frac{1}{10}(2t-1)\right) + x_1^3.$$

Define a diffeomorphism $\widetilde{\varphi}_t\colon \widetilde{\Pi}_1 \to \mathbb{R}^3$ as

$$\widetilde{\varphi}_t(x_1,x_2,x_3) = (\varphi_t(x_1), -2x_2, 2x_3).$$

By construction, the diffeomorphism $\widetilde{\varphi}_0$ has three periodic points: the sources $P_1(-1/\sqrt{10},0)$ and $P_2(1/\sqrt{10},0)$ of period 2 and the fixed saddle point $P_3(0,0)$. For $\varepsilon>0$ and $\delta>0$, set

$$J_\varepsilon=\left[-\frac{1}{\sqrt{10}}-\varepsilon,\frac{1}{\sqrt{10}}+\varepsilon\right]\subset OX_1,\qquad V_{\varepsilon,\delta}=\left\{(x_1,x_2)\in\mathbb{R}^3\colon\ |x_1|<\frac{1}{\sqrt{10}}+\varepsilon,\ x_2^2+x_3^2<\delta^2\right\}.$$

Take a neighborhood $\Pi_1$ of the arc $\operatorname{cl}W^{\mathrm{s}}_{p_i}$ and a diffeomorphism $\beta\colon\Pi_1\to\widetilde{\Pi}_1$ such that $\beta(\alpha_i)=P_1$, $\beta(\alpha_{i+1})=P_2$, $\beta(p_i)=P_3$, and $\beta(l\cap\Pi_1)=Ox_1\cap\widetilde{\Pi}_1$. Then the following diffeomorphism is well defined in some neighborhood $V_{\varepsilon_1,\delta_1}$:

$$\widetilde{f}=\beta f^{m_{pq}/2}\beta^{-1}.$$

By Proposition 4.3, we can assume that in the neighborhoods

$$V_{P_1}=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3\colon\ \left|x_1+\frac{1}{\sqrt{10}}\right|<\varepsilon_1,\ x_2^2+x_3^2<\delta_1^2\right\},$$
$$V_{P_2}=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3\colon\ \left|x_1-\frac{1}{\sqrt{10}}\right|<\varepsilon_1,\ x_2^2+x_3^2<\delta_1^2\right\},$$
$$V_{P_3}=\big\{(x_1,x_2,x_3)\in\mathbb{R}^3\colon\ |x_i|<\delta_1\big\}$$

of the points $P_1$, $P_2$, and $P_3$, the diffeomorphism $\widetilde{f}$ coincides with $(D\widetilde{\varphi}_0)_{P_1}$, $(D\widetilde{\varphi}_0)_{P_2}$, and $(D\widetilde{\varphi}_0)_{P_3}$, respectively. Denote by $\phi$ the restriction of $\widetilde{f}$ to the interval $J_{\varepsilon_1}$. On the cylinder $V_{\varepsilon_1,\delta_1}$, define a diffeomorphism $\widetilde{\phi}$ by setting

$$\widetilde{\phi}(x_1,x_2,x_3)=(\phi(x_1),-2x_2,2x_3).$$

For $\delta>0$, define a cutoff function $\rho_\delta(r)$, $r\geq0$, equal to 1 for $r\in[0,\delta]$ and to 0 for $r\geq2\delta$. For $\delta_2=\delta_1/2$, define a family of diffeomorphisms $\widetilde{a}_t$, $t\in[0,1]$, on the cylinder $V_{\varepsilon_1,\delta_1}$ as

$$\widetilde{a}_t(x_1,x_2,x_3)=t\rho_{\delta_2}\big(\sqrt{x_2^2+x_3^2}\,\big)\widetilde{\phi}(x_1,x_2,x_3)+\big(1-t\rho_{\delta_2}\big(\sqrt{x_2^2+x_3^2}\,\big)\big)\widetilde{f}(x_1,x_2,x_3).$$

By construction, $\widetilde{a}_t$, $t\in[0,1]$, coincides with $\widetilde{f}$ on $\partial V_{\varepsilon_1,\delta_1}$, $\widetilde{a}_0=\widetilde{f}$ on $V_{\varepsilon_1,\delta_1}$, and $\widetilde{a}_1=\widetilde{\phi}$ on $V_{\varepsilon_1,\delta_2}$. Since the diffeomorphisms $\widetilde{f}$ and $\widetilde{\phi}$ coincide on the interval $J_{\varepsilon_1}$ and cylinders $V_{P_1}$, $V_{P_2}$, and $V_{P_3}$, we can assume without loss of generality that the value of $\delta_2$ is chosen so that the diffeomorphism $\widetilde{a}_t$ has no nonwandering points different from $P_1$, $P_2$, and $P_3$.

For $\varepsilon_2=\varepsilon_1/4$ and $t\in[0,1]$, set

$$v_t(x_1)=\rho_{\varepsilon_2}(|x_1|)\varphi_t(x_1)+(1-\rho_{\varepsilon_2}(|x_1|))\phi(x_1),\qquad x_1\in J_{\varepsilon_1}.$$

By construction, the diffeomorphism $v_t$ coincides with $\varphi_t$ on $J_{\varepsilon_2}$ and with $\phi$ on $J_{\varepsilon_1}\setminus J_{2\varepsilon_2}$. For $t\in[0,1]$, set $\nu_t(x_1)=tv_0(x_1)+(1-t)\phi(x_1)$. By construction, $\nu_0$ coincides with $\phi$ and $\nu_1$ coincides with $v_0$. Let $w_t=\nu_t*v_t$ and $\widetilde{w}_t(x_1,x_2)=(w_t(x_1),-2x_2,2x_3)$ for $(x_1,x_2,x_3)\in V_{\varepsilon_1,\delta_1}$.

For $\delta_3=\delta_2/2$, we define the following family of diffeomorphisms $\widetilde{b}_t$, $t\in[0,1]$, on the cylinder $V_{\varepsilon_1,\delta_1}$:

$$\widetilde{b}_t(x_1,x_2,x_3)=\rho_{\delta_3}\big(\sqrt{x_2^2+x_3^2}\,\big)\widetilde{w}_t(x_1,x_2,x_3)+\big(1-\rho_{\delta_3}\big(\sqrt{x_2^2+x_3^2}\,\big)\big)\widetilde{a}_1(x_1,x_2,x_3).$$

By construction, $\widetilde{b}_t$, $t\in[0,1]$, coincides with $\widetilde{f}$ on $\partial V_{\varepsilon_1,\delta_1}$, $\widetilde{b}_0=\widetilde{a}_1$ on $V_{\varepsilon_1,\delta_1}$, $\widetilde{b}_t=\widetilde{w}_t$ on $V_{\varepsilon_1,\delta_3}$, and $\widetilde{b}_1=\widetilde{v}_1$ on $V_{\varepsilon_1,\delta_3}$.

Set $\widetilde{c}_t=\widetilde{a}_t*\widetilde{b}_t$ and $U_2=\beta^{-1}(V_{\varepsilon_1,\delta_1})$. Then, outside $\bigcup_{k=0}^{m-1}f^k(U_2)$, the sought arc $\Gamma_{f_\mu,f_{\mu-1}}$ coincides with $f$; $f_t(z)=f(z)$ for $z\in f^k(U_2)$, $k=0,\dots,m_{pq}/2-2$; and $f_t(z)=\beta^{-1}(\widetilde{c}_t(\beta(f^{1-m_{pq}/2}(z))))$ for $z\in f^{m_{pq}/2-1}(U_2)$. $\square$

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