

# DELTA-INVARIANTS FOR FANO VARIETIES WITH LARGE AUTOMORPHISM GROUPS

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ABSTRACT. For a polarized variety  $(X, L)$  and a closed connected subgroup  $G \subset \text{Aut}(X, L)$  we define a  $G$ -invariant version of the  $\delta$ -threshold. We prove that for a Fano variety  $(X, -K_X)$  and a connected subgroup  $G \subset \text{Aut}(X)$  this invariant characterizes  $G$ -equivariant  $K$ -stability. We also use this invariant to investigate  $G$ -equivariant  $K$ -stability of some Fano varieties with large groups of symmetries, including spherical Fano varieties. We also consider the case of  $G$  being a finite group.

## 1. INTRODUCTION

The problem of constructing Kähler–Einstein metrics on Fano varieties (over the field  $\mathbb{C}$  of complex numbers) has been extensively studied in recent years. In particular, for smooth Fano manifolds the existence of Kähler–Einstein metrics was shown by Chen, Donaldson and Sun to be equivalent to an algebro-geometric condition of  $K$ -*polystability* [CDS15]. Another approach to this problem is the *variational* one, developed in [BBGZ13, BBEGZ11, Ber16]. For a Fano variety with finite automorphism group the existence of a Kähler–Einstein metric is equivalent to a stronger property of *uniform  $K$ -stability*. This was shown in [BBJ18] for  $X$  smooth and in [Ber16, LTW19] for a Fano variety with klt singularities.

In view of the above results, it is important to be able to check if a given Fano variety  $X$  is  $K$ -polystable or uniformly  $K$ -stable. A priori this requires computing certain numerical invariants for all polarized one-parameter degenerations of  $X$  (see Definition 3.6 below for the precise definition of  $K$ -stability). Ideally, one would like to have a numerical invariant, depending on the variety  $X$  and an ample (or, more generally, big) polarization  $L$  only, such that the  $K$ -stability of  $(X, L)$  is detected by this invariant. The first example of such invariant was the  $\alpha$ -invariant (or its version  $\alpha_G(X)$  for a compact group  $G$  of symmetries of  $X$ ) introduced by Tian [Ti87, p. 229] via analytic methods. Tian gave a sufficient condition for the existence of a Kähler–Einstein metric on a Fano manifold in terms of  $\alpha_G(X)$ .

**Theorem 1.1** ([Ti87, Theorems 2.1 and 4.1]). Let  $X$  be a Fano manifold of dimension  $n$  and  $G \subset \text{Aut}(X)$  a compact subgroup. If  $\alpha_G(X) > \frac{n}{n+1}$  then  $X$  admits a Kähler–Einstein metric.

In early 2000s it became evident to experts that Tian’s  $\alpha$ -invariant coincides with the global log canonical threshold of  $X$  (see [C08] and [CS08, Appendix A]). An algebraic counterpart of Tian’s result was given by Odaka and Sano [OS12, Theorem 1.4]. They have shown by purely algebraic methods that a klt Fano variety  $X$  satisfying  $\alpha(X) > n/(n+1)$  is  $K$ -stable. Moreover, for a Fano variety  $X$  and an closed subgroup  $G \subset \text{Aut}(X)$  Odaka and Sano proved [OS12, Theorem 1.10] that the condition  $\alpha_G(X) > n/(n+1)$  implies  $K$ -stability of  $X$  along  $G$ -equivariant degenerations ( $G$ -equivariant  $K$ -stability, see Definition 3.6).

Automorphism groups have been successfully used to establish the existence (or nonexistence) of Kähler–Einstein metrics for many particular examples of Fano varieties. To list a few examples, we should mention the obstructions for the existence of such metrics [Mat57, Fut83]. Also, for smooth del Pezzo surfaces the necessary and sufficient condition for being Kähler–Einstein is reductivity of  $\text{Aut}(X)$  [Ti90]. For toric Fano varieties  $K$ -stability was studied in [WZ04, Don02, Don08]. The case of varieties with torus action of complexity one was considered in [Su13, IS17]. In fact, by [LWX18, Theorem 1.4] torus-equivariant  $K$ -polystability of a Fano variety  $X$  with a torus action is equivalent to  $K$ -polystability of  $X$ . Tian’s result was applied in [Nad90, C08, CS09] to prove the Kähler–Einstein property for certain Fano threefolds, including those of types  $V_1, V_5$  (see e.g. [IP99, §12] for the classification of Fano threefolds). Equivariant  $K$ -polystability of Fano threefolds of type  $V_{22}$  was studied in [Don08, Section 5], [CS12,

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Appendix A] and in [CS18]. For a criterion of equivariant  $K$ -stability of spherical Fano varieties see [Del16]. Also an extremely important general result was proved by Datar and Székelyhidi.

**Theorem 1.2** ([DS16, Theorem 1]). Let  $X$  be a smooth Fano manifold and  $G \subset \text{Aut}(X)$  a reductive subgroup. If  $X$  is  $K$ -polystable with respect to  $G$ -equivariant test configurations then  $X$  is Kähler–Einstein.

These results show the significance of equivariant  $K$ -stability for the study of Fano varieties.

Another invariant, the so-called  $\delta$ -invariant, was defined for Fano varieties by Fujita and Odaka [FO16, Definition 0.2] using log canonical thresholds of basis-type divisors. The inequality  $\delta(X) > 1$  implies uniform  $K$ -stability of  $X$  by [FO16, Theorem 0.3] and in fact turns out to be equivalent to it (see [BJ17, Theorem B]).

Since uniform  $K$ -stability forces the automorphism group of  $X$  to be finite [BBEGZ11, Theorem 5.4], we cannot use  $\delta$ -invariant to study Kähler–Einstein property of Fano varieties with  $\text{Aut}(X)$  infinite. However, if we restrict to  $G$ -equivariant degenerations for a suitable subgroup  $G \subset \text{Aut}(X)$  (which is sufficient by Theorem 1.2), then in many examples uniform  $K$ -stability holds. Therefore it is reasonable to expect that there is a version of  $\delta$ -invariant for a variety  $X$  with a reductive subgroup  $G \subset \text{Aut}(X)$ .

The main goal of the present paper is to define the  $\delta_G$ -invariant for a polarized variety  $(X, L)$  and a closed connected subgroup  $G \subset \text{Aut}(X, L)$ . To do so, we follow the approach to  $\delta$ -invariants and  $K$ -stability via valuations on the field of rational functions on  $X$ , developed in [Fuj16, Fuj17, BJ17, BHJ17]. Note also that an alternative valuative criterion for  $K$ -semistability was proved in [Li15]. In Section 2 we give the necessary definitions. The main object we consider is the space  $\text{DivVal}_X^G$  of  $G$ -invariant divisorial valuations on  $X$ . Up to a multiplicative constant, every such valuation  $v$  is given by the order of vanishing at the generic point of a  $G$ -stable prime divisor  $E$  on a birational model  $\varphi: Y \rightarrow X$ . We consider the log discrepancy  $A_X(v) = 1 + \text{ord}_E(K_{Y/X})$  and the expected vanishing order

$$S_L(v) = \frac{1}{\text{Vol}(X)} \int_0^\infty \text{Vol}(\varphi^*L - tE) dt$$

of the valuation. Being inspired by [BJ17, Theorem 4.4], we define (see Definition 2.19):

$$\delta_G(X, L) = \inf_{v \in \text{DivVal}_X^G} \frac{A_X(v)}{S_L(v)}.$$

Also in Section 2 we study basic properties of this invariant and compare it to  $\alpha_G$ -invariant of Tian. In Section 3 we discuss equivariant  $K$ -stability and prove our main result.

**Theorem 1.3** (see Theorem 3.7 below). Let  $(X, -K_X)$  be a klt  $\mathbb{Q}$ -Fano variety with the anticanonical polarization. Let  $G \subset \text{Aut}(X)$  be a connected reductive subgroup. Then  $(X, -K_X)$  is uniformly  $K$ -stable with respect to  $G$ -equivariant degenerations if and only if the  $\delta_G$ -invariant of  $(X, -K_X)$  is greater than one.

It is of course desirable to generalize this theorem to the case of an arbitrary closed reductive subgroup  $G \subset \text{Aut}(X)$  (e. g. a finite group  $G$ ).

In Section 4 we investigate  $\delta$ -invariants of varieties with an action of a torus  $T = (\mathbb{G}_m)^k$ . We prove that  $\delta$ -invariant can be computed using only  $T$ -invariant valuations (Proposition 4.1), generalizing a result of Blum and Jonsson, who considered the case of a toric variety  $X$  and a maximal torus  $T$ .

Section 5 is devoted to  $\delta_G$ -invariants of spherical Fano varieties. If  $X$  is a spherical Fano variety under the action of a connected reductive group  $G$ , we give a formula for  $\delta_G(X)$ . We choose a Borel subgroup  $B \subset G$  and a maximal torus  $T \subset B$ . This formula uses the description of  $\text{Val}_X^G$  as the cone  $\mathcal{V}$  in a finite-dimensional vector space. The log discrepancy  $A_X(v)$  identifies with a certain piecewise linear function  $h_C$  on  $\mathcal{V}$ . The function  $S_L(v)$  can be expressed, following [Del16], via the moment polytope  $\Delta^+$ , the Duistermaat–Heckman measure  $\text{DH}$  on  $\Delta^+$  (see [Del16, Theorem 4.5]) and the vector  $2\rho_Q$ , determined by  $\Delta^+$  and the root system of  $(G; T)$ .

**Proposition 1.4** (see Proposition 5.4 below). Let  $X$  be a Fano variety which is spherical under the action of a connected reductive group  $G$ . Then  $\delta_G$ -invariant of  $X$  can be expressed as follows:

$$\delta_G(X) = \min_{v \in \mathcal{V}} \frac{h_C(v)}{h_C(v) - V \cdot \langle 2\rho_Q - \text{bar}_{\text{DH}}(\Delta^+), \tilde{v} \rangle}.$$

Here  $\text{bar}_{DH}(\Delta^+)$  is the barycenter of  $\Delta^+$  with respect to the Duistermaat–Heckman measure DH and  $V$  is a constant depending on  $\Delta^+$  and DH only.

In Section 6 we consider the case of a variety with an action of a finite group  $G$ ; we give an alternative definition of  $\delta_G$  using  $G$ -invariant divisors and prove the ramification formula.

**Proposition 1.5** (see Proposition 6.3 below). Let  $X$  be a variety with klt singularities and  $-K_X$  big. Let  $G \subset \text{Aut}(X)$  be a finite group. Denote by  $Y = X/G$  the quotient variety and let  $B = \sum_i (1 - 1/m_i) B_i$  be the branch divisor on  $Y$ . Then we have

$$\delta_G(X) = \delta(Y, B)$$

where  $\delta(Y, B)$  is the  $\delta$ -invariant of the klt pair  $(Y, B)$ .

We expect that there is a unified definition of  $\delta_G$  for any closed subgroup  $G \subset \text{Aut}(X, L)$ . We also hope for a generalization of Theorem 3.7 to the case of a Fano variety  $X$  with an arbitrary reductive subgroup  $G \subset \text{Aut}(X)$ .

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## 2. GENERAL DEFINITIONS

**2.1. Notation and conventions.** We work over the field  $\mathbb{C}$  of complex numbers. All varieties are projective and  $\mathbb{Q}$ -Gorenstein. A  $\mathbb{Q}$ -Fano variety is a variety such that  $K_X$  is an ample  $\mathbb{Q}$ -Cartier divisor. We restrict ourselves to  $\mathbb{Q}$ -Fano varieties (or pairs) with Kawamata log terminal (klt) singularities. For all basic information regarding singularities we refer to [Kol97]. A  $\mathbb{Q}$ -line bundle  $L$  is a reflexive sheaf of rank 1 such that some tensor power of  $L$  is locally free.

In this section we recall the definitions of  $\alpha$  and  $\delta$ -invariants via log canonical thresholds and valuations. Then we study the space of  $G$ -invariant valuations and define  $\delta_G$ .

### 2.2. Log canonical thresholds.

**Definition 2.1.** Let  $X$  be a normal projective variety and let  $\Delta$  be an effective  $\mathbb{Q}$ -divisor on  $X$  such that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier and the pair  $(X, \Delta)$  has klt singularities. For an effective  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor  $D$  on  $X$  we define the *log canonical threshold* of  $D$  with respect to  $(X, \Delta)$  by the formula

$$\text{lct}(X, \Delta; D) = \sup\{t \in \mathbb{R} \mid (X, \Delta + tD) \text{ is log canonical}\}.$$

**Definition 2.2.** Let  $(X, \Delta)$  be a klt log Fano pair. We define the  *$\alpha$ -invariant* of  $(X, \Delta)$  by

$$\alpha(X, \Delta) = \inf\{\text{lct}(X, \Delta; D) \mid D \sim_{\mathbb{Q}} -(K_X + \Delta) \text{ and } D \text{ is an effective } \mathbb{Q}\text{-divisor}\}.$$

We also define the  *$\delta$ -invariant* of the pair  $(X, \Delta)$  as follows. For every  $m \in \mathbb{N}$  such that  $m(K_X + \Delta)$  is a Weil divisor we look at the space  $H^0(X, -m(K_X + \Delta))$ ; put  $N_m = h^0(X, -m(K_X + \Delta))$ . For every basis  $(s_1, \dots, s_{N_m})$  of  $H^0(X, -m(K_X + \Delta))$  we denote  $D(s_1), \dots, D(s_{N_m})$  the corresponding divisors. We call  $\mathbb{Q}$ -divisors of the form

$$D = \frac{1}{mN_m}(D(s_1) + \dots + D(s_{N_m}))$$

(anticanonical)  $\mathbb{Q}$ -divisors of  *$m$ -basis type* on  $(X, \Delta)$ . We define for  $m \in \mathbb{N}$  the invariant  $\delta_m(X, \Delta)$  by

$$\delta_m(X, \Delta) = \inf\{\text{lct}(X, \Delta; D) \mid D \text{ is of } m\text{-basis type}\}$$

and finally we define the  $\delta$ -invariant by

$$\delta(X, \Delta) = \limsup_{m \in \mathbb{N}} \delta_m(X, \Delta).$$

If  $\Delta = 0$  then we simply write  $\alpha(X)$  for  $\alpha(X, 0)$  and analogously for  $\delta(X)$ .

**2.3. The space of valuations.** In this subsection we recollect some basic information about the space of valuations on the function field of a variety. We refer to [JM12, BdFFU15, BJ17] for more details. In this subsection  $X$  is a normal and  $\mathbb{Q}$ -Gorenstein projective variety over  $\mathbb{C}$ .

**Definition 2.3.** A valuation on  $X$  is a real valuation  $v: \mathbb{C}(X)^* \rightarrow \mathbb{R}$  on the function field of  $X$  which is trivial on  $\mathbb{C}$ . We denote by  $\text{Val}_X$  the set of all valuations on  $X$ . The latter is endowed with the topology of pointwise convergence, i. e. the weakest topology with the property that all evaluation maps  $\text{ev}_f: \text{Val}_X \rightarrow \mathbb{R}, v \mapsto v(f)$  are continuous for every  $f \in \mathbb{K}(X)$ .

**Definition 2.4.** A valuation on  $X$  is called divisorial (or geometric) if it has the form  $v = c \cdot \text{ord}_E(\cdot)$  where  $E$  is a prime divisor on a birational model of  $X$  and  $c \in \mathbb{R}$ . The set of divisorial valuations on  $X$  is denoted by  $\text{DivVal}_X$ .

**Proposition 2.5.** The set of divisorial valuations is dense in the space of all valuations on  $\mathbb{C}(X)$  in the topology of pointwise convergence.

**Remark 2.6.** The above Proposition 2.5 is proved using the theory of Berkovich spaces. We sketch the proof for reader's convenience. We can associate to the variety  $X$  its Berkovich analytification  $X^{an}$ . The set of points of  $X^{an}$  is, by definition, the set of pairs  $(x, |\cdot|_x)$  where  $x \in X$  is a scheme point and  $|\cdot|_x: k(x) \rightarrow \mathbb{R}$  is a valuation on the residue field of  $x$ . Then the set  $\text{Val}_X$  is identified with the preimage of the generic point of  $X$  under the projection  $\pi: X^{an} \rightarrow X$ . The topology induced on  $\text{Val}_X$  from  $X^{an}$  is precisely the topology of pointwise convergence. Thus Proposition 2.5 follows from the density theorem for divisorial points in a Berkovich space (see e.g. [Gub98, Theorem 7.12]).

**Definition 2.7.** Let  $v = \text{ord}_E$  be a divisorial valuation on  $X$  where  $E \subset Y$  is a prime divisor on a birational model  $f: Y \rightarrow X$ . Let  $K_{Y/X} = K_Y - f^*K_X$  be a relative canonical divisor. Then the log discrepancy of  $v$  is defined by  $A_X(v) = 1 + \text{ord}_E(K_{Y/X})$ .

**Proposition 2.8.** ([BdFFU15, Theorem 3.1]) The log discrepancy extends to a function

$$A_X: \text{Val}_X \rightarrow \mathbb{R} \cup \{\infty\}$$

which is lower semicontinuous and homogeneous of order 1, that is,  $A_X(\lambda \cdot v) = \lambda \cdot A_X(v)$  for  $\lambda \in \mathbb{R}_{\geq 0}$ .

We recall the definition of the volume function for  $\mathbb{Q}$ -divisors (see e.g. [BJ17, 2.1]).

**Definition 2.9.** Let  $D$  be a  $\mathbb{Q}$ -divisor on a variety  $X$  of dimension  $n$ . Let  $k \in \mathbb{N}$  be such that  $kD$  is a Weil divisor. Then the volume of  $D$  is defined by

$$\text{Vol}(D) = \limsup_{m \rightarrow \infty} \frac{n!}{k^n \cdot m^n} h^0(X, \mathcal{O}_X(mkD)).$$

**Definition 2.10.** Let  $v = \text{ord}_E$  be a divisorial valuation on  $X$  where  $E \subset Y$  is a prime divisor on a birational model  $f: Y \rightarrow X$ . We define the pseudoeffective threshold of  $v$  (or the maximal vanishing order) with respect to the big  $\mathbb{Q}$ -divisor  $L$  by

$$T_L(v) = \sup\{t \in \mathbb{R} \mid \text{Vol}(f^*L - tE) > 0\}.$$

We also define the expected vanishing order of  $v$  with respect to  $L$  by

$$S_L(v) = \frac{1}{\text{Vol}(L)} \int_0^\infty \text{Vol}(f^*L - tE) dt.$$

**Proposition 2.11.** [BJ17, Proposition 3.13] The functions  $T_L$  and  $S_L$  can be uniquely extended to functions  $\text{Val}_X \rightarrow \mathbb{R}$  which are lower semicontinuous and homogeneous of order 1, that is,  $T(\lambda \cdot v) = \lambda \cdot T(v)$  and  $S(\lambda \cdot v) = \lambda \cdot S(v)$  for  $\lambda \in \mathbb{R}_{\geq 0}$ .

Now we give the valuative definition of the  $\alpha$  and  $\delta$ -thresholds, following [BJ17]. This definition is equivalent to Definition 2.2, as shown in [BJ17, Theorem C].

**Definition 2.12.** We define the  $\alpha$ - and  $\delta$ -thresholds of  $(X, L)$  (or the  $\alpha$ - and  $\delta$ -invariants of  $(X, L)$ ) by the formulas

$$\alpha(X, L) = \inf_{v \in \text{Val}_X} \frac{A_X(v)}{T_L(v)} = \inf_{v \in \text{DivVal}_X} \frac{A_X(v)}{T_L(v)}$$

and analogously

$$\delta(X, L) = \inf_{v \in \text{Val}_X} \frac{A_X(v)}{S_L(v)} = \inf_{v \in \text{DivVal}_X} \frac{A_X(v)}{S_L(v)}.$$

**2.4. The automorphism group of a polarized variety.** A *polarized variety* is a pair  $(X, L)$  where  $X$  is a normal complex projective variety and  $L$  is a Weil divisor which is  $\mathbb{Q}$ -Cartier and big. We denote by  $\text{Aut}(X, L) \subset \text{Aut}(X)$  the subgroup of automorphisms of  $X$  preserving the class  $[L] \in \text{Cl}(X)$ .

**Proposition 2.13** (see e.g. [KPS18, Lemma 3.1.2]). Let  $(X, L)$  be a polarized variety and let  $m \in \mathbb{N}$  be such that  $mL$  is Cartier and the map

$$\varphi_{|mL|}: X \rightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(mL))^*)$$

is birational onto its image. Then the above map is  $\text{Aut}(X, L)$ -equivariant and the group  $\text{Aut}(X, L)$  embeds into  $\text{PGL}(H^0(X, \mathcal{O}_X(mL))^*)$ . Therefore  $\text{Aut}(X, L)$  is a linear algebraic group.

We are mostly interested in the case when  $X$  is a Fano variety and  $L = -K_X$ . Since the automorphism group  $\text{Aut}(X)$  preserves the anticanonical class, we have the following corollary.

**Corollary 2.14.** The automorphism group of a  $\mathbb{Q}$ -Fano variety is a linear algebraic group.

We will also consider polarized pairs  $(X, \Delta; L)$ ; here the group  $\text{Aut}(X, \Delta; L) \subset \text{Aut}(X)$  is a stabilizer of  $\Delta$  and the class of  $L$ . By Proposition 2.13 the group  $\text{Aut}(X, \Delta; L)$  is a linear algebraic group as well.

**2.5.  $G$ -invariant valuations.** In this subsection we work with a closed connected subgroup  $G \subset \text{Aut}(X, L)$ ; then  $G$  is a linear algebraic group as well. The group  $G$  acts on the function field  $\mathbb{C}(X)$  by  $f \mapsto \gamma \cdot f$  and therefore on  $\text{Val}_X$ . We describe the space of  $G$ -invariant valuations mainly following [Kn93, Tim11].

**Definition 2.15.** We denote by  $\text{Val}_X^G$  the set of all valuations  $v: \mathbb{C}(X)^* \rightarrow \mathbb{R}$  which are invariant under the action of (the real form  $G_{\mathbb{R}}$  of)  $G$ . That is, a valuation  $v$  belongs to  $\text{Val}_X^G$  if and only if for all  $\gamma \in G$  and for all  $f \in \mathbb{C}(X)^*$  we have  $v(\gamma \cdot f) = v(f)$ . We also denote by  $\text{DivVal}_X^G$  the set of all *divisorial*  $G$ -invariant valuations. It has a topology induced from  $\text{Val}_X$ .

**Proposition 2.16.** (see e. g. [Tim11, Proposition 19.8]) Let  $X$  be a  $G$ -variety with  $G$  connected. Then every  $G$ -invariant divisorial valuation  $v \in \text{DivVal}_X^G$  is proportional to a valuation  $\text{ord}_D$  where  $D$  is a  $G$ -invariant prime divisor on some birational  $G$ -model of  $X$ .

The following approximation theorem of Sumihiro is crucial for the study of  $G$ -invariant valuations.

**Theorem 2.17** ([Sum74, Lemma 10]). Let  $G$  be a connected algebraic group and let  $X$  be an irreducible  $G$ -variety. For any nontrivial divisorial valuation  $v$  on  $\mathbb{C}(X)$  there exists a  $G$ -invariant divisorial valuation  $\bar{v}$  with the following property: for any  $f \in \mathbb{C}(X)$  there exists a nonempty Zariski-open subset  $U_f \subset G$  such that for any  $\gamma \in U_f$  we have  $v(\gamma \cdot f) = \bar{v}(f)$ .

**Proposition 2.18.** Let  $X$  be a  $G$ -variety. The set  $\text{DivVal}_X^G$  of  $G$ -invariant divisorial valuations is dense in the set  $\text{Val}_X^G$  in the topology of pointwise convergence.

*Proof.* We assume that  $\text{Val}_X^G$  is nonempty; otherwise there is nothing to prove. Suppose that  $v \in \text{Val}_X^G$  is  $G$ -invariant. By Proposition 2.5 we can find a sequence  $\{v_i\}$  of divisorial valuations converging to  $v$ . For every  $i \in \mathbb{N}$  let  $\bar{v}_i$  be the  $G$ -invariant divisorial valuation associated to  $v_i$  by Theorem 2.17. It suffices to prove that the sequence  $\{\bar{v}_i\}$  converges to  $v$ . Take a rational function  $f \in \mathbb{C}(X)$ ; then for every  $i \in \mathbb{N}$  we have the subset

$$U_{f,i} = \{\gamma \in G \mid v_i(\gamma \cdot f) = \bar{v}_i(f)\}.$$

Let  $U_f = \bigcap_{i \in \mathbb{N}} U_{f,i}$  be the intersection of these subsets; it is nonempty since the field  $\mathbb{C}$  is uncountable. Then for any  $\gamma \in U_f$  we obtain that

$$\bar{v}_i(f) = v_i(\gamma \cdot f)$$

converge to  $v(\gamma \cdot f) = v(f)$  by definition of  $\bar{v}_i$  and  $G$ -invariance of  $v$ . Therefore for any  $f \in \mathbb{C}(X)$  the sequence  $\bar{v}_i(f)$  converges to  $v(f)$ , as desired. This proves density of  $G$ -invariant divisorial valuations in the space  $\text{Val}_X^G$ .  $\square$

Now let us introduce the definitions for the  $\alpha_G$  and  $\delta_G$ -thresholds via  $G$ -invariant valuations.

**Definition 2.19.** Let  $X$  be a variety and let  $L$  be a big  $\mathbb{Q}$ -divisor. Let also  $G \subset \text{Aut}(X, L)$  be a closed connected subgroup; we define the  $G$ -invariant  $\alpha$ - and  $\delta$ -thresholds of  $(X, L)$  (we will also call them the  $\alpha$ - and  $\delta$ -invariants of  $(X, L)$  with the action of  $G$ ) by the formulas

$$\alpha_G(X, L) = \inf_{v \in \text{Val}_X^G} \frac{A_X(v)}{T_L(v)} = \inf_{v \in \text{DivVal}_X^G} \frac{A_X(v)}{T_L(v)}$$

and analogously

$$\delta_G(X, L) = \inf_{v \in \text{Val}_X^G} \frac{A_X(v)}{S_L(v)} = \inf_{v \in \text{DivVal}_X^G} \frac{A_X(v)}{S_L(v)}.$$

For a variety  $X$  with  $L = -K_X$  big we write  $\delta_G(X, -K_X) = \delta_G(X)$  and the same for  $\alpha_G$ . In the case of a pair  $(X, \Delta; L)$  and a subgroup  $G \subset \text{Aut}(X, \Delta; L)$  we can define  $\alpha_G(X, \Delta; L)$  and  $\delta_G(X, \Delta; L)$  in the same way by introducing the log discrepancy  $A_{X, \Delta}(v)$  with respect to the pair  $(X, \Delta)$ .

**Remark 2.20.** If a variety  $X$  is a homogeneous space under the action of an algebraic group  $G$  then both  $\alpha_G$  and  $\delta_G$  are infinite. Indeed, since the action of  $G$  is transitive, the space  $\text{Val}_X^G$  is empty.

**Remark 2.21.** In [CS08, Appendix A] the algebraic version of Tian's invariant  $\alpha_G(X)$  was defined by

$$\alpha_G(X, L) = \inf_{m \in \mathbb{N}} \inf_{\substack{|\Sigma| \subset |mL| \\ |\Sigma|^G = |\Sigma|}} \text{lct}(X, \frac{1}{m}|\Sigma|)$$

where the second infimum is taken over all  $G$ -invariant linear subsystems  $|\Sigma| \subset |mL|$ .

**Proposition 2.22.** The definition of  $\alpha_G(X, L)$  given in Remark 2.21 is equivalent to Definition 2.19.

*Proof.* By definition from Remark 2.21, we have  $\alpha_G(X, L) = \inf_{m \in \mathbb{N}} \alpha_{G, m}(X, L)$  where

$$\alpha_{G, m}(X, L) = \inf_{\substack{|\Sigma| \subset |mL| \\ |\Sigma|^G = |\Sigma|}} \text{lct}(X, \frac{1}{m}|\Sigma|).$$

Note that  $\text{lct}(X, |\Sigma|)$  is defined as the log canonical threshold of the base ideal  $\mathcal{I}_\Sigma$  of  $|\Sigma|$ . Since  $|\Sigma|$  is  $G$ -invariant, the base ideal is also  $G$ -invariant. There exists a divisorial valuation computing  $\text{lct}(X, \mathcal{I}_\Sigma)$  by [BLJ17, 1.7]; moreover, by  $G$ -equivariant log resolution (see e.g. [Kol07, 3.9.1]) and connectedness of  $G$  this valuation can be chosen  $G$ -invariant. Note also that for  $v \in \text{Val}_X^G$  we have

$$\sup_{m \in \mathbb{N}} \sup_{\substack{|\Sigma| \subset |mL| \\ |\Sigma|^G = |\Sigma|}} \frac{1}{m} v(\mathcal{I}_\Sigma) = T_L(v).$$

Indeed, by definition,  $T(v) = \sup_{m \in \mathbb{N}} \sup_{D \in |mL|} \frac{1}{m} v(D)$ . For any  $D_0 \in |mL|$  with  $v(D_0) = \lambda$  the linear system

$$|\Sigma| = \{D \in |mL| \mid v(D) \geq \lambda/m\}$$

is nonempty and  $G$ -invariant by invariance of  $v$ ; moreover,

$$v(\frac{1}{m}|\Sigma|) = \inf_{D \in |\Sigma|} v(D) \geq \lambda/m.$$

The reverse inequality is obvious. Expanding the definitions and switching the order of infima we can write

$$\inf_{m \in \mathbb{N}} \inf_{\substack{|\Sigma| \subset |mL| \\ |\Sigma|^G = |\Sigma|}} \text{lct}(X, \frac{1}{m}|\Sigma|) = \inf_{m \in \mathbb{N}} \inf_{\substack{|\Sigma| \subset |mL| \\ |\Sigma|^G = |\Sigma|}} \inf_{v \in \text{Val}_X^G} \frac{A_X(v)}{\frac{1}{m}v(\mathcal{I}_\Sigma)} = \inf_{v \in \text{Val}_X^G} \frac{A_X(v)}{T_L(v)}$$

and thus we obtain the equivalence of two definitions.  $\square$

**Proposition 2.23.** Let  $(X, L)$  be a polarized variety and  $G \subset \text{Aut}(X, L)$  a closed connected subgroup. We have the following inequalities for  $\alpha_G(X, L)$  and  $\delta_G(X, L)$  where  $X$  has dimension  $n$ :

$$0 < \alpha_G(X, L) \leq \delta_G(X, L) \leq (n+1)\alpha_G(X, L).$$

If, moreover,  $L$  is ample then we have a stronger inequality

$$\alpha_G(X, L)(1 + \frac{1}{n}) \leq \delta_G(X, L).$$

*Proof.* The first inequalities follow from [BJ17, Lemma 2.6]. The second inequality follows from the fact that  $\frac{S_L(v)}{T_L(v)} \leq \frac{n}{n+1}$  for  $L$  an ample  $\mathbb{Q}$ -divisor and all valuations  $v \in \text{Val}_X$  [Fuj17, Proposition 2.1].  $\square$

### 3. EQUIVARIANT $K$ -STABILITY

In this section we collect basic information on  $G$ -equivariant  $K$ -stability and prove Theorem 1.3 (see Theorem 3.7 below). The following definitions are well-known and can be found e.g. in [Ti97, Don02, BHJ17].

**Definition 3.1.** A test configuration for a polarized pair  $(X, L)$  is a pair  $(\mathcal{X}, \mathcal{L})$  consisting of a variety  $\mathcal{X}$  with a projective surjective morphism  $\pi: \mathcal{X} \rightarrow \mathbb{A}^1$  and a  $\pi$ -semiample  $\mathbb{Q}$ -line bundle  $\mathcal{L}$  together with a  $\mathbb{G}_m$ -action on  $\mathcal{X}$  preserving the class of  $\mathcal{L}$  such that

- The morphism  $\pi: \mathcal{X} \rightarrow \mathbb{A}^1$  is  $\mathbb{G}_m$ -equivariant with respect to the given action on  $\mathcal{X}$  and the multiplicative action on  $\mathbb{A}^1$ ;
- The pair  $(\mathcal{X} \setminus \mathcal{X}_0, \mathcal{L}|_{\mathcal{X} \setminus \mathcal{X}_0})$  is  $\mathbb{G}_m$ -equivariantly isomorphic to  $(X \times (\mathbb{A}^1 \setminus \{0\}), p_1^*L)$ .

**Definition 3.2** ([DS16]). Let  $G$  be a closed subgroup of  $\text{Aut}(X, L)$ . A test configuration  $(\mathcal{X}, \mathcal{L})$  is called  $G$ -equivariant if there exists an action of  $G$  on the pair  $(\mathcal{X}, \mathcal{L})$  which commutes with the  $\mathbb{G}_m$ -action, preserves the fibers and restricts to the given  $G$ -action on  $(\mathcal{X}_t, \mathcal{L}|_{\mathcal{X}_t}) \simeq (X, L)$  for  $t \neq 0$ .

**Definition 3.3.** Let  $(\mathcal{X}, \mathcal{L})$  be a test configuration for  $(X, L)$ . The compactification  $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$  of  $(\mathcal{X}, \mathcal{L})$  is the degeneration of  $(X, L)$  over  $\mathbb{P}^1$  defined by gluing  $(\mathcal{X}, \mathcal{L})$  with  $(X \times (\mathbb{P}^1 \setminus \{0\}), p_1^*L)$  along the subset  $X \times (\mathbb{P}^1 \setminus \{0, 1\})$ . The compactification is  $G$ -equivariant if  $(\mathcal{X}, \mathcal{L})$  is.

**Definition 3.4** ([LX14]). A test configuration  $(\mathcal{X}, \mathcal{L})$  is trivial if the relative canonical model  $(\mathcal{X}^{\text{can}}, \mathcal{L}^{\text{can}})$  of the normalization of  $(\mathcal{X}, \mathcal{L})$  is  $\mathbb{G}_m$ -equivariantly isomorphic to  $(X \times \mathbb{A}^1, p_1^*L + c\mathcal{X}_0)$  for some  $c \in \mathbb{Q}$ . We say that  $(\mathcal{X}, \mathcal{L})$  is of product type if  $\mathcal{X}$  is isomorphic to  $X \times \mathbb{A}^1$ .

**Definition 3.5** ([BHJ17, LX14]). Consider a test configuration  $(\mathcal{X}, \mathcal{L})$  for  $(X, L)$ . Let us denote by  $\mathcal{Z}$  the normalization of the graph of the map  $\bar{\mathcal{X}} \dashrightarrow X \times \mathbb{P}^1$ ; it has natural maps  $\pi: \mathcal{Z} \rightarrow X \times \mathbb{P}^1$  and  $\varphi: \mathcal{Z} \rightarrow \mathcal{X}$ . We define the following invariants:

$$\lambda_{\max}(\mathcal{X}, \mathcal{L}) = \frac{(p_1 \circ \pi)^* L^n \cdot \varphi^* \bar{\mathcal{L}}}{L^n} \quad \text{and} \quad J^{NA}(\mathcal{X}, \mathcal{L}) = \lambda_{\max}(\mathcal{X}, \mathcal{L}) - \frac{\bar{\mathcal{L}}^{n+1}}{(n+1)L^n}$$

where the latter is called the norm of  $(\mathcal{X}, \mathcal{L})$ . Also for a normal test configuration  $(\mathcal{X}, \mathcal{L})$  for a  $\mathbb{Q}$ -Fano variety  $(X, -K_X)$  we define the Donaldson–Futaki invariant  $\text{DF}(\mathcal{X}, \mathcal{L})$  by

$$\text{DF}(\mathcal{X}, \mathcal{L}) = \frac{n}{n+1} \cdot \frac{\bar{\mathcal{L}}^{n+1}}{(-K_X)^n} + \frac{\bar{\mathcal{L}}^n \cdot K_{\mathcal{X}/\mathbb{P}^1}}{(-K_X)^n}.$$

We also define the Ding invariant of  $(\mathcal{X}, \mathcal{L})$  as follows. Let  $D_{(\mathcal{X}, \mathcal{L})}$  be the unique  $\mathbb{Q}$ -divisor defined by the conditions  $\text{Supp}(D_{(\mathcal{X}, \mathcal{L})}) \subset \mathcal{X}_0$  and  $D_{(\mathcal{X}, \mathcal{L})} \sim_{\mathbb{Q}} -K_{\mathcal{X}/\mathbb{P}^1} - \bar{\mathcal{L}}$ . The Ding invariant  $\text{Ding}(\mathcal{X}, \mathcal{L})$  is defined by

$$\text{Ding}(\mathcal{X}, \mathcal{L}) = -\frac{\bar{\mathcal{L}}^{n+1}}{(n+1)(-K_X)^n} - 1 + \text{lt}(\mathcal{X}, D_{(\mathcal{X}, \mathcal{L})}; \mathcal{X}_0).$$

**Definition 3.6.** A  $\mathbb{Q}$ -Fano variety  $(X, -K_X)$  is called

- $K$ -stable if  $\text{DF}(\mathcal{X}, \mathcal{L}) > 0$  for every normal test configuration  $(\mathcal{X}, \mathcal{L})$  for  $(X, L)$ ;
- uniformly  $K$ -stable if there exists  $\varepsilon > 0$  such that  $\text{DF}(\mathcal{X}, \mathcal{L}) \geq \varepsilon J^{NA}(\mathcal{X}, \mathcal{L})$  for every normal test configuration  $(\mathcal{X}, \mathcal{L})$  for  $(X, L)$ .
- uniformly Ding-stable if there exists  $\varepsilon > 0$  such that  $\text{Ding}(\mathcal{X}, \mathcal{L}) \geq \varepsilon J^{NA}(\mathcal{X}, \mathcal{L})$  for every normal test configuration  $(\mathcal{X}, \mathcal{L})$  for  $(X, L)$ .

For a closed subgroup  $G \subset \text{Aut}(X)$  we say that  $(X, -K_X)$  is  $G$ -equivariantly  $K$ -stable (or  $K$ -stable along  $G$ -equivariant test configurations) if the corresponding inequalities hold for  $G$ -equivariant test configurations; the same for uniform  $K$ - or Ding-stability.

**Theorem 3.7.** Let  $(X, L) = (X, -K_X)$  be a  $\mathbb{Q}$ -Fano variety with the anticanonical polarization and let  $G \subset \text{Aut}(X)$  be a connected reductive subgroup. Then the following conditions are equivalent:

- (1) The variety  $X$  is uniformly  $K$ -stable along  $G$ -equivariant test configurations;
- (2) The variety  $X$  is uniformly Ding-stable along  $G$ -equivariant test configurations;

(3) The  $G$ -invariant  $\delta$ -threshold of  $X$  is greater than one:  $\delta_G(X) > 1$ .

The proof of Theorem 3.7 mainly follows [Fuj16, Theorem 1.4] and [LX14]. We divide the proof into Propositions 3.8 and 3.10 below. Our goal is to check that the constructions of test configurations used in [Fuj16, Theorem 4.1] and [LX14, Theorem 4] can be made  $G$ -equivariantly, for  $G$  a connected reductive group.

**Proposition 3.8** ( $G$ -equivariant version of [Fuj16, Theorem 4.1]). Let  $(X, L) = (X, -K_X)$  be a  $\mathbb{Q}$ -Fano variety and  $G \subset \text{Aut}(X)$  a closed connected subgroup; assume that for all  $G$ -equivariant normal test configurations  $(\mathcal{X}, \mathcal{L})$  for  $(X, -K_X)$  we have the inequality

$$\text{Ding}(\mathcal{X}, \mathcal{L}) \geq \varepsilon J^{NA}(\mathcal{X}, \mathcal{L})$$

for some  $\varepsilon > 0$ . Then we have for every  $G$ -invariant divisorial valuation on  $\mathbb{C}(X)$  the inequality

$$\beta_X(v) = 1 - \frac{S_{-K_X}(v)}{A_X(v)} \geq \varepsilon'$$

for some  $\varepsilon' > 0$ . In particular,  $\delta_G(X) > 1$ .

*Proof.* Let us first outline the idea of the proof of [Fuj16, Theorem 4.1]. Starting from a divisorial valuation  $v = \text{ord}_E$ , we construct a sequence of test configurations  $(\mathcal{X}^r, \mathcal{L}^r)$  using *flag ideals* (introduced in [Od13]) associated to the valuation  $v$ . Then from the computations in [Fuj16, Claim 4.4] and [Fuj17, Claim 2.5] we conclude that uniform boundedness from below of the Ding invariants  $\text{Ding}(\mathcal{X}^r, \mathcal{L}^r)$  implies positivity of  $\beta(v)$ . We show that this construction can be made  $G$ -equivariantly provided that we start from a  $G$ -invariant divisorial valuation. We assume that  $\text{Val}_X^G$  is nonempty; otherwise there is nothing to prove.

Let  $r_0 \in \mathbb{N}$  be the Cartier index of  $K_X$ . To a divisorial valuation  $v = \text{ord}_E$  where  $E$  is a  $G$ -stable prime divisor on a birational model  $\varphi: Y \rightarrow X$  we associate the *order filtration*  $\mathcal{F}_v$  on the graded algebra

$$R(X, -r_0 K_X) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(-mr_0 K_X)) = \bigoplus_{m \in \mathbb{N}} V_m.$$

The filtration is defined by

$$\mathcal{F}_v^t V_m = H^0(Y, \mathcal{O}_Y(-mr_0 \varphi^* K_X - tE)) \subset H^0(X, \mathcal{O}_X(-mr_0 K_X)).$$

It is a filtration by  $G$ -invariant linear subspaces of  $R(X, -r_0 K_X)$ . To this filtration and to any given  $t \in \mathbb{R}, m \in \mathbb{N}$  we can associate a nontrivial  $G$ -invariant ideal

$$\mathcal{I}_{(m,t)} = \text{Im}(\mathcal{F}_v^t V_m \otimes \mathcal{O}_X(-mr_0 K_X) \rightarrow \mathcal{O}_X).$$

By [Fuj16, Claim 4.2], we have

$$\mathcal{F}_v^t V_m = H^0(X, -mr_0 K_X \cdot \mathcal{I}_{(m,t)}).$$

We now define for appropriate  $e_+, e_- \in \mathbb{Z}$  and  $r \in \mathbb{N}$  large enough (as in [Fuj16, Theorem 4.1]) the flag ideal  $\mathcal{I}_r \subset \mathcal{O}_{X \times \mathbb{A}^1}$  by

$$\mathcal{I}_r = \mathcal{I}_{(r, re_+)} + \mathcal{I}_{(r, re_+ - 1)} \cdot t + \cdots + \mathcal{I}_{(r, re_- + 1)} \cdot t^{r(e_+ - e_-) - 1} + (t^{r(e_+ - e_-)}).$$

This ideal is invariant under the action of  $G$  on  $X \times \mathbb{A}^1$ . We construct a test configuration by blowing up the flag ideal  $\Phi_r: \text{Bl}_{\mathcal{I}_r}(X \times \mathbb{A}^1) \rightarrow X \times \mathbb{A}^1$ . Let  $E_r$  be the exceptional divisor of the blow-up; we set

$$\mathcal{X}^r = \text{Bl}_{\mathcal{I}_r}(X \times \mathbb{A}^1) \quad \text{and} \quad \mathcal{L}^r = \Phi_r^*(-K_{X \times \mathbb{A}^1}) - \frac{1}{rr_0} E_r.$$

Then the  $G$ -action lifts to the blow-up leaving  $\mathcal{L}^r$  invariant; therefore  $(\mathcal{X}^r, \mathcal{L}^r)$  is a  $G$ -equivariant test configuration for  $(X, L)$ . By assumption, for the (still  $G$ -equivariant) normalization  $(\mathcal{X}^{r,\nu}, \nu^* \mathcal{L}^r)$  of any test configuration  $(\mathcal{X}^r, \mathcal{L}^r), r \in \mathbb{N}$  we have the inequality

$$\text{Ding}(\mathcal{X}^{r,\nu}, \nu^* \mathcal{L}^r) \geq \varepsilon J^{NA}(\mathcal{X}^{r,\nu}, \nu^* \mathcal{L}^r)$$

for some fixed  $\varepsilon > 0$ . By the proof of [Fuj16, Theorem 4.1] and [Fuj17, Claims 2.4, 2.5] we obtain (in the notation of [Fuj17]) the inequality

$$\beta(E) \geq A_X(E) - \frac{1 + \varepsilon'}{\text{Vol}(-K_X)} \int_0^\infty \text{Vol}(-\varphi^* K_X - tE) dt$$



for some  $\varepsilon' > 0$  depending on  $\varepsilon > 0$  only and not on  $E$ . Since  $v = \text{ord}_E \in \text{DivVal}_X^G$  was arbitrary, we obtain

$$\delta_G(X, L) = \inf_{v \in \text{DivVal}_X^G} \frac{A_X(v)}{S_L(v)} \geq 1 + \varepsilon' > 1$$

as it was to be shown.  $\square$

**Proposition 3.9** ([LX14, Theorem 4] and [Fuj16, Theorem 5.2]). Let  $(\mathcal{X}, \mathcal{L})$  be a normal test configuration for a  $\mathbb{Q}$ -Fano variety  $(X, -K_X)$ . Then there exist

- a finite base change  $(\mathcal{X}^{(d)}, \mathcal{L}^{(d)}) \rightarrow (\mathcal{X}, \mathcal{L})$ ;
- a  $\mathbb{G}_m$ -equivariant birational map  $\mathcal{X}^{(d)} \rightarrow \mathcal{X}^s$  obtained by running a relative MMP with scaling of an ample divisor  $\mathcal{H}$

such that the resulting test configuration  $(\mathcal{X}^s, \mathcal{L}^s)$  is special, that is,  $\mathcal{X}_0^s$  is irreducible and reduced. Moreover, for every special test configuration  $(\mathcal{X}^s, \mathcal{L}^s)$  obtained from  $(\mathcal{X}, \mathcal{L})$  the following properties hold true:

- (a) There exists  $d \in \mathbb{N}$  such that  $\text{DF}(\mathcal{X}^s, \mathcal{L}^s) \leq d \cdot \text{DF}(\mathcal{X}, \mathcal{L})$  and for any  $\varepsilon \in [0, 1]$  we have

$$\text{Ding}(\mathcal{X}^s, \mathcal{L}^s) - \varepsilon J^{NA}(\mathcal{X}^s, \mathcal{L}^s) \leq d \cdot (\text{Ding}(\mathcal{X}, \mathcal{L}) - \varepsilon J^{NA}(\mathcal{X}, \mathcal{L}));$$

- (b) The Donaldson–Futaki invariant  $\text{DF}(\mathcal{X}^s, \mathcal{L}^s)$  is equal to the Ding invariant of  $(\mathcal{X}^s, \mathcal{L}^s)$  and can be expressed via the invariants of the divisorial valuation  $v = v_{\mathcal{X}_0^s}$  as follows:

$$\text{DF}(\mathcal{X}^s, \mathcal{L}^s) = A_X(v_{\mathcal{X}_0^s}) - S_L(v_{\mathcal{X}_0^s}).$$

The above proposition states that to check uniform  $K$ - or Ding-stability of  $(X, -K_X)$  one only needs to compute the invariants for all *special* test configurations  $(\mathcal{X}^s, \mathcal{L}^s)$  as above. The next proposition says that one can pass from a  $G$ -equivariant test configuration to a  $G$ -equivariant *special* test configuration (cf. an analogous statement in [Del16, Theorem B] for spherical Fano varieties).

Note that for any test configuration  $(\mathcal{X}, \mathcal{L})$  the  $\mathbb{G}_m$ -equivariant birational map  $\mathcal{X} \dashrightarrow X \times \mathbb{A}^1$  gives an isomorphism  $\mathbb{C}(\mathcal{X}) = \mathbb{C}(X)(t)$ . This isomorphism is  $G$ -equivariant if the test configuration  $(\mathcal{X}, \mathcal{L})$  is. The projection  $X \times \mathbb{A}^1 \rightarrow X$  gives a  $G$ -equivariant embedding  $\mathbb{C}(X) \subset \mathbb{C}(X)(t)$ . If the central fiber  $\mathcal{X}_0$  is irreducible then the restriction of the divisorial valuation  $v_{\mathcal{X}_0}$  to  $\mathbb{C}(X)$  is either divisorial or trivial (see e.g. [BHJ17, Lemma 4.1] or [Tim11, Proposition B.8]).

**Proposition 3.10.** Let  $(\mathcal{X}, \mathcal{L})$  be a normal test configuration for a  $\mathbb{Q}$ -Fano variety  $(X, -K_X)$ . If  $(\mathcal{X}, \mathcal{L})$  is  $G$ -equivariant for a closed connected subgroup  $G \subset \text{Aut}(X)$  then every special test configuration  $(\mathcal{X}^s, \mathcal{L}^s)$  constructed from  $(\mathcal{X}, \mathcal{L})$  as in Proposition 3.9 is  $G$ -equivariant and satisfies the properties (a) and (b) from Proposition 3.9. In particular, if  $(\mathcal{X}^s, \mathcal{L}^s)$  is non-trivial then the restriction of the divisorial valuation  $v_{\mathcal{X}_0^s}$  to the subfield

$$\mathbb{C}(X) \subset \mathbb{C}(X)(t) \simeq \mathbb{C}(\mathcal{X}^s)$$

is a  $G$ -invariant divisorial valuation.

*Proof.* We start from a normal  $G$ -equivariant test configuration  $(\mathcal{X}, \mathcal{L})$  and compactify it to  $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ . Taking an equivariant log resolution of the pair  $(\bar{\mathcal{X}}, \bar{\mathcal{X}}_0)$  (by [Kol07, 3.9.1]) and a  $G$ -equivariant finite base change  $t \mapsto t^d$  (by [LX14, Lemma 5]) we can assume that the pair  $(\bar{\mathcal{X}}, \bar{\mathcal{X}}_0)$  is log canonical.

Then by Proposition 3.9 a special test configuration can be obtained from  $(\bar{\mathcal{X}}, \bar{\mathcal{X}}_0)$  by running a relative  $K_{\bar{\mathcal{X}}/\mathbb{P}^1}$ -MMP with scaling of an ample  $\mathbb{Q}$ -divisor  $\mathcal{H}$ . We can take  $\mathcal{H}$  such that the class of  $\mathcal{H}$  lies in  $\text{Pic}^G(\bar{\mathcal{X}})$ . We recall that  $\overline{\text{NE}}(\bar{\mathcal{X}}) = \overline{\text{NE}}(\bar{\mathcal{X}})^G$  by [And01, Lemma 1.5] since  $G$  is connected. Therefore for every divisorial extremal ray of  $\overline{\text{NE}}(\bar{\mathcal{X}})$  of the Mori cone the contraction is  $G$ -equivariant. For flips, the statement follows since the Proj of an algebra with a  $G$ -action has an induced structure of a  $G$ -variety.

Thus at the final step of the MMP we obtain a  $G$ -equivariant test configuration  $(\bar{\mathcal{X}}^s, \bar{\mathcal{L}}^s)$  such that  $\mathcal{X}_0^s$  is a  $G$ -stable prime divisor. So Proposition 3.9 applies to  $(\mathcal{X}^s, \mathcal{L}^s)$  and ensures that the properties (a) and (b) are satisfied. Moreover, the valuation  $v_{\mathcal{X}_0^s}$  on  $\mathbb{C}(\mathcal{X}^s)$  is  $G$ -invariant. If  $(\mathcal{X}^s, \mathcal{L}^s)$  is nontrivial then the restriction of  $v_{\mathcal{X}_0^s}$  to  $\mathbb{C}(X)$  is divisorial by [Tim11, Proposition B.8] or [BHJ17, Lemma 4.1] and invariant by the induced  $G$ -action on  $X$ .  $\square$

*Proof of Theorem 3.7.* The implication (2)  $\Rightarrow$  (3) is precisely Proposition 3.8. To show (3)  $\Rightarrow$  (1) we use Propositions 3.9 and 3.10 in order to pass from a  $G$ -equivariant test configuration  $(\mathcal{X}, \mathcal{L})$  to a special test

configuration  $(\mathcal{X}^s, \mathcal{L}^s)$  such that the valuation  $v_{\mathcal{X}_0^s}$  is  $G$ -invariant. We have  $\beta(v_{\mathcal{X}_0^s}) \geq \varepsilon$  by assumption and thus

$$\mathrm{DF}(\mathcal{X}^s, \mathcal{L}^s) \geq \varepsilon J^{NA}(\mathcal{X}^s, \mathcal{L}^s).$$

Therefore  $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq \varepsilon' J^{NA}(\mathcal{X}, \mathcal{L})$  by Proposition 3.9. Finally, since the Donaldson–Futaki and Ding invariants of special test configurations coincide, we get the implication (1)  $\Rightarrow$  (2).  $\square$

**Remark 3.11.** Suppose that the space  $\mathrm{Val}_X^G$  is empty (e.g.  $X$  is a  $G$ -homogeneous variety). Then for every  $G$ -equivariant special test configuration  $(\mathcal{X}^s, \mathcal{L}^s)$  the corresponding valuation on  $X$  is trivial. Therefore we have  $J^{NA}(\mathcal{X}^s, \mathcal{L}^s) = 0$  and the test configuration  $(\mathcal{X}^s, \mathcal{L}^s)$  is trivial by [BHJ17, Theorem 7.9]. Thus the  $G$ -equivariant  $K$ -stability condition is trivially satisfied. Note that the existence of Kähler–Einstein metrics on compact Kähler homogeneous manifolds was established by Matsushima in [Mat72, Theorem 3].

#### 4. VARIETIES WITH TORUS ACTION

In this section we consider the case of a variety  $X$  with an action of a torus  $T = (\mathbb{G}_m)^k \subset \mathrm{Aut}(X)$ .

**4.1. Varieties with an action of a torus.** Toric varieties form a well-understood class of varieties with large groups of symmetries. The formula for the  $\alpha$ -invariant of a toric variety in terms of the associated fan was given in [BS99] and [CS08, Lemma 5.1]. The analogous formula for the  $\delta$ -invariant first appeared in [BlJ17, Corollary 7.16]. The underlying idea in these formulas is that it suffices to consider only torus-invariant divisors (or valuations) in the computation of  $\alpha(X)$  and  $\delta(X)$ . We show that the same is true for a subtorus  $T \subset \mathrm{Aut}(X)$  of any dimension.

**Proposition 4.1.** Let  $(X, L)$  be a polarized variety and let  $T \subset \mathrm{Aut}(X, L)$  be a subtorus. Then the following equalities hold true:

$$\alpha_T(X, L) = \alpha(X, L) \quad \text{and} \quad \delta_T(X, L) = \delta(X, L).$$

*Proof.* We apply the degeneration to the initial filtration argument from [BlJ17]. Let us give an outline of their argument. Attached to a valuation  $v \in \mathrm{Val}_X$  is the filtration  $\mathcal{F}_v$  on  $R(X, L)$ . Using a construction from [KK12] we can associate to  $\mathcal{F}_v$  the so-called initial filtration  $\mathrm{in}(\mathcal{F}_v)$ . Its crucial property is that the sequence of base ideals

$$\mathfrak{b}_\bullet(\mathrm{in}(\mathcal{F}_v)) = \{\mathfrak{b}_{t,m}(\mathrm{in}(\mathcal{F}_v))\}, t \in \mathbb{R}, m \in \mathbb{N}$$

consists of monomial ideals and therefore is invariant under the action of  $T$ . Moreover, by [BlJ17, Proposition 7.13] it is possible to degenerate  $\mathcal{F}_v$  to  $\mathrm{in}(\mathcal{F}_v)$  in a one-parameter family in such a way that the log canonical threshold of the sequence defined by

$$\mathrm{lct}(\mathfrak{b}_\bullet) = \inf_{v \in \mathrm{Val}_X^*} \frac{A_X(v)}{v(\mathfrak{b}_\bullet)}$$

does not increase after passing to  $\mathrm{in}(\mathcal{F}_v)$ , that is

$$\mathrm{lct}(\mathfrak{b}_\bullet(\mathrm{in}(\mathcal{F}_v))) \leq \mathrm{lct}(\mathfrak{b}_\bullet(\mathcal{F}_v)).$$

By [BlJ17, Proposition 7.3] we can associate to  $\mathrm{in}(\mathcal{F}_v)$  a  $T$ -invariant valuation  $\bar{v}$  computing the log canonical threshold  $\mathrm{lct}(\mathfrak{b}_\bullet(\mathrm{in}(\mathcal{F}_v)))$ . The valuation  $\bar{v}$  has the property that

$$\mathrm{lct}(\mathfrak{b}_\bullet(\mathrm{in}(\mathcal{F}_v))) = A_X(\bar{v}) \leq A_X(v) = \mathrm{lct}(\mathfrak{b}_\bullet(\mathcal{F}_v))$$

by [BlJ17, Lemma 1.1]. Moreover, we have  $S_L(\bar{v}) \geq S_L(v)$  by [BlJ17, Proposition 6.8]. Therefore, the infimum in the definition of  $\delta(X, L)$  is at most the infimum over the subspace  $\mathrm{Val}_X^T$ . The statement now follows from the definition of  $\delta_T(X, L)$ . The proof for  $\alpha_T(X, L)$  is the same.  $\square$

**Remark 4.2.** In the case of  $\alpha(X)$  this result was proved in [OS12, Corollary 1.8] for any connected solvable group using the Borel fixed point theorem. We expect Proposition 4.1 to hold in the case of a connected solvable group  $G$ . It would follow from the proof of Proposition 4.1 if we knew that  $\mathrm{lct}(\mathfrak{b}_\bullet(\mathrm{in}(\mathcal{F}_v)))$  is computed by a  $G$ -invariant valuation.

This result shows that for varieties with an action of a torus  $T = (\mathbb{G}_m)^k$  it is necessary to consider additional symmetries in order to use Tian’s criterion (or Theorem 3.7). This method was implemented e.g. in [Su13] for  $T$ -varieties of complexity one and in [CS18] for Fano threefolds from the  $V_{22}$  family having automorphism groups  $\mathbb{G}_m \rtimes \mathbb{Z}_2$  (cf. [DKK17] where the additional symmetries were not used).

## 5. SPHERICAL FANO VARIETIES

A natural generalization of toric varieties is the class of spherical varieties. A variety  $X$  is spherical if it has an action of a connected reductive group  $G$  such that a Borel subgroup  $B \subset G$  acts on  $X$  with an open orbit. For general information on spherical varieties we refer to e.g. [Kn91, Per14, Tim11]. Log canonical thresholds of spherical varieties were investigated in [Pas16, Smi17, Del15]. An extensive study of  $K$ -stability of spherical Fano varieties was undertaken by Delcroix in [Del16]. For a Fano variety  $X$ , spherical under the action of a connected reductive group  $G$  we give a formula for  $\delta_G$  in terms of the combinatorial data defined by  $X$ .

We fix the notation, mostly following [Del16]. Let  $X$  be a Fano variety, spherical under the action of a connected reductive group  $G$ . Let  $B \subset G$  be a Borel subgroup and let  $T \subset B$  be a maximal torus. We denote by  $\mathfrak{X}(T)$  the group of algebraic characters of  $T$ . Let also  $\Phi$  be the root system of  $(G; T)$  and  $\Phi^+$  be the set of positive roots given by  $B$ . Let also  $\mathfrak{N}(T)$  be the group of 1-parameter subgroups of  $T$ .

We describe the set of  $G$ -invariant valuations on  $X$ . This set depends on the open  $G$ -orbit  $U$  only. Let us fix a Borel subgroup  $B \subset G$ . Associated to  $B$  is the space

$$M_B(U) = \{\chi \in \mathfrak{X}(B) \mid b \cdot f = \chi(b)f \text{ for all } b \in B \text{ and some } f \in \mathbb{C}(X)^*\}.$$

We denote by  $N_B(U)$  the dual of  $M_B(U)$ ; it is a free abelian group of finite rank. To every  $v \in \text{Val}_X^G$  we can associate a vector  $\rho_v \in N_B(U) \otimes \mathbb{R}$  by the rule  $\rho_v(\chi) = v(f)$  where  $f \in \mathbb{C}(X)^*$  is a rational function as in definition of  $M_B(U)$ . This is a well defined map exactly because  $B$  has an open orbit. The space  $N_B \otimes \mathbb{R}$  is a quotient of  $\mathfrak{N}(T)$ ; we denote by  $\pi: \mathfrak{N}(T) \otimes \mathbb{R} \rightarrow N_B \otimes \mathbb{R}$  the quotient map. The next result gives a particularly nice description of the set  $\text{Val}_X^G$ .

**Theorem 5.1** (see e.g. [Kn91, Corollary 1.8]). The map  $\rho: \text{Val}_X^G \rightarrow N_B(U) \otimes \mathbb{R}$  is injective and identifies the set  $\text{Val}_X^G$  with a convex cone  $\mathcal{V}$  in the finite-dimensional space  $N_B(U) \otimes \mathbb{R}$ .

In [Pas17] Pasquier described the log discrepancy as a piecewise linear function on the cones of the complete colored fan  $\mathbb{F}_X \subset N_B \otimes \mathbb{R}$  of  $X$  (see e.g. [Pas17, 2.4]).

**Proposition 5.2** ([Pas17, Proposition 5.2]). Let  $X$  be a spherical variety and let  $\mathbb{F}_X$  be the complete colored fan associated to  $X$ . There exists a unique function  $h: N_B \otimes \mathbb{R} \rightarrow \mathbb{R}$  such that its restriction  $h_C$  to every cone  $C \in \mathbb{F}_X$  is linear and for every  $v \in \mathcal{V}$  we have  $A_X(v) = h_C(v)$  for some  $C \in \mathbb{F}_X$ .

In [Del16] Delcroix investigated  $G$ -equivariant  $K$ -stability of spherical Fano varieties. He gave a construction of  $G$ -equivariant special test configurations  $(\mathcal{X}, \mathcal{L})$  corresponding to vectors in the valuation cone  $\mathcal{V}$ . Moreover, he computed the Donaldson–Futaki invariants of these test configurations in terms of the moment polytope  $\Delta^+ \subset \mathfrak{X}(T)$  (see [Del16, Definition 3.14]) and the so-called Duistermaat–Heckman measure  $\text{DH}$  on  $\Delta^+$  (see [Del16, Theorem 4.5]). The moment polytope determines a subsystem  $\Phi_L \subset \Phi$ ; we denote by  $2\rho_Q$  the sum of elements in  $\Phi^+ \setminus \Phi_L$ . Also we denote by  $\bar{v}$  the preimage of  $v \in \mathcal{V}$  under the projection  $\pi: \mathfrak{N}(T) \otimes \mathbb{R} \rightarrow N_B \otimes \mathbb{R}$ .

**Theorem 5.3** ([Del16, Theorems B and C]). Let  $(X, -K_X)$  be a spherical Fano variety. To every vector  $v \in \mathcal{V}$  corresponds a  $G$ -equivariant test configuration  $(\mathcal{X}^v, \mathcal{L}^v)$  for  $(X, -K_X)$  with irreducible central fiber  $\mathcal{X}_0^v$ . Moreover, there exists  $m \in \mathbb{N}$  such that the test configuration constructed from  $mv$  is special and, conversely, every  $G$ -equivariant special test configuration for  $(X, -K_X)$  can be constructed in this way. The Donaldson–Futaki invariant of a special test configuration corresponding to  $v$  is given by the formula

$$\text{DF}(\mathcal{X}^v, \mathcal{L}^v) = V \cdot \langle 2\rho_Q - \text{bar}_{\text{DH}}(\Delta^+), \bar{v} \rangle.$$

Here  $\text{bar}_{\text{DH}}(\Delta^+)$  is the barycenter of the moment polytope with respect to the Duistermaat–Heckman measure  $\text{DH}$  and  $V$  is a universal constant depending on  $\Delta^+$  and  $\text{DH}$  only. The above expression does not depend on the choice of the lift  $\bar{v}$  of  $v$ .

This computation allows to express the  $\delta_G$ -invariant of  $X$  in terms of the combinatorial data associated to  $X$  and  $B \subset G$  and prove Proposition 1.4.

**Proposition 5.4.** Let  $X$  be a Fano variety which is spherical under the action of  $G$ ; let  $B \subset G$  be a Borel subgroup. In the above notation, the following formula holds for the  $\delta_G$ -invariant of  $X$ :

$$\delta_G(X) = \min_{v \in \mathcal{V}} \frac{h_C(v)}{h_C(v) - V \cdot \langle 2\rho_Q - \text{bar}_{\text{DH}}(\Delta^+), \bar{v} \rangle}.$$

*Proof.* By Theorem 5.1 we can identify  $\text{Val}_X^G$  with the cone  $\mathcal{V}$ . Since the functions  $A_X$  and  $S_L$  are homogeneous of order 1 by Propositions 2.8 and 2.11, their ratio depends only on the line generated by  $v \in \mathcal{V}$ . Thus for any  $v \in \mathcal{V}$  we can consider the special test configuration  $(\mathcal{X}^v, \mathcal{L}^v)$  from Theorem 5.3. Using Proposition 3.9 we find

$$\text{DF}(\mathcal{X}^v, \mathcal{L}^v) = V \cdot \langle 2\rho_Q - \text{bar}_{DH}(\Delta^+), \tilde{v} \rangle = A_X(v) - S_L(v).$$

By Proposition 5.2 the log discrepancy function identifies with the piecewise linear function  $h_C$ . Therefore the  $\delta_G$ -invariant is the minimum over  $\mathcal{V}$  of the two piecewise linear functions  $h_C$  and  $h_C - S_L$ , so the formula follows.  $\square$

**Example 5.5.** In the case when  $X$  is a toric variety and  $G = (\mathbb{G}_m)^n$  the polytope  $\Delta^+$  is the usual polytope  $\Delta$  associated to  $X$ . The cone of  $G$ -invariant valuations is generated by the valuations  $v_{D_i}$  corresponding to the torus-invariant divisors  $D_1, \dots, D_d$ . The log discrepancy function is linear on the cones of the fan of  $X$ . Therefore, the infimum is attained on one of the valuations  $v_{D_i}$  (see [BLJ17, Corollary 7.4]). Thus, by Proposition 4.1 we recover the formula from [BLJ17, Corollary 7.16]:

$$\delta(X) = \delta_G(X) = \min_{1 \leq i \leq d} \frac{1}{1 + \langle \text{bar}(\Delta), v_i \rangle}.$$

**Example 5.6.** Let  $X$  be a spherical homogeneous space under the action of  $G$ , for example a Grassmannian  $\text{Gr}(k, n)$ . Then, as in Remark 2.20, we obtain  $\delta_G(X) = \infty$ .

## 6. FINITE AUTOMORPHISM GROUPS

In this section we show that in case of a variety  $X$  with a finite group action, we can adapt Definition 2.2 to  $G$ -equivariant setting. Also we compare the invariant  $\delta_G$  defined below to the  $\delta$ -invariant of the quotient  $Y = X/G$  with the orbifold pair structure. We do not know if there is an analogue of Theorem 3.7 for the case of a finite group  $G$ .

**Definition 6.1.** Let  $(X, \Delta; L)$  be a pair with a big  $\mathbb{Q}$ -divisor  $L$ . Let  $G$  be a finite subgroup of the automorphism group  $\text{Aut}(X, \Delta; L)$ . Then we define the  $\alpha_{G,m}$ -invariant for the pair  $(X, \Delta; L)$  by

$$\alpha_{G,m}(X, \Delta; L) = \inf\{\text{lct}(X, \Delta; D) \mid D \sim mL \text{ and } D \text{ is effective and } G\text{-invariant}\}.$$

Analogously, let us define

$$\delta_{G,m}(X, \Delta; L) = \inf\{\text{lct}(X, \Delta; D) \mid D \sim mL \text{ is a } G\text{-invariant } \mathbb{Q}\text{-divisor of } m\text{-basis type}\}.$$

We define

$$\alpha_G(X, \Delta; L) = \inf_{m \in \mathbb{N}} \alpha_{G,m}(X, \Delta; L) \quad \text{and} \quad \delta_G(X, \Delta; L) = \lim_{m \rightarrow \infty} \delta_{G,m}(X, \Delta; L).$$

**Example 6.2.** Consider the case  $X = \mathbb{P}^1$  and  $G = \mathbb{Z}_m$ . Then the minimal length of a  $G$ -orbit on  $X$  is 1, so for any  $G$ -invariant divisor  $D$  we have  $T(D) \leq 2$  and  $S(D) \leq \frac{1}{2} \int_0^2 (2-t) dt = 1$ . Thus we get

$$\alpha_G(\mathbb{P}^1) = \frac{1}{2} \quad \text{and} \quad \delta_G(\mathbb{P}^1) = 1.$$

More generally, if  $G$  is a finite group acting faithfully on  $\mathbb{P}^1$  then by Proposition 2.23 we have

$$\delta_G(\mathbb{P}^1) = 2\alpha_G(\mathbb{P}^1)$$

which is equal to the minimal length of a  $G$ -orbit on  $\mathbb{P}^1$  (see e.g. [CPS18, Example 2]).

We now prove the ramification formula (Proposition 1.5 from the introduction).

**Proposition 6.3.** Let  $X$  be a normal and  $\mathbb{Q}$ -Gorenstein variety with at most log terminal singularities and  $-K_X$  big. Consider a finite subgroup  $G \subset \text{Aut}(X)$ . Let us denote by  $Y = X/G$  the quotient variety, by  $\pi: X \rightarrow Y$  the quotient map and by  $B = \sum (1 - \frac{1}{m_i}) B_i$  the branch  $\mathbb{Q}$ -divisor on  $Y$ . Then the following ramification formulas hold for  $\alpha_G(X)$  and for  $\delta_G(X)$ :

$$\alpha_G(X) = \alpha(Y, B) \quad \text{and} \quad \delta_G(X) = \delta(Y, B).$$

*Proof.* For a  $\mathbb{Q}$ -divisor  $D_Y$  on  $Y$  its pullback  $\pi^*D_Y$  is a  $G$ -invariant  $\mathbb{Q}$ -divisor on  $X$ . We also define the crepant preimage  $D_X$  of  $D_Y$  via  $\pi$  by the formula

$$K_X + D_X \sim_{\mathbb{Q}} \pi^*(K_Y + B + D_Y).$$

By the canonical bundle formula for a finite morphism we have  $-K_X \sim_{\mathbb{Q}} -\pi^*(K_Y + B)$ , in particular, the pair  $(Y, B)$  is klt. Consider a  $\mathbb{Q}$ -divisor  $D_Y \sim_{\mathbb{Q}} -(K_Y + B)$ . Then its pullback  $\pi^*D_Y$  is a  $G$ -invariant  $\mathbb{Q}$ -divisor  $\mathbb{Q}$ -linearly equivalent to  $-K_X$ . Conversely, every  $G$ -invariant  $\mathbb{Q}$ -divisor  $D \sim_{\mathbb{Q}} -K_X$  is a pullback of a  $\mathbb{Q}$ -divisor  $D_Y = \pi(D)$  on  $Y$ . In particular,  $D_Y \sim_{\mathbb{Q}} -(K_Y + B)$  is of  $m$ -basis type in if and only if  $\pi^*D_Y$  is of  $m$ -basis type. Moreover by [Kol97, Proposition 3.16] the pair  $(Y, D_Y)$  is klt if and only if the pair  $(X, D_X)$  is klt. Therefore we can write

$$\begin{aligned} \delta_m(Y, B) &= \inf\{\text{lct}(Y, B; D_Y) \mid D_Y \sim_{\mathbb{Q}} -(K_Y + B) \text{ is of } m\text{-basis type}\} \\ &= \inf\{\text{lct}(X; D_X) \mid D_X \text{ is the crepant preimage of } D_Y \text{ as above}\} \\ &= \inf\{\text{lct}(X; \pi^*D_Y) \mid \pi^*D_Y \sim_{\mathbb{Q}} -K_X \text{ and } \pi^*D_Y \text{ is } G\text{-invariant of } m\text{-basis type}\} = \delta_{G,m}(X) \end{aligned}$$

for all  $m \in \mathbb{N}$  and thus  $\delta_G(X) = \delta(Y, B)$ . Analogously, for the  $\alpha$ -invariant we have

$$\begin{aligned} \alpha(Y, B) &= \inf\{\text{lct}(Y, B; D_Y) \mid D_Y \sim_{\mathbb{Q}} -(K_Y + B) \text{ and } D_Y \text{ is an effective } \mathbb{Q}\text{-divisor}\} \\ &= \inf\{\text{lct}(X; D) \mid D \sim_{\mathbb{Q}} -K_X \text{ and } D \text{ is a } G\text{-invariant effective } \mathbb{Q}\text{-divisor}\} = \alpha_G(X) \end{aligned}$$

as desired.  $\square$

**Example 6.4.** In Example 6.2 consider the quotient map  $\pi: \mathbb{P}^1 \rightarrow \mathbb{P}^1$  by the cyclic group  $G = \mathbb{Z}_m$ . The branch divisor is equal to

$$B = \frac{m-1}{m}(p_1 + p_2).$$

To compute  $\alpha(\mathbb{P}^1, B)$  we note that for any effective  $D \sim_{\mathbb{Q}} -K_{\mathbb{P}^1}$  and for any point  $p \in \mathbb{P}^1$  we have

$$\text{lct}_p(\mathbb{P}^1, D) \geq 1/2$$

since  $\alpha(\mathbb{P}^1) = 1/2$ . Therefore for any  $p$  and for any  $D$  we have  $\text{mult}_p(D) \leq 2$ . Thus for any effective  $E \sim_{\mathbb{Q}} -(K_{\mathbb{P}^1} + B)$  we obtain that  $\text{mult}_p(E) \leq 2/n$ . This implies that

$$\text{lct}_p(\mathbb{P}^1, B; E) \geq 1/2$$

and moreover the value  $1/2$  is attained; so that  $\alpha(\mathbb{P}^1, B) = 1/2$ . The same computation together with the fact that  $\delta(\mathbb{P}^1) = 1$  shows that  $\delta(\mathbb{P}^1, B) = 1$ .

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