

# DOUBLE AFFINE GRASSMANNIANS AND COULOMB BRANCHES OF $3d\mathfrak{n} = 4$ QUIVER GAUGE THEORIES

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## Abstract

We propose a conjectural construction of various slices for double affine Grassmannians as Coulomb branches of 3-dimensional  $\mathfrak{n} = 4$  supersymmetric affine quiver gauge theories. It generalizes the known construction for the usual affine Grassmannians, and makes sense for arbitrary symmetric Kac-Mody algebras.

## 1 Introduction

**1.1 Historical background.** The geometric Satake equivalence Lusztig [1983], Ginzburg [1995], Beilinson and Drinfeld [2000], and Mirković and Vilonen [2007] proposed by V. Drinfeld for the needs of the Geometric Langlands Program proved very useful for the study of representation theory of reductive algebraic groups (starting from G. Lusztig’s construction of  $q$ -analogues of weight multiplicities). About 15 years ago, I. Frenkel and I. Grojnowski envisioned an extension of the geometric Satake equivalence to the case of loop groups. The affine Grassmannians (the main objects of the geometric Satake equivalence) are ind-schemes of ind-finite type. Their loop analogues (double affine Grassmannians) are much more infinite, beyond our current technical abilities. We are bound to settle for some provisional substitutes, such as transversal slices to the smaller strata in the closures of bigger strata. These substitutes still carry quite powerful geometric information.

Following I. Frenkel’s suggestion, some particular slices for the double affine Grassmannians were constructed in terms of Uhlenbeck compactifications of instanton moduli spaces on Kleinian singularities about 10 years ago. More recently, H. Nakajima’s approach to Coulomb branches of 3-dimensional  $\mathfrak{n} = 4$  supersymmetric gauge theories, applied to affine quiver gauge theories, paved a way for the construction of the most general slices.

**1.2 Contents.** We recall the geometric Satake equivalence in [Section 2](#). The (generalized) slices for the affine Grassmannians are reviewed in [Section 3](#). The problem of constructing (the slices for) the double affine Grassmannians is formulated in [Section 4](#). The mathematical construction of Coulomb branches of  $3d$   $\mathfrak{N} = 4$  gauge theories and its application to slices occupies [Section 5](#). Some more applications are mentioned in [Section 6](#).

**Acknowledgments.** This report is based mostly on the works of A. Braverman and H. Nakajima, some joint with the author. I was incredibly lucky to have an opportunity to learn mathematics from them. Before meeting them, I was introduced to some semiinfinite ideas sketched below by A. Beilinson, V. Drinfeld, B. Feigin, V. Ginzburg and I. Mirković. It is also a pleasure to acknowledge my intellectual debt to R. Bezrukavnikov, D. Gaiotto, D. Gaitsgory, J. Kamnitzer and V. Pestun.

## 2 Geometric Satake equivalence

Let  $\mathcal{O}$  denote the formal power series ring  $\mathbb{C}[[z]]$ , and let  $\mathcal{K}$  denote its fraction field  $\mathbb{C}((z))$ . Let  $G$  be an almost simple complex algebraic group with a Borel and a Cartan subgroup  $G \supset B \supset T$ , and with the Weyl group  $W_{\text{fin}}$  of  $(G, T)$ . Let  $\Lambda$  be the coweight lattice, and let  $\Lambda^+ \subset \Lambda$  be the submonoid of dominant coweights. Let also  $\Lambda_+ \subset \Lambda$  be the submonoid spanned by the simple coroots  $\alpha_i$ ,  $i \in I$ . We denote by  $G^\vee \supset T^\vee$  the Langlands dual group, so that  $\Lambda$  is the weight lattice of  $G^\vee$ .

The affine Grassmannian  $\text{Gr}_G = G_{\mathcal{K}}/G_{\mathcal{O}}$  is an ind-projective scheme, the union  $\bigsqcup_{\bar{\lambda} \in \Lambda^+} \text{Gr}_G^{\bar{\lambda}}$  of  $G_{\mathcal{O}}$ -orbits. The closure of  $\text{Gr}_G^{\bar{\lambda}}$  is a projective variety  $\overline{\text{Gr}}_G^{\bar{\lambda}} = \bigsqcup_{\bar{\mu} \leq \bar{\lambda}} \text{Gr}_G^{\bar{\mu}}$ . The fixed point set  $\text{Gr}_G^T$  is naturally identified with the coweight lattice  $\Lambda$ ; and  $\bar{\mu} \in \Lambda$  lies in  $\text{Gr}_G^{\bar{\lambda}}$  iff  $\bar{\mu} \in W_{\text{fin}}\bar{\lambda}$ .

One of the cornerstones of the Geometric Langlands Program initiated by V. Drinfeld is an equivalence  $\mathbb{S}$  of the tensor category  $\text{Rep}(G^\vee)$  and the category  $\text{Perv}_{G_{\mathcal{O}}}(\text{Gr}_G)$  of  $G_{\mathcal{O}}$ -equivariant perverse constructible sheaves on  $\text{Gr}_G$  equipped with a natural monoidal convolution structure  $\star$  and a fiber functor  $H^\bullet(\text{Gr}_G, -)$  [Lusztig \[1983\]](#), [Ginzburg \[1995\]](#), [Beilinson and Drinfeld \[2000\]](#), and [Mirković and Vilonen \[2007\]](#). It is a categorification of the classical Satake isomorphism between  $K(\text{Rep}(G^\vee)) = \mathbb{C}[T^\vee]^{W_{\text{fin}}}$  and the spherical affine Hecke algebra of  $G$ . The geometric Satake equivalence  $\mathbb{S}$  sends an irreducible  $G^\vee$ -module  $V^{\bar{\lambda}}$  with highest weight  $\bar{\lambda}$  to the Goresky-MacPherson sheaf  $\text{IC}(\overline{\text{Gr}}_G^{\bar{\lambda}})$ .

In order to construct a commutativity constraint for  $(\text{Perv}_{G_{\mathcal{O}}}(\text{Gr}_G), \star)$ , Beilinson and Drinfeld introduced a relative version  $\text{Gr}_{G, BD}$  of the Grassmannian over the Ran space of a smooth curve  $X$ , and a fusion monoidal structure  $\Psi$  on  $\text{Perv}_{G_{\mathcal{O}}}(\text{Gr}_G)$  (isomorphic to  $\star$ ). One of the main discoveries of [Mirković and Vilonen \[2007\]](#) was a  $\Lambda$ -grading of

the fiber functor  $H^\bullet(\mathrm{Gr}_G, \mathcal{F}) = \bigoplus_{\bar{\lambda} \in \Lambda} \Phi_{\bar{\lambda}}(\mathcal{F})$  by the hyperbolic stalks at  $T$ -fixed points. For a  $G^\vee$ -module  $V$ , its weight space  $V_{\bar{\lambda}}$  is canonically isomorphic to the hyperbolic stalk  $\Phi_{\bar{\lambda}}(\mathcal{S}V)$ .

Various geometric structures of a perverse sheaf  $\mathcal{S}V$  reflect some fine representation theoretic structures of  $V$ , such as Brylinski-Kostant filtration and the action of dynamical Weyl group, see [Ginzburg and Riche \[2015\]](#). One of the important technical tools of studying  $\mathrm{Perv}_{G_\Theta}(\mathrm{Gr}_G)$  is the embedding  $\mathrm{Gr}_G \hookrightarrow \mathbf{Gr}_G$  into Kashiwara infinite type scheme  $\mathbf{Gr}_G = G_{\mathbb{C}((z^{-1}))}/G_{\mathbb{C}[[z]]}$  [Kashiwara \[1989\]](#) and [Kashiwara and Tanisaki \[1995\]](#). The quotient  $G_{\mathbb{C}[[z^{-1}]]} \backslash \mathbf{Gr}_G$  is the moduli stack  $\mathrm{Bun}_G(\mathbb{P}^1)$  of  $G$ -bundles on the projective line  $\mathbb{P}^1$ . The  $G_{\mathbb{C}[[z^{-1}]]}$ -orbits on  $\mathbf{Gr}_G$  are of finite codimension; they are also numbered by the dominant coweights of  $G$ , and the image of an orbit  $\mathbf{Gr}_G^{\bar{\lambda}}$  in  $\mathrm{Bun}_G(\mathbb{P}^1)$  consists of  $G$ -bundles of isomorphism type  $\bar{\lambda}$  [Grothendieck \[1957\]](#). The stratifications  $\mathrm{Gr}_G = \bigsqcup_{\bar{\lambda} \in \Lambda^+} \mathrm{Gr}_G^{\bar{\lambda}}$  and  $\mathbf{Gr}_G = \bigsqcup_{\bar{\lambda} \in \Lambda^+} \mathbf{Gr}_G^{\bar{\lambda}}$  are transversal, and their intersections and various generalizations thereof are the subject of the next section.

### 3 Generalized slices

**3.1 The dominant case.** We denote by  $K_1$  the first congruence subgroup of  $G_{\mathbb{C}[[z^{-1}]]}$ : the kernel of the evaluation projection  $\mathrm{ev}_\infty: G_{\mathbb{C}[[z^{-1}]]} \twoheadrightarrow G$ . The transversal slice  $\mathcal{W}_{\bar{\mu}}^{\bar{\lambda}}$  (resp.  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$ ) is defined as the intersection of  $\mathrm{Gr}_G^{\bar{\lambda}}$  (resp.  $\overline{\mathrm{Gr}}_G^{\bar{\lambda}}$ ) and  $K_1 \cdot \bar{\mu}$  in  $\mathbf{Gr}_G$ . It is known that  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$  is nonempty iff  $\bar{\mu} \leq \bar{\lambda}$ , and  $\dim \overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$  is an affine irreducible variety of dimension  $\langle 2\bar{\rho}^\vee, \bar{\lambda} - \bar{\mu} \rangle$ . Following an idea of I. Mirković, [Kamnitzer, Webster, Weekes, and Yacobi \[2014\]](#) proved that  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}} = \bigsqcup_{\bar{\mu} \leq \bar{\nu} \leq \bar{\lambda}} \mathcal{W}_{\bar{\mu}}^{\bar{\nu}}$  is the decomposition of  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$  into symplectic leaves of a natural Poisson structure.

The only  $T$ -fixed point of  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$  is  $\bar{\mu}$ . We consider the cocharacter  $2\bar{\rho}: \mathbb{C}^\times \rightarrow T$ , and denote by  $R_{\bar{\mu}}^{\bar{\lambda}} \subset \overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$  the corresponding repellent: the closed affine subvariety formed by all the points that flow into  $\bar{\mu}$  under the action of  $2\bar{\rho}(t)$ , as  $t$  goes to  $\infty$ . Let  $r$  stand for the closed embedding of  $R_{\bar{\mu}}^{\bar{\lambda}}$  into  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$ , and let  $\iota$  stand for the closed embedding of  $\bar{\mu}$  into  $R_{\bar{\mu}}^{\bar{\lambda}}$ . Then the hyperbolic stalk  $\Phi_{\bar{\mu}}^{\bar{\lambda}} \mathcal{F}$  of a  $T$ -equivariant constructible complex  $\mathcal{F}$  on  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$  is defined as  $\iota^! r^* \mathcal{F}$ , see [Braden \[2003\]](#) and [Drinfeld and Gaitsgory \[2014\]](#).

Recall that the geometric Satake equivalence takes an irreducible  $G^\vee$ -module  $V^{\bar{\lambda}}$  to the IC-sheaf  $\mathrm{IC}(\overline{\mathrm{Gr}}_G^{\bar{\lambda}})$ , and the weight space  $V_{\bar{\mu}}^{\bar{\lambda}}$  is realized as  $V_{\bar{\mu}}^{\bar{\lambda}} = \Phi_{\bar{\mu}} \mathrm{IC}(\overline{\mathrm{Gr}}_G^{\bar{\lambda}}) = \Phi_{\bar{\mu}}^{\bar{\lambda}} \mathrm{IC}(\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}})$ . The usual stalks of both  $\mathrm{IC}(\overline{\mathrm{Gr}}_G^{\bar{\lambda}})$  and  $\mathrm{IC}(\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}})$  at  $\bar{\mu}$  are isomorphic up to shift to the associated graded  $\mathrm{gr} V_{\bar{\mu}}^{\bar{\lambda}}$  with respect to the Brylinski-Kostant filtration.

**3.2 The general case.** If we want to reconstruct the whole of  $V^{\bar{\lambda}}$  from the various slices  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$ , we are missing the weight spaces  $V_{\bar{\mu}}^{\bar{\lambda}}$  with nondominant  $\bar{\mu}$ . To take care of the remaining weight spaces, for arbitrary  $\bar{\mu}$  we consider the moduli space  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$  of the following data:

- (a) A  $G$ -bundle  $\mathcal{P}$  on  $\mathbb{P}^1$ .
- (b) A trivialization  $\sigma: \mathcal{P}_{\text{triv}}|_{\mathbb{P}^1 \setminus \{0\}} \xrightarrow{\sim} \mathcal{P}|_{\mathbb{P}^1 \setminus \{0\}}$  having a pole of degree  $\leq \bar{\lambda}$  at  $0 \in \mathbb{P}^1$  (that is defining a point of  $\overline{\text{Gr}}_G^{\bar{\lambda}}$ ).
- (c) A  $B$ -structure  $\phi$  on  $\mathcal{P}$  of degree  $w_0\bar{\mu}$  with the fiber  $B_- \subset G$  at  $\infty \in \mathbb{P}^1$  (with respect to the trivialization  $\sigma$  of  $\mathcal{P}$  at  $\infty \in \mathbb{P}^1$ ). Here  $G \supset B_- \supset T$  is the Borel subgroup opposite to  $B$ , and  $w_0 \in W_{\text{fin}}$  is the longest element.

This construction goes back to [Finkelberg and Mirković \[1999\]](#). The space  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$  is nonempty iff  $\bar{\mu} \leq \bar{\lambda}$ . In this case it is an irreducible affine normal Cohen-Macaulay variety of dimension  $\langle 2\bar{\rho}^\vee, \bar{\lambda} - \bar{\mu} \rangle$ , see [Braverman, Finkelberg, and Nakajima \[2016a\]](#). In case  $\bar{\mu}$  is dominant, the two definitions of  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$  agree. At the other extreme, if  $\bar{\lambda} = 0$ , then  $\overline{\mathcal{W}}_{-\alpha}^0$  is nothing but the open zastava space  $\check{Z}^{-w_0\alpha}$ . The  $T$ -fixed point set  $(\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}})^T$  is nonempty iff the weight space  $V_{\bar{\mu}}^{\bar{\lambda}}$  is not 0; in this case  $(\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}})^T$  consists of a single point denoted  $\bar{\mu}$ . We consider the repellent  $R_{\bar{\mu}}^{\bar{\lambda}} \subset \overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$ . It is a closed subvariety of dimension  $\langle \bar{\rho}^\vee, \bar{\lambda} - \bar{\mu} \rangle$  (equidimensional). We have  $V_{\bar{\mu}}^{\bar{\lambda}} = \Phi_{\bar{\mu}}^{\bar{\lambda}} \text{IC}(\overline{\text{Gr}}_G^{\bar{\lambda}}) = \Phi_{\bar{\mu}}^{\bar{\lambda}} \text{IC}(\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}})$ , so that  $V^{\bar{\lambda}} = \bigoplus_{\bar{\mu} \in \Lambda} \Phi_{\bar{\mu}}^{\bar{\lambda}} \text{IC}(\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}})$  (see [Krylov \[2017\]](#)). Similarly to [Braverman and Gaietsgory \[2001\]](#), one can introduce a crystal structure on the set of irreducible components  $\bigsqcup_{\bar{\mu} \in \Lambda} \text{Irr } R_{\bar{\mu}}^{\bar{\lambda}}$  (see [Krylov \[2017\]](#)), so that the resulting crystal is isomorphic to the integrable crystal  $\mathbf{B}(\bar{\lambda})$  (for a beautiful survey on crystals, see [Kashiwara \[1995\]](#)).

**3.3 Beilinson-Drinfeld slices.** Let  $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_N)$  be a collection of dominant coweights of  $G$ . We consider the moduli space  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$  of the following data:

- (a) A collection of points  $(z_1, \dots, z_N) \in \mathbb{A}^N$  on the affine line  $\mathbb{A}^1 \subset \mathbb{P}^1$ .
- (b) A  $G$ -bundle  $\mathcal{P}$  on  $\mathbb{P}^1$ .
- (c) A trivialization  $\sigma: \mathcal{P}_{\text{triv}}|_{\mathbb{P}^1 \setminus \{z_1, \dots, z_N\}} \xrightarrow{\sim} \mathcal{P}|_{\mathbb{P}^1 \setminus \{z_1, \dots, z_N\}}$  with a pole of degree  $\leq \sum_{s=1}^N \bar{\lambda}_s \cdot z_s$  on the complement.
- (d) A  $B$ -structure  $\phi$  on  $\mathcal{P}$  of degree  $w_0\bar{\mu}$  with the fiber  $B_- \subset G$  at  $\infty \in \mathbb{P}^1$  (with respect to the trivialization  $\sigma$  of  $\mathcal{P}$  at  $\infty \in \mathbb{P}^1$ ).

$\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$  is nonempty iff  $\bar{\mu} \leq \bar{\lambda} := \sum_{s=1}^N \bar{\lambda}_s$ . In this case it is an irreducible affine normal Cohen-Macaulay variety flat over  $\mathbb{A}^N$  of relative dimension  $\langle 2\bar{\rho}^\vee, \bar{\lambda} - \bar{\mu} \rangle$ , see [Braverman, Finkelberg, and Nakajima \[2016a\]](#). The fiber over  $N \cdot 0 \in \mathbb{A}^N$  is nothing but  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$ . We

can consider the Verdier specialization  $\mathrm{Sp} \, \mathrm{IC}(\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}})$  to the special fiber  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$ . It is a perverse sheaf on  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}} \times \mathbb{A}^N$  smooth along the diagonal stratification of  $\mathbb{A}^N$ . We denote by  $\Psi \mathrm{IC}(\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}})$  its restriction to  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}} \times \underline{z}$  where  $\underline{z}$  is a point of  $\mathbb{A}_{\mathbb{R}}^N$  such that  $z_1 > \dots > z_N$ . Then

$$\Psi \mathrm{IC}(\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}) \simeq \bigoplus_{\bar{\mu} \leq \bar{\nu} \leq \bar{\lambda}, \bar{\nu} \in \Lambda^+} M_{\bar{\nu}}^{\bar{\lambda}} \otimes \mathrm{IC}(\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\nu}}),$$

where  $M_{\bar{\nu}}^{\bar{\lambda}}$  is the multiplicity  $\mathrm{Hom}_{G^{\vee}}(V^{\bar{\nu}}, V^{\bar{\lambda}_1} \otimes \dots \otimes V^{\bar{\lambda}_N})$ .

**3.4 Convolution diagram over slices.** In the setup of [Section 3.3](#) we consider the moduli space  $\widetilde{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$  of the following data:

- (a) A collection of points  $(z_1, \dots, z_N) \in \mathbb{A}^N$  on the affine line  $\mathbb{A}^1 \subset \mathbb{P}^1$ .
- (b) A collection of  $G$ -bundles  $(\mathcal{P}_1, \dots, \mathcal{P}_N)$  on  $\mathbb{P}^1$ .
- (c) A collection of isomorphisms  $\sigma_s: \mathcal{P}_{s-1}|_{\mathbb{P}^1 \setminus \{z_s\}} \xrightarrow{\sim} \mathcal{P}_s|_{\mathbb{P}^1 \setminus \{z_s\}}$  with a pole of degree  $\leq \bar{\lambda}_s$  at  $z_s$ . Here  $1 \leq s \leq N$ , and  $\mathcal{P}_0 := \mathcal{P}_{\mathrm{triv}}$ .
- (d) A  $B$ -structure  $\phi$  on  $\mathcal{P}_N$  of degree  $w_0 \bar{\mu}$  with the fiber  $B_{\infty} \subset G$  at  $\infty \in \mathbb{P}^1$  (with respect to the trivialization  $\sigma_N \circ \dots \circ \sigma_1$  of  $\mathcal{P}_N$  at  $\infty \in \mathbb{P}^1$ ).

A natural projection  $\varpi: \widetilde{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}} \rightarrow \overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$  sends  $(\mathcal{P}_1, \dots, \mathcal{P}_N, \sigma_1, \dots, \sigma_N)$  to  $(\mathcal{P}_N, \sigma_N \circ \dots \circ \sigma_1)$ . We denote  $\varpi^{-1}(\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}})$  by  $\widetilde{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$ . Then  $\varpi: \widetilde{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}} \rightarrow \overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$  is stratified semismall, and

$$\varpi_* \mathrm{IC}(\widetilde{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}) = \bigoplus_{\bar{\mu} \leq \bar{\nu} \leq \bar{\lambda}, \bar{\nu} \in \Lambda^+} M_{\bar{\nu}}^{\bar{\lambda}} \otimes \mathrm{IC}(\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\nu}}).$$

## 4 Double affine Grassmannian

In this section  $G$  is assumed to be a simply connected almost simple complex algebraic group.

**4.1 The affine group and its Langlands dual.** We consider the minimal integral even positive definite  $W_{\mathrm{fin}}$ -invariant symmetric bilinear form  $(\cdot, \cdot)$  on the coweight lattice  $\Lambda$ . It gives rise to a central extension  $\widehat{G}$  of the polynomial version  $G_{\mathbb{C}[t^{\pm 1}]}$  of the loop group:

$$1 \rightarrow \mathbb{C}^{\times} \rightarrow \widehat{G} \rightarrow G_{\mathbb{C}[t^{\pm 1}]} \rightarrow 1.$$

The loop rotation group  $\mathbb{C}^{\times}$  acts naturally on  $G_{\mathbb{C}[t^{\pm 1}]}$ , and this action lifts to  $\widehat{G}$ . We denote the corresponding semidirect product  $\mathbb{C}^{\times} \ltimes \widehat{G}$  by  $G_{\mathrm{aff}}$ . It is an untwisted affine Kac-Moody group ind-scheme.

We denote by  $G_{\text{aff}}^\vee$  the corresponding Langlands dual group. Note that if  $G$  is not simply laced, then  $G_{\text{aff}}^\vee$  is a twisted affine Kac-Moody group, not to be confused with  $(G^\vee)_{\text{aff}}$ . However, we have a canonical embedding  $G^\vee \hookrightarrow G_{\text{aff}}^\vee$ .

We fix a Cartan torus  $\mathbb{C}^\times \times T \times \mathbb{C}^\times \subset G_{\text{aff}}$  and its dual Cartan torus  $\mathbb{C}^\times \times T^\vee \times \mathbb{C}^\times \subset G_{\text{aff}}^\vee$ . Here the first copy of  $\mathbb{C}^\times$  is the central  $\mathbb{C}^\times$ , while the second copy is the loop rotation  $\mathbb{C}^\times$ . Accordingly, the weight lattice  $\Lambda_{\text{aff}}$  of  $G_{\text{aff}}^\vee$  is  $\mathbb{Z} \oplus \Lambda \oplus \mathbb{Z}$ : the first copy of  $\mathbb{Z}$  is the central charge (level), and the second copy is the energy. A typical element  $\lambda \in \Lambda_{\text{aff}}$  will be written as  $\lambda = (k, \bar{\lambda}, n)$ . The subset of dominant weights  $\Lambda_{\text{aff}}^+ \subset \Lambda_{\text{aff}}$  consists of all the triples  $(k, \bar{\lambda}, n)$  such that  $\bar{\lambda} \in \Lambda^+$  and  $\langle \bar{\lambda}, \bar{\theta}^\vee \rangle \leq k$ . Here  $\bar{\theta}^\vee = \sum_{i \in I} a_i \alpha_i^\vee$  is the highest root of  $G \supset B \supset T$ . We denote by  $\Lambda_{\text{aff},k}^+ \subset \Lambda_{\text{aff}}^+$  the finite subset of dominant weights of level  $k$ ; we also denote by  $\Lambda_{\text{aff},k} \subset \Lambda_{\text{aff}}$  the subset of all the weights of level  $k$ . We say that  $\lambda \geq \mu$  if  $\lambda - \mu$  is an element of the submonoid generated by the positive roots of  $G_{\text{aff}}^\vee$  (in particular,  $\lambda$  and  $\mu$  must have the same level). Finally, let  $\bar{\omega}_i$ ,  $i \in I$ , be the fundamental coweights of  $G$ , and  $\rho := (1, 0, 0) + \sum_{i \in I} (a_i, \bar{\omega}_i, 0) \in \Lambda_{\text{aff}}$ .

The affine Weyl group  $W_{\text{aff}}$  is the semidirect product  $W_{\text{fin}} \ltimes \Lambda$ . For  $k \in \mathbb{Z}_{>0}$ , we also consider its version  $W_{\text{aff},k} = W_{\text{fin}} \ltimes k\Lambda$ ; it acts naturally on  $\Lambda_{\text{aff},k} = \{k\} \times \Lambda \oplus \mathbb{Z}$  (trivially on  $\mathbb{Z}$ ). Every  $W_{\text{aff},k}$ -orbit on  $\Lambda_{\text{aff},k}$  contains a unique representative in  $\Lambda_{\text{aff},k}^+$ . It follows that if we denote by  $\Gamma_k$  the group of roots of unity of order  $k$ , then there is a natural isomorphism  $\Lambda_{\text{aff},k}^+ / \mathbb{Z} = W_{\text{aff},k} \backslash \Lambda \xrightarrow{\sim} \text{Hom}(\Gamma_k, G) / \text{Ad}_G$ .

**4.2 The quest.** We would like to have a double affine Grassmannian  $\text{Gr}_{G_{\text{aff}}}$  and a geometric Satake equivalence between the category of integrable representations  $\text{Rep}(G_{\text{aff}}^\vee)$  and an appropriate category of perverse sheaves on  $\text{Gr}_{G_{\text{aff}}}$ . Note that the affine Satake isomorphism at the level of functions is established in [Braverman and Kazhdan \[2013\]](#) and [Braverman, Kazhdan, and Patnaik \[2016\]](#) (and in [Gaussent and Rousseau \[2014\]](#) for arbitrary Kac-Moody groups).

Such a quest was formulated by I. Grojnowski in his talk at ICM-2006 in Madrid. At approximately the same time, I. Frenkel suggested that the integrable representations of level  $k$  should be realized in cohomology of certain instanton moduli spaces on  $\mathbb{A}^2 / \Gamma_k$ . Here  $\Gamma_k$  acts on  $\mathbb{A}^2$  in a hyperbolic way:  $\zeta(x, y) = (\zeta x, \zeta^{-1} y)$ .

Note that the set of dominant coweights  $\Lambda^+$  is well ordered, which reflects the fact that the affine Grassmannian  $\text{Gr}_G$  is an ind-projective scheme. However, the set of affine dominant coweights  $\Lambda_{\text{aff}}^+$  is not well ordered: it does not have a minimal element. In fact, it has an automorphism group  $\mathbb{Z}$  acting by the energy shifts:  $(k, \bar{\lambda}, n) \mapsto (k, \bar{\lambda}, n + n')$  (we add a multiple of the minimal imaginary coroot  $\delta$ ). This indicates that the sought for double affine Grassmannian  $\text{Gr}_{G_{\text{aff}}}$  is an object of semiinfinite nature.

At the moment, the only technical possibility of dealing with semiinfinite spaces is via transversal slices to strata. Following I. Frenkel's suggestion, in the series [Braverman and](#)

Finkelberg [2010, 2012, 2013] we developed a partial affine analogue of slices of Section 3 defined in terms of Uhlenbeck spaces  $\mathcal{U}_G(\mathbb{A}^2/\Gamma_k)$ .

**4.3 Dominant slices via Uhlenbeck spaces.** The Uhlenbeck space  $\mathcal{U}_G^d(\mathbb{A}^2)$  is a partial closure of the moduli space  $\text{Bun}_G^d(\mathbb{A}^2)$  of  $G$ -bundles of second Chern class  $d$  on the projective plane  $\mathbb{P}^2$  trivialized at the infinite line  $\mathbb{P}_\infty^1 \subset \mathbb{P}^2$ , see Braverman, Finkelberg, and Gaitsgory [2006]. It is known that  $\text{Bun}_G^d(\mathbb{A}^2)$  is smooth quasiaffine, and  $\mathcal{U}_G^d(\mathbb{A}^2)$  is a connected affine variety of dimension  $2dh_G^\vee$  (where  $h_G^\vee$  is the dual Coxeter number of  $G$ ). Conjecturally,  $\mathcal{U}_G^d(\mathbb{A}^2)$  is normal; in this case  $\mathcal{U}_G^d(\mathbb{A}^2)$  is the affinization of  $\text{Bun}_G^d(\mathbb{A}^2)$ .

The group  $G \times \text{GL}(2)$  acts naturally on  $\mathcal{U}_G^d(\mathbb{A}^2)$ : the first factor via the change of trivialization at  $\mathbb{P}_\infty^1$ , and the second factor via its action on  $(\mathbb{P}^2, \mathbb{P}_\infty^1)$ . The group  $\Gamma_k$  is embedded into  $\text{GL}(2)$ . Given  $\mu = (k, \bar{\mu}, m) \in \Lambda_{\text{aff}, k}^+$  we choose its lift to a homomorphism from  $\Gamma_k$  to  $G$ ; thus  $\Gamma_k$  embeds diagonally into  $G \times \text{GL}(2)$  and acts on  $\text{Bun}_G^d(\mathbb{A}^2)$ . The fixed point subvariety  $\text{Bun}_G^d(\mathbb{A}^2)^{\Gamma_k}$  consists of  $\Gamma_k$ -equivariant bundles and is denoted  $\text{Bun}_{G, \mu}^d(\mathbb{A}^2/\Gamma_k)$ ; another choice of lift above leads to an isomorphic subvariety. Since  $0 \in \mathbb{A}^2$  is a  $\Gamma_k$ -fixed point, for any  $\Gamma_k$ -equivariant  $G$ -bundle  $\mathcal{O} \in \text{Bun}_{G, \mu}^d(\mathbb{A}^2/\Gamma_k)$  the group  $\Gamma_k$  acts on the fiber  $\mathcal{O}_0$ . This action defines an element of  $\text{Hom}(\Gamma_k, G)/\text{Ad}_G$  to be denoted  $[\mathcal{O}_0]$ .

Now given  $\lambda = (k, \bar{\lambda}, l) \in \Lambda_{\text{aff}, k}^+$  we define  $\text{Bun}_{G, \mu}^\lambda(\mathbb{A}^2/\Gamma_k)$  as the subvariety of  $\text{Bun}_{G, \mu}^d(\mathbb{A}^2/\Gamma_k)$  formed by all  $\mathcal{P}$  such that the class  $[\mathcal{O}_0] \in \text{Hom}(\Gamma_k, G)/\text{Ad}_G$  is the image of  $\lambda$ , and  $d = k(l - m) + \frac{(\bar{\lambda}, \bar{\lambda}) - (\bar{\mu}, \bar{\mu})}{2}$ . It is a union of connected components of  $\text{Bun}_{G, \mu}^d(\mathbb{A}^2/\Gamma_k)$ . Conjecturally,  $\text{Bun}_{G, \mu}^\lambda(\mathbb{A}^2/\Gamma_k)$  is connected. This conjecture is proved if  $G = \text{SL}(N)$ , or  $k = 1$ , or  $k$  is big enough for arbitrary  $G$  and fixed  $\bar{\lambda}, \bar{\mu}$ .

Finally, we define the dominant slice  $\overline{\mathcal{W}}_\mu^\lambda$  as the closure  $\mathcal{U}_{G, \mu}^\lambda(\mathbb{A}^2/\Gamma_k)$  of  $\text{Bun}_{G, \mu}^\lambda(\mathbb{A}^2/\Gamma_k)$  in the Uhlenbeck space  $\mathcal{U}_G^d(\mathbb{A}^2)$ .

**4.4 (Hyperbolic) stalks.** The Cartan torus  $T_{\text{aff}} = \mathbb{C}^\times \times T \times \mathbb{C}^\times$  maps into  $G \times \text{GL}(2)$ . Here the first copy of  $\mathbb{C}^\times$  goes to the diagonal torus of  $\text{SL}(2) \subset \text{GL}(2)$ , while the second copy of  $\mathbb{C}^\times$  goes to the center of  $\text{GL}(2)$ . So  $T_{\text{aff}}$  acts on  $\overline{\mathcal{W}}_\mu^\lambda$ , and we denote by  $\mu \in \overline{\mathcal{W}}_\mu^\lambda$  the only fixed point. The corresponding repellent  $R_\mu^\lambda$  is the closed affine subvariety formed by all the points that flow into  $\mu$  under the action of  $2\rho(t)$ , as  $t$  goes to  $\infty$ . The corresponding hyperbolic stalk  $\Phi_\mu^\lambda \text{IC}(\overline{\mathcal{W}}_\mu^\lambda)$  is conjecturally isomorphic to the weight space  $V_\mu^\lambda$  of the integrable  $G_{\text{aff}}^\vee$ -module  $V^\lambda$  with highest weight  $\lambda$ . In type  $A$  this conjecture follows from the identification of  $\overline{\mathcal{W}}_\mu^\lambda$  with a Nakajima cyclic quiver variety and I. Frenkel's level-rank duality between the weight multiplicities and the tensor product multiplicities Frenkel [1982], Nakajima [2002, 2009], and Braverman and Finkelberg

[2010]. In type  $ADE$  at level 1 this conjecture follows from Braverman, Finkelberg, and Nakajima [2016b]. Also, as the notation suggests, the hyperbolic stalk  $\Phi_\mu^\lambda \text{IC}(\overline{\mathcal{W}}_\mu^\lambda)$  is isomorphic to the vanishing cycles of  $\text{IC}(\overline{\mathcal{W}}_\mu^\lambda)$  at  $\mu$  with respect to a general function vanishing at  $\mu$  Finkelberg and Kubrak [2015]. The usual stalk of  $\text{IC}(\overline{\mathcal{W}}_\mu^\lambda)$  at  $\mu$  is conjecturally isomorphic to the associated graded of  $V_\mu^\lambda$  with respect to the the affine Brylinski-Kostant filtration Slofstra [2012]. At level 1, this conjecture follows from the computation of the IC-stalks of Uhlenbeck spaces in Braverman, Finkelberg, and Gaiitsgory [2006].

The affine analogs of generalized slices of Sections 3.2, 3.3, 3.4 were constructed in type  $A$  in Braverman and Finkelberg [2012, 2013] in terms of Nakajima cyclic quiver varieties mentioned above. For arbitrary  $G$ , the desired generalized slices are expected to be the Uhlenbeck partial compactifications of the moduli spaces of  $\Gamma_k$ -equivariant  $G_c$ -instantons (where  $G_c \subset G$  is a maximal compact subgroup) on multi Taub-NUT spaces (for a physical explanation via a supersymmetric conformal field theory in 6 dimensions, see Witten [2010]). Unfortunately, we are still lacking a modular definition of the Uhlenbeck compactification Baranovsky [2015], and the existing *ad hoc* constructions are not flexible enough. Another approach via the Coulomb branches of framed affine quiver gauge theories following Nakajima [2016a] and Braverman, Finkelberg, and Nakajima [2016c,a, 2017] is described in the remaining sections. For a beautiful short introduction to the Coulomb branches, the reader may consult Nakajima [2016b, 2015].

## 5 Coulomb branches of $3d \mathfrak{N} = 4$ quiver gauge theories

**5.1 General setup.** Let  $\mathbf{N}$  be a finite dimensional representation of a complex connected reductive group  $\mathbb{G}$  (having nothing to do with  $G$  of previous sections). We consider the moduli space  $\mathcal{R}_{\mathbb{G}, \mathbf{N}}$  of triples  $(\mathcal{P}, \sigma, s)$  where  $\mathcal{P}$  is a  $\mathbb{G}$ -bundle on the formal disc  $D = \text{Spec } \mathcal{O}$ ;  $\sigma$  is a trivialization of  $\mathcal{P}$  on the punctured formal disc  $D^* = \text{Spec } \mathbb{K}$ ; and  $s$  is a section of the associated vector bundle  $\mathcal{P}_{\text{triv}} \otimes \mathbf{N}$  on  $D^*$  such that  $s$  extends to a regular section of  $\mathcal{P}_{\text{triv}} \otimes \mathbf{N}$  on  $D$ , and  $\sigma(s)$  extends to a regular section of  $\mathcal{P} \otimes \mathbf{N}$  on  $D$ . In other words,  $s$  extends to a regular section of the vector bundle associated to the  $\mathbb{G}$ -bundle glued from  $\mathcal{P}$  and  $\mathcal{P}_{\text{triv}}$  on the non-separated formal scheme glued from 2 copies of  $D$  along  $D^*$  (*raviolo*). The group  $\mathbb{G}_{\mathcal{O}}$  acts on  $\mathcal{R}_{\mathbb{G}, \mathbf{N}}$  by changing the trivialization  $\sigma$ , and we have an evident  $\mathbb{G}_{\mathcal{O}}$ -equivariant projection  $\mathcal{R}_{\mathbb{G}, \mathbf{N}} \rightarrow \text{Gr}_{\mathbb{G}}$  forgetting  $s$ . The fibers of this projection are profinite dimensional vector spaces: the fiber over the base point is  $\mathbf{N} \otimes \mathcal{O}$ , and all the other fibers are subspaces in  $\mathbf{N} \otimes \mathcal{O}$  of finite codimension. One may say that  $\mathcal{R}_{\mathbb{G}, \mathbf{N}}$  is a  $\mathbb{G}_{\mathcal{O}}$ -equivariant “constructible profinite dimensional vector bundle” over  $\text{Gr}_{\mathbb{G}}$ . The  $\mathbb{G}_{\mathcal{O}}$ -equivariant Borel-Moore homology  $H_\bullet^{\mathbb{G}_{\mathcal{O}}}(\mathcal{R}_{\mathbb{G}, \mathbf{N}})$  is well-defined, and forms an associative algebra with respect to a convolution operation. This algebra is commutative,



finitely generated and integral, and its spectrum  $\mathfrak{M}_C(\mathbb{G}, \mathbf{N}) = \text{Spec } H_{\bullet}^{\mathbb{G}^\circ}(\mathcal{R}_{\mathbb{G}, \mathbf{N}})$  is an irreducible normal affine variety of dimension  $2 \text{rk}(\mathbb{G})$ , the *Coulomb branch*. It is supposed to be a (singular) hyper-Kähler manifold [Seiberg and Witten \[1997\]](#).

Let  $\mathbb{T} \subset \mathbb{G}$  be a Cartan torus with Lie algebra  $\mathfrak{t} \subset \mathfrak{g}$ . Let  $W = N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$  be the corresponding Weyl group. Then the equivariant cohomology  $H_{\mathbb{G}^\circ}^\bullet(\text{pt}) = \mathbb{C}[\mathfrak{t}/W]$  forms a subalgebra of  $H_{\bullet}^{\mathbb{G}^\circ}(\mathcal{R}_{\mathbb{G}, \mathbf{N}})$  (a *Cartan subalgebra*), so we have a projection  $\Pi: \mathfrak{M}_C(\mathbb{G}, \mathbf{N}) \rightarrow \mathfrak{t}/W$ .

Finally, the algebra  $H_{\bullet}^{\mathbb{G}^\circ}(\mathcal{R}_{\mathbb{G}, \mathbf{N}})$  comes equipped with quantization: a  $\mathbb{C}[\hbar]$ -deformation  $\mathbb{C}_\hbar[\mathfrak{M}_C(\mathbb{G}, \mathbf{N})] = H_{\bullet}^{\mathbb{C}^\times \times \mathbb{G}^\circ}(\mathcal{R}_{\mathbb{G}, \mathbf{N}})$  where  $\mathbb{C}^\times$  acts by loop rotations, and  $\mathbb{C}[\hbar] = H_{\mathbb{C}^\times}^\bullet(\text{pt})$ . It gives rise to a Poisson bracket on  $\mathbb{C}[\mathfrak{M}_C(\mathbb{G}, \mathbf{N})]$  with an open symplectic leaf, so that  $\Pi$  becomes an integrable system:  $\mathbb{C}[\mathfrak{t}/W] \subset \mathbb{C}[\mathfrak{M}_C(\mathbb{G}, \mathbf{N})]$  is a Poisson-commutative polynomial subalgebra with  $\text{rk}(\mathbb{G})$  generators.

**5.2 Flavor symmetry.** Suppose we have an extension  $1 \rightarrow \mathbb{G} \rightarrow \tilde{\mathbb{G}} \rightarrow \mathbb{G}_F \rightarrow 1$  where  $\mathbb{G}_F$  is a connected reductive group (a *flavor group*), and the action of  $\mathbb{G}$  on  $\mathbf{N}$  is extended to an action of  $\tilde{\mathbb{G}}$ . Then the action of  $\mathbb{G}_\theta$  on  $\mathcal{R}_{\mathbb{G}, \mathbf{N}}$  extends to an action of  $\tilde{\mathbb{G}}_\theta$ , and the convolution product defines a commutative algebra structure on the equivariant Borel-Moore homology  $H_{\bullet}^{\tilde{\mathbb{G}}^\circ}(\mathcal{R}_{\mathbb{G}, \mathbf{N}})$ . We have the restriction homomorphism  $H_{\bullet}^{\tilde{\mathbb{G}}^\circ}(\mathcal{R}_{\mathbb{G}, \mathbf{N}}) \rightarrow H_{\bullet}^{\mathbb{G}^\circ}(\mathcal{R}_{\mathbb{G}, \mathbf{N}}) = H_{\bullet}^{\mathbb{G}^\circ}(\mathcal{R}_{\mathbb{G}, \mathbf{N}}) \otimes_{H_{\mathbb{G}_F}^\bullet(\text{pt})} \mathbb{C}$ . In other words,  $\underline{\mathfrak{M}}_C(\mathbb{G}, \mathbf{N}) := \text{Spec } H_{\bullet}^{\tilde{\mathbb{G}}^\circ}(\mathcal{R}_{\mathbb{G}, \mathbf{N}})$  is a deformation of  $\mathfrak{M}_C(\mathbb{G}, \mathbf{N})$  over  $\text{Spec } H_{\mathbb{G}_F}^\bullet(\text{pt}) = \mathfrak{t}_F/W_F$ .

We will need the following version of this construction. Let  $Z \subset \mathbb{G}_F$  be a torus embedded into the flavor group. We denote by  $\tilde{\mathbb{G}}^Z$  the pullback extension  $1 \rightarrow \mathbb{G} \rightarrow \tilde{\mathbb{G}}^Z \rightarrow Z \rightarrow 1$ . We define  $\underline{\mathfrak{M}}_C^Z(\mathbb{G}, \mathbf{N}) := \text{Spec } H_{\bullet}^{\tilde{\mathbb{G}}^Z}(\mathcal{R}_{\mathbb{G}, \mathbf{N}})$ : a deformation of  $\mathfrak{M}_C(\mathbb{G}, \mathbf{N})$  over  $\mathfrak{z} := \text{Spec } H_Z^\bullet(\text{pt})$ .

Since  $\mathfrak{M}_C(\mathbb{G}, \mathbf{N})$  is supposed to be a hyper-Kähler manifold, its flavor deformation should come together with a (partial) resolution. To construct it, we consider the obvious projection  $\tilde{\pi}: \mathcal{R}_{\tilde{\mathbb{G}}, \mathbf{N}} \rightarrow \text{Gr}_{\tilde{\mathbb{G}}} \rightarrow \text{Gr}_{\mathbb{G}_F}$ . Given a dominant coweight  $\lambda_F \in \Lambda_F^+ \subset \text{Gr}_{\mathbb{G}_F}$ , we set  $\mathcal{R}_{\tilde{\mathbb{G}}, \mathbf{N}}^{\lambda_F} := \tilde{\pi}^{-1}(\lambda_F)$ , and consider the equivariant Borel-Moore homology  $H_{\bullet}^{\tilde{\mathbb{G}}^Z}(\mathcal{R}_{\tilde{\mathbb{G}}, \mathbf{N}}^{\lambda_F})$ . It carries a convolution module structure over  $H_{\bullet}^{\tilde{\mathbb{G}}^Z}(\mathcal{R}_{\mathbb{G}, \mathbf{N}})$ . We consider  $\widetilde{\mathfrak{M}}_C^{Z, \lambda_F}(\mathbb{G}, \mathbf{N}) := \text{Proj}(\bigoplus_{n \in \mathbb{N}} H_{\bullet}^{\tilde{\mathbb{G}}^Z}(\mathcal{R}_{\tilde{\mathbb{G}}, \mathbf{N}}^{n\lambda_F})) \xrightarrow{\varpi} \underline{\mathfrak{M}}_C^Z(\mathbb{G}, \mathbf{N})$ . We denote  $\varpi^{-1}(\mathfrak{M}_C(\mathbb{G}, \mathbf{N}))$  by  $\widetilde{\mathfrak{M}}_C^{\lambda_F}(\mathbb{G}, \mathbf{N})$ . We have  $\widetilde{\mathfrak{M}}_C^{\lambda_F}(\mathbb{G}, \mathbf{N}) = \text{Proj}(\bigoplus_{n \in \mathbb{N}} H_{\bullet}^{\mathbb{G}^\circ}(\mathcal{R}_{\tilde{\mathbb{G}}, \mathbf{N}}^{n\lambda_F}))$ .

More generally, for a strictly convex (i.e. not containing nontrivial subgroups) cone  $V \subset \Lambda_F^+$ , we consider the multi projective spectra  $\widetilde{\mathfrak{M}}_C^{\mathbb{Z}, V}(\mathcal{G}, \mathbf{N}) := \text{Proj}(\bigoplus_{\lambda_F \in V} H_\bullet^{\tilde{\mathcal{G}}^\mathbb{Z}}(\mathcal{R}_{\tilde{\mathcal{G}}, \mathbf{N}}^{\lambda_F})) \xrightarrow{\varpi} \mathfrak{M}_C^{\mathbb{Z}}(\mathcal{G}, \mathbf{N})$  and  $\widetilde{\mathfrak{M}}_C^V(\mathcal{G}, \mathbf{N}) := \text{Proj}(\bigoplus_{\lambda_F \in V} H_\bullet^{\mathcal{G}^\mathbb{O}}(\mathcal{R}_{\tilde{\mathcal{G}}, \mathbf{N}}^{\lambda_F})) \xrightarrow{\varpi} \mathfrak{M}_C(\mathcal{G}, \mathbf{N})$ .

**5.3 Quiver gauge theories.** Let  $Q$  be a quiver with  $Q_0$  the set of vertices, and  $Q_1$  the set of arrows. An arrow  $e \in Q_1$  goes from its tail  $t(e) \in Q_0$  to its head  $h(e) \in Q_0$ . We choose a  $Q_0$ -graded vector spaces  $V := \bigoplus_{j \in Q_0} V_j$  and  $W := \bigoplus_{j \in Q_0} W_j$ . We set  $\mathcal{G} = \text{GL}(V) := \prod_{j \in Q_0} \text{GL}(V_j)$ . We choose a second grading  $W = \bigoplus_{s=1}^N W^{(s)}$  compatible with the  $Q_0$ -grading of  $W$ . We set  $\mathcal{G}_F$  to be a Levi subgroup  $\prod_{s=1}^N \prod_{j \in Q_0} \text{GL}(W_j^{(s)})$  of  $\text{GL}(W)$ , and  $\tilde{\mathcal{G}} := \mathcal{G} \times \mathcal{G}_F$ . Finally, we define a central subgroup  $\mathbb{Z} \subset \mathcal{G}_F$  as follows:  $\mathbb{Z} := \prod_{s=1}^N \Delta_{\mathbb{C}^\times}^{(s)} \subset \prod_{s=1}^N \prod_{j \in Q_0} \text{GL}(W_j^{(s)})$ , where  $\mathbb{C}^\times \cong \Delta_{\mathbb{C}^\times}^{(s)} \subset \prod_{j \in Q_0} \text{GL}(W_j^{(s)})$  is the diagonally embedded subgroup of scalar matrices. The reductive group  $\tilde{\mathcal{G}}$  acts naturally on  $\mathbf{N} := \bigoplus_{e \in Q_1} \text{Hom}(V_{t(e)}, V_{h(e)}) \oplus \bigoplus_{j \in Q_0} \text{Hom}(W_j, V_j)$ .

The Higgs branch of the corresponding quiver gauge theory is the Nakajima quiver variety  $\mathfrak{M}_H(\mathcal{G}, \mathbf{N}) = \mathfrak{M}(V, W)$ . We are interested in the Coulomb branch  $\mathfrak{M}_C(\mathcal{G}, \mathbf{N})$ .

**5.4 Back to slices in an affine Grassmannian.** Let now  $G$  be an adjoint simple simply laced algebraic group. We choose an orientation  $\Omega$  of its Dynkin graph (of type  $ADE$ ), and denote by  $I$  its set of vertices. Given an  $I$ -graded vector space  $W$  we encode its dimension by a dominant coweight  $\bar{\lambda} := \sum_{i \in I} \dim(W_i) \bar{\omega}_i \in \Lambda^+$  of  $G$ . Given an  $I$ -graded vector space  $\tilde{W}$  we encode its dimension by a positive coroot combination  $\alpha := \sum_{i \in I} \dim(V_i) \alpha_i \in \Lambda_+$ . We set  $\bar{\mu} := \bar{\lambda} - \alpha \in \Lambda$ . Given a direct sum decomposition  $W = \bigoplus_{s=1}^N W^{(s)}$  compatible with the  $I$ -grading of  $W$  as in Section 5.3, we set  $\bar{\lambda}_s := \sum_{i \in I} \dim(W_i^{(s)}) \bar{\omega}_i \in \Lambda^+$ , and finally,  $\bar{\lambda} := (\bar{\lambda}_1, \dots, \bar{\lambda}_N)$ .

Recall the notations of Section 5.2. Since the flavor group  $\mathcal{G}_F$  is a Levi subgroup of  $\text{GL}(W)$ , its weight lattice is naturally identified with  $\mathbb{Z}^{\dim W}$ . More precisely, we choose a basis  $w_1, \dots, w_{\dim W}$  of  $W$  such that any  $W_i$ ,  $i \in I$ , and  $W^{(s)}$ ,  $1 \leq s \leq N$ , is spanned by a subset of the basis, and we assume the following monotonicity condition: if for  $1 \leq a < b < c \leq \dim W$  we have  $w_a, w_b \in W^{(s)}$  for certain  $s$ , then  $w_b \in W^{(s)}$  as well. We define a strictly convex cone  $V = \{(n_1, \dots, n_{\dim W})\} \subset \Lambda_F^+ \subset \mathbb{Z}^{\dim W}$  by the following conditions: (a) if  $w_k \in W^{(s)}$ ,  $w_l \in W^{(t)}$ , and  $s < t$ , then  $n_k \geq n_l \geq 0$ ; (b) if  $w_k, w_l \in W^{(s)}$ , then  $n_k = n_l$ . The following isomorphisms are constructed in Braverman, Finkelberg, and Nakajima [2016a] (notations of Section 3):

$$\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}} \xrightarrow{\sim} \mathfrak{M}_C(\mathcal{G}, \mathbf{N}), \quad \overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}} \xrightarrow{\sim} \mathfrak{M}_C^{\mathbb{Z}}(\mathcal{G}, \mathbf{N}),$$

(we learned of their existence from V. Pestun). We also expect the following isomorphisms:

$$\widetilde{\mathcal{W}}_{\mu}^{\bar{\lambda}} \xrightarrow{\sim} \widetilde{\mathcal{M}}_C^{\mathbb{Z}, \vee}(\mathbb{G}, \mathbf{N}), \quad \widetilde{\mathcal{W}}_{\mu}^{\bar{\lambda}} \xrightarrow{\sim} \widetilde{\mathcal{M}}_C^{\vee}(\mathbb{G}, \mathbf{N}).$$

In case  $G$  is an adjoint simple non simply laced algebraic group, it can be obtained by folding from a simple simply laced group  $\tilde{G}$  (i.e. as the fixed point set of an outer automorphism of  $\tilde{G}$ ). The corresponding automorphism of the Dynkin quiver of  $\tilde{G}$  acts on the above Coulomb branches, and the slices for  $G$  can be realized as the fixed point sets of these Coulomb branches.

**5.5 Back to slices in a double affine Grassmannian.** We choose an orientation of an *affine* Dynkin graph of type  $A^{(1)}, D^{(1)}, E^{(1)}$  with the set of vertices  $\tilde{I} = I \sqcup \{i_0\}$ , and repeat the construction of [Section 5.4](#) for an affine dominant coweight  $\lambda = \sum_{i \in \tilde{I}} \dim(W_i) \omega_i = (k, \bar{\lambda}, 0) \in \Lambda_{\text{aff}}^+$ , a positive coroot combination  $\alpha = \sum_{i \in \tilde{I}} \dim(V_i) \alpha_i \in \Lambda_{\text{aff}, +}$ , and  $\mu := \lambda - \alpha = (k, \bar{\mu}, n) \in \Lambda_{\text{aff}}$ .

We define the slices in  $\text{Gr}_{G_{\text{aff}}}$  (where  $G$  is the corresponding adjoint simple simply laced algebraic group) as

$$\overline{\mathcal{W}}_{\mu}^{\lambda} := \mathcal{M}_C(\mathbb{G}, \mathbf{N}), \quad \overline{\mathcal{W}}_{\mu}^{\bar{\lambda}} := \mathcal{M}_C^{\mathbb{Z}}(\mathbb{G}, \mathbf{N}), \quad \widetilde{\mathcal{W}}_{\mu}^{\lambda} := \widetilde{\mathcal{M}}_C^{\mathbb{Z}, \vee}(\mathbb{G}, \mathbf{N}), \quad \widetilde{\mathcal{W}}_{\mu}^{\bar{\lambda}} := \widetilde{\mathcal{M}}_C^{\vee}(\mathbb{G}, \mathbf{N}).$$

If  $\mu$  is dominant, the slices  $\overline{\mathcal{W}}_{\mu}^{\lambda}$  conjecturally coincide with the ones of [Section 4.3](#). In type  $A$  this conjecture follows from the computation [Nakajima and Takayama \[2017\]](#) of Coulomb branches of the cyclic quiver gauge theories and their identification with the Nakajima cyclic quiver varieties.

Note that  $\pi_0(\mathcal{R}_{\mathbb{G}, \mathbf{N}}) = \pi_0(\text{Gr}_{\text{GL}(V)}) = \pi_1(\text{GL}(V)) = \mathbb{Z}^{\tilde{I}}$ , so that  $H_{\bullet}^{\mathbb{G}^{\circ}}(\mathcal{R}_{\mathbb{G}, \mathbf{N}}) = \mathbb{C}[\mathcal{M}_C(\mathbb{G}, \mathbf{N})] = \mathbb{C}[\overline{\mathcal{W}}_{\mu}^{\lambda}]$  is  $\mathbb{Z}^{\tilde{I}}$ -graded. We identify  $\mathbb{Z}^{\tilde{I}}$  with the root lattice of  $T_{\text{aff}} \subset G_{\text{aff}}$ :  $\mathbb{Z}^{\tilde{I}} = \mathbb{Z}\langle \alpha_i^{\vee} \rangle_{i \in \tilde{I}}$ . Then the  $\mathbb{Z}^{\tilde{I}}$ -grading on  $\mathbb{C}[\overline{\mathcal{W}}_{\mu}^{\lambda}]$  corresponds to a  $T_{\text{aff}}$ -action on  $\overline{\mathcal{W}}_{\mu}^{\lambda}$ . Composing with the cocharacter  $2\rho: \mathbb{C}^{\times} \rightarrow T_{\text{aff}}$ , we obtain an action of  $\mathbb{C}^{\times}$  on  $\overline{\mathcal{W}}_{\mu}^{\lambda}$ . Conjecturally, the fixed point set  $(\overline{\mathcal{W}}_{\mu}^{\lambda})^{\mathbb{C}^{\times}}$  is nonempty iff the  $V_{\mu}^{\lambda} \neq 0$ , and in this case the fixed point set consists of a single point denoted by  $\mu$ . We consider the corresponding repellent  $R_{\mu}^{\lambda} \subset \overline{\mathcal{W}}_{\mu}^{\lambda}$  and the hyperbolic stalk  $\Phi_{\mu}^{\lambda} \text{IC}(\overline{\mathcal{W}}_{\mu}^{\lambda})$ .

Similarly to [Section 5.4](#), in case  $G$  is an adjoint simple non simply laced group, the Dynkin diagram of its affinization can be obtained by folding of a Dynkin graph of type  $A^{(1)}, D^{(1)}, E^{(1)}$ , and the above slices for  $G$  are defined as the fixed point sets of the corresponding slices for the unfolding of  $G$ . The repellents and the hyperbolic stalks are thus defined for arbitrary simple  $G$  too, and we expect the conclusions of [Sections 3.2, 3.3, 3.4](#) to hold in the affine case as well.

**5.6 Warning.** In order to formulate the statements about multiplicities for fusion and convolution as in 3.3 and 3.4, we must have closed embeddings of slices  $\overline{\mathcal{W}}_{\mu}^{\lambda'} \hookrightarrow \overline{\mathcal{W}}_{\mu}^{\lambda}$  for  $\lambda' \leq \lambda \in \Lambda_{\text{aff}}^+$ . Certainly we do have the natural closed embeddings of generalized slices in  $\text{Gr}_G$ :  $\overline{\mathcal{W}}_{\mu}^{\bar{\lambda}'} \hookrightarrow \overline{\mathcal{W}}_{\mu}^{\bar{\lambda}}$ ,  $\bar{\lambda}' \leq \bar{\lambda} \in \Lambda^+$ , but these embeddings have no manifest interpretation in terms of Coulomb branches (see Section 6.4 below for a partial advance, though). For a slice in  $\text{Gr}_G$ , the collection of closures of symplectic leaves in  $\overline{\mathcal{W}}_{\mu}^{\bar{\lambda}}$  coincides with the collection of smaller slices  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}'} \subset \overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$ ,  $\bar{\mu} \leq \bar{\lambda}' \leq \bar{\lambda}$ ,  $\bar{\lambda}' \in \Lambda^+$ . However, in the affine case, in general there are *more* symplectic leaves in  $\overline{\mathcal{W}}_{\mu}^{\lambda}$  than the cardinality of  $\{\lambda' \in \Lambda_{\text{aff}}^+ : \mu \leq \lambda' \leq \lambda\}$ . For example, if  $k = 1$ , and  $\mu = 0$ , so that  $\overline{\mathcal{W}}_{\mu}^{\lambda} \simeq \mathcal{U}_G^d(\mathbb{A}^2)$ , the symplectic leaves are numbered by the partitions of size  $\leq d$ : they are all of the form  $\mathcal{S} \times \text{Bun}_G^{d'}(\mathbb{A}^2)$  where  $0 \leq d' \leq d$ , and  $\mathcal{S}$  is a stratum of the diagonal stratification of  $\text{Sym}^{d-d'} \mathbb{A}^2$ .

Thus we expect that the slice  $\overline{\mathcal{W}}_{\mu}^{\lambda'}$  for  $\lambda' \in \Lambda_{\text{aff}}^+$ ,  $\mu \leq \lambda' \leq \lambda$ , is isomorphic to the closure of a symplectic leaf in  $\overline{\mathcal{W}}_{\mu}^{\lambda}$ . We also do expect the multiplicity of  $\text{IC}(\overline{\mathcal{W}}_{\mu}^{\lambda'})$  in  $\Psi\text{IC}(\overline{\mathcal{W}}_{\mu}^{\lambda}) = \varpi_* \text{IC}(\widetilde{\mathcal{W}}_{\mu}^{\lambda})$  to be  $M_{\lambda'}^{\lambda} = \text{Hom}_{G_{\text{aff}}^{\vee}}(V^{\lambda'}, V^{\lambda_1} \otimes \dots \otimes V^{\lambda_N})$  for any  $\lambda' \in \Lambda_{\text{aff}}^+$  such that  $\mu \leq \lambda' \leq \lambda$ . However, it is possible that the IC sheaves of *other* symplectic leaves' closures also enter  $\Psi\text{IC}(\widetilde{\mathcal{W}}_{\mu}^{\lambda}) = \varpi_* \text{IC}(\widetilde{\mathcal{W}}_{\mu}^{\lambda})$  with nonzero multiplicities. We should understand the representation-theoretic meaning of these extra multiplicities, cf. Nakajima [2009, Theorem 5.15 and Remark 5.17(3)] for  $G$  of type  $A$ .

Also, the closed embeddings of slices (for Levi subgroups of  $G_{\text{aff}}$ ) seem an indispensable tool for constructing a  $\mathfrak{g}_{\text{aff}}^{\vee}$ -action on  $\bigoplus_{\mu} \Phi_{\mu}^{\lambda} \text{IC}(\overline{\mathcal{W}}_{\mu}^{\lambda})$  or a structure of  $\mathfrak{g}_{\text{aff}}^{\vee}$ -crystal on  $\bigsqcup_{\mu} \text{Irr}(R_{\mu}^{\lambda})$  (via reduction to Levi subgroups), cf. Krylov [2017].

**5.7 Further problems.** Note that the construction of Section 5.5 uses no specific properties of the affine Dynkin graphs, and works in the generality of arbitrary graph  $Q$  without edge loops and the corresponding Kac-Moody Lie algebra  $\mathfrak{g}_Q$ . We still expect the conclusions of Sections 3.2, 3.3, 3.4 to hold in this generality, see Braverman, Finkelberg, and Nakajima [2016a, 3(x)].

The only specific feature of the affine case is as follows. Recall that the category  $\text{Rep}_k(G_{\text{aff}}^{\vee})$  of integrable  $G_{\text{aff}}^{\vee}$ -modules at level  $k \in \mathbb{Z}_{>0}$  is equipped with a braided balanced tensor *fusion* structure Moore and Seiberg [1989] and Bakalov and Kirillov [2001]. Unfortunately, I have no clue how this structure is reflected in the geometry of  $\text{Gr}_{G_{\text{aff}}}$ . I believe this is one of the most pressing problems about  $\text{Gr}_{G_{\text{aff}}}$ .

## 6 Applications

**6.1 Hikita conjecture.** We already mentioned in [Section 5.5](#) that in case  $V_\mu^\lambda \neq 0$  we expect the fixed point set  $(\overline{\mathcal{W}}_\mu^\lambda)^{T_{\text{aff}}}$  to consist of a single point  $\mu$ . This point is the support of a nilpotent scheme  $\mu$  defined as follows: we choose a  $T_{\text{aff}}$ -equivariant embedding  $\overline{\mathcal{W}}_\mu^\lambda \hookrightarrow \mathbb{A}^N$  into a representation of  $T_{\text{aff}}$ , and define  $\mu$  as the scheme-theoretic intersection of  $\overline{\mathcal{W}}_\mu^\lambda$  with the zero weight subspace  $\mathbb{A}_0^N$  inside  $\mathbb{A}^N$ . The resulting subscheme  $\mu \subset \overline{\mathcal{W}}_\mu^\lambda$  is independent of the choice of a  $T_{\text{aff}}$ -equivariant embedding  $\overline{\mathcal{W}}_\mu^\lambda \hookrightarrow \mathbb{A}^N$ . According to the Hikita conjecture [Hikita \[2017\]](#), the ring  $\mathbb{C}[\mu]$  is expected to be isomorphic to the cohomology ring  $H^\bullet(\mathfrak{M}(V, W))$  of the corresponding Nakajima affine quiver variety, see [Section 5.3](#). This is an instance of *symplectic duality* (3d mirror symmetry) between Coulomb and Higgs branches. The Hikita conjecture for the slices  $\overline{\mathcal{W}}_\mu^\lambda$  in  $\text{Gr}_G$  and the corresponding finite type Nakajima quiver varieties is proved in [Kannitzer, Tingley, Webster, Weekes, and Yacobi \[2015\]](#) for types  $A, D$  (and conditionally for types  $E$ ).

**6.2 Monopole formula.** We return to the setup of [Section 5.1](#). Recall that  $\mathcal{R}_{\mathbb{G}, \mathbb{N}}$  is a union of (profinite dimensional) vector bundles over  $\mathbb{G}_\Theta$ -orbits in  $\text{Gr}_G$ . The corresponding Cousin spectral sequence converging to  $H_\bullet^{\mathbb{G}_\Theta}(\mathcal{R}_{\mathbb{G}, \mathbb{N}})$  degenerates and allows to compute the equivariant Poincaré polynomial (or rather Hilbert series)

$$(1) \quad P_t^{\mathbb{G}_\Theta}(\mathcal{R}_{\mathbb{G}, \mathbb{N}}) = \sum_{\theta \in \Lambda_{\mathbb{G}}^+} t^{d_\theta - 2\langle \rho_{\mathbb{G}}, \theta \rangle} P_{\mathbb{G}}(t; \theta).$$

Here  $\deg(t) = 2$ ,  $P_{\mathbb{G}}(t; \theta) = \prod_i (1 - t^{d_i})^{-1}$  is the Hilbert series of the equivariant cohomology  $H_\bullet^{\text{Stab}_{\mathbb{G}}(\theta)}(\text{pt})$  ( $d_i$  are the degrees of generators of the ring of  $\text{Stab}_{\mathbb{G}}(\theta)$ -invariant functions on its Lie algebra), and  $d_\theta = \sum_{\chi \in \Lambda_{\mathbb{G}}^\vee} \max(-\langle \chi, \theta \rangle, 0) \dim \mathbf{N}_\chi$ . This is a slight variation of the *monopole formula* of [Cremonesi, Hanany, and Zaffaroni \[2014\]](#). Note that the series (1) may well diverge (even as a formal Laurent series: the space of homology of given degree may be infinite-dimensional), e.g. this is always the case for unframed quiver gauge theories. To ensure its convergence (as a formal Taylor series with the constant term 1) one has to impose the so called ‘good’ or ‘ugly’ assumption on the theory. In this case the resulting  $\mathbb{N}$ -grading on  $H_\bullet^{\mathbb{G}_\Theta}(\mathcal{R}_{\mathbb{G}, \mathbb{N}})$  gives rise to a  $\mathbb{C}^\times$ -action on  $\mathfrak{M}_{\mathbb{C}}(\mathbb{G}, \mathbb{N})$ , making it a conical variety with a single (attracting) fixed point.

Now recall the setup of [Sections 5.3, 5.4](#); in particular, the isomorphism  $\overline{\mathcal{W}}_\mu^\lambda \xrightarrow{\sim} \mathfrak{M}_{\mathbb{C}}(\mathbb{G}, \mathbb{N})$ . In case  $\bar{\mu}$  is dominant, the slice  $\overline{\mathcal{W}}_\mu^\lambda \subset \text{Gr}_G$  is conical with respect to the loop rotation  $\mathbb{C}^\times$ -action. However, this action is *not* the one of the previous paragraph. They differ by a hamiltonian  $\mathbb{C}^\times$ -action (preserving the Poisson structure). The Hilbert series of  $\overline{\mathcal{W}}_\mu^\lambda$  graded by the loop rotation  $\mathbb{C}^\times$ -action is given by

$$(2) \quad P_t(\mathbb{C}[\overline{\mathcal{W}}_\mu^\lambda]) = \sum_{\theta \in \Lambda_{\mathbb{G}}^+} t^{d_\theta - 2\langle \rho_{\mathbb{G}}, \theta \rangle - \frac{1}{2} \bar{\theta}^\dagger \cdot \det \mathbf{N}_{\text{hor}} + \frac{1}{2} \bar{\theta}^\dagger \cdot C \cdot \alpha} P_{\mathbb{G}}(t; \theta).$$

Here  $\deg(t) = 1$ ;  $\alpha = \bar{\lambda} - \bar{\mu} \in \Lambda_+ = \mathbb{N}^I$ ;  $\bar{\theta}$  is the class of  $\theta \in \Lambda_G = \Lambda_{\mathrm{GL}(V)}$  in  $\pi_0 \mathrm{Gr}_{\mathrm{GL}(V)} = \mathbb{Z}^I$ ;  $\bar{\theta}^\dagger$  is the transposed row-vector;  $C$  is the  $I \times I$  Cartan matrix of  $G$ ; and  $\mathbf{N}_{\mathrm{hor}} = \bigoplus_{i \rightarrow j \in \Omega} \mathrm{Hom}(V_i, V_j)$  is the “horizontal” summand of  $\mathrm{GL}(V)$ -module  $\mathbf{N}$ , so that  $\det \mathbf{N}_{\mathrm{hor}}$  is a character of  $\mathrm{GL}(V)$ , i.e. an element of  $\mathbb{Z}^I$ .

Finally, we consider a double affine Grassmannian slice  $\overline{\mathcal{W}}_\mu^\lambda$  with dominant  $\mu$  as in [Section 4.3](#). The analogue of the loop rotation action of the previous paragraph is the action of the second copy of  $\mathbb{C}^\times$  (the center of  $\mathrm{GL}(2)$ ) in [Section 4.4](#). We expect that the Hilbert series of  $\overline{\mathcal{W}}_\mu^\lambda$  graded by this  $\mathbb{C}^\times$ -action is given by the evident affine analogue of the formula (2) (with the  $\tilde{I} \times \tilde{I}$  Cartan matrix  $C_{\mathrm{aff}}$  of  $G_{\mathrm{aff}}$  replacing  $C$ ). In particular, in case of level 1, this gives a formula for the Hilbert series of the coordinate ring  $\mathbb{C}[\mathcal{U}_G^d(\mathbb{A}^2)]$  of the Uhlenbeck space proposed in [Cremonesi, Mekareeya, Hanany, and Ferlito \[2014\]](#). Note that the latter formula works for arbitrary  $G$ , not necessarily simply laced one. In type  $A$  it follows from the results of [Nakajima and Takayama \[2017\]](#).

**6.3 Zastava.** Let us consider the Coulomb branch  $\mathfrak{M}_C(\mathcal{G}, \mathbf{N})$  of an *unframed* quiver gauge theory for an  $ADE$  type quiver:  $W_i = 0 \ \forall i \in I$ , so that  $\mathbf{N} = \mathbf{N}_{\mathrm{hor}}$ . An isomorphism  $\mathfrak{M}_C(\mathcal{G}, \mathbf{N}) \xrightarrow{\sim} \check{Z}^\alpha$  with the open zastava<sup>1</sup> (the moduli space of degree  $\alpha$  based maps from the projective line  $\mathbb{P}^1 \ni \infty$  to the flag variety  $\mathcal{B} \ni B_-$  of  $G$ , where  $\alpha = \sum_{i \in I} (\dim V_i) \alpha_i$ ), is constructed in [Braverman, Finkelberg, and Nakajima \[2016a\]](#) (we learned of its existence from V. Pestun). As the name suggests, the open zastava is a (dense smooth symplectic) open subvariety in the zastava space  $Z^\alpha$ , a normal Cohen-Macaulay affine Poisson variety.

Note that there is another version of zastava  $\mathbf{Z}^\alpha$  that is the solution of a moduli problem ( $G$ -bundles on  $\mathbb{P}^1$  with a generalized  $B$ -structure and an extra  $U_-$ -structure transversal at  $\infty \in \mathbb{P}^1$ ) [Braverman, Finkelberg, Gaitsgory, and Mirković \[2002\]](#) given by a scheme cut out by the Plücker equations. This scheme is not reduced in general (the first example occurs in type  $A_4$ ) [E. Feigin and Makedonskyi \[2017\]](#), and  $Z^\alpha$  is the corresponding variety:  $Z^\alpha := \mathbf{Z}_{\mathrm{red}}^\alpha$ .

We already mentioned that the open zastava is a particular case of a generalized slice:  $\check{Z}^\alpha = \overline{\mathcal{W}}_{w_0\alpha}^0$ . The zastava space  $Z^\alpha$  is the limit of slices in the following sense: for any  $\bar{\lambda} \geq \bar{\mu}$  such that  $w_0\bar{\mu} - w_0\bar{\lambda} = \alpha$ , there is a loop rotation equivariant regular birational morphism  $s_{\bar{\mu}}^{\bar{\lambda}}: \overline{\mathcal{W}}_{-w_0\bar{\mu}}^{-w_0\bar{\lambda}} \rightarrow Z^\alpha$ , and for any  $N \in \mathbb{N}$  and big enough dominant  $\bar{\mu}$ , the corresponding morphism of the coordinate rings graded by the loop rotations  $(s_{\bar{\mu}}^{\bar{\lambda}})^*: \mathbb{C}[Z^\alpha] \rightarrow \mathbb{C}[\overline{\mathcal{W}}_{-w_0\bar{\mu}}^{-w_0\bar{\lambda}}]$  is an isomorphism in degrees  $\leq N$  (both  $\mathbb{C}[Z^\alpha]$  and  $\mathbb{C}[\overline{\mathcal{W}}_{-w_0\bar{\mu}}^{-w_0\bar{\lambda}}]$  for dominant  $\bar{\mu}$  are positively graded).

<sup>1</sup>Zastava = *flags* in Croatian.

Now  $\mathbb{C}[Z^\alpha]$  is obtained by the following version of the Coulomb branch construction. Given a vector space  $U$  we define the positive part of the affine Grassmannian  $\mathrm{Gr}_{\mathrm{GL}(U)}^+ \subset \mathrm{Gr}_{\mathrm{GL}(U)}$  as the moduli space of vector bundles  $\mathcal{U}$  on the formal disc  $D = \mathrm{Spec}(\mathcal{O})$  equipped with trivialization  $\sigma: \mathcal{U}|_{D^*} \xrightarrow{\sim} U \otimes \mathcal{O}_{D^*}$  on the formal punctured disc  $D^* = \mathrm{Spec}(\mathbb{K})$  such that  $\sigma$  extends through the puncture as an embedding  $\sigma: \mathcal{U} \hookrightarrow U \otimes \mathcal{O}_D$ . Since  $\mathbb{G} = \mathrm{GL}(V) = \prod_{i \in I} \mathrm{GL}(V_i)$ , we have  $\mathrm{Gr}_{\mathrm{GL}(V)} = \prod_{i \in I} \mathrm{Gr}_{\mathrm{GL}(V_i)}$ , and we define  $\mathrm{Gr}_{\mathrm{GL}(V)}^+ = \prod_{i \in I} \mathrm{Gr}_{\mathrm{GL}(V_i)}^+$ . Finally, we define  $\mathcal{R}_{\mathbb{G}, \mathbf{N}}^+$  as the preimage of  $\mathrm{Gr}_{\mathrm{GL}(V)}^+ \subset \mathrm{Gr}_{\mathrm{GL}(V)}$  under  $\mathcal{R}_{\mathbb{G}, \mathbf{N}} \rightarrow \mathrm{Gr}_{\mathrm{GL}(V)}$ . Then  $H_\bullet^{\mathbb{G}^\circ}(\mathcal{R}_{\mathbb{G}, \mathbf{N}}^+)$  forms a convolution subalgebra of  $H_\bullet^{\mathbb{G}^\circ}(\mathcal{R}_{\mathbb{G}, \mathbf{N}})$ , and an isomorphism  $\mathfrak{M}_\mathbb{C}^+(\mathbb{G}, \mathbf{N}) := \mathrm{Spec} H_\bullet^{\mathbb{G}^\circ}(\mathcal{R}_{\mathbb{G}, \mathbf{N}}^+) \xrightarrow{\sim} Z^\alpha$  is constructed in [Braverman, Finkelberg, and Nakajima \[2016a\]](#).

An analogue of the monopole formula (2) gives the character of the  $T \times \mathbb{C}^\times$ -module  $\mathbb{C}[Z^\alpha]$ :

$$(3) \quad \chi(\mathbb{C}[Z^\alpha]) = \sum_{\Lambda_\mathbb{G}^{++}} z^{\bar{\theta}} t^{d_\theta - 2(\rho_\mathbb{G}, \theta) - \frac{1}{2}\bar{\theta}^\vee \cdot \det \mathbf{N} + \frac{1}{2}\bar{\theta}^\vee \cdot \mathbf{C} \cdot \alpha} P_\mathbb{G}(t; \theta).$$

Here  $\Lambda_\mathbb{G}^{++}$  is the set of  $I$ -tuples of partitions;  $i$ -th partition having length at most  $\dim V_i$  (recall that the cone of dominant coweights  $\Lambda_\mathbb{G}^+$  is formed by the  $I$ -tuples of nonincreasing sequences  $(\lambda_1^{(i)} \geq \lambda_2^{(i)} \geq \dots \geq \lambda_{\dim V_i}^{(i)})$  of integers, and for  $\Lambda_\mathbb{G}^{++} \subset \Lambda_\mathbb{G}^+$  we require these integers to be nonnegative). Also,  $z$  denotes the coordinates on the Cartan torus  $T \subset G$  identified with  $(\mathbb{C}^\times)^I$  via  $z_i = \alpha_i^\vee$ .

The character of the  $T \times \mathbb{C}^\times$ -module  $\mathbb{C}[Z^\alpha]$  for  $G$  of type  $ADE$  was also computed in [Braverman and Finkelberg \[2014\]](#). Namely, it is given by the *fermionic formula* of [B. Feigin, E. Feigin, Jimbo, Miwa, and Mukhin \[2009\]](#), and the generating function of these characters for all  $\alpha \in \Lambda_+$  is an eigenfunction of the  $q$ -difference Toda integrable system. It would be interesting to find a combinatorial relation between the monopole and fermionic formulas.

In the affine case, the zastava space  $Z_{\mathfrak{gaff}}^\alpha$  was introduced in [Braverman, Finkelberg, and Gaitsgory \[2006\]](#). It is an irreducible affine algebraic variety containing a (dense smooth symplectic) open subvariety  $\check{Z}_{\mathfrak{gaff}}^\alpha$ : the moduli space of degree  $\alpha$  based maps from the projective line  $\mathbb{P}^1$  to the Kashiwara flag scheme  $\mathbf{Fl}_{\mathfrak{gaff}}$ . Contrary to the finite case, the open subvariety  $\check{Z}_{\mathfrak{gaff}}^\alpha$  is not affine, but only quasiaffine, and we denote by  $\underline{\check{Z}}_{\mathfrak{gaff}}^\alpha$  its affine closure. We do not know if the open embedding  $\check{Z}_{\mathfrak{gaff}}^\alpha \hookrightarrow Z_{\mathfrak{gaff}}^\alpha$  extends to an open embedding  $\underline{\check{Z}}_{\mathfrak{gaff}}^\alpha \hookrightarrow Z_{\mathfrak{gaff}}^\alpha$ : it depends on the normality property of  $Z_{\mathfrak{gaff}}^\alpha$  that is established only for  $\mathfrak{g}$  of types  $A, C$  [Braverman and Finkelberg \[2014\]](#) and [Finkelberg and Rybnikov \[2014\]](#) at the moment (but is expected for all types). For an  $A^{(1)}, D^{(1)}, E^{(1)}$  type quiver and an unframed quiver gauge theory with  $\alpha = \sum_{i \in \bar{I}} (\dim V_i) \alpha_i$ , we have an isomorphism  $\mathfrak{M}_\mathbb{C}(\mathbb{G}, \mathbf{N}) \xrightarrow{\sim} \underline{\check{Z}}_{\mathfrak{gaff}}^\alpha$  [Braverman, Finkelberg, and Nakajima \[2016a\]](#). If  $Z_{\mathfrak{gaff}}^\alpha$  is



normal, this isomorphism extends to  $\mathfrak{M}_C^+(\mathbb{G}, \mathbf{N}) \xrightarrow{\sim} Z_{\text{gaff}}^\alpha$ , and the fermionic formula for the character  $\chi(\mathbb{C}[Z^\alpha])$  holds true.

Finally, for an arbitrary quiver  $Q$  without edge loops we can consider an unframed quiver gauge theory, and a coroot  $\alpha := \sum_{i \in Q_0} (\dim V_i) \alpha_i$  of the corresponding Kac-Moody Lie algebra  $\mathfrak{g}_Q$ . The moduli space  $\check{Z}_{\mathfrak{g}_Q}^\alpha$  of based maps from  $\mathbb{P}^1$  to the Kashiwara flag scheme  $\mathbf{Fl}_{\mathfrak{g}_Q}$  was studied in [Braverman, Finkelberg, and Gaitsgory \[2006\]](#). It is a smooth connected variety. We expect that it is quasiasfine, and its affine closure  $\check{Z}_{\mathfrak{g}_Q}^{\circ\alpha}$  is isomorphic to the Coulomb branch  $\mathfrak{M}_C(\mathbb{G}, \mathbf{N})$ . It would be interesting to find a modular interpretation of  $\mathfrak{M}_C^+(\mathbb{G}, \mathbf{N})$  and its stratification into symplectic leaves. In the affine case such an interpretation involves Uhlenbeck spaces  $\mathcal{U}_G^d(\mathbb{A}^2)$ .

The Jordan quiver corresponds to the Heisenberg Lie algebra. The computations of [Finkelberg, Ginzburg, Ionov, and Kuznetsov \[2016\]](#) suggest that the Uhlenbeck compactification of the Calogero-Moser phase space plays the role of zastava for the Heisenberg Lie algebra.

**6.4 Multiplication and quantization.** The multiplication of slices in the affine Grassmannian  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}} \times \overline{\mathcal{W}}_{\bar{\mu}'}^{\bar{\lambda}'} \rightarrow \overline{\mathcal{W}}_{\bar{\mu}+\bar{\mu}'}^{\bar{\lambda}+\bar{\lambda}'}$  was constructed in [Braverman, Finkelberg, and Nakajima \[2016a\]](#) via multiplication of scattering matrices for singular monopoles (we learned of its existence from T. Dimofte, D. Gaiotto and J. Kamnitzer). The corresponding comultiplication  $\mathbb{C}[\overline{\mathcal{W}}_{\bar{\mu}+\bar{\mu}'}^{\bar{\lambda}+\bar{\lambda}'}] \rightarrow \mathbb{C}[\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}] \otimes \mathbb{C}[\overline{\mathcal{W}}_{\bar{\mu}'}^{\bar{\lambda}'}]$  can not be seen directly from the Coulomb branch construction of slices. However, its quantization  $\mathbb{C}_{\hbar}[\overline{\mathcal{W}}_{\bar{\mu}+\bar{\mu}'}^{\bar{\lambda}+\bar{\lambda}'}] \rightarrow \mathbb{C}_{\hbar}[\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}] \otimes \mathbb{C}_{\hbar}[\overline{\mathcal{W}}_{\bar{\mu}'}^{\bar{\lambda}'}]$  (recall from the end of [Section 5.1](#) that  $\mathbb{C}_{\hbar}[\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}]$  is the loop rotation equivariant Borel-Moore homology of the corresponding variety of triples) already can be realized in terms of Coulomb branches. The reason for this is that the quantized Coulomb branch  $\mathbb{C}_{\hbar}[\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}]$  is likely to have a presentation by generators and relations of a *truncated shifted Yangian*  $Y_{\bar{\mu}}^{\bar{\lambda}} \simeq \mathbb{C}_{\hbar}[\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}]$  of [Braverman, Finkelberg, and Nakajima \[ibid., Appendix B\]](#). Also, it seems likely that the comultiplication of [Finkelberg, Kamnitzer, Pham, Rybnikov, and Weekes \[2018\]](#) descends to a homomorphism  $\Delta: Y_{\bar{\mu}+\bar{\mu}'}^{\bar{\lambda}+\bar{\lambda}'} \rightarrow Y_{\bar{\mu}}^{\bar{\lambda}} \otimes Y_{\bar{\mu}'}^{\bar{\lambda}'}$ . Finally, we expect that the desired comultiplication  $\mathbb{C}[\overline{\mathcal{W}}_{\bar{\mu}+\bar{\mu}'}^{\bar{\lambda}+\bar{\lambda}'}] \rightarrow \mathbb{C}[\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}] \otimes \mathbb{C}[\overline{\mathcal{W}}_{\bar{\mu}'}^{\bar{\lambda}'}]$  is obtained by setting  $\hbar = 0$  in  $\Delta$ .

Returning to the question of constructing the closed embeddings of slices  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}'} \hookrightarrow \overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$  in terms of Coulomb branches (see the beginning of [Section 5.6](#)), we choose dominant coweights  $\bar{v}, \bar{v}', \bar{\mu}'$  such that  $\bar{v}' + \bar{v} = \bar{\lambda}'$ ,  $\bar{\mu}' + \bar{v} = \bar{\lambda}$ , and set  $\bar{\mu}'' := \bar{\mu} - \bar{\mu}'$ . Then we have the multiplication morphism  $\overline{\mathcal{W}}_{\bar{\mu}''}^{\bar{v}} \times \overline{\mathcal{W}}_{\bar{\mu}'}^{\bar{v}'} \rightarrow \overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$ , and we restrict it to  $\overline{\mathcal{W}}_{\bar{\mu}''}^{\bar{v}} = \overline{\mathcal{W}}_{\bar{\mu}''}^{\bar{v}} \times \{\bar{\mu}'\} \rightarrow \overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$  where  $\bar{\mu}' \in \overline{\mathcal{W}}_{\bar{\mu}'}^{\bar{v}'}$  is the fixed point. Then the desired closed subvariety  $\overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}'} \subset \overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$  is nothing but the closure of the image of  $\overline{\mathcal{W}}_{\bar{\mu}''}^{\bar{v}} \rightarrow \overline{\mathcal{W}}_{\bar{\mu}}^{\bar{\lambda}}$ .



The similar constructions are supposed to work for the slices in  $\mathrm{Gr}_{G_{\mathrm{aff}}}$ . They are based on the comultiplication for *affine Yangians* constructed in [Guay, Nakajima, and Wendlandt \[2017\]](#).

**6.5 Affinization of KLR algebras.** We recall the setup of [Section 5.3](#), and set  $W = 0$  (no framing). We choose a sequence  $\mathbf{j} = (j_1, \dots, j_N)$  of vertices such that any vertex  $j \in Q_0$  enters  $\dim V_j$  times; thus,  $N = \dim V$ . The set of all such sequences is denoted  $\mathbf{J}(V)$ . We choose a  $Q_0$ -graded flag  $V = V^0 \supset V^1 \supset \dots \supset V^N = 0$  such that  $V^{n-1}/V^n$  is a 1-dimensional vector space supported at the vertex  $j_n$  for any  $n = 1, \dots, N$ . It gives rise to the following flag of  $Q_0$ -graded lattices in  $V_{\mathcal{K}} = V \otimes \mathcal{K} : \dots \supset L^{-1} \supset L^0 \supset L^1 \supset \dots$ , where  $L^{r+N} = zL^r$  for any  $r \in \mathbb{Z}$ ;  $L^0 = V_{\mathcal{O}}$ , and  $L^n/L^N = V^n \subset V = L^0/L^N$  for any  $n = 1, \dots, N$ . Let  $\mathfrak{l}_{\mathbf{j}} \subset \mathcal{G}_{\mathcal{O}}$  be the stabilizer of the flag  $L^\bullet$  (an Iwahori subgroup). Then the  $\mathfrak{l}_{\mathbf{j}}$ -module  $\mathbf{N}_{\mathcal{O}}$  contains a submodule  $\mathbf{N}_{\mathbf{j}}$  formed by the  $\mathcal{K}$ -linear homomorphisms  $b_e : V_{t(e), \mathcal{K}} \rightarrow V_{h(e), \mathcal{K}}$  such that for any  $e \in Q_1$  and  $r \in \mathbb{Z}$ ,  $b_e$  takes  $L_{t(e)}^r$  to  $L_{h(e)}^{r+1}$ .

We consider the following version of the variety of triples:  $\mathcal{R}_{\mathbf{j}, \mathbf{j}} := \{[g, s] \in \mathcal{G}_{\mathcal{K}} \times^{\mathfrak{l}_{\mathbf{j}}} \mathbf{N}_{\mathbf{j}} : gs \in \mathbf{N}_{\mathbf{j}}\}$ , cf. [Braverman, Etingof, and Finkelberg \[2016\]](#) and [Webster \[2016, Section 4\]](#). Then the equivariant Borel-Moore homology  $\mathcal{H}_{\mathbf{j}, \mathbf{j}} := H_{\bullet}^{\mathbb{C}^\times \times \mathfrak{l}_{\mathbf{j}}}(\mathcal{R}_{\mathbf{j}, \mathbf{j}})$  forms an associative algebra with respect to a convolution operation. Moreover, if we take another sequence  $\mathbf{j}' \in \mathbf{J}(V)$  and consider  $\mathcal{R}_{\mathbf{j}', \mathbf{j}} := \{[g, s] \in \mathcal{G}_{\mathcal{K}} \times^{\mathfrak{l}_{\mathbf{j}}} \mathbf{N}_{\mathbf{j}} : gs \in \mathbf{N}_{\mathbf{j}'}\}$ , then  $\mathcal{H}_{\mathbf{j}', \mathbf{j}} := H_{\bullet}^{\mathbb{C}^\times \times \mathfrak{l}_{\mathbf{j}'}}(\mathcal{R}_{\mathbf{j}', \mathbf{j}})$  forms a  $\mathcal{H}_{\mathbf{j}', \mathbf{j}'} - \mathcal{H}_{\mathbf{j}, \mathbf{j}}$ -bimodule with respect to convolution, and we have convolutions  $\mathcal{H}_{\mathbf{j}'', \mathbf{j}'} \otimes \mathcal{H}_{\mathbf{j}', \mathbf{j}} \rightarrow \mathcal{H}_{\mathbf{j}'', \mathbf{j}}$ . In other words,  $\mathcal{H}_V := \bigoplus_{\mathbf{j}, \mathbf{j}' \in \mathbf{J}(V)} \mathcal{H}_{\mathbf{j}', \mathbf{j}}$  forms a convolution algebra.

Furthermore, given  $\mathbf{j}_1 \in \mathbf{J}(V)$ ,  $\mathbf{j}_2 \in \mathbf{J}(V')$ , the concatenated sequence  $\mathbf{j}_1 \mathbf{j}_2$  lies in  $\mathbf{J}(V \oplus V')$ , and one can define the morphisms  $\mathcal{H}_{\mathbf{j}_1', \mathbf{j}_1} \otimes \mathcal{H}_{\mathbf{j}_2', \mathbf{j}_2} \rightarrow \mathcal{H}_{\mathbf{j}_1' \mathbf{j}_2', \mathbf{j}_1 \mathbf{j}_2}$  summing up to a homomorphism  $\mathcal{H}_V \otimes \mathcal{H}_{V'} \rightarrow \mathcal{H}_{V \oplus V'}$ .

Similarly to the classical theory of Khovanov-Lauda-Rouquier algebras (see a beautiful survey [Rouquier \[2012\]](#) and references therein), we expect that in case  $Q$  has no loop edges, the categories of finitely generated graded projective  $\mathcal{H}_V$ -modules provide a categorification of the positive part  $U_Q^{++}$  of the quantum toroidal algebra  $U_Q$  (where  $U_Q^{++}$  is defined as the subalgebra generated by the positive modes of the positive generators  $e_{j,r} : j \in Q_0, r \geq 0$ ).

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