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Ergodic Properties of Tame Dynamical Systems

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Abstract—The problem of the $*$-weak decomposability into ergodic components of a topological $\mathbb{N}_0$-dynamical system (Ω, φ), where φ is a continuous endomorphism of a compact metric space Ω, is considered in terms of the associated enveloping semigroups. It is shown that, in the tame case (where the Ellis semigroup $E(\Omega, \varphi)$ consists of endomorphisms of Ω of the first Baire class), such a decomposition exists for an appropriately chosen generalized sequential averaging method. A relationship between the statistical properties of (Ω, φ) and the mutual structure of minimal sets and ergodic measures is discussed.

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1. INTRODUCTION

The object of our attention is topological $\mathbb{N}_0$-dynamical systems, that is, semicascades (Ω, φ) generated by a continuous endomorphism φ of a compact metric space Ω. The purpose of this paper is to develop a unified view of the three aspects of the theory of such systems:

(1) the $*$-weak convergence of various ergodic means (averages over orbits of the system) for scalar test functions $x \in X \doteq C(\Omega)$ or Borel measures $\mu \in X^*$; as applied to Cesàro means, this approach goes back to works of Kryloff and Bogoliouboff [1] and of Oxtoby [2];

(2) the relationship between minimal sets and ergodic measures;

(3) the decomposability of a dynamical system (Ω, φ) into irreducible (ergodic) subsystems, depending on the choice of an averaging method.

The main results are obtained for the class of tame systems, which were introduced (under a different name) by Köhler in [3] and studied in detail in [4]–[8]. There are several equivalent definitions of a tame dynamical system; e.g., a system (Ω, φ) is said to be *tame* if its Ellis semigroup consists of endomorphisms of Ω belonging to the first Baire class. The class of tame systems is denoted by $\mathscr{D}_{\mathrm{tm}}$. The interest in such objects is due to the relatively simple topology of their enveloping semigroups against the background of the often involved phase dynamics. A number of results concerning the convergence of generalized ergodic means for $(\Omega, \varphi) \in \mathscr{D}_{\mathrm{tm}}$ were obtained in [9] and [10]; in the latter paper, the more general case of the action on X of arbitrary amenable operator semigroups was considered. There are grounds for believing that there is a connection between the tame–untame dichotomy and the absence or presence of chaotic phase dynamics; in any case, any untame semicascade on $[0, 1]$ turns out to be chaotic in the Lie–Yorke sense [3].

We discuss the following properties of $*$-weakly ergodic (see Sec. 2.1) operator nets and sequences $\mathscr{V} \subset \mathscr{L}(X^*)$, which we identify with the corresponding averaging methods:

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(a) the convergence of *all* nets (sequences) $\mathscr{V}$; the convergence of *some* ergodic sequences $\mathscr{V}$;

(b) the possibility of a statistical description of the behavior of orbits of (Ω, φ) by using ergodic measures.

We consider properties (a) of a semicascade (Ω, φ) in relation to the following dynamical characteristics:

(i) the orbital subsystems are uniquely ergodic;

(ii) the supports of ergodic measures are minimal;

(iii) the minimal subsystems are uniquely ergodic.

In Sec. 3, we systematize and strengthen related results of the recent papers [9]–[11]. In particular, we prove Theorem 3.3, which asserts that, for $(\Omega, \varphi) \in \mathscr{D}_{\text{tm}}$, each ergodic sequence $\mathscr{V}$ contains a convergent subsequence; in particular, there exists a convergent subsequence of Cesàro means.

The main results of this paper are presented in Sec. 4. We prove (see Theorem 4.5) that a tame topological dynamical system (Ω, φ) admits various decompositions into ergodic components, depending on the sequential averaging method, and describe all such decompositions in terms of a certain operator semigroup $\mathscr{K}_c \subseteq \mathscr{L}(X^*)$ related to (Ω, φ). The decomposability $(\Omega, \varphi) \in \mathscr{D}_{\text{tm}}$ into ergodic component means the existence of an ergodic sequence $\mathscr{V}$ such that the asymptotic $\mathscr{V}$-distributions of all orbits are determined by ergodic measures. Thus, tame semicascades have property (b).

Section 5 contains a brief review of typical examples of tame and untame $\mathbb{N}_0$-systems. In particular, we give a criterion obtained recently in [12], which effectively distinguishes between tame and untame affine semicascades on the tori $\mathbb{T}^d$, $d \geq 1$.

2. PRELIMINARIES

Thus, we consider semicascades (Ω, φ), where φ is a continuous endomorphism of a compact metric space Ω. For a fixed space Ω, we sometimes identify (Ω, φ) with φ and use terms such as "minimal endomorphism." Let $X = C(\Omega)$, and let U be the Koopman operator, which is defined by $Ux = x \circ \varphi$ for $x \in X$; then, for $V = U^*$, $V \in \mathscr{L}(X^*)$, we have

$$\|U\|_{\mathscr{L}(X)} = \|V\|_{\mathscr{L}(X^*)} = 1.$$

By $\mathscr{P}(\Omega)$ we denote the convex set of Borel probability measures on Ω, which is compact in the w^*-topology of X^*, and by X_1, the subspace of X^{**} formed by bounded functions of the first Baire class. Below we give the necessary information about ergodic means, the enveloping semigroups associated with (Ω, φ), and tame dynamical systems.

2.1. Ergodic Means

We slightly modify the classical definition of [13] for the case of a cyclic semigroup $\{V^n\}$ of shift operators, which we consider in this paper; we say that a net $\{V_\alpha\} \subseteq \operatorname{co}\{V^n,\, n \in \mathbb{N}_0\}$ in $\mathscr{L}(X^*)$ is *ergodic* if

$$(\text{Id} - V)V_\alpha \xrightarrow{\text{W}^*\text{O}} 0: \qquad (x, (\text{Id} - V)V_\alpha \mu) \to 0, \qquad x \in X, \quad \mu \in X^*. \tag{2.1}$$

Here $V_\alpha = U_\alpha^*$ for $U_\alpha \in \mathscr{L}(X)$; the net $\{U_\alpha\} \subseteq \operatorname{co}\, \{U^n,\, n \in \mathbb{N}_0\}$ is said to be ergodic as well. If $V_\alpha \xrightarrow{\text{W}^*\text{O}} Q$, $Q \in \mathscr{L}(X^*)$, then $Q^2 = Q$. Thanks to the duality

$$(Ux, \mu) = (x, V\mu), \qquad x \in X, \quad \mu \in X^*,$$

the convergence $V_\alpha \xrightarrow{\text{W}^*\text{O}} Q$ in $\mathscr{L}(X^*)$ is equivalent to the convergence $U_\alpha x \xrightarrow{w^*} Q^* x$ in X^{**}, where $x \in X$ and $Q^* \in \mathscr{L}(X^{**})$. For ergodic sequences $\{U_n\} \subset \mathscr{L}(X)$, this convergence is equivalent to the pointwise convergence $U_n x \to \overline{x} \in X_1$ of functions. Note that ergodicity is preserved under the passage to subnets and subsequences. In what follows, talking about ergodic means, we usually mean operator nets (sequences) in $\mathscr{L}(X^*)$.

Various ergodic sequences $\mathscr{V} = \{V_n\} \subset \mathscr{L}(X^*)$ can be obtained by applying methods for the summation of number sequences with infinite number matrices $S = \{s_{n,k}\}$ satisfying the following conditions:

(1) $s_{n,k} \geq 0$ and $\sum_{k=0}^{\infty} s_{n,k} = 1$, $n \geq 0$;

(2) each row of S contains finitely many $s_{n,k} > 0$;

(3) $\lim_{n\to\infty}(s_{n,0} + \sum_{k=1}^{\infty} |s_{n,k} - s_{n,k-1}|) = 0$.

The sequence of operators $V_n = \sum_{k=0}^{\infty} s_{n,k} V^k$ turns out to be ergodic, because $\|(\mathrm{Id} - V)V_n\|_{\mathscr{L}(X^*)} \to 0$ as $n \to \infty$. For example, the weights corresponding to appropriate Riesz means are

$$s_{n,k} = \frac{p_k}{p_0 + p_1 + \cdots + p_n}, \qquad 0 \leq k \leq n, \qquad s_{n,k} = 0, \qquad k > n,$$

where $p_n \geq p_{n+1} > 0$ and $\sum_{n=0}^{\infty} p_n = \infty$ ($p_n \equiv 1$ for the Cesàro means).

2.2. Enveloping Semigroups

The Ellis semigroup [14] $E(\Omega, \varphi)$ of a semicascade (Ω, φ) is the closure of the set $\{\varphi^n,\ n \in \mathbb{N}_0\}$ of transformations in the direct product topology of Ω^Ω. The Köhler semigroup $\mathscr{K}(\Omega, \varphi)$ is the closure of the set

$$\mathscr{K}^0 = \{V^n,\ n \in \mathbb{N}_0\}$$

of operators in the W*O-topology of $\mathscr{L}(X^*)$ [3]. Finally, the semigroup $\mathscr{K}_{\mathrm{c}}(\Omega, \varphi)$ was defined is [11] as the W*O-closure of the convex hull $\mathrm{co}\mathscr{K}^0$. The right-topological semigroups $E(\Omega, \varphi)$, $\mathscr{K}(\Omega, \varphi)$, and $\mathscr{K}_{\mathrm{c}}(\Omega, \varphi)$ are compact. In fact, $\mathscr{K}_{\mathrm{c}}(\Omega, \varphi)$ is the enveloping semigroup of the action $\mathscr{P} \times W \xrightarrow{V} \mathscr{P}$ on $\mathscr{P} = \mathscr{P}(\Omega)$ of the polynomial Abelian semigroup $W = \mathrm{co}\{t^n,\ n \geq 0\}$ with usual multiplication.

Below we list some useful properties of the semigroup $\mathscr{K}_{\mathrm{c}} = \mathscr{K}_{\mathrm{c}}(\Omega, \varphi)$ (see [11, Sec. 1]). The nonempty kernel (the intersection of two-sided ideals) $\mathrm{Ker}\,\mathscr{K}_{\mathrm{c}}$ of the semigroup $\mathscr{K}_{\mathrm{c}}$ consists precisely of the projections $Q \in \mathscr{K}_{\mathrm{c}}$ with unit norm satisfying the condition $VQ = Q$ or, equivalently,

$$QX^* = \mathrm{fix}(V) \doteq \{\mu \in X^* : V\mu = \mu\}.$$

An operator net $V_\alpha \in \mathrm{co}\mathscr{K}^0$ such that $V_\alpha \xrightarrow{\mathrm{W^*O}} T \in \mathscr{K}_{\mathrm{c}}$ is ergodic if and only if $T \in \mathrm{Ker}\,\mathscr{K}_{\mathrm{c}}$. Each element $Q \in \mathrm{Ker}\,\mathscr{K}_{\mathrm{c}}$ is the limit of some ergodic operator net; i.e., for each $\varphi \in C(\Omega, \Omega)$, there exist W*O-convergent ergodic nets.

Remark 2.1. According to Theorem 3.2 of [11], all ergodic nets (2.1) converge if and only if $\mathrm{Ker}\,\mathscr{K}_{\mathrm{c}}$ consists of a single element, which is necessarily the zero element of the semigroup $\mathscr{K}_{\mathrm{c}}$. In [10], a slightly different (more general and more traditional) definition of an ergodic net was used; namely, it was assumed that, in (2.1), $V_\alpha \in \overline{\mathrm{co}}\,\mathscr{K}^0 = \mathscr{K}_{\mathrm{c}}$. Nevertheless, the condition $\mathrm{card}\,\mathrm{Ker}\,\mathscr{K}_{\mathrm{c}} = 1$ implies the convergence of all such nets as well [10, Theorem 4.3].

2.3. Tame Dynamical Systems

In the language of function theory, tame $\mathbb{N}_0$-systems can be defined as follows (see [3]).

Definition 2.2. A semicascade (Ω, φ) is said to be *tame* if, for any $x \in X$ and any subsequence $\{n(k)\} \subseteq \mathbb{N}_0$,

$$\inf_a \left\| \sum_{k=0}^{\infty} a_k x_{n(k)} \right\|_X = 0,$$

where $x_{n(k)} = x \circ \varphi^{n(k)}$ and the infimum is taken over all sequences $a \in \ell^1$ with finitely many nonzero terms such that $\sum_{k=0}^{\infty} |a_k| = 1$.

In essence, this condition is related to the problem of the isomorphic embeddability of ℓ^1 in a Banach space, which goes back to Rosenthal [15]. Let Π_b and Π_1 be, respectively, the sets of Borel endomorphisms of Ω and of endomorphisms of Ω of the first Baire class. Each of the following properties is equivalent to Definition 2.2:

(a) $E(\Omega, \varphi)$ is a Fréchet–Urysohn compact space;

(b) $\operatorname{card} E(\Omega, \varphi) \leq \mathfrak{c}$;

(c) $\mathscr{K}_c(\Omega, \varphi)$ is a Fréchet–Urysohn compact space;

(d) $E(\Omega, \varphi) \subset \Pi_1$;

(e) $E(\Omega, \varphi) \subset \Pi_b$.

Properties (c) and (e) appeared in [10, Proposition 3.11] and [9, Theorem 3.4], respectively, as equivalent definitions of a tame dynamical system; for the other properties, see the survey [4] and references therein. In fact, the semigroups $E(\Omega, \varphi)$ and $\mathscr{K}_c(\Omega, \varphi)$ in conditions (a) and (c) are sequentially compact. According to condition (a), a dynamical system is tame if its Ellis semigroup is metrizable. The compact subsystems and direct products of tame systems are tame as well [4]. We also mention two striking properties of the phase dynamics of *minimal* tame systems:

(1) the topological entropy of such systems vanishes [4, p. 2356];

(2) any such system (Ω, φ) is point distal, i.e., there exists a point $\omega_0 \in \Omega$ such that none of the pairs of points (ω, ω_0), $\omega \neq \omega_0$, is proximal [5, Proposition 4.4].

3. CONVERGENCE OF ERGODIC MEANS

A criterion for the $*$-weak convergence of the Cesàro means

$$U_n = \frac{1}{n+1}(I + U + \cdots + U^n), \qquad V_n = \frac{1}{n+1}(I + V + \cdots + V^n)$$

was obtained in [16, Theorem 1] and extended to arbitrary ergodic nets

$$\{U_\alpha\} \subset \mathscr{L}(X) \qquad \text{and} \qquad \{V_\alpha\} \subset \mathscr{L}(X^*)$$

in [11, Theorem 1.5]. Namely, the following theorem is valid.

Theorem 3.1 (separation principle). *If* $X_0 = \{x \in X : U_\alpha x \xrightarrow{w^*} \overline{x} \in X^{**}\}$, *then* $X_0 = X$ *if and only if the limit elements* $\overline{x}$ *separate* $\operatorname{fix}(V)$.

The latter condition means that, for any invariant measure $\mu \in \text{fix}(V)$, there exist continuous functions $x_1, x_2 \in X_0$ such that $(\overline{x}_1, \mu) \neq (\overline{x}_2, \mu)$. Moreover, X_0 is a nonempty closed U-invariant linear subspace of X and

$$\overline{x} = Tx, \qquad T \in \mathscr{L}(X_0, X^{**}), \quad \|T\| = 1.$$

In the case of ergodic sequences, we have $\overline{x} \in X_1$.

We use the following notation for a $\mathbb{N}_0$-dynamical system (Ω, φ):

- $m \subseteq \Omega$ is a minimal set;
- $\mu_e \in \mathscr{P}(\Omega)$ is an ergodic measure;
- $\overline{o}(\omega)$ is the closure of the orbit $o(\omega) = \{\varphi^n\omega,\ n \geq 0\}$ of $\omega \in \Omega$.

We consider the following properties of the dynamics of (Ω, φ) and of ergodic operator nets $\mathscr{V} \subset \mathscr{L}(X^*)$:

- (single m in $\overline{o}$): each $\overline{o}(\omega)$ contains a unique m;
- (supp $\mu_\mathrm{e} = m$): the supports of μ_e are minimal;
- (single μ_e on m): the minimal subsystems (m, φ) are uniquely ergodic;
- UE($\overline{o}$): the orbital subsystems $(\overline{o}(\omega), \varphi)$ are uniquely ergodic;
- (AEN): all nets $\mathscr{V}$ converge;
- (AES): all sequences $\mathscr{V}$ converge;
- (SES): some sequence $\mathscr{V}$ converges.

The following lemma describes some general relations between these properties.

Lemma 3.2. *The following implications hold for any semicascade* (Ω, φ):

i) (AEN) $\Rightarrow$ $UE(\overline{o})$ $\Rightarrow$ (AES);

ii) (SES) $\Rightarrow$ (single μ_e on m);

iii) UE($\overline{o}$) $\Leftrightarrow$ (single m in $\overline{o}$) + (supp $\mu_\mathrm{e} = m$) + (single μ_e on m).

Proof. The implication (AEN) $\Rightarrow$ UE($\overline{o}$) follows from Lemma 5.9 of [10] and Remark 2.1. The implication UE($\overline{o}$) $\Rightarrow$ (AES) was proved in [11, Theorem 3.2]. Assertion (ii) is a minor generalization of Theorem 5.4 of [2]. If there exists a convergent ergodic sequence of operators $V_n = U_n^*$ and a set $m \subseteq \Omega$ is minimal, then $(U_n x)(\omega) \to \overline{x}(\omega)$ for any $x \in X$ and $\omega \in m$, and, moreover, $\overline{x}(\varphi\omega) \equiv \overline{x}(\omega)$ on m. Since the orbits $o(\omega) \subseteq m$ are dense, it follows that the restriction $\overline{x}|_m$ is either constant or everywhere discontinuous. A function of the first Baire class cannot be everywhere discontinuous; therefore, by virtue of, e.g., the separation principle (Theorem 3.1), the dynamical system (m, φ) is uniquely ergodic. Assertion (iii) is trivial. ☐

We see that if a minimal set supports more than one ergodic measure, then there exist no convergent ergodic sequences (although there always exist convergent ergodic nets). This effect occurs for certain minimal analytic diffeomorphisms of the torus $\mathbb{T}^2$ which admit uncountably many ergodic measures [17, Corollary 12.6.4].

In the tame case, much more can be said about the convergence of ergodic means.

Theorem 3.3. *For a tame* $\mathbb{N}_0$*-system* (Ω, φ)*, the following assertions hold*:

i) *each ergodic operator net* $\{V_\alpha\} \subset \mathscr{L}(X^*)$ *contains a convergent ergodic sequence* $V_{\alpha(n)}$;

(ii) *each ergodic operator sequence* $\{V_n\} \subset \mathscr{L}(X^*)$ *contains a convergent ergodic subsequence; in particular, the sequence of Cesàro means contains a convergent subsequence.*

Proof. Since $\mathscr{K}_c = \mathscr{K}_c(\Omega, \varphi)$ is compact, we can assume without loss of generality that $V_\alpha \xrightarrow{\mathrm{W^*O}} Q$, where $Q \in \operatorname{Ker} \mathscr{K}_c$. The topological space $\mathscr{K}_c$ is compact and Fréchet–Urysohn, since (Ω, φ) is a tame semicascade; therefore, the net $\{V_\alpha\}$ contains a sequence $V_{\alpha(n)} \xrightarrow{\mathrm{W^*O}} Q$, and this sequence is ergodic, because $Q \in \operatorname{Ker} \mathscr{K}_c$.

Assertion (ii) follows from the sequential compactness of $\mathscr{K}_c$ and the preservation of ergodicity under the passage to a subsequence. □

The ergodic sequence $\{V_{\alpha(n)}\}$ in 3.3(i) is not generally a subsequence of the ergodic net $\{V_\alpha\}$. The following theorem describes the relationship between the ergodic and dynamical properties of tame systems.

Theorem 3.4. *Any tame* $\mathbb{N}_0$*-system* (Ω, φ) *has property* (SES). *Moreover,*

$$(\mathrm{AEN}) \iff \mathrm{UE}(\overline{o}) \iff (\mathrm{AES}) \iff (\text{single } m \text{ in } \overline{o}).$$

Proof. Convergent ergodic operator sequences for a tame semicascade exist by Theorem 3.3. Suppose that all such sequences converge and there exist two different elements $Q_1, Q_2 \in \operatorname{Ker} \mathscr{K}_c(\Omega, \varphi)$. According to 3.3(i), there exist ergodic sequences

$$V_n^{(1)} \xrightarrow{\mathrm{W^*O}} Q_1, \qquad V_n^{(2)} \xrightarrow{\mathrm{W^*O}} Q_2.$$

The mixed sequence $V_{2n-1} = V_n^{(1)}$, $V_{2n} = V_n^{(2)}$ is ergodic but divergent. Thus, property (AES) implies

$$\operatorname{card} \operatorname{Ker} \mathscr{K}_c = 1,$$

which is equivalent (by Theorem 3.2 of [11]) to property (AEN), and by virtue of Lemma 3.2(i), we have (AEN) $\Leftrightarrow$ UE($\overline{o}$) $\Leftrightarrow$ (AES) for tame systems. Finally, Theorem 4.6 of [9] gives the implication (single m in $\overline{o}$) $\Rightarrow$ (AES), and it remains to note that UE($\overline{o}$) $\Rightarrow$ (single m in $\overline{o}$). □

The equivalence (AEN) $\Leftrightarrow$ (single m in $\overline{o}$) was proved independently in [10, Theorem 5.10]. Theorem 3.4 implies, in particular, the unique ergodicity of minimal tame semicascades; this fact was proved for a larger class of tame systems more than ten years ago in [7] and [8]. In [10, Lemma 5.12], it was strengthened: the uniqueness of a minimal set $m \subseteq \Omega$ implies the unique ergodicity of $(\Omega, \varphi) \in \mathscr{D}_{\mathrm{tm}}$. In this connection, it is useful to make the following remark.

Remark 3.5. The supports of ergodic measures of tame $\mathbb{N}_0$-systems either are minimal or contain more than one minimal set.

On the other hand, according to Theorem 3.1 of [18], if a semicascade (Ω, φ) has a unique minimal set and the Cesàro means $*$-weakly converge, then there exists either precisely one ergodic measure or uncountably many such measures. Moreover, the second possibility can indeed be realized [18, Sec. 4].

Remark 3.6. Even for tame systems, the convergence of one ergodic sequence does not imply that of all other ergodic sequences, i.e., (SES) $\not\Rightarrow$ (AES). Namely, a tame Bernoulli subshift has been constructed for which the Cesàro means converge but condition (single m in $\overline{o}$) does not hold [10, Example 5.14]. According to Theorem 3.4, for this semicascade, condition (AES) does not hold either.

4. THE ASYMPTOTIC DISTRIBUTION OF ORBITS

In this section, we carry over some constructions of [1] and [2] related to the pointwise convergence on Ω of the Cesàro means $U_n x$ for continuous test functions $x \in X = C(\Omega)$ to arbitrary ergodic sequences. Instead of the individual ergodic theorem (which does not hold for general averaging methods), we use a priori information about the pointwise convergence of certain generalized ergodic means. Our main task is to prove the possibility of decomposing tame dynamical systems into irreducible (ergodic) components. By $\mathscr{P}_{\text{in}}(\Omega)$ and $\mathscr{P}_{\text{e}}(\Omega)$ we denote the subsets of $\mathscr{P}(\Omega)$ formed by all φ-invariant and all φ-ergodic measures, respectively, and by X_1 we denote the set of bounded scalar functions of the first Baire class on Ω. A set $\Theta \subseteq \Omega$ is said to be *bi-invariant* if $\varphi^{-1}\Theta = \Theta$. We also use the notation $D(\Omega)$ for the set of Dirac measures δ_ω on Ω and

$$\mathscr{K}_{\text{c}} = \mathscr{K}_{\text{c}}(\Omega, \varphi) \subseteq \mathscr{L}(X^*)$$

for the operator semigroup defined in Sec. 2.2.

We assume that there exists a convergent ergodic operator sequence

$$\mathscr{V} = \{V_n\} \subset \mathscr{L}(X^*), \qquad V_n \xrightarrow{\text{W}^*\text{O}} Q \in \operatorname{Ker} \mathscr{K}_{\text{c}};$$

for this convergence we use the shorthand notation $\mathscr{V} \to Q$. Under this assumption, for the dual ergodic sequence $\{U_n\} \subset \mathscr{L}(X)$, $U_n^* = V_n$, we have $(U_n x)(\omega) \to \overline{x}(\omega)$ for all $\omega \in \Omega$ and $x \in X$; moreover, the function $\overline{x} \in X_1$ is invariant (that is, $\overline{x} \circ \varphi = \overline{x}$), and to each point $\omega \in \Omega$ there corresponds the measure

$$\mu_\omega = Q\delta_\omega \in \mathscr{P}_{\text{in}}(\Omega), \qquad \overline{x}(\omega) = (x, \mu_\omega),$$

which determines the asymptotic $\mathscr{V}$-distribution of the orbit $o(\omega)$. This means essentially that $V_n\delta_\omega \xrightarrow{w^*} \mu_\omega$. A linear projection Q in X^* induces a mapping $\Psi_{\mathscr{V}} \colon \Omega \to \mathscr{P}_{\text{in}}(\Omega)$ of the first Baire class. This mapping is completely determined by the limit element Q of the sequence $\mathscr{V}$, but it is convenient to use the notation $\Psi_{\mathscr{V}}$.

Lemma 4.1. *If an ergodic sequence $\mathscr{V} = \{V_n\}$ converges, then $\Psi_{\mathscr{V}}\Omega \supseteq \mathscr{P}_{\text{e}}(\Omega)$.*

In other words, for any convergent ergodic sequence $\mathscr{V}$, each ergodic measure determines the asymptotic $\mathscr{V}$-distribution of some orbit.

Proof. Let $\mu \in \mathscr{P}_{\text{e}}(\Omega)$, and let $x \in X$. We set $c(x) = (x, \mu)$. Under the assumptions of the lemma, we have

$$(U_n x, \mu) = (x, V_n\mu) = (x, \mu),$$

and the dual sequence $(U_n x)(\omega)$ converges to $\overline{x}(\omega)$ for all $\omega \in \Omega$; moreover, $\overline{x} = \overline{x} \circ \varphi$ and $(\overline{x}, \mu) = c(x)$ by Lebesgue's theorem. An argument similar to that used to prove the implication (i) $\Rightarrow$ (iv) in Proposition 7.15 of [19] shows that the ergodicity of μ implies that the bounded invariant function $\overline{x} \in X_1$ equals identically the constant $c(x)$ on a Borel set $\Theta_{x,\mu} \subseteq \Omega$ of full μ-measure. Therefore, for all points $\omega \in \Theta_{x,\mu}$, we have

$$(x, V_n\delta_\omega) = (U_n x, \delta_\omega) \to (\overline{x}, \delta_\omega) = (\overline{x}, \mu) = (x, \mu). \tag{4.1}$$

Choosing x in any countable set Y dense in X, we obtain relation (4.1) for $x \in Y$ and $\omega \in \Theta_\mu$, where $\Theta_\mu = \bigcap_{x \in Y} \Theta_{x,\mu}$ and $\mu(\Theta_\mu) = 1$. Since $\|V_n\| \le 1$ for all $n \in \mathbb{N}_0$, it follows that this relation also holds for any $x \in X$ and $\omega \in \Theta_\mu$. Thus,

$$V_n\delta_\omega \xrightarrow{\text{w}^*} \mu \qquad \text{for} \quad \omega \in \Theta_\mu. \quad \square$$

Given a convergent ergodic sequence $\mathscr{V} = \{V_n\}$, we set

$$\Omega_{\mathscr{V}} = \{\omega \in \Omega : \mu_\omega \in \mathscr{P}_{\text{e}}(\Omega)\},$$

where $V_n\delta_\omega \xrightarrow{\text{w}^*} \mu_\omega$; then, for ergodic measures μ, the sets $\Omega_{\mu,\mathscr{V}} = \Psi_{\mathscr{V}}^{-1}\mu$ form a partition of $\Omega_{\mathscr{V}}$ into $\mathscr{V}$*-quasi-ergodic* components. The sets $\Omega_{\mathscr{V}}$ and $\Omega_{\mu,\mathscr{V}}$ are bi-invariant. Moreover, they are Borel; this follows from purely topological considerations [2, pp. 78–79] in no way related to the specifics of Cesàro averaging. Since $\Omega_{\mu,\mathscr{V}} \supseteq \Theta_\mu$, where Θ_μ is the set from the proof of Lemma 4.1, we obtain the following result.

Corollary 4.2. *If an ergodic sequence $\mathscr{V}$ converges, then to each ergodic measure μ there corresponds a Borel $\mathscr{V}$-quasi-ergodic set $\Omega_{\mu,\mathscr{V}}$ of full μ-measure.*

Now we proceed to the main topic of this paper.

Definition 4.3. We say that an $\mathbb{N}_0$-system (Ω,φ) is *ergodically decomposable* if there exists a convergent ergodic operator sequence $\mathscr{V}$ such that $\Omega_{\mathscr{V}}=\Omega$ or, equivalently, $\Psi_{\mathscr{V}}\Omega=\mathscr{P}_{\rm e}(\Omega)$.

In fact, an ergodically decomposable *topological* dynamical system (Ω,φ) admits a decomposition into ergodic subsystems $(\Omega_{\mu,\mathscr{V}},\varphi)$, $\mu\in\mathscr{P}_{\rm e}(\Omega)$. In this situation, we have $\mathscr{V}\to Q\in\operatorname{Ker}\mathscr{K}_{\rm c}$, and to any continuous function $x\in X$ there corresponds a function $\overline{x}=Q^*x\in X_1$ taking the constant value $(\overline{x},\mu)=(x,\mu)$ on each quasi-ergodic set $\Omega_{\mu,\mathscr{V}}$. Thus, for each measure $\mu\in\mathscr{P}_{\rm e}(\Omega)$, the *metric* dynamical system $(\Omega_{\mu,\mathscr{V}},\varphi)$ is ergodic with respect to μ in the standard sense [19, Definition 6.18]. In the interpretation of [17, Sec. 4.1], the ergodic decomposability of a semicascade (Ω,φ) means that the asymptotic $\mathscr{V}$-distributions of all orbits are determined by ergodic measures. Since the mapping $\Psi_{\mathscr{V}}\colon\Omega\to\mathscr{P}_{\rm e}(\Omega)$ inducing the decomposition of (Ω,φ) is of the first Baire class, it follows that the points of continuity of this mapping form a dense G_δ-set in Ω.

It turns out that the ergodic decomposability of an $\mathbb{N}_0$-dynamical system is related to the existence of ergodic operator sequences converging to extreme points of the kernel of the semigroup $\mathscr{K}_{\rm c}=\mathscr{K}_{\rm c}(\Omega,\varphi)$.

Proposition 4.4. *If an ergodic sequence $\mathscr{V}$ converges to $Q\in\operatorname{ex}\operatorname{Ker}\mathscr{K}_{\rm c}$, then the dynamical system (Ω,φ) is ergodically decomposable.*

Proof. Let $\mathscr{V}=\{V_n\}$. According to Proposition 2.10 of [9], in this case, we have $Q\colon D(\Omega)\to\mathscr{P}_{\rm e}(\Omega)$, and since $V_n\xrightarrow{\mathrm{W^*O}}Q$, it follows that $V_n\delta_\omega\xrightarrow{\mathrm{w}^*}\mu\in\mathscr{P}_{\rm e}(\Omega)$ for each $\omega\in\Omega$. Thus, $\Omega_{\mathscr{V}}=\Omega$, and the system (Ω,φ) is ergodically decomposable. □

Theorem 4.5 (main theorem). *Any tame $\mathbb{N}_0$-system (Ω,φ) is ergodically decomposable.*

Proof. Each projection $Q\in\operatorname{ex}\operatorname{Ker}\mathscr{K}_{\rm c}$ is the W*O-limit of some ergodic net $\{V_\alpha\}\subset\mathscr{L}(X^*)$; by Theorem 3.3(i), there exists an ergodic operator sequence $V_{\alpha(n)}\xrightarrow{\mathrm{W^*O}}Q$. Proposition 4.4 implies ergodic decomposability with $\mathscr{V}=\{V_{\alpha(n)}\}$. □

Now, let us describe the structure of all possible decompositions of tame systems into ergodic components.

Lemma 4.6. *For any tame $\mathbb{N}_0$-system (Ω,φ), the operators $T\in\mathscr{K}_{\rm c}(\Omega,\varphi)$ are determined by their values at the Dirac measures.*

Proof. In the situation under consideration, the semigroup $\mathscr{K}_{\rm c}$ is a Fréchet–Urysohn compact space; therefore, for any $T\in\mathscr{K}_{\rm c}$, there exists a sequence

$$\{V_n\}\subseteq\operatorname{co}\{V^n,\,n\ge 0\}$$

converging to T in the W*O-topology of $\mathscr{L}(X^*)$. For $U_n^*=V_n$, $x\in X$, and $\omega\in\Omega$, we have $(x,V_n\delta_\omega)=(U_nx)(\omega)$ and

$$(U_nx)(\omega)\to\overline{x}(\omega),\qquad\text{where}\quad \overline{x}(\omega)=(x,T\delta_\omega).$$

Here $\overline{x}\in X_1$ and, by Lebesgue's theorem,

$$(x,V_n\mu)=(U_nx,\mu)\to(\overline{x},\mu)=(x,T\mu)$$

for each measure $\mu\in\mathscr{P}(\Omega)$. At the same time, $\overline{x}(\omega)=(x,T\delta_\omega)$ for $\omega\in\Omega$, and hence the operator T is completely determined by its values on $D(\Omega)$. □

Lemma 4.6 strengthens a similar assertion (Theorem 3.5, (a1) ⇒ (a4)) in [9], in which, instead of the assumption $(\Omega,\varphi)\in\mathscr{D}_{\rm tm}$, it was required that the semigroup $E(\Omega,\varphi)$ be metrizable.

Corollary 4.7. *If* $(\Omega, \varphi) \in \mathscr{D}_{\mathrm{tm}}$ *and* $Q_1|_{D(\Omega)} = Q_2|_{D(\Omega)}$ *for* $Q_1, Q_2 \in \operatorname{Ker} \mathscr{K}_{\mathrm{c}}$*, then* $Q_1 = Q_2$.

This readily implies that, in the tame case, the condition $Q \in \operatorname{ex} \operatorname{Ker} \mathscr{K}_{\mathrm{c}}$ is not only sufficient but also necessary for the relation $Q\colon D(\Omega) \to \mathscr{P}_{\mathrm{e}}(\Omega)$, $Q \in \operatorname{Ker} \mathscr{K}_{\mathrm{c}}$, to hold. For $(\Omega, \varphi) \in \mathscr{D}_{\mathrm{tm}}$, it is natural to define quasi-ergodic sets based on elements $Q \in \operatorname{ex} \operatorname{Ker} \mathscr{K}_{\mathrm{c}}$ rather than on convergent ergodic sequences $\mathscr{V}$. Namely, we set

$$\Omega_{\mu,Q} = \{\omega \in \Omega : Q\delta_\omega = \mu\}, \qquad \mu \in \mathscr{P}_{\mathrm{e}}(\Omega).$$

We see that the Borel bi-invariant quasi-ergodic sets $\Omega_{\mu,Q}$ of full μ-measure form a partition Φ_Q of the phase space Ω. The set Λ of all ergodic sequences $\mathscr{V} \to Q$ decomposes into disjoint classes Λ_Q corresponding to different Q. The elements $\mathscr{V} \in \Lambda_Q$ relate the dynamics of the semicascade (Ω, φ) to ergodic measures; namely, the asymptotic $\mathscr{V}$-distribution of each orbit $o(\omega)$ is determined by the measure $\mu = Q\delta_\omega$. Moreover, there is a one-to-one (by Corollary 4.7) correspondence between the projections $Q \in \operatorname{ex} \operatorname{Ker} \mathscr{K}_{\mathrm{c}}$, the partitions Φ_Q of the phase space Ω into quasi-ergodic sets, and the partitions Λ_Q of the set Λ of ergodic operator sequences converging to extreme points of the kernel of the semigroup $\mathscr{K}_{\mathrm{c}}(\Omega, \varphi)$.

5. ADDENDUM

In this section, we consider several typical examples of tame and untame $\mathbb{N}_0$-dynamical systems. We set $I = [0, 1]$.

Example 1. According to [20, Proposition 10.5] and [4, Sec. 9], any semicascade generated by a self-homeomorphism of I or $\mathbb{S}^1$ has metrizable Ellis semigroup and, hence, is tame.

Example 2. The left Bernoulli shift on the set $\Omega = \{0, 1\}^{\mathbb{N}_0}$ of sequences $\omega_0, \omega_1, \dots$ with standard metric

$$\rho(\omega, \nu) = (1 + \min\{k : \omega_k \neq \nu_k\})^{-1}$$

generates an untame $\mathbb{N}_0$-system (Ω, φ), which, however, has tame subsystems (Θ, φ). There exists an elegant description of these subsystems: each infinite set $L \subseteq \mathbb{N}_0$ contains an infinite subset $K \subseteq L$ such that the projection $\pi_K(\Theta)$ is a countable subset of $\{0, 1\}^K$ [6, Theorem 4.7].

Example 3. For the semicascade (I, φ) defined in the example in [21, pp. 147–149], the set of periodic point is not closed, and, given any orbit $o(\omega)$, $\omega \in I$, either this orbit is eventually periodic (that is, $\varphi^k \omega = \varphi^{k+p} \omega$ for some $k \geq 0$ and $p \geq 1$) or its limit points fill the classical Cantor set. This semicascade turns out to be tame [3, Example 5.8(c)].

Example 4. On the other hand, any semicascade (I, φ) having periodic points with period not equal to a power of 2 is untame [3, Example 5.8(e)].

Example 5. A minor modification of an argument in [4, p. 2354] shows that the projective action of any invertible operator $T \in GL(n, \mathbb{R}^n)$, $n \geq 2$, induces a tame semicascade on the sphere $\mathbb{S}^{n-1}$.

Examples 3 and 5 show that tame systems may exhibit nontrivial phase dynamics.

A very simple and constructive equivalent definition of tame dynamics was given in [12]: *a semicascade* (Ω, φ) *is tame if any sequence* $\varphi^{n(k)}$*,* $\{n(k)\} \subseteq \mathbb{N}_0$*, of iterations contains a pointwise convergent subsequence.* Based on this definition, the author of [12] obtained a criterion distinguishing between tame and untame affine torus endomorphisms $\varphi\colon \omega \to A\omega + b$, $\omega \in \mathbb{T}^d$, $d \geq 1$, with integer matrix A and any shift $b \in \mathbb{T}^d$. If $\det A = \pm 1$, then φ is an automorphism.

Theorem 5.1 (Lebedev [12]). *A semicascade* $(\mathbb{T}^d, \varphi)$ *is tame if and only if* $A^k = A^l$ *for some* $k, l \in \mathbb{N}_0$*,* $k \neq l$.

The eigenvalues $\lambda(A)$ of such a matrix are either zeros or roots of unity, and

$$\varphi^k = \varphi^l + b_1,$$

where b_1 is a shift on $\mathbb{T}^d$. If $\det A = \pm 1$, then the assumption of the theorem takes the form $A^k = \mathrm{Id}$. In particular, the automorphism $\varphi\colon (\omega_1, \omega_2) \to (\omega_1 + \omega_2, \omega_2)$ of the torus $\mathbb{T}^2$ is not tame.

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