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Proceedings of the Fourth Russian Finnish Symposium on Discrete Mathematics

Turku Centre for Computer Science

TUCS Lecture Notes
No 26, May 2017

Proceedings of the Fourth Russian Finnish Symposium on Discrete Mathematics

Organized in Turku on 16.-19.5.2017

Juhani Karhumäki, Yuri Matiyasevich, Aleksi Saarela (editors)

Preface

The fourth RuFiDiM conference, Russian-Finnish Symposium on Discrete Mathematics, took place in Turku in May, from 16th til 19th, 2017. This meeting was organized as a part of research activities between Steklov Institute of Mathematics of St. Petersburg and Department of Mathematics and Statistics of University of Turku. The goal of the conference series is to increase cooperation between Finnish and Russian mathematicians in discrete mathematics, but the symposium is open for a broader international audience. In the present event there were contributions from 10 different nations.

RuFiDiM 2017 consisted of six invited talks and 25 contributed presentations. The invited speakers were Volker Diekert (University of Stuttgart), Alexandr Kostochka (University of Illinois at Urbana-Champaign), Alexei Miasnikov (Stevens institute of Technology), Igor Potapov (University of Liverpool), Aleksi Saarela (University of Turku) and Jouko Väänänen (University of Helsinki). The program was chosen by the international program committee. Abstracts or extended abstracts of the lectures are presented in these preproceedings.

The organizers are grateful to the supporters of the symposium: Magnus Ehrnrooth Foundation, Turku University Foundation, Finnish Academy of Sciences (Mathematics Fund), and the University of Turku and Turku Centre for Computer Science.

Turku and St. Petersburg, April 2017

Juhani Karhumäki, Yuri Matiyasevich, Aleksi Saarela

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Network analysis based on a typology of nodes

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Abstract

Commonly in network analysis an undirected graph (network) is represented by its adjacency matrix, and while the latter may have an enormous order. We show that in many cases instead of the adjacency matrix a much smaller matrix (referred as type adjacency matrix) may be used. We introduce notions of the types of nodes of graph and of the type adjacency matrix, propose an algorithm for division of the set of nodes into types. We study properties of the type adjacency matrix and some of its applications in the social and economic network analysis.

Keywords: undirected graph, network, type of node, centrality, growth of network, production, knowledge, externality, network game, Nash equilibrium, network formation.

1 Introduction

Commonly in network analysis and its economic and social applications an undirected graph (network) is represented by its adjacency matrix, **A**. A drawback of the adjacency matrix is that it can have an enormous size, and this may trouble working with graphs possessing big size but simple structure. For example, for the star graph with ν peripheral nodes the adjacency matrix is of size $(\nu+1)\times(\nu+1)$, and this matrix, by itself, does not rely on the simple structure of the graph.

This drawback may be somehow rectified if another kind of matrix, referred further as type adjacency matrix, **T**, is used. The type adjacency matrix usually has much smaller size in comparison with the adjacency matrix and reflects important features of the structure of the graph in an aggregate form. For example, for the str graph the type adjacency matrix has the size 2×2, independently on the number of peripheral nodes. Due to the small size, the type adjacency matrix can considerably simplify the network analysis.

The presence of the type adjacency matrix corresponds to the fact that the set of the nodes of undirected graph can be divided into disjoint subsets (types) in such way that each node of definite type has definite numbers of neighbors (adjacent nodes) of each type. We will consider such division with the minimal possible number of types.

For example, for the network shown in Figure 1, whose adjacency matrix is

$$\mathbf{A}=\begin{pmatrix}0&1&1&1&1\\1&0&1&1&0\\1&1&0&0&1\\1&1&0&0&1\\1&0&1&1&0\end{pmatrix},$$

the set of nodes is divided into two types: the 1^{st} type consists of node 1, and the 2^{nd} type includes nodes 2, 3, 4, 5 (see Figure 2). The 1^{st} type node is adjacent to 0 first-type nodes

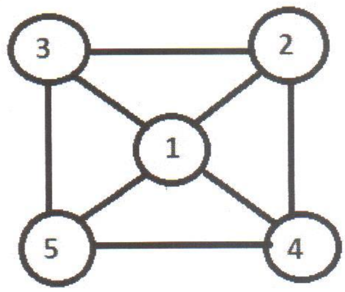

Figure 1: Graph with 5×5 adjacency matrix.

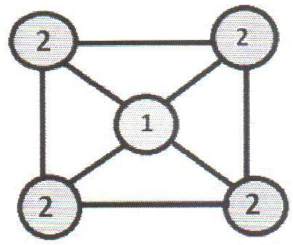

Figure 2: Division of the set of nodes into two types.

and to 4 second-type nodes, while each 2^{nd} type node is adjacent to 1 first-type node and 2 second-type nodes. These numbers of neighbors of different types can be written in form of the type adjacency matrix:

$$\mathbf{T} = \begin{pmatrix} 0 & 4 \\ 1 & 2 \end{pmatrix}.$$

We will see that in many respects the type adjacency matrix may serve as a substitute for the adjacency matrix.

In the present paper we provide a definition of the types of nodes and the type adjacency matrix, propose an algorithm for division of the set of nodes into types and construction of the type adjacency matrix, and study its properties and some of its applications.

2 Types of nodes

Let G be undirected graph (network) of order n and $\mathbf{A}$ be its adjacency matrix. Let us remind that $\mathbf{A}$ is $n \times n$ matrix such that $a_{ij} = a_{ji} = 1$ if in the graph there is an edge connecting nodes i and j, and $a_{ij} = a_{ji} = 0$ otherwise; $a_{ii} = 0$ for all $i = 1, 2, \ldots, n$.

The concept of types of nodes may be explained informally in the following way. The nodes of graph can be colored in S colors in such way that each node of color j has a definite number $l_i(j)$ of neighbors of color i (for each $i = 1, 2, \ldots, S$).

More formally, the set of nodes of graph may be decomposed into minimal number S of disjoint classes $j = 1, 2, \ldots, S$ in such way that any node belonging class j has $l_i(j)$ neighbors from class i (for $i = 1, 2, \ldots S$). The classes will be referred as *types of nodes*. Type j is characterized by vector $\mathbf{l}(j) = (l_1(j); l_2(j); \ldots; l_S(j))$, where $l_i(j)$ is the number of neighbors in class i for each node of class j.

2.1 Type adjacency matrix

Let the graph have S types of nodes. We construct a $S \times S$-matrix $\mathbf{T}$ in the following way. The first row of this matrix is the row vector $\mathbf{l}(1)$, the second row is $\mathbf{l}(2)$, ..., the S-th row is $\mathbf{l}(S)$:

$$\mathbf{T} = \begin{pmatrix} \mathbf{l}(1) \\ \mathbf{l}(2) \\ \dots \\ \mathbf{l}(S) \end{pmatrix} = \begin{pmatrix} l_1(1) & l_2(1) & \dots & l_S(1) \\ l_1(2) & l_2(2) & \dots & l_S(2) \\ \dots & \dots & \dots & \dots \\ l_1(S) & l_2(S) & \dots & l_S(S) \end{pmatrix}.$$

This matrix will be referred as *type adjacency matrix* of the graph. Our type adjacency matrix $\mathbf{T}$ is the same as, in spectral graph theory, a quotient matrix for the symmetric matrix $\mathbf{A}$ (e.g. [6], [2]).

Figures 3 and 4 provide an example of two graphs which have the same size and the same distribution of degrees, but different typology. Their type adjacency matrices are, correspondingly,

$$\begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Later we will see that there exist graphs of different size, but with the same typology.

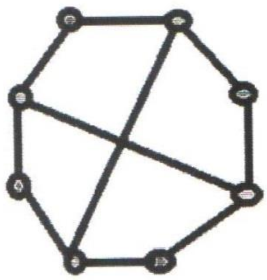

Figure 3: Graph with $\mathbf{l}(1) = (0; 2)$, $\mathbf{l}(2) = (2; 1)$.

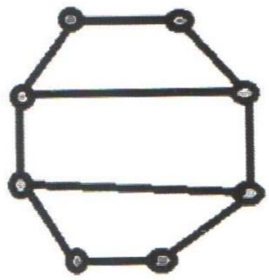

Figure 4: Graph with $\mathbf{l}(1) = (1; 1)$, $\mathbf{l}(2) = (1; 2)$.

2.2 Algorithm

Let us describe an algorithm of subdivision of the set of nodes of undirected graph into types. Let s be a current number of subsets of subdivision. Initially $s = 1$.

Iteration of the algorithm. Consider nodes of the first subset. If all of them have the same numbers of neighbors in each subset $1, 2, \dots, s$, then the first subset is not changed.

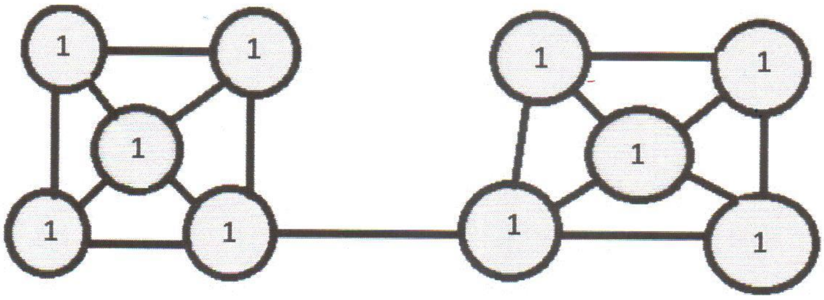

Figure 5: Start of the algorithm: $s = 1$.

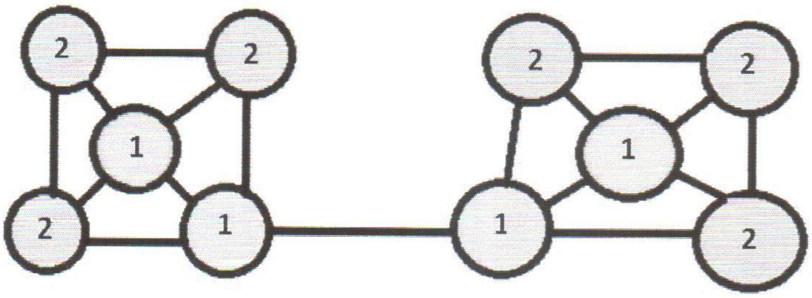

Figure 6: Result of the first iteration: $s = 2$.

In the opposite case, we divide the first subset into new subsets in such way that all nodes of each new subset have the same numbers of neighbors in subsets.

We proceed in precisely same way with the second, the third, ..., the s-th subset. If on the present iteration the number of subsets s has not changed, then the algorithm finishes its work. If s has increased, then the new iteration starts.

The number of subsets s does not decrease in process of the algorithm. Since s is bounded from above by the number n of nodes in the graph, the algorithm converges. It is clear that the algorithm divides the set of nodes into the minimal possible number of classes.

Example. Let us apply the algorithm to the graph depicted in Figure 2. Initially $s = 1$, all nodes constitute the same one set (Figure 5).

After the first iteration we obtain the division corresponding to degrees, depicted in Figure 6. Then, on the first step of the second iteration, we obtain the division depicted in Figure 7. On the second step of the second iteration we obtain the division shown in Figure 8.

On the third iteration nothing changes, and the algorithm stops.

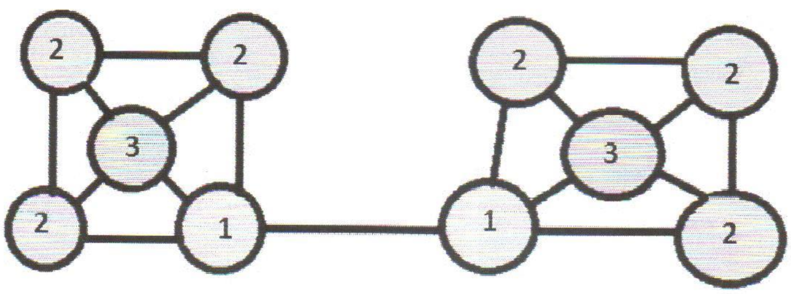

Figure 7: The first step of the second iteration: $s = 3$.

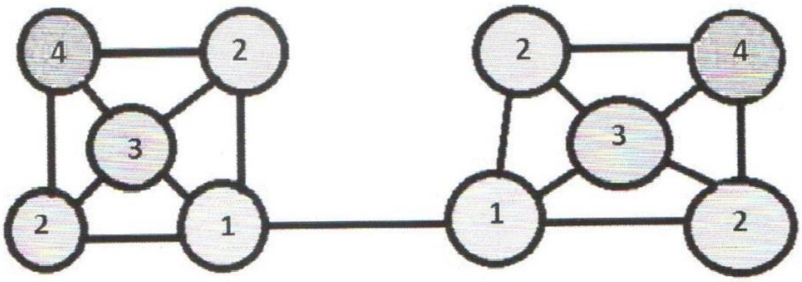

Figure 8: The second step of the second iteration: $s = 4$.

We have obtained a subdivision of the set of nodes of the graph into four types which are characterized by the vectors of numbers of neighbors:

$$\mathbf{l}(1) = (1;2;1;0), \mathbf{l}(2) = (1;0;1;1), \mathbf{l}(3) = (1;2;0;1), \mathbf{l}(4) = (0;2;1;0).$$

The corresponding type adjacency matrix is

$$\mathbf{T} = \begin{pmatrix} \mathbf{l}(1) \\ \mathbf{l}(2) \\ \mathbf{l}(3) \\ \mathbf{l}(4) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix}.$$

2.3 Relation to modular division

Our subdivision of the set of nodes of graph into types is a generalization of the modular aggregation of Allouch [1]. Allouch [1] defines *module* as a subset of nodes, such that each node in a module has the same neighbors. It can be seen that each division into modules is a division into types, but not the opposite.

3 Some applications of the type adjacency matrix in network analysis

3.1 Numbers of routes

It is known that each element a_{ij}^k of the k-th power of the adjacency matrix, $\mathbf{A}^k$, is equal to the number of the k-step routes between nodes i and j. In a parallel way, each element t_{ij}^k of the k-th power of the type adjacency matrix, $\mathbf{T}^k$, shows the number of the k-step routes between any i-th type node and all j-th type nodes.

E.g., for the graph shown in Figure 1 the 3^{rd} power of the adjacency matrix,

$$\mathbf{A}^3 = \begin{pmatrix} 8 & 8 & 8 & 8 & 8 \\ 8 & 4 & 8 & 8 & 4 \\ 8 & 8 & 4 & 4 & 8 \\ 8 & 8 & 4 & 4 & 8 \\ 8 & 4 & 8 & 8 & 4 \end{pmatrix},$$

shows that in the graph there are 8 three-step routes from node 3 to node 2 and 4 three-step routes from node 3 to node 4. The same graph is characterized by the type adjacency matrix

$$\mathbf{T} = \begin{pmatrix} 0 & 4 \\ 1 & 2 \end{pmatrix}.$$

Matrix $\mathbf{T}$ shows, in particular, that the degree of the 1^{st} type node is $0+4=4$, and the degree of any 2^{nd} type node is $1+2=3$. The 3^{rd} power, e.g.,

$$\mathbf{T}^3 = \begin{pmatrix} 8 & 32 \\ 8 & 24 \end{pmatrix},$$

shows that in the graph there are 24 three-step routes from each 2^{nd} type node to all 2^{nd} type nodes.

3.2 Centrality measures

Network analysis uses various centrality measures, among them degree centrality, Katz-Boncicz centralities [12],[4], eigenvector centrality, and α- centrality [5].

The *degree centrality* of the node (or of each node of the type) may be calculated in a similar way by use of either the adjacency matrix, $\mathbf{A}$, or the type adjacency matrix $\mathbf{T}$:

$$C_D(i) = \frac{\sum_{j=1}^{n} a_{ij}}{n-1}, \tag{1}$$

$$C_D(\tilde{i}) = \frac{\sum_{j=1}^{S} t_{\tilde{i}j}}{n-1},$$

where $\tilde{i}$ is the type of node i.

Let us remind how the numbers of routes are used in calculation of the *Katz-Bonacich centrality measures.* Katz [12] and Bonacich [4] propose to use in definition of centrality measure not the degree of node i, as in formula (1), but the total discounted number of routes outgoing from the node i. All such routes are accounted for, and the longer the route is (the more is its length, k), the smaller discounting multiplier, α^k (where $0<\alpha<1$), it receives in calculation of the discounted number of routes.

Because, as was already mentioned, the number of the k-step routes is an element of the power of adjacency matrix, $\mathbf{A}^k$, the total discounted number of the routes between nodes i and j (where $i \neq j$) is equal to

$$m_{ij} = \sum_{k=1}^{\infty} \alpha^k a_{ij}^k.$$

Assuming for the sake of simplicity of further calculation that there is a fictitious zero-length route from node i to itself, one obtains the formula

$$\bar{m}_{ii} = \sum_{k=0}^{\infty} \alpha^k a_{ij}^k.$$

Using the notation for the identity matrix, $\mathbf{I} = (\alpha\mathbf{A})^0$, one comes to the formula for the matrix of the total discounted numbers of routes between pairs of nodes:

$$\bar{\mathbf{M}} = \sum_{k=0}^{\infty} (\alpha\mathbf{A})^k = (\mathbf{I} - \alpha\mathbf{A})^{-1}.$$

The total discounted number of routes between node i and all possible nodes j, called sometimes (e.g. [8]) *Bonacich centrality* of node i, is

$$C_B(i) = \sum_{j=1}^{n} \bar{m}_{ij}.$$

The vector of the Bonacich centrality measures for nodes is

$$\mathbf{C}_B = \begin{pmatrix} C_B(1) \\ C_B(2) \\ \dots \\ C_B(n) \end{pmatrix} = \bar{\mathbf{M}} \cdot \mathbf{1} = (\mathbf{I} - \alpha\mathbf{A})^{-1} \cdot \mathbf{1},$$

where $\mathbf{1}$ is the vector of all ones.

We have seen, that in this version of the formula for Bonacich centralities an unwanted 1 enters the measure for each node – it is the number of fictitious "zero-length" routes which was added for the sake of simplicity of calculation. Some authors, to improve this "fault", do subtract these unwanted units; then the formula turns into

$$\mathbf{C}_K = ((\mathbf{I} - \alpha\mathbf{A})^{-1} - \mathbf{I}) \cdot \mathbf{1}.$$

This version is often referred as *Katz centrality*. Both versions are met in the literature, often under similar names, what commonly puzzles readers.

Since only the summary numbers of the routes from node i (but not the numbers of routes to particular nodes) are needed in calculation of both Katz-Bonacich centrality measures, the type adjacency matrix, $\mathbf{T}$, can be, naturally, used instead of the adjacency matrix, $\mathbf{A}$.

The total discounted number of routes from any node of type i to all nodes of type j (where $i, j = 1, 2, \dots, S$; $i \neq j$) is equal to

$$\tilde{m}_{ij} = \sum_{k=1}^{\infty} \alpha^k t_{ij}^k.$$

A fictitious "zero-length" route is interpreted now as a route from a node of type i to nodes of the same type. Accounting for this route, we obtain the following formula for the discounted number of routes from any node of type i to all nodes of the same type:

$$\bar{\tilde{m}}_{ii} = \sum_{k=0}^{\infty} \alpha^k t_{ij}^k,$$

Hence, the Bonacich centrality measure of any node of type i is equal to

$$\tilde{C}_B(i) = \sum_{j=1}^{S} \bar{\tilde{m}}_{ij}.$$

If node i_0 is of type i, then $C_B(i_0) = \tilde{C}_B(i)$. We come to the formula for the matrix of summary discounted numbers of routes between any nodes of type i and all nodes of type j $(i, j = 1, 2, \dots, S)$:

$$\bar{\tilde{\mathbf{M}}} = \sum_{k=0}^{\infty} (\alpha\mathbf{T})^k = (\tilde{\mathbf{I}} - \alpha\mathbf{T})^{-1},$$

where $\tilde{\mathbf{I}}$ is the $S \times S$ unit matrix.

The vector of Bonacich centralities for types is

$$\tilde{\mathbf{C}}_B = \begin{pmatrix} \tilde{C}_B(1) \\ \tilde{C}_B(2) \\ \dots \\ \tilde{C}_B(S) \end{pmatrix} = \bar{\tilde{\mathbf{M}}} \cdot \tilde{\mathbf{1}} = (\tilde{\mathbf{I}} - \alpha\mathbf{T})^{-1} \cdot \tilde{\mathbf{1}},$$

where $\bar{\bar{\mathbf{M}}}, \tilde{\mathbf{I}}, \mathbf{T}$ are $S \times S$-matrices, and $\tilde{\mathbf{1}}$ is the S-vector of all ones. Correspondingly, the vector of Katz centralities for types is

$$\tilde{\mathbf{C}}_K = ((\tilde{\mathbf{I}} - \alpha\mathbf{T})^{-1} - \tilde{\mathbf{I}})\tilde{\mathbf{1}}.$$

As an example, let us calculate the vector of Bonacich centralities for the star graph with ν peripheral nodes. The type adjacency matrix is

$$\mathbf{T} = \begin{pmatrix} 0 & \nu \\ 1 & 0 \end{pmatrix},$$

hence,

$$(\tilde{\mathbf{I}} - \alpha\mathbf{T}) = \begin{pmatrix} 1 & -\alpha\nu \\ -\alpha & 1 \end{pmatrix},$$

$$(\tilde{\mathbf{I}} - \alpha\mathbf{T})^{-1} = \frac{1}{1-\alpha^2\nu}\begin{pmatrix} 1 & \alpha\nu \\ \alpha & 1 \end{pmatrix},$$

$$\tilde{\mathbf{C}}_B = (\tilde{\mathbf{I}} - \alpha\mathbf{T})^{-1} \cdot \tilde{\mathbf{1}} = \frac{1}{1-\alpha^2\nu}(1+\alpha\nu; 1+\alpha)^T.$$

The positive values of centralities are received under values of parameter α sufficiently small in comparison with the size of the star:

$$\alpha < \frac{1}{\sqrt{\nu}}.$$

In other words, the parameter α has to be lower than the inverse of the Frobenius eigenvalue of matrix $\mathbf{T}$.

We see that the Bonacich centrality of the center of the star is $(1+\alpha\nu)/(1+\alpha)$ times higher than the Bonacich centrality of the peripheral node. If $\alpha \to 1/\sqrt{\nu}$, i.e. if α converges to the inverse of the Frobenius eigenvalue, the ratio of the Bonacich centralities of the center and the periphery converges to $\sqrt{\nu}$, i.e. to the ratio of their eigenvalue centralities.

The following two theorems show that the type adjacency matrix $\mathbf{T}$ may be used instead of the adjacency matrix $\mathbf{A}$ for calculation of *eigenvalue centralities* and *α – centralities*.

Let the first type have n_1 nodes, the second type have n_2 nodes, ..., the S-th type have n_S nodes, so that

$$n = n_1 + n_2 + \ldots + n_S.$$

Let the nodes of graph G be numbered in the following way. The nodes of the first type are numbered from 1 to n_1, the nodes of the second type – from $n_1 + 1$ to $n_1 + n_2$, ..., the nodes of the S-th type – from $n_1 + n_2 + \ldots + n_{S-1} + 1$ to n.

Theorem 162. *1. Let λ be an eigenvalue of the type adjacency matrix $\mathbf{T}$ and $\tilde{\mathbf{g}}$ be corresponding eigenvector*

$$\tilde{\mathbf{g}} = (\beta_1; \beta_2; \ldots; \beta_S)^T.$$

Then λ is also an eigenvalue of the adjacency matrix $\mathbf{A}$ and the corresponding eigenvector is

$$\mathbf{g} = (\beta_1; \beta_1; \ldots; \beta_1; \beta_2; \beta_2; \ldots; \beta_2; \ldots; \beta_S; \beta_S; \ldots; \beta_S)^T$$

(where β_i is repeated n_i times; $i = 1, 2, \ldots, S$)).

2. Let λ^* be the Frobenius eigenvalue of matrix $\mathbf{T}$. Then λ^* is also the Frobenius eigenvalue of matrix $\mathbf{A}$.

Proof. 1. For any $i = 1, 2, \ldots, S$ we have for i-th row of matrix $\mathbf{T}$:

$$\beta_1 l_1(i) + \beta_2 l_2(i) + \ldots + \beta_S l_S(i) = \lambda \beta_i.$$

Then for any $i = 1, 2, \ldots, S$ and $j = 1, 2, \ldots, n_i$ we obtain for the $(n_1+n_2+\ldots+n_{i-1}+j)$-th row of matrix $\mathbf{A}$ the equality

$$\underbrace{\beta_1 + \beta_1 + \ldots + \beta_1}_{l_1(i)} + \underbrace{\beta_2 + \beta_2 + \ldots + \beta_2}_{l_2(i)} + \ldots + \underbrace{\beta_S + \beta_S + \ldots + \beta_S}_{l_S(i)} = \lambda \beta_i.$$

2. Assume that the Frobenius eigenvalue of matrix $\mathbf{A}$ is $\mu \neq \lambda^*$. The first part of the theorem implies that $\mu > \lambda^*$. Let $\mathbf{e}$ be the Frobenius eigenvector of matrix $\mathbf{A}$, and $\hat{\mathbf{e}}$ be the vector constructed in the following way. Each of the first n_1 components of vector $\hat{\mathbf{e}}$ is equal to the maximum of the first n_1 components of vector $\mathbf{e}$; each of the next n_2 components of $\hat{\mathbf{e}}$ is equal to the maximum of the corresponding n_2 components of $\mathbf{e}$;...; each of the last n_S components of $\hat{\mathbf{e}}$ is equal to the maximum of the last n_S components of $\mathbf{e}$. Let $\hat{\mathbf{f}}$ be the S-vector corresponding to $\hat{\mathbf{e}}$ (i.e. i-th component of $\hat{\mathbf{f}}$ $(i = 1, 2, \ldots, S)$ is equal to the maximum of n_i corresponding components of $\mathbf{e}$). Evidently,

$$\mathbf{A}\hat{\mathbf{e}} \geq \mu \hat{\mathbf{e}}.$$

Correspondingly,

$$\mathbf{T}\hat{\mathbf{f}} \geq \mu \hat{\mathbf{f}}. \tag{2}$$

But, according to the Perron-Frobenius theorem, since λ^* is the Frobenius eigenvalue, (2) implies that $\lambda^* \geq \mu$. Contradiction!

□ □

Theorem 163. *Let* $\mathbf{A}$ *be adjacency matrix,* $\mathbf{T}$ *be type adjacency matrix,* α *be any number,* $\tilde{\mathbf{e}} = (e_1; e_2; \ldots; e_S)^T$ *be any* S*-vector and let the matrix* $\tilde{\mathbf{I}} + \alpha\mathbf{T}$ *be invertible. Then the matrix* $\mathbf{I} + \alpha\mathbf{A}$ *is also invertible. Let* κ *be the* S*-vector of* α*-centralities,*

$$\kappa = (\tilde{\mathbf{I}} + \alpha\mathbf{T})^{-1}\tilde{\mathbf{e}}. \tag{3}$$

Let $\mathbf{e}$ *be the* n*-vector,* n_1 *first components of which are equal to the component* e_1*;* n_2 *following components are equal to* e_2*; ...; the last* n_S *components are equal to* e_S*. Let* $\mathbf{k}$ *be the* n*-vector of* α*-centralities,*

$$\mathbf{k} = (\mathbf{I} + \alpha\mathbf{A})^{-1}\mathbf{e}. \tag{4}$$

Then the first n_1 *components of vector* $\mathbf{k}$ *are equal to* κ_1*; the next* n_2 *components are equal to* κ_2*; ...; the last* n_S *components are equal to* κ_S*.*

Proof. The system of equations (3) is equivalent to

$$\kappa + \alpha\mathbf{T}\kappa = \tilde{\mathbf{e}}. \tag{5}$$

Writing the system (5) for individual components, we obtain

$$\kappa_i = \alpha(l_1(i)\kappa_1 + l_2(i)\kappa_2 + \ldots + l_S(i)\kappa_S) = \tilde{e}_i, \tag{6}$$

where $i = 1, 2, \ldots, S$. But for each i equation (6) coincides with each of the equations with numbers from $n_1 + n_2 + \ldots + n_{i-1} + 1$ until $n_1 + n_2 + \ldots + n_i$ of the system

$$\mathbf{k} + \alpha\mathbf{A}\mathbf{k} = \mathbf{e},$$

which is equivalent to system (4). The invertibility of the matrix $\mathbf{I} + \alpha\mathbf{A}$ follows from invertibility of the matrix $\tilde{\mathbf{I}} + \mathbf{T}$. □ □

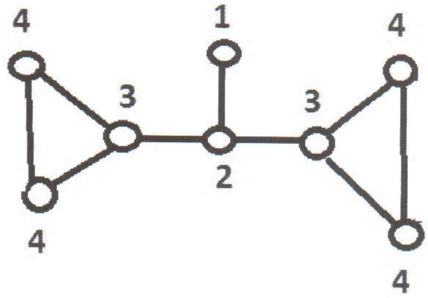

Figure 9: Graph with 2 peripheral components ($c = 2$).

3.3 Growth of a core-periphery network

The graph shown in Figure 9 has the following structure: the central node (type 2), two symmetric peripheral components (with nodes of types 3 and 4) and a node close to the center (type 1). This graph can be described either by the adjacency matrix $\mathbf{M}$ of order 8 or by the type adjacency matrix of order 4:

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Any of matrices $\mathbf{M}$ and $\mathbf{T}$ can be used for calculation of the eigenvector centralities (which are approximately proportional to 0.25, 0.61, 0.61, 0.43 for the types 1, 2, 3, 4, correspondingly).

Now, let the number of symmetric peripheral components, c, increase (Figure 10). The type adjacency matrix becomes

$$\mathbf{T}(c) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & c & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Despite the increase of the order of the adjacency matrix, $(3c+2) \times (3c+2)$, the order of the type adjacency matrix, 4×4, does not change; moreover, the only element which changes in the type adjacency matrix is $c = l_3(2)$ – the number of the 3^{rd} type nodes.

Below we list the eigenvectors of the type adjacency matrices with the numbers of peripheral components $c = 2, 3, 4, 5, 6$.

$$\mathbf{x}(2) = \begin{pmatrix} 1 \\ 2.44 \\ 2.44 \\ 1.72 \end{pmatrix}, \mathbf{x}(3) = \begin{pmatrix} 1 \\ 2.67 \\ 1.96 \\ 1.22 \end{pmatrix}, \mathbf{x}(4) = \begin{pmatrix} 1 \\ 2.82 \\ 1.71 \\ 0.96 \end{pmatrix}, \mathbf{x}(5) = \begin{pmatrix} 1 \\ 2.96 \\ 1.54 \\ 0.79 \end{pmatrix}, \mathbf{x}(6) = \begin{pmatrix} 1 \\ 3.07 \\ 1.43 \\ 0.68 \end{pmatrix}.$$

It is interesting to observe how the eigenvector centrality of the center (type 2) increases with respect to c, and how at the same time the relative centrality of type 1 (which is a neighbor of the center) increases in relation not only to type 4 but even to type 3. The type adjacency matrix $\mathbf{T}$ allows a simple and visible analysis without any increase in the order of the matrix.

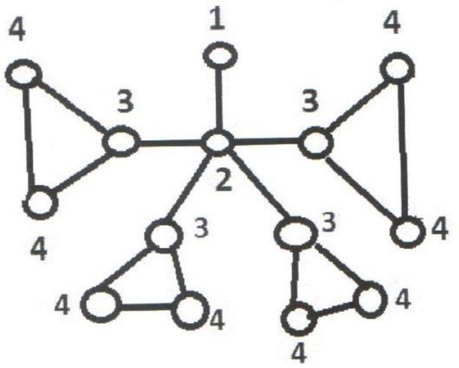

Figure 10: Graph with 4 peripheral components ($c = 4$).

Theorem 164. *With increase of c, if the eigenvector centrality of the* 1^{st} *type node is taken constant equal to 1, then the centrality of the* 2^{nd} *type node increases slower than* $\sqrt{c}$ *and goes to infinity; the centrality of each* 3^{rd} *type node decreases and converges to 1; the centrality of each* 4^{th} *type node decreases faster than* $1/\sqrt{c}$ *and converges to 0.*

Proof. The characteristic equation of the matrix $\mathbf{T}$ is

$$(\lambda - 1)[\lambda^3 - (c+3)\lambda - 2] = 0.$$

To the eigenvalue $\lambda = 1$ the eigenvector $(1, 1, 0, -1/2)^T$ corresponds. The equation for other three eigenvalues is

$$\lambda^3 - (c+3)\lambda - 2 = 0. \quad (7)$$

This equation is of the form

$$\lambda^3 + 3p\lambda + 2q = 0,$$

where

$$p = -\frac{c+3}{3}, \ q = -1.$$

Since the discriminant is

$$D = q^2 + p^3 = 1 - \frac{(c+3)^3}{27} < 0$$

(because $c \geq 2$), the equation has three real roots:

$$\lambda_1 = -2rcos\frac{\phi}{3}, \ \lambda_2 = 2rcos\frac{\pi - \phi}{3}, \ \lambda_3 = 2r\frac{\pi + \phi}{3},$$

where $r = \pm\sqrt{|p|} = \pm\sqrt{(c+3)/3}$ (the sign of r is chosen to coincide with the sign of q); the angle ϕ is determined by the relation

$$cos\phi = \frac{q}{r^3} = \left(\frac{3}{c+3}\right)^{3/2}.$$

Under $c = 2:\ \phi = \arccos{(3/5)^{3/2}} \approx 1.087$. With increase in c the angle ϕ also increases and $\lim_{c\to\infty} \phi = \pi/2$. Thus, of the three roots of the equation, only the first one is positive:

$$\lambda_1 = 2\sqrt{\frac{c+3}{3}} \cos\left(\frac{1}{3} \arccos\left(\frac{3}{c+3}\right)^{3/2}\right). \quad (8)$$

Under $c = 2$:

$$\lambda_1 = 2\sqrt{\frac{5}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{3}{5}\right)^{\frac{3}{2}}\right) \approx 2.414 > 1;$$

with increase in c the value λ_1 also increases and $\lim_{c\to\infty}\lambda_1 = +\infty$. Thus, under any c the Frobenius eigenvalue of matrix $\mathbf{T}(c)$ is given by (8). Let us find the general form of the dependence of the coordinates of the eigenvectors corresponding the eigenvalues obtained from equation (7). The eigenvector is defined by the equation

$$\mathbf{T}(c)\mathbf{X} = \lambda\mathbf{X},$$

or, in coordinate form:

$$\begin{cases} \lambda x_1 = x_2, \\ x_1 + cx_3 = \lambda x_2, \\ x_2 + 2x_4 = \lambda x_3, \\ x_3 + (1-\lambda)x_4 = 0. \end{cases}$$

Solving these system of equations and using the fact that $\lambda \neq 1$, we find the eigenvector corresponding to the eigenvalue λ:

$$\mathbf{X} = \left(1, \lambda, \frac{\lambda^2 - 1}{c}, \frac{\lambda + 1}{c}\right)^T.$$

If λ is given by (7), then the components of the vector $\mathbf{X}$ are the values of eigenvector centrality of the nodes of the corresponding types. This implies the demanded result. □

□

4 Game equilibria in a model of production and externalities in network

One more field of application of the type adjacency matrix is analysis of game equilibria in economic networks. An example is the network model of production with knowledge externalities [14], [15].

Behavior of agents in a network structure is defined in much by actions of their neighbors or by information received from them. Network economics and network games theory consider questions of network formation, spreading of information in networks, positive and negative externalities, complementarity and substitutability of activities (see reviews [9], [7], [10]). Externalities, i.e. influence of other agents which does not go through the price mechanism, possess properties of public goods and are not fully paid. Positive externalities, and among them externalities of knowledge and human capital, spring up both in processes of production [16], [13] and consumption [3], and it is important to account for them in economic and sociological analysis, forecasting, and mechanism design.

Matveenko and Korolev [14], [15] continue the line of research of Nash equilibria in networks in presence of positive externalities and introduce several new elements in comparison to the previous literature. Firstly, production externalities are studied; agents' efforts have meaning of investments, in particular, investments into knowledge. The presence of production block allows to interpret game-theoretic concepts of strategic complementarity (supermodularity) and strategic substitutability (submodularity) as, correspondingly, absence and presence of productivity and to analyse these concepts within the same model.

Secondly, the model, for the first time in the network literature, uses the notion of the Jacobian production externality [11], [16], [13] in definition of the concept of game equilibrium. The essence of this notion is that any agent makes her decision staying in a particular environment which depends on actions produced by the agent herself and by her neighbors in network. When making her decision, the agent considers the state of the environment as exogenous; this means that the agent does not take into account that her actions can directly influence the state of the environment. As a simplest example, imagine a game equilibrium in a collective of smokers and non-smokers. A smoker, when making in equilibrium a decision to continue or to give up smoking, makes it staying in an environment relating already to her smoking.

The third novation of the model is the use of dynamic approach. Essentially, the model is a network generalization of the simple two-period model of endogenous growth and knowledge externalities of Romer [16].

We will see that an important place in the model is played by the typology of nodes described above. The typology defines behavior of agents in equilibrium and allows to consider a possibility of transplantation of equilibrium among networks of different size but the same typology.

4.1 Description of the model

There is an undirected graph (network), G, with n nodes $i = 1, 2, \ldots, n$; each node represents an agent. In time period 1 each agent i possesses endowment e of good and can use it partly (or wholly) for consumption in the 1^{st} period, c_i^1, and partly (or wholly) for investment into knowledge, k_i. The investment is immediately transformed into the stock of knowledge and is used in production of good for consumption in the 2^{nd} period, c_i^2. Agent's preferences are described by quadratic utility function

$$U(c_i^1, c_i^2) = c_i^1(e - ac_i^1) + dc_i^2,$$

where a is a satiation coefficient; $d > 0$. It is assumed that under $c_i^1 \in [0, e]$ the utility increases in c_i^1. These assumptions imply that $0 < a < 1/2$. The production in node i is described by function

$$F(k_i, K_i) = gk_iK_i \ (g > 0),$$

which depends on the state of knowledge, k_i, and the environment, K_i. The environment, by definition, is the sum of investments of the agent herself and her neighbors (the agents in the adjacent nodes of the graph, $j \in N(i)$).

Since increase in each of the parameters d and g promotes increase in the 2^{nd} period consumption, we denote $dg = b$ and talk about parameter b as a productivity. We assume $b > a$. If $b > 2a$ we say that *productivity presents*, and if $b < 2a$ we say that *productivity absents.*

Let us consider the following game. Players are the agents $i = 1, 2, \ldots, n$. Strategies of player i are her feasible volumes of investment, $k_i \in [0, e]$. *Nash equilibrium with externalities* (for shortness, *equilibrium* is a profile of players' strategies, $k_1^*, k_2^*, \ldots, k_n^*$, such that each k_i^* solves the agent's problem:

$$\max_{c_1^i, c_2^i, k_i} U(c_i^1, c_i^2)$$

s.t.

$$c_i^1 \le e - k_i,$$

$$c_i^2 \le F(k_i, K_i),$$

$$c_i^1 \geq 0, c_i^2 \geq 0, k_i \geq 0,$$

given $K_i = k_i^* + \tilde{K}_i$, where $\tilde{K}_i = \Sigma_{j \in N(i)} k_j^*$ is the sum of investments by the player i's neighbors in the network. If $k_i^* \in (0, e), i = 1, 2, \ldots, n$, the equilibrium is called *inner*.

The first order conditions imply (see details in [14], [15]) that the inner equilibrium (when it exists given values of parameters) is uniquely defined by the system of equations

$$(b - 2a)\mathbf{k} + b\mathbf{A}\mathbf{k} = e(1 - 2a)\mathbf{1},$$

where $\mathbf{k} = (k_1, k_2, \ldots, k_n)^T$, $\mathbf{A}$ is the adjacency matrix of graph G, $\mathbf{1}$ is the vector of all ones. It follows that

$$k^* = (\mathbf{I} - \alpha\mathbf{A}^{-1})\tilde{\mathbf{e}},$$

where $\alpha = b/(2a - b)$, $\tilde{\mathbf{e}} = [e(1 - 2a)/(b - 2a)]\mathbf{1}$. Thus, agents' strategies are defined by their α-centralities in the network.

4.2 Usage of the type adjacency matrix in analysis of game equilibria

An alternative way to find the inner equilibrium is by use of the type adjacency matrix. We have

$$(b - 2a)\hat{\mathbf{k}} + b\mathbf{T}\hat{\mathbf{k}} = e(1 - 2a)\hat{\mathbf{1}}, \tag{9}$$

where $\hat{\mathbf{k}} = (\hat{k}_1, \hat{k}_2, \ldots, \hat{k}_s)^T$ is the vector of investments by types, $\hat{\mathbf{1}}$ is the S-vector with all ones. This implies

$$\hat{\mathbf{k}}^* = (\hat{\mathbf{I}} - \alpha\mathbf{T}^{-1})\hat{\mathbf{e}},$$

where $\alpha = b/(2a - b)$, $\hat{\mathbf{e}} = [e(1 - 2a)/(b - 2a)]\hat{\mathbf{1}}$.

In the inner equilibrium (which is unique) all agents of the same type use the same strategy, i.e. make the same investment (defined by the α-centrality of the type). Moreover, if two networks are characterized by the same type adjacency matrix $\mathbf{T}$, then their inner equilibria do coincide, in the sense that agents in the nodes of the same type make the same investment.

Thus, the characterization of the inner equilibrium in terms of the types of nodes is especially important because it shows a possibility of transplantation of the inner equilibrium from one network into another network with different size but the same typology.

For example, in Figure 11 three graphs are shown which have different sizes but the same typology: the type adjacency matrix is

$$\mathbf{T} = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}.$$

Transplantation of equilibrium among these graphs is possible.

Generally, let there be two types of nodes characterized by the type adjacency matrix

$$\mathbf{T} = \begin{pmatrix} s_1 & s_2 \\ t_1 & t_2 \end{pmatrix}.$$

Then (9) implies the system of linear equations

$$\begin{cases} (b - 2a + s_1 b)k_1 + s_2 b k_2 = e(1 - 2a), \\ t_1 b k_1 + (b - 2a + t_2 b)k_2 = e(1 - 2a), \end{cases}$$

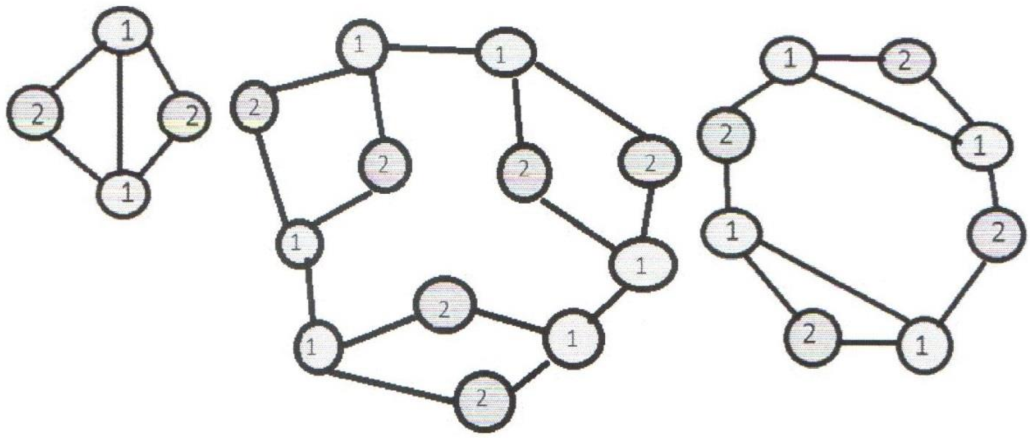

Figure 11: Three graphs with the same typology.

where k_1, k_2 are investments of the types. The solution of the system is the pair

$$k_1 = \frac{e(1-2a)[b-2a+(t_2-s_2)b}{(b-2a)^2+(s_1+t_2)(b-2a)b+(s_1t_2-t_1s_2)b^2}, \tag{10}$$

$$k_2 = \frac{e(1-2a)[b-2a+(s_1-t_1)b}{(b-2a)^2+(s_1+t_2)(b-2a)b+(s_1t_2-t_1s_2)b^2}, \tag{11}$$

If $0 < k_i < e,\ i = 1, 2$, then the values k_1, k_2 define the inner equilibrium of the game.

Here we limit ourselves by several examples of the networks with two types of nodes.

For the chain of four nodes (with the order of types: 2–1–1–2) the type adjacency matrix is

$$\mathbf{T} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Formulas (10)-(11) take the form

$$k_1 = \frac{2ae(1-2a)}{6ab-4a^2-b^2},$$

$$k_2 = \frac{e(1-2a)(2a-b)}{6ab-4a^2-b^2}.$$

The conditions of inner equilibrium $0 < k_i < e, i = 1, 2$ are fulfilled under absence of productivity ($b < 2a$.)

A generalization of the previous case is a fan, i.e. a dyad to each node of which a bundle of ν hanging nodes is adjoined. The type adjacency matrix of the fan is

$$\mathbf{T} = \begin{pmatrix} 1 & \nu \\ 1 & 0 \end{pmatrix}.$$

An important example of network with two types of nodes is the star network; let us remind that its type adjacency matrix is

$$\mathbf{T} = \begin{pmatrix} 0 & \nu \\ 1 & 0 \end{pmatrix}.$$

Equations (9)-(10) turn into

$$k_1 = \frac{e(1-2a)[(\nu-1)b+2a]}{\nu b^2-(b-2a)^2},$$

$$k_2 = \frac{2ea(1-2a)}{\nu b^2 - (b-2a)^2}.$$

The pair k_1, k_2 defines inner equilibrium if $0 < k_i < e, i = 1, 2$, i.e. if

$$\begin{cases} \nu b^2 - (b-2a)^2 > 0, \\ \nu b^2 - (b-2a)^2 > (1-2a)[(\nu-1)b + 2a], \\ \nu b^2 - (b-2a)^2 > 2a(1-2a). \end{cases}$$

The third and the first inequalities follow from the second one, and the latter is fulfilled for all ν if $b + 2a > 1$ and $b > (-6a + 1 + \sqrt{36a^2 - 4a + 1})/2$.

The following proposition identifies agents interested in growth of the star network.

Theorem 165. *In star network, if the number of peripheral nodes, ν , increases, then knowledge and utility in the central node decrease under absence of productivity, but increase under presence of productivity. Knowledge and utility in each peripheral node always decrease.*

Proof. Derivative of k_1 in ν (if ν is considered as a continuous variable) is

$$\frac{2bae(1-2a)(b-2a)}{[(b-2a)^2 - \nu b^2]^2}.$$

Hence, knowledge in the central node decreases in ν if $b < 2a$ and increases if $b > 2a$. It is directly seen that k_2 decreases in ν. According to Theorem 2.2 in [15], utility increases in knowledge. □ □

Acknowledgements

This research is supported by the Russian Foundation for Basic Research (project 17-06-00618).

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