EA-Matrix integrals of associative algebras and equivariant localization.

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To R.K.Gordin on the occasion of his 70th birthday

Keywords Mirror symmetry, Gromov-Witten invariants, Noncommutative varieties, Batalin-Vilkovisky formalism
Abstract The EA-matrix integrals, introduced in [B1], are studied in the case of graded associative algebras with odd or even scalar product. I prove that the EA-matrix integrals for associative algebras with scalar product are integrals of equivariantly closed differential forms with respect to Lie algebra $gl_N(A)$.

Introduction

The theory of periods of noncommutative varieties, depending on commutative parameters, was introduced in [B5]. The analogue of top-degree holomorphic form in this setting was shown in *loc.cit.* to be certain element of semi-infinite subspace of negative cyclic homology. The integrals of this element satisfy the second order equation with respect to the parameters of deformations of the varieties. It was proven in *loc.cit.* that the generating function of genus zero Gromov-Witten invariants of complete intersection in $\mathbb{CP}^d$ with trivial canonical class coincides with the coefficient of this second order equation for the family of mirror varieties. This approach had singled out the $A_\infty-$ algebras/categories, satisfying cyclic homology analogue of degeneration of Hodge to de Rham spectral sequence, as the proper definition of (smooth and compact) noncommutative varieties.

The EA-matrix integrals were introduced in [B1] as a set of periods of associative, more generally $A_\infty-$ algebras or noncommutative varieties, depending on *noncommutative* parameters:

$$\mathcal{F}(Y)=\int_{\Gamma}\exp(Tr\langle Y,X\rangle+\frac{1}{3!}m_{\tilde{A}}(X,\frac{\partial}{\partial X}))\vdash\prod_{\alpha,i,j}dX_i^{\alpha,j}$$

$\tilde{A}=A\otimes q_N$ / $\tilde{A}=A\otimes gl_N$ in even/odd scalar product case, here q_N is the odd matrix algebra, see *loc.cit.* It was shown in theorem 3 in *loc.cit.* that the matrix Airy integral from [K] corresponds in this way to the simplest associative algebra of one dimension $A=\{e|e^2=e\}$.

The usual varieties correspond here to $A_\infty-$ algebras of endomorphisms of generators of their $D^b(\mathrm{Coh})-$ categories.

The asymptotic expansion of EA-matrix integrals via BV formalism was shown in [B1, B3] to define, as a sum over generalized ribbon graphs, a generating function for series of cohomology classes of compactified moduli spaces of curves of all genus. A particular example is the formula for cohomology-valued generating function for products of ψ-classes, $\psi_i=c_1(T^*_{p_i})$, in the cohomology $H^*(\bar{\mathcal{M}}_{g,n})$ calculated by the stable ribbon graph complex ([B3]):

$$\sum_{\sum d_i=d}\psi_1^{d_1}\dots\psi_n^{d_n}\prod_{i=1}^{n}\frac{(2d_i-1)!!}{\lambda_i^{(2d_i+1)}}=\left[\sum_{G\in\Gamma_{g,n}^{dec,odd}}G\,\frac{2^{-\chi(G)}}{|\mathrm{Aut}(G)|}\prod_{e\in\mathrm{Edge}(G)}\frac{1}{\lambda_{i(e)}+\lambda_{j(e)}}\right]\tag{0.1}$$

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where the sum on the right is over *stable ribbon* oriented graphs of genus g with n numbered punctures, with $2d+n$ edges, and such that at each vertex the cyclically ordered subsets of edges have arbitrary *odd* cardinality.

In this paper it is proven that the EA-matrix integrals for associative algebras with scalar product are integrals of equivariantly closed differential forms with respect to the Lie algebra $gl_N(A)$. This generalizes and clarifies the similar result with respect to the Lie algebra gl_N from [B2]. The localization formula for the $gl_N(A)-$ action then leads in [B4] to calculation of these EA matrix integrals via determinants and $\tau-$ functions of integrable hierarchies.

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Notations

For a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $A=A_0\oplus A_1$ denote via ΠA the parity inverted vector space, $(\Pi A)_0=A_1$, $(\Pi A)_1=A_0$. For an element a from $\mathbb{Z}/2\mathbb{Z}$-graded vector space A denote by $\pi a\in\Pi A$ the same element considered with parity reversed.

1 Equivariantly closed De Rham differential form.

Let $A=A_0\oplus A_1$ denotes a $\mathbb{Z}/2\mathbb{Z}$-graded associative algebra, $\dim_k A_0=r<\infty$, $char(k)=0$, with multiplication denoted by $m_2: A^{\otimes 2}\to A$. Let A be endowed with *odd* invariant non-degenerate scalar product $\langle\cdot,\cdot\rangle: A_0\otimes A_1\to k$. The multiplication tensor can be viewed then as the $\mathbb{Z}/3\mathbb{Z}$ - cyclically invariant linear function on $(\Pi A)^{\otimes 3}$

$$m_A:(\pi a_1,\pi a_2,\pi a_3)\mapsto (-1)^{\bar a_2+1}\langle m_2(a_1,a_2),a_3\rangle,\ m_A\in(\mathrm{Hom}((\Pi A)^{\otimes 3},k))^{\mathbb{Z}/3\mathbb{Z}}.$$

The odd symmetric scalar product on A corresponds to the odd anti-symmetric product $\langle\cdot,\cdot\rangle^\pi$ on ΠA:

$$\langle \pi a_1,\pi a_2\rangle^\pi=(-1)^{\bar a_1+1}\langle a_1,a_2\rangle$$

The tensor product with the matrix algebra gl_N is again naturally a $\mathbb{Z}/2\mathbb{Z}-$ graded associative algebra with the odd scalar product $A\otimes gl_N$. The cyclic tensor

$$m_{A\otimes gl_N}\in(\mathrm{Hom}((\Pi A\otimes gl_N)^{\otimes 3},k))^{\mathbb{Z}/3\mathbb{Z}} \tag{1.1}$$

restricted to the diagonal $\Pi A\otimes gl_N\subset(\Pi A\otimes gl_N)^{\otimes 3}$ is $GL(N)$-invariant cubic polynomial, denoted by $m_{A\otimes gl_N}(Z)$, $Z\in\Pi A\otimes gl_N$. The associativity of the algebra A translates into the equation

$$\{m_{A\otimes gl_N}(Z),m_{A\otimes gl_N}(Z)\}=0, \tag{1.2}$$

where $\{\cdot,\cdot\}$ is the odd Poisson bracket corresponding to the odd anti-symmetric product $Tr|_{gl_N^{\otimes 2}}\otimes\langle\cdot,\cdot\rangle^\pi$ on $\Pi A\otimes gl_N$.

Proposition 1 *The algebra of functions on $\Pi A\otimes gl_N$ is identified naturally, preserving the odd Poisson bracket, with the algebra of polyvectors on the even affine space $\Pi A_1\otimes gl_N$.* $\square$

This is analogous to algebra of functions on symplectic space being identified naturally, preserving Poisson bracket, with algebra of functions on cotangent bundle of given lagrangian subspace.

Denote by $X^\alpha \in gl_N$, $P_\alpha \in \Pi gl_N$ the matrices of coordinates on $\Pi A \otimes gl_N$ corresponding to a choice of a dual pair of bases $\{e^\alpha\}$, $\{\xi_\alpha\}$ on A_0 and A_1 so that

$$Z = \sum_\alpha \pi\xi_\alpha \otimes X^\alpha + \pi e^\alpha \otimes P_\alpha \tag{1.3}$$

Then $(P_\alpha)^i_j$ corresponds to the vector field $\frac{\partial}{\partial (X^\alpha)^j_i}$ on $\Pi A_1 \otimes gl_N$. The cubic polynomial $\frac{1}{3!}m_{A\otimes gl_N}(Z)$ corresponds to the sum of the function and the bivector,

$$\frac{1}{3!}\sum_{\alpha,\beta,\gamma}(m_A)_{\alpha\beta\gamma}Tr(X^\alpha X^\beta X^\gamma) + \frac{1}{2}\sum_{\alpha,\beta,\gamma}(m_A)^{\beta\gamma}_\alpha Tr(X^\alpha P_\beta P_\gamma). \tag{1.4}$$

The odd Poisson bracket is generated by the odd second order Batalin-Vilkovisky differential Δ acting on the algebra of functions on $\Pi A \otimes gl_N$

$$\{f_1, f_2\} = (-1)^{\bar{f}_1}(\Delta(f_1 f_2) - \Delta(f_1)f_2 + (-1)^{\bar{f}_1}f_1\Delta(f_2)) \tag{1.5}$$

$$\Delta = \sum_{\alpha,i,j}\frac{\partial^2}{\partial X^{\alpha,j}_i \partial P^i_{\alpha,j}}$$

1.1 Divergence-free condition

Let us assume from now on that the Lie algebra A_0 is unimodular.

Condition 2 *(unimodularity of A_0) For any $a \in A_0$*

$$tr([a,\cdot]|_{A_0}) = 0 \tag{1.6}$$

Proposition 3 *The unimodularity of A_0 (1.6) implies*

$$\Delta m_{A\otimes gl_N}(Z) = 0. \tag{1.7}$$

□

Next proposition is the standard corollary of the equations (1.2),(1.7) and the relation (1.5).

Proposition 4 *The exponent of the sum (1.4) is closed under the Batalin-Vilkovisky differential*

$$\Delta \exp\left(\frac{1}{3!}m_{A\otimes gl_N}(Z)\right) = 0.$$

□

1.2 Closed De Rham differential form

The affine space $\Pi A_1 \otimes gl_N$ has a holomorphic volume element, defined canonically up to a multiplication by a constant

$$\varpi = \lambda \prod_{\alpha,i,j} dX^{\alpha,j}_i .$$

It identifies the polyvectorfields on $\Pi A_1 \otimes gl_N$ with the de Rham differential forms $\Omega_{\Pi A_1 \otimes gl_N}$ on the same affine space via

$$\gamma \mapsto \gamma \vdash \varpi$$

The Batalin-Vilkovisky differential Δ corresponds then to the De Rham differential d_{DR} acting on the differential forms. By the proposition 4 the polyvector $\exp\frac{1}{3!}\left(m_{A\otimes gl_N}(Z)\right)$ defines the closed differential form

$$\Psi(X)=\exp\left(\frac{1}{3!}\sum_{\alpha,\beta,\gamma}(m_A)_{\alpha\beta\gamma}Tr(X^{\alpha}X^{\beta}X^{\gamma})+\frac{1}{2}\sum_{\alpha,\beta,\gamma}(m_A)_{\alpha}^{\beta\gamma}Tr(X^{\alpha}\frac{\partial}{\partial X^{\beta}}\wedge\frac{\partial}{\partial X^{\gamma}})\right)\vdash\lambda\prod_{\alpha,i,j}dX_i^{\alpha,j} \quad (1.8)$$

$$d_{DR}\Psi(X)=0$$

It is a sum of the closed differential forms of degrees rN^2 , rN^2-2 ,....

1.3 Equivariantly closed differential form

The unimodularity (1.6) implies the invariance of ϖ under the co-adjoint action of the Lie algebra $A_0\otimes gl_N$

$$X\mapsto[Y,X],$$

$Y\in A_0\otimes gl_N$. Consider the $A_0\otimes gl_N$ -equivariant differential forms on $\Pi A_1\otimes gl_N$:

$$\Omega_{\Pi A_1\otimes gl_N}^{A_0\otimes gl_N}=(\Omega_{\Pi A_1\otimes gl_N}\otimes\mathcal{O}_{A_0\otimes gl_N})^{A_0\otimes gl_N}.$$

The $A_0\otimes gl_N$ -equivariant differential is given by

$$d_{A_0\otimes gl_N}\Phi(Y)=d_{DR}\Phi-\sum_{\alpha,l,j}Y_{a,j}^{l}(i_{[E_l^j\otimes e^a,\cdot]}\Phi)$$

$\Phi\in\Omega_{\Pi A_1\otimes gl_N}^{A_0\otimes gl_N}$, where i_γ denotes the contraction operator with respect to the vector field γ, see e.g. [DKV]. This differential corresponds, when passing to functions on $\Pi A\otimes gl_N$, to the sum

$$\Delta_{A_0\otimes gl_N}\ :\ f(Z,Y)\mapsto\Delta f-\frac{1}{2}Tr\langle[Y,Z],Z\rangle^{\pi}f,$$
$$f(Z,Y)\in(\mathcal{O}_{\Pi A\otimes gl_N}\otimes\mathcal{O}_{A_0\otimes gl_N})^{A_0\otimes gl_N}$$

of the Batalin-Vilkovisky differential and the operator of multiplication by the odd quadratic function

$$\frac{1}{2}Tr\langle[Y,Z],Z\rangle^{\pi}=m_{A\otimes gl_N}(Y\otimes Z\otimes Z). \quad (1.9)$$

The function depends on the equivariant parameters $Y\in A_0\otimes gl_N$.

Theorem 1 *The product of the closed de Rham differential form* $\Psi(X)$ *(1.8) with the function* $\exp Tr\langle Y,X\rangle$, $Y\in A_0\otimes gl_N$, $X\in\Pi A_1\otimes gl_N$, *is* $A_0\otimes gl_N$ *-equivariantly closed differential form:*

$$d_{A_0\otimes gl_N}(\exp(Tr\langle Y,X\rangle+\frac{1}{3!}m_{A\otimes gl_N}(X,\frac{\partial}{\partial X}))\vdash\lambda\prod_{\alpha,i,j}dX_i^{\alpha,j})=0$$

Proof Denote by $i_{m(X\frac{\partial}{\partial X}\frac{\partial}{\partial X})}$ the operator of contraction with the bivector field $\frac{1}{2}\sum_{\alpha,\beta,\gamma}(m_A)_{\alpha}^{\beta\gamma}Tr(X^{\alpha}\frac{\partial}{\partial X^{\beta}}\wedge\frac{\partial}{\partial X^{\gamma}})$ and by $R_{Tr(YdX)}$ the operator of exterior multiplication by the 1-form $Tr\langle Y,dX\rangle$ acting on differential forms,

$$R_{Tr(YdX)}=[d_{DR},i_{Tr\langle Y,X\rangle}]$$

where $i_{Tr\langle Y,X\rangle}$ is the multiplication by the linear function $Tr\langle Y,X\rangle$. Then

$$[i_{m(X\frac{\partial}{\partial X}\frac{\partial}{\partial X})},R_{Tr(YdX)}]=i_{[\cdot,Y]}$$

This is simply a particular case of the standard relation

$$[i_{\gamma_1},Lie_{\gamma_2}]=i_{[\gamma_1,\gamma_2]}$$

for the action of polyvector fields. Notice that

$$d_{DR}e^{Tr\langle Y,X\rangle}=e^{Tr\langle Y,X\rangle}\left(d_{DR}+R_{Tr(YdX)}\right)$$

and that

$$R_{Tr(YdX)}\exp\left(i_{m(X\frac{\partial}{\partial X}\frac{\partial}{\partial X})}\right)=\exp\left(i_{m(X\frac{\partial}{\partial X}\frac{\partial}{\partial X})}\right)\left(R_{Tr(YdX)}+i_{[\cdot,Y]}\right) \tag{1.10}$$

Since $d_{DR}\Psi(X)=0$, and $R_{Tr(YdX)}\prod_{\alpha,i,j}dX_i^{\alpha,j}=0$, therefore

$$d_{DR}(e^{Tr\langle Y,X\rangle}\Psi(X))=i_{[\cdot,Y]}e^{Tr\langle Y,X\rangle}\Psi(X).\ \square$$

2 The integral.

The closed differential form $\Psi(X)$ is integrated over the cycles, which are standard in the theory of exponential integrals $\int_\Gamma \exp f$, see ([AVG] and references therein):

$$\Gamma\in H_*(M,\mathrm{Re}(f)\to-\infty),\ \ M=\Pi A_1\otimes gl_N(\mathbb{C}). \tag{2.1}$$

Here f is the first term in (1.4), which is the restriction of the cubic polynomial $\frac{1}{3!}m_{A\otimes gl_N}(Z)$ on M .

The relative homology are the same for such f , $f\neq 0$, and for $f+Tr\langle Y,X\rangle$ since linear term is dominated by the cubic term when $|X|\to+\infty$. Choosing a real form of $A_0\otimes gl_N(\mathbb{C})$ and taking the cycles in $H_*(M,\mathrm{Re}(f)\to-\infty)$ invariant with respect to this Lie algebra gives natural cycles for integration of the equivariantly closed differential form $e^{Tr\langle Y,X\rangle}\Psi(X)$

$$\mathcal{F}(Y)=\int_\Gamma \exp(Tr\langle Y,X\rangle+\frac{1}{3!}m_{A\otimes gl_N}(X,\frac{\partial}{\partial X}))\vdash\prod_{\alpha,i,j}dX_i^{\alpha,j}$$

In general the integration cycles are the elements of the equivariant homology

$$H_{*,A_0\otimes gl_N}(M,\mathrm{Re}(\frac{1}{3!}\sum_{\alpha,\beta,\gamma}(m_A)_{\alpha\beta\gamma}Tr(X^\alpha X^\beta X^\gamma))\to-\infty)$$

One can consider also the normalized integral

$$\widehat{\mathcal{F}}(Y)=\int_\Gamma \exp(Tr\langle Y,X\rangle+\frac{1}{3!}m_{A\otimes gl_N}(X,\frac{\partial}{\partial X}))\vdash\prod_{\alpha,i,j}dX_i^{\alpha,j}/\mathcal{F}_{[2]}(Y) \tag{2.2}$$

where $\mathcal{F}_{[2]}(Y)$ is the corresponding Gaussian integral of the quadratic part of $f+Tr\langle Y,X\rangle$ at a critical point $(-Y)^{\frac{1}{2}}$.

Let the associative algebra A_0 has an *anti-involution* $a\to a^\dagger$

$$(ab)^\dagger=b^\dagger a^\dagger,(ca)^\dagger=\bar{c}a^\dagger,\mathrm{tr}\left(a^\dagger\right)=\overline{\mathrm{tr}\left(a\right)},\left(a^\dagger\right)^\dagger=a$$

The anti-involution defines the natural cycle for the equivariant integration. This anti-involution extends naturally to $A_0\otimes gl_N(\mathbb{C})$. Then the Lie subalgebra of anti-hermitian elements in $A_0\otimes gl_N(\mathbb{C})$

$$u_N(A_0)=\left\{Y^\dagger=-Y\mid Y\in A_0\otimes gl_N(\mathbb{C})\right\}$$

is a real form of $A_0\otimes gl_N(\mathbb{C})$. And the space of hermitian elements in the dual space

$$\Gamma=\left\{X^\dagger=X\mid X\in A_0^\vee\otimes gl_N(\mathbb{C})\right\} \tag{2.3}$$

is invariant under the action of $u_N(A_0)$. Then the "real-slice" Γ is the natural choice of the cycle for the equivariant integration.

The localization formula for equivariant cohomology reduces the integral of the equivariantly closed form Ω over Γ to the integral over the fixed locus F ,

$$\int_\Gamma\Omega=\int_F\frac{\Omega}{eu\left(N_F\right)} \tag{2.4}$$

where $eu(N_F)$ is the euler class of the normal bundle of F in Γ, see [AB], [DKV]. Calculating the integral using the equivariant localization leads to generalized Vandermond determinants and $\tau-$ functions.

Let for simplicity the algebra A with odd scalar product is the tensor product $A = A_0 \otimes q_1$ of the even associative algebra A_0 with scalar product, denoted $\eta(y_1, y_2)$, and the algebra $q_1 = \{1, \xi \mid \xi^2 = 1\}$ with the odd scalar product $\langle 1, \xi \rangle = 1$.

Assume that the natural scalar product on the Lie algebra of anti-hermitian elements in A_0 is positive definite

$$-\eta(y, y) = \eta\left(y, y^{\dagger}\right) > 0.$$

Otherwise one can apply to Γ a partial Wick rotation.

Then the localization formula (2.4) after some calculations leads to the following result ([B4]):

Proposition 5 *The integral (2.2), written in variables* $t \in HC^*(A)$, *is a* $\tau-$ *function of KP-type hierarchy and, in particular, satisfies the Hirota quadratic equations.*

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