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Supersymmetric extension of qKZ-Ruijsenaars correspondence

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Abstract

We describe the correspondence of the Matsuo–Cherednik type between the quantum n-body Ruijsenaars–Schneider model and the quantum Knizhnik–Zamolodchikov equations related to supergroup $GL(N|M)$. The spectrum of the Ruijsenaars–Schneider Hamiltonians is shown to be independent of the $\mathbb{Z}_2$-grading for a fixed value of $N+M$, so that $N+M+1$ different qKZ systems of equations lead to the same n-body quantum problem. The obtained results can be viewed as a quantization of the previously described quantum-classical correspondence between the classical n-body Ruijsenaars–Schneider model and the supersymmetric $GL(N|M)$ quantum spin chains on n sites.
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1. Introduction

The KZ-Calogero and qKZ-Ruijsenaars correspondences are the Matsuo–Cherednik type constructions [12,10,18,19] for solutions of the Calogero–Moser–Sutherland [4] and Ruijsenaars–Schneider [14] quantum problems by means of solutions of the Knizhnik–Zamolodchikov (KZ) [8] and quantum Knizhnik–Zamolodchikov (qKZ) equations [11] respectively. Consider, for example, the qKZ equations[1] related to the Lie group $GL(K)$:

$$e^{\eta\hbar\partial_{x_i}}\Big|\Phi\Big\rangle = \mathbf{K}_i^{(\hbar)}\Big|\Phi\Big\rangle, \qquad i=1,\dots,n, \tag{1.1}$$

$$\mathbf{K}_i^{(\hbar)} = \mathbf{R}_{i\,i-1}(x_i-x_{i-1}+\eta\hbar)\dots\mathbf{R}_{i1}(x_i-x_1+\eta\hbar)\mathbf{g}^{(i)}\mathbf{R}_{in}(x_i-x_n)\dots\mathbf{R}_{i\,i+1}(x_i-x_{i+1}), \tag{1.2}$$

where $\mathbf{g}=\mathrm{diag}(g_1,\dots,g_K)$ is a diagonal $K\times K$ (twist) matrix, and $\mathbf{g}^{(i)}$ acts by $\mathbf{g}$ multiplication in the i-th tensor component of the Hilbert space $\mathcal{V}=(\mathbb{C}^K)^{\otimes n}$. The quantum R-matrices $\mathbf{R}_{ij}$ are in the fundamental representation of $GL(K)$. They act in the i-th and j-th tensor components of $\mathcal{V}$ and satisfy the quantum Yang–Baxter equation, which guarantees compatibility of equations (1.1). The twist matrix $\mathbf{g}$ is the symmetry of $\mathbf{R}_{ij}$: $\mathbf{g}^{(i)}\mathbf{g}^{(j)}\mathbf{R}_{ij}=\mathbf{R}_{ij}\mathbf{g}^{(i)}\mathbf{g}^{(j)}$. In the rational case we deal with the Yang's R-matrix [17]:

$$\mathbf{R}_{ij}(x)=\frac{x\mathbf{I}+\eta\mathbf{P}_{ij}}{x+\eta}, \tag{1.3}$$

where $\mathbf{I}$ is identity operator in $\mathrm{End}(\mathcal{V})$, and $\mathbf{P}_{ij}$ is the permutation operator, which interchanges the i-th and j-th tensor components in $\mathcal{V}$. The operators[2]

$$\mathbf{M}_a=\sum_{l=1}^{n} e_{aa}^{(l)} \tag{1.4}$$

commute with $\mathbf{K}_i^{(\hbar)}$ and provide the weight decomposition of the Hilbert space $\mathcal{V}$ into the direct sum

$$\mathcal{V}=V^{\otimes n} = \bigoplus_{M_1,\dots,M_K} \mathcal{V}(\{M_a\}) \tag{1.5}$$

of eigenspaces of operators $\mathbf{M}_a$ with the eigenvalues $M_a\in\mathbb{Z}_{\geq 0}$, $a=1,\dots,K$: $M_1+\ldots+M_K=n$. Using the standard basis $\{e_a\}$ in $\mathbb{C}^K$ introduce the basis vectors in $\mathcal{V}(\{M_a\})$ as the vectors

$$\Big|J\Big\rangle = e_{j_1}\otimes e_{j_2}\otimes\ldots\otimes e_{j_n}, \tag{1.6}$$

where the number of indices j_k such that $j_k=a$ is equal to M_a for all $a=1,\dots,K$. The dual vectors $\Big\langle J\Big|$ are defined in so that $\Big\langle J\Big|J'\Big\rangle=\delta_{J,J'}$.

Then the statement of the qKZ-Ruijsenaars correspondence is as follows [19]. For any solution of the qKZ equations (1.1) $\Big|\Phi\Big\rangle=\sum_J \Phi_J\Big|J\Big\rangle$ from the weight subspace $\mathcal{V}(\{M_a\})$ the function

$$\Psi=\sum_J \Phi_J\,, \qquad \Phi_J=\Phi_J(x_1,\dots,x_n) \tag{1.7}$$

---

[1] The quantum R-matrices entering (1.2) are assumed to be unitary: $\mathbf{R}_{ij}(x)\mathbf{R}_{ji}(-x)=\mathrm{id}$.

[2] The set $\{e_{ab}\,|\,a,b=1...K\}$ is the standard basis in $\mathrm{Mat}(K,\mathbb{C})$: $(e_{ab})_{ij}=\delta_{ia}\delta_{jb}$.

or

$$\Psi = \Big\langle \Omega \Big| \Phi \Big\rangle, \qquad \Big\langle \Omega \Big| = \sum_{J:\,|J\rangle \in \mathcal{V}(\{M_a\})} \Big\langle J \Big| \tag{1.8}$$

with the property

$$\Big\langle \Omega \Big| \mathbf{P}_{ij} = \Big\langle \Omega \Big| \tag{1.9}$$

is an eigenfunction of the Macdonald difference operator:

$$\sum_{i=1}^{n} \prod_{j\neq i}^{n} \frac{x_i - x_j + \eta}{x_i - x_j} \Psi(x_1, \dots, x_i + \eta\hbar, \dots, x_n) = E\Psi(x_1, \dots, x_n), \qquad E = \sum_{a=1}^{K} M_a g_a . \tag{1.10}$$

The eigenvalues of the higher rational Macdonald–Ruijsenaars Hamiltonians

$$\hat{\mathcal{H}}_d = \sum_{I\subset\{1,\dots,n\},|I|=d} \Big(\prod_{s\in I, r\notin I} \frac{x_s - x_r + \eta}{x_s - x_r} \Big) \prod_{i\in I} e^{\eta\hbar\partial_{x_i}} \tag{1.11}$$

are given by the elementary symmetric polynomial of n variables $e_d(\underbrace{g_1, \dots, g_1}_{M_1}, \dots \underbrace{g_N, \dots, g_K}_{M_K})$.

QC-duality. Using the asymptotics of solutions to the (q)KZ equations [15] it was also argued in [18,19] that the qKZ-Ruijsenaars correspondence can be viewed as a quantization of the quantum-classical duality [1,7,2] (see also [13,5]), which relates the generalized inhomogeneous quantum spin chains and the classical Ruijsenaars–Schneider model. Consider the classical K-body Ruijsenaars–Schneider model, where the positions of particles $\{x_i\}$ are identified with the inhomogeneity parameters of the spin chain which is described by its transfer matrix

$$\mathbf{T}(x) = \mathrm{tr}_0\Big(\widetilde{\mathbf{R}}_{0n}(x - x_n) \dots \widetilde{\mathbf{R}}_{02}(x - x_2) \widetilde{\mathbf{R}}_{01}(x - x_1)(\mathbf{g} \otimes \mathbf{I}) \Big) \tag{1.12}$$

with the R-matrix

$$\widetilde{\mathbf{R}}(x) = \frac{x + \eta}{x} \mathbf{R}(x) = \mathbf{I} + \frac{\eta}{x} \mathbf{P}. \tag{1.13}$$

The quantum spin chain Hamiltonians are defined as follows:

$$\mathbf{H}_i = \operatorname*{Res}_{x=x_i} \mathbf{T}(x) = \widetilde{\mathbf{R}}_{i\,i-1}(x_i - x_{i-1}) \dots \widetilde{\mathbf{R}}_{i1}(x_i - x_1) \mathbf{g}^{(i)} \widetilde{\mathbf{R}}_{in}(x_i - x_n) \dots \widetilde{\mathbf{R}}_{i\,i+1}(x_i - x_{i+1}). \tag{1.14}$$

Therefore,

$$\mathbf{H}_i = \mathbf{K}_i^{(0)} \prod_{j\neq i}^{n} \frac{x_i - x_j + \eta}{x_i - x_j}, \qquad \mathbf{K}_i^{(0)} = \mathbf{K}_i^{(\hbar)} \,|_{\hbar=0} \,. \tag{1.15}$$

Identify also the generalized velocities $\{\dot{x}_i\}$ with the eigenvalues of (1.14). Then the action variables $\{I_i \,|\, i = 1, ..., K\}$ of the classical model (eigenvalues of the Lax matrix) are given by the values of $g_1, ..., g_K$ with multiplicities $M_1, ..., M_K$:

$$\{I_i \,|\, i = 1, ..., K\} = \Big\{ \underbrace{g_1, \dots, g_1}_{M_1}, \dots \underbrace{g_N, \dots, g_K}_{M_K} \Big\}. \tag{1.16}$$

See details in [7], where this statement was proved using the algebraic Bethe ansatz technique.

QC-correspondence. On the other hand, the quantum-classical duality possesses a generalization to the so-called quantum-classical correspondence [16], where the classical Ruijsenaars–Schneider model is related not to a single spin chain but to the set of $K+1$ supersymmetric spin chains [9] associated with supergroups

$$GL(K|0)\,,\; GL(K-1|1)\,,\quad \dots\quad ,GL(1|K-1)\,,\; GL(0|K)\,. \tag{1.17}$$

More precisely, it was shown in [16] that the previous statement (1.16) is valid for all supersymmetric chains with supergroups (1.17).

The aim of this paper is to quantize the (supersymmetric) quantum-classical correspondence, that is to establish supersymmetric version of the qKZ-Ruijsenaars correspondence for the qKZ equations related to the supergroups $GL(N|M)$. We construct generalizations of the vector $\Big\langle\Omega\Big|$ (1.8) and show that the quantum K-body Ruijsenaars–Schneider model follows from all $K+1$ qKZ systems of equations related to the supergroups $GL(N|M)$ with $N+M=K$ (1.17). The skew-symmetric vectors $\Big\langle\Omega_-\Big|$ with the property $\Big\langle\Omega_-\Big|\mathbf{P}_{ij}=-\Big\langle\Omega_-\Big|$ (instead of symmetric vector (1.9)) are described as well. They lead to the Ruijsenaars–Schneider model with different sign of the coupling constant η and $\hbar$.

The paper is organized as follows. For simplicity we start with the rational KZ-Calogero correspondence. Then we proceed to the rational and trigonometric qKZ-Ruijsenaars relations. Most of notations are borrowed from [18,19,16]. We briefly describe the notations and definitions related to the graded Lie algebras (groups) in the Appendix.

2. SUSY KZ-Calogero correspondence

The rational Knizhnik–Zamolodchikov (KZ) equations [8] have the form

$$\hbar\partial_{x_i}\Big|\Phi\Big\rangle=\left(\mathbf{g}^{(i)}+\kappa\sum_{j\neq i}^{n}\frac{\mathbf{P}_{ij}}{x_i-x_j}\right)\Big|\Phi\Big\rangle, \tag{2.1}$$

where $\Big|\Phi\Big\rangle=\Big|\Phi\Big\rangle(x_1,\dots,x_n)$ belongs to the tensor product $\mathcal{V}=V\otimes V\otimes\ldots\otimes V=V^{\otimes n}$ of the vector spaces $V=\mathbb{C}^{N|M}$, $\mathbf{P}_{ij}$ is the (graded) permutation operator (A.7) of the i-th and j-th tensor components, $\mathbf{g}=\mathrm{diag}(g_1,\dots,g_{N+M})$ is a diagonal $(N+M)\times(N+M)$ matrix and $\mathbf{g}^{(i)}$ is the operator in $\mathcal{V}$ acting as $\mathbf{g}$ on the i-th component (and identically on the rest of the components). The operators

$$\mathbf{H}_i=\mathbf{g}^{(i)}+\kappa\sum_{j\neq i}^{n}\frac{\mathbf{P}_{ij}}{x_i-x_j} \tag{2.2}$$

form the commutative set of Gaudin Hamiltonians [6]. Similarly to non-supersymmetric case they also commute with the operators:

$$\mathbf{M}_a=\sum_{l=1}^{n}\mathbf{e}_{aa}^{(l)}\,, \tag{2.3}$$

where $\mathbf{e}_{ab}$ are basis elements of $\mathrm{End}(\mathbb{C}^{N|M})$ (A.2)–(A.4). In what follows we restrict ourselves to the subspace $\mathcal{V}(\{M_a\})$ corresponding to a component of decomposition (1.5) with the fixed set of eigenvalues M_a for the operators $\mathbf{M}_a$. We fix a basis in $\mathcal{V}(\{M_a\})$:

$$\Big|J\Big\rangle = e_{a_1} \otimes e_{a_2} \otimes \ldots \otimes e_{a_n} = \Big|a_1 ... a_n\Big\rangle,$$

where e_a are basis vectors in V and the number of indices a_k such that $a_k = a$ is equal to M_a for all $a = 1, \ldots, N + M$. A general solution to (2.1) can be written as

$$\Big|\Phi\Big\rangle = \sum_J \Phi_J \Big|J\Big\rangle, \tag{2.4}$$

where the coefficients Φ_J are functions of all parameters entering (2.1).

To proceed further we need to find a (co)vector

$$\Big\langle\Omega\Big| = \sum_J \Big\langle J\Big|\Omega_J \tag{2.5}$$

similar to (1.8) with the property

$$\Big\langle\Omega\Big|\mathbf{P}_{ij} = \Big\langle\Omega\Big|, \tag{2.6}$$

where in contrast to (1.9) the permutation operator $\mathbf{P}_{ij}$ acts in the graded space (it has the form (A.7)). Having such a vector and taking into account the identities (A.11) and (A.12), we can repeat all the calculations from [18] without any changes. They lead to the eigenvalue equation for the second Calogero–Moser Hamiltonian:

$$\left(\hbar^2 \sum_{i=1}^{n} \partial_{x_i}^2 - \sum_{i\neq j}^{n} \frac{\kappa(\kappa-\hbar)}{(x_i - x_j)^2}\right)\Psi = E\Psi, \tag{2.7}$$

where

$$\Psi = \Big\langle\Omega\Big|\Phi\Big\rangle = \sum_J \Omega_J \Phi_J \tag{2.8}$$

and

$$E = \sum_{a=1}^{N+M} M_a g_a^2. \tag{2.9}$$

Let us construct the vector $\Big\langle\Omega\Big|$. Due to (A.9) the basis vector $\Big\langle J\Big|$ entering $\Big\langle\Omega\Big|$ can not contain two identical fermions (vectors e_a with $\mathsf{p}(a) = 1$). Otherwise we get a contradiction with (2.6). Keeping this in mind choose a vector $\Big|J\Big\rangle$ with $a_1 \leq a_2 \leq ... \leq a_n$ from $\mathcal{V}(\{M_a\})$, and fix the coefficient $\Omega_{a_1 \leq a_2 \leq ... \leq a_n} = 1$ for this set. Next, generate the rest of vectors $\Big|J\Big\rangle$ by the rule that the permutation of two nearby indices multiplies the coefficient by the standard parity factor:

$$\Omega_{a_1\, a_2 ... a_{m+1}\, a_m ... a_n} = (-1)^{\mathsf{p}(a_m)\mathsf{p}(a_{m+1})} \Omega_{a_1\, a_2 ... a_m\, a_{m+1} ... a_n} \tag{2.10}$$

By repeating this procedure and summing up all the resultant vectors $\Big|J\Big\rangle$ (in the orbit of the action of permutation operators with the corresponding coefficients Ω_J) we get the final answer for $\Big|\Omega\Big\rangle$. Here are some examples.

Example 2.1. Let $N+M=2$, $n=3$, $M_1=2$, $M_2=1$, $\mathsf{p}(1)=0$, $\mathsf{p}(2)=1$. Then

$$\Big|\Omega\Big\rangle = \Big|112\Big\rangle + \Big|121\Big\rangle + \Big|211\Big\rangle . \tag{2.11}$$

Example 2.2. Let $N+M=3$, $n=3$, $M_1=M_2=M_3=1$. Then

$$\begin{aligned}\Big|\Omega\Big\rangle &= \Big|123\Big\rangle + (-1)^{\mathsf{p}(1)\mathsf{p}(2)}\Big|213\Big\rangle + (-1)^{\mathsf{p}(2)\mathsf{p}(3)}\Big|132\Big\rangle + \\ &+ (-1)^{\mathsf{p}(1)\mathsf{p}(3)+\mathsf{p}(2)\mathsf{p}(3)}\Big|312\Big\rangle + (-1)^{\mathsf{p}(1)\mathsf{p}(2)+\mathsf{p}(1)\mathsf{p}(3)}\Big|231\Big\rangle + \\ &+ (-1)^{\mathsf{p}(1)\mathsf{p}(2)+\mathsf{p}(2)\mathsf{p}(3)+\mathsf{p}(1)\mathsf{p}(3)}\Big|321\Big\rangle .\end{aligned} \tag{2.12}$$

Example 2.3. Let $N+M=3$, $n=4$, $M_1=2$, $M_2=M_3=1$, $\mathsf{p}(1)=0$, $\mathsf{p}(2)=\mathsf{p}(3)=1$. Then

$$\begin{aligned}\Big|\Omega\Big\rangle &= \Big|1123\Big\rangle + \Big|1213\Big\rangle + \Big|2113\Big\rangle + \Big|1231\Big\rangle + \\ &+ \Big|2311\Big\rangle + \Big|2131\Big\rangle + \Big|2113\Big\rangle - (2 \leftrightarrow 3) .\end{aligned} \tag{2.13}$$

Note that in the case when $\mathsf{p}(a)=0$ for all a we return back to the non-supersymmetric case: $\Omega_J=1$ for all J. On the other hand, when $\mathsf{p}(a)=1$ for all a we get completely antisymmetric tensor $\Omega_{a_1\ldots a_n}=\epsilon_{a_1\ldots a_n}$. Thus different choices of $\mathfrak{B}$ (A.1) provide different eigenfunctions (2.8). At the same time the eigenvalues are the same (2.9), so that we get a degeneracy of the spectrum for the Hamiltonian (2.7).

It is also worth noting that in order to change the sign of κ in the Hamiltonian (2.7) we need to construct vector $\Big|\Omega_-\Big\rangle$, which is antisymmetric under the action of permutations:

$$\Big\langle\Omega_-\Big|\mathbf{P}_{ij} = -\Big\langle\Omega_-\Big| , \tag{2.14}$$

where the sign is opposite to the one in (2.6). Such a vector can not contain two identical bosons because the permutation of them contradicts assumption (2.14). In other situations it can be constructed. The example is given below.

Example 2.4. Let $N+M=3$, $n=3$, $M_1=M_2=M_3=1$ as in (2.12) and $\mathsf{p}(1)=\mathsf{p}(2)=\mathsf{p}(3)=1$. Then

$$\Big|\Omega_-\Big\rangle = \Big|123\Big\rangle + \Big|213\Big\rangle + \Big|132\Big\rangle + \Big|312\Big\rangle + \Big|231\Big\rangle + \Big|321\Big\rangle . \tag{2.15}$$

3. SUSY qKZ-Ruijsenaars correspondence: rational case

In this section we generalize the correspondence between KZ equations and Calogero–Moser systems to the case of SUSY qKZ equations and the Ruijsenaars–Schneider systems. The qKZ equations have the form

$$e^{\eta\hbar\partial_{x_i}}\Big|\Phi\Big\rangle = \mathbf{K}_i^{(\hbar)}\Big|\Phi\Big\rangle, \qquad i=1,\ldots,n , \tag{3.1}$$

where the operators in the r.h.s.

$$\mathbf{K}_i^{(\hbar)} = \mathbf{R}_{i\,i-1}(x_i - x_{i-1} + \eta\hbar)\dots\mathbf{R}_{i1}(x_i - x_1 + \eta\hbar)\mathbf{g}^{(i)}\mathbf{R}_{in}(x_i - x_n)\dots\mathbf{R}_{i\,i+1}(x_i - x_{i+1}) \tag{3.2}$$

are constructed by means of the quantum R-matrix $\mathbf{R}$, which is a (unitary) solution of the graded Yang–Baxter equation. We start with the rational one

$$\mathbf{R}_{ij}(x) = \frac{x\mathbf{I} + \eta\mathbf{P}_{ij}}{x+\eta}, \tag{3.3}$$

where $\mathbf{P}_{ij}$ is the graded permutation operator (A.7). Similarly to the non-supersymmetric case introduce the rescaled R-matrix:

$$\widetilde{\mathbf{R}}(x) = \frac{x+\eta}{x}\mathbf{R}(x) = \mathbf{I} + \frac{\eta}{x}\mathbf{P}. \tag{3.4}$$

The transfer matrix of the corresponding supersymmetric spin chain

$$\mathbf{T}(x) = \mathrm{str}_0\Big(\widetilde{\mathbf{R}}_{0n}(x - x_n)\dots\widetilde{\mathbf{R}}_{02}(x - x_2)\widetilde{\mathbf{R}}_{01}(x - x_1)\,(\mathbf{g}\otimes\mathbf{I})\Big) \tag{3.5}$$

provides non-local Hamiltonians as its residues:

$$\mathbf{T}(x) = \mathrm{str}\,\mathbf{g}\cdot\mathbf{I} + \sum_{j=1}^{n}\frac{\eta\mathbf{H}_j}{x - x_j}. \tag{3.6}$$

Explicitly,

$$\mathbf{H}_i = \widetilde{\mathbf{R}}_{i\,i-1}(x_i - x_{i-1})\dots\widetilde{\mathbf{R}}_{i1}(x_i - x_1)\mathbf{g}^{(i)}\widetilde{\mathbf{R}}_{in}(x_i - x_n)\dots\widetilde{\mathbf{R}}_{i\,i+1}(x_i - x_{i+1}). \tag{3.7}$$

Alternatively,

$$\mathbf{H}_i = \mathbf{K}_i^{(0)}\prod_{j\neq i}^{n}\frac{x_i - x_j + \eta}{x_i - x_j}\,. \tag{3.8}$$

From comparison of expansions of the transfer matrix as $x \to \infty$ in the forms (3.5) and (3.6)

$$\mathrm{str}\,\mathbf{g}\cdot\mathbf{I} + \frac{\eta}{x}\sum_{i=1}^{n}\mathrm{str}_0\Big(\mathbf{P}_{0i}\,\mathbf{g}^{(0)}\Big) + \dots = \mathrm{str}\,\mathbf{g}\cdot\mathbf{I} + \frac{\eta}{x}\sum_{i=1}^{n}\mathbf{H}_i + \dots \tag{3.9}$$

we obtain:

$$\sum_{i=1}^{n}\mathbf{H}_i = \sum_{i=1}^{n}\mathbf{g}^{(i)} = \sum_{a=1}^{N+M} g_a\mathbf{M}_a, \tag{3.10}$$

where the property (A.12) was used. To obtain the correspondence we project the qKZ-equations on the vector $\Big|\Omega\Big\rangle$ (2.6), constructed in the previous section:

$$e^{\eta\hbar\partial_{x_i}}\Big\langle\Omega\Big|\Phi\Big\rangle = e^{\eta\hbar\partial_{x_i}}\,\Psi = \Big\langle\Omega\Big|\mathbf{K}_i^{(\hbar)}\Big|\Phi\Big\rangle = \Big\langle\Omega\Big|\mathbf{K}_i^{(0)}\Big|\Phi\Big\rangle, \tag{3.11}$$

and repeat all calculations from [19]. This yields:

$$\sum_{i=1}^{n}\left(\prod_{j\neq i}^{n}\frac{x_i-x_j+\eta}{x_i-x_j}\right)e^{\eta\hbar\partial_{x_i}}\Psi=\sum_{i=1}^{n}\prod_{j\neq i}^{n}\frac{x_i-x_j+\eta}{x_i-x_j}\left\langle\Omega\middle|\mathbf{K}_i^{(0)}\middle|\Phi\right\rangle$$
$$=\sum_{i=1}^{n}\left\langle\Omega\middle|\mathbf{H}_i\middle|\Phi\right\rangle=\sum_{i=1}^{n}\left\langle\Omega\middle|\mathbf{g}^{(i)}\middle|\Phi\right\rangle=\sum_{a=1}^{N+M}g_a\left\langle\Omega\middle|\mathbf{M}_a\middle|\Phi\right\rangle=\left(\sum_{a=1}^{N+M}g_aM_a\right)\Psi,$$

where

$$\Psi=\left\langle\Omega\middle|\Phi\right\rangle \tag{3.12}$$

is the eigenfunction and

$$E=\sum_{a=1}^{N+M}g_aM_a \tag{3.13}$$

is the eigenvalue.

Remark 3.1. To obtain the Macdonald–Ruijsenaars Hamiltonian with the opposite sign of the coupling constant η and $\hbar$ one should start with the R-matrix

$$\mathbf{R}_{ij}(x)=\frac{x\mathbf{I}+\eta\mathbf{P}_{ij}}{x-\eta} \tag{3.14}$$

in (3.1) instead of (3.3). The R-matrix (3.14) is still unitary and acts identically on the antisymmetric vector $\left|\Omega_-\right\rangle$ (2.14) which is to be used instead of $\left|\Omega\right\rangle$.

3.1. Higher Hamiltonians

Following the construction in the non-supersymmetric case, it can be shown that the wave function $\Psi=\left\langle\Omega\middle|\Phi\right\rangle$ satisfies the equations

$$\prod_{s=1}^{d}e^{\eta\hbar\frac{\partial}{\partial x_{i_s}}}\Psi=\left\langle\Omega\middle|\mathbf{K}_{i_1}^{(0)}\dots\mathbf{K}_{i_d}^{(0)}\middle|\Phi\right\rangle\quad\text{for}\quad i_k\neq i_m\,. \tag{3.15}$$

The proof of this statement is the same as in [19]. One more point needed for the correspondence is the determinant identity

$$\det_{1\leq i,j\leq n}\left(z\delta_{ij}-\frac{\eta\mathbf{H}_i}{x_j-x_i+\eta}\right)=\prod_{a=1}^{N}(z-g_a)^{\mathbf{M}_a}. \tag{3.16}$$

It was proven for the supersymmetric case in [16]. Therefore, the correspondence works in the supersymmetric case as well. Namely, given a solution $|\Phi\rangle$ of the qKZ equations the wave function of the rational Ruijsenaars–Schneider quantum problem is given by (3.12). The eigenvalues are the same symmetric polynomials as in the non-supersymmetric case (1.11).

4. SUSY qKZ-Ruijsenaars correspondence, trigonometric case

The trigonometric (hyperbolic) solution to the graded Yang–Baxter equation has the following form [3]:

$$
\begin{aligned}
\mathbf{R}_{12}(x) \;=\; & \frac{1}{2\sinh(x+\eta)} \sum_{a=1}^{N+M} \left(e^{x+\eta} q^{-2\mathsf{p}(a)} - e^{-x-\eta} q^{2\mathsf{p}(a)} \right) e_{aa} \otimes e_{aa} \\
& + \frac{\sinh x}{\sinh(x+\eta)} \sum_{a\neq b}^{N+M} e_{aa} \otimes e_{bb} \\
& + \frac{\sinh \eta}{\sinh(x+\eta)} \sum_{a<b}^{N+M} \left(e^{x} (-1)^{\mathsf{p}(b)} e_{ab} \otimes e_{ba} + e^{-x} (-1)^{\mathsf{p}(a)} e_{ba} \otimes e_{ab} \right),
\end{aligned}
\tag{4.1}
$$

where $q = e^{\eta}$. It can be rewritten as follows:

$$
\mathbf{R}_{12}(x) = \mathbf{P}_{12} + \frac{\sinh x}{\sinh(x+\eta)} \left(\mathbf{I} - \mathbf{P}^{q}_{12} \right) + \mathbf{G}^{+}_{12}\,,
\tag{4.2}
$$

where $\mathbf{P}_{12}$ is the graded permutation operator (A.7), $\mathbf{P}^{q}_{12}$ – its q-deformation (the quantum permutation operator)

$$
\mathbf{P}^{q}_{12} = \sum_{a=1}^{N+M} (-1)^{\mathsf{p}(a)} e_{aa} \otimes e_{aa} + q \sum_{a>b}^{N+M} (-1)^{\mathsf{p}(b)} e_{ab} \otimes e_{ba} + q^{-1} \sum_{a<b}^{N+M} (-1)^{\mathsf{p}(b)} e_{ab} \otimes e_{ba}
\tag{4.3}
$$

and

$$
\begin{aligned}
\mathbf{G}^{+}_{12} &= \sum_{a=1}^{N+M} \Big(\frac{\sinh(x+\eta-2\eta\mathsf{p}(a))}{\sinh(x+\eta)} - (-1)^{\mathsf{p}(a)} + \frac{\sinh(x)}{\sinh(x+\eta)} ((-1)^{\mathsf{p}(a)} - 1) \Big) e_{aa} \otimes e_{aa} \\
&= 2 \sum_{a\in\mathfrak{F}} \frac{(\cosh\eta - 1)\sinh x}{\sinh(x+\eta)}\, e_{aa} \otimes e_{aa}
\end{aligned}
\tag{4.4}
$$

or

$$
\mathbf{G}^{+}_{12} = \sum_{a=1}^{N+M} \mathbf{G}^{+}_{a}\, e_{aa} \otimes e_{aa}\,, \qquad \mathbf{G}^{+}_{a} = \frac{(1-(-1)^{\mathsf{p}(a)})(\cosh\eta - 1)\sinh x}{\sinh(x+\eta)}\,.
\tag{4.5}
$$

The R-matrix entering the transfer matrix differs from (4.1) by a scalar factor:

$$
\widetilde{\mathbf{R}}_{12}(x) = \frac{\sinh(x+\eta)}{\sinh x} \mathbf{R}_{12}(x)\,,
\tag{4.6}
$$

and the transfer matrix itself is defined similarly to (3.5). The Hamiltonians are introduced through the expansion

$$
\mathbf{T}(x) = \mathbf{C} + \sinh\eta \sum_{k=1}^{n} \mathbf{H}_k \coth(x - x_k)\,.
\tag{4.7}
$$

They are related to the operators in the r.h.s. of the qKZ-equations by the same formulae as in non-supersymmetric case:

$$\mathbf{H}_i = \mathbf{K}_i^{(0)} \prod_{j\neq i}^{n} \frac{\sinh(x_i - x_j + \eta)}{\sinh(x_i - x_j)} . \tag{4.8}$$

4.1. Construction of q-symmetric vectors

Our strategy is as follows. Following the non-supersymmetric construction [19], we now need to find a vector $\left\langle \Omega_q \right|$ with the property

$$\left\langle \Omega_q \right| \mathbf{R}_{i\,i-1}(x) = \left\langle \Omega_q \right| \mathbf{P}_{i\,i-1}, \qquad i = 2, \dots, n . \tag{4.9}$$

Let us show that this vector has the form:

$$\left\langle \Omega_q \right| = \sum_J q^{\ell(J)} \Omega_J \left\langle J \right| , \tag{4.10}$$

where Ω_J is the same as in the rational case (2.7), (2.10), while $\ell(J)$ is defined to be the minimal number of elementary permutations required to get the multi-index $J = (j_1, j_2, \dots, j_n)$ starting from the "minimal" one. The "minimal" order implies that the j_k's are ordered as $1 \le j_1 \le j_2 \le \dots \le j_n \le N$ (see [19]). The proof is straightforward. First, by the construction we see that

$$\left\langle \Omega_q \right| \mathbf{P}^q_{i,i-1} = \left\langle \Omega_q \right| . \tag{4.11}$$

In contrast to the non-supersymmetric case we have additional terms $\mathbf{G}^+_{i,i-1}$ in R-matrices (4.2). However, they do not provide any effect when acting on $\left\langle \Omega_q \right|$:

$$\left\langle \Omega_q \right| \mathbf{G}^+_{i,i-1} = 0 . \tag{4.12}$$

It happens because of the tensor structure (4.4). Indeed,

$$\mathbf{G}^+_{i,i-1} \left| J \right\rangle = \mathbf{G}^+_{a_i} \delta_{a_i, a_{i-1}} \left| J \right\rangle , \tag{4.13}$$

so that only the same basis vectors e_{a_i} entering $\left| J \right\rangle$ may contribute. But we have already assumed that our vector $\left\langle \Omega_q \right|$ does not contain two identical fermions, and for bosons $\mathbf{G}^+_a = 0$. Finally, using (4.2) we arrive at (4.9).

Example 4.1. Let $N + M = 3$, $n = 3$, $M_1 = M_2 = M_3 = 1$. Then

$$\begin{aligned} \left| \Omega_q \right\rangle = & \left| 123 \right\rangle + q\,(-1)^{\mathsf{p}(1)\mathsf{p}(2)} \left| 213 \right\rangle + q\,(-1)^{\mathsf{p}(2)\mathsf{p}(3)} \left| 132 \right\rangle + \\ & + q^2 (-1)^{\mathsf{p}(1)\mathsf{p}(3)+\mathsf{p}(2)\mathsf{p}(3)} \left| 312 \right\rangle + q^2\,(-1)^{\mathsf{p}(1)\mathsf{p}(2)+\mathsf{p}(1)\mathsf{p}(3)} \left| 231 \right\rangle + \\ & + q^3 (-1)^{\mathsf{p}(1)\mathsf{p}(2)+\mathsf{p}(2)\mathsf{p}(3)+\mathsf{p}(1)\mathsf{p}(3)} \left| 321 \right\rangle . \end{aligned} \tag{4.14}$$

4.2. Calculation of the eigenvalue

Coming back to the proof of the correspondence we need the identity

$$\Big\langle \Omega_q \Big| \mathbf{K}_i^{(\hbar)} = \Big\langle \Omega_q \Big| \mathbf{K}_i^{(0)} = \Big\langle \Omega_q \Big| \mathbf{P}_{i\,i-1} \dots \mathbf{P}_{i1}\,, \tag{4.15}$$

which follows from $\mathbf{P}_{i\,i-1}\mathbf{P}^q_{i\,i-2} = \mathbf{P}^q_{i-1\,i-2}\mathbf{P}_{i\,i-1}$ and an analogue of the identity

$$\mathbf{T}(\pm\infty) = \mathbf{C} \pm \sinh\eta \sum_k \mathbf{H}_k = \sum_{a=1}^N g_a e^{\pm\eta \mathbf{M}_a}$$

for the supersymmetric case. It is as follows.

Proposition 4.1.

$$\begin{aligned} \mathbf{T}(\infty) &= \sum_{a\in\mathfrak{B}} g_a e^{\eta\mathbf{M}_a} - \sum_{a\in\mathfrak{F}} g_a e^{-\eta\mathbf{M}_a}, \\ \mathbf{T}(-\infty) &= \sum_{a\in\mathfrak{B}} g_a e^{-\eta\mathbf{M}_a} - \sum_{a\in\mathfrak{F}} g_a e^{\eta\mathbf{M}_a}. \end{aligned} \tag{4.16}$$

Proof. We will prove the first equality. The proof of the second one is similar. Let us first find the asymptotics of the R-matrix:

$$\begin{aligned} \widetilde{\mathbf{R}}(\infty) = \mathbf{I} &+ (q-q^{-1}) \sum_{a<b}^{N+M} (-1)^{\mathsf{p}(b)} e_{ab}\otimes e_{ba} + (q-1)\sum_{a=1}^{N+M} (-1)^{\mathsf{p}(a)} e_{aa}\otimes e_{aa} \\ &+ \sum_{a=1}^{N+M} \Big(q^{1-2\mathsf{p}(a)} - (-1)^{\mathsf{p}(a)} q + ((-1)^{\mathsf{p}(a)}-1)\Big) e_{aa}\otimes e_{aa}\,. \end{aligned} \tag{4.17}$$

This expression can be rewritten in the following form:

$$\widetilde{\mathbf{R}}(\infty) = \mathbf{I} + (q-q^{-1}) \sum_{a<b}^{N+M} (-1)^{\mathsf{p}(b)} e_{ab}\otimes e_{ba} + \sum_{a=1}^{N+M} \Big(q^{1-2\mathsf{p}(a)} - 1\Big) e_{aa}\otimes e_{aa}\,. \tag{4.18}$$

The off-diagonal part does not contribute to the trace in (3.5). Therefore,

$$\begin{aligned} \mathbf{T}(\infty) &= \sum_{a=1}^{N+M} (-1)^{\mathsf{p}(a)} g_a \prod_{j=1}^n \Big(1 + (q^{1-2\mathsf{p}(a)}-1) e_{aa}^{(j)}\Big) = \\ &= \sum_{a=1}^{N+M} (-1)^{\mathsf{p}(a)} g_a \prod_{j=1}^n \Big(1 + \sum_{N_j=1}^{\infty} \frac{\eta^{N_j}(1-2\mathsf{p}(a))^{N_j}}{N_j!} e_{aa}^{(j)}\Big) = \\ &= \sum_{a=1}^{N+M} (-1)^{\mathsf{p}(a)} g_a \prod_{j=1}^n \Big(\sum_{N_j=0}^{\infty} \frac{\eta^{N_j}(1-2\mathsf{p}(a))^{N_j}}{N_j!} (e_{aa}^{(j)})^{N_j}\Big) \end{aligned} \tag{4.19}$$

and, finally,

$$\mathbf{T}(\infty) = \sum_{a=1}^{N+M} (-1)^{\mathsf{p}(a)} g_a \prod_{j=1}^{n} \left(e^{\eta(1-2\mathsf{p}(a)) e_{aa}^{(j)}} \right) = \sum_{a=1}^{N+M} (-1)^{\mathsf{p}(a)} g_a \left(e^{\eta(1-2\mathsf{p}(a)) \sum_{j=1}^{n} e_{aa}^{(j)}} \right) =$$
$$= \sum_{a=1}^{N+M} (-1)^{\mathsf{p}(a)} g_a \left(e^{\eta(1-2\mathsf{p}(a)) \mathbf{M}_a} \right) = \sum_{a \in \mathfrak{B}} g_a e^{\eta \mathbf{M}_a} - \sum_{a \in \mathfrak{F}} g_a e^{-\eta \mathbf{M}_a} . \qquad \square \tag{4.20}$$

Notice that although this expression depends on the choice of $\mathfrak{B}$ and $\mathfrak{F}$ the eigenvalue of the Ruijsenaars–Schneider Hamiltonian is independent of it:

$$\begin{aligned}
\sum_{i=1}^{n} \left(\prod_{j \neq i}^{n} \frac{\sinh(x_i - x_j + \eta)}{\sinh(x_i - x_j)} \right) e^{\eta \hbar \partial_{x_i}} \Psi \;&=\; \sum_{i=1}^{n} \prod_{j \neq i}^{n} \frac{\sinh(x_i - x_j + \eta)}{\sinh(x_i - x_j)} \left\langle \Omega_q \middle| \mathbf{K}_i^{(0)} \middle| \Phi \right\rangle \\
&= \sum_{i=1}^{n} \left\langle \Omega_q \middle| \mathbf{H}_i \middle| \Phi \right\rangle \;=\; \left\langle \Omega_q \middle| \frac{\mathbf{T}(\infty) - \mathbf{T}(-\infty)}{2 \sinh \eta} \middle| \Phi \right\rangle \\
&= \left\langle \Omega_q \middle| \sum_{a \in \mathfrak{B}} g_a \frac{\sinh(\eta \mathbf{M}_a)}{\sinh \eta} + \sum_{a \in \mathfrak{F}} g_a \frac{\sinh(\eta \mathbf{M}_a)}{\sinh \eta} \middle| \Phi \right\rangle \\
&= \sum_{a=1}^{N+M} g_a \left\langle \Omega_q \middle| \frac{\sinh(\eta \mathbf{M}_a)}{\sinh \eta} \middle| \Phi \right\rangle \;=\; \left(\sum_{a=1}^{N+M} g_a \frac{\sinh(\eta M_a)}{\sinh \eta} \right) \Psi .
\end{aligned} \tag{4.21}$$

Therefore,

$$\Psi = \left\langle \Omega_q | \Phi \right\rangle \tag{4.22}$$

is indeed an eigenfunction of the Ruijsenaars–Schneider Hamiltonian with the eigenvalue

$$E = \sum_{a=1}^{N+M} g_a \frac{\sinh(\eta M_a)}{\sinh \eta} . \tag{4.23}$$

4.3. Construction of q-antisymmetric vectors

In order to extend the correspondence to the case of the Hamiltonian with the opposite sign of η we should start with a different R-matrix:

$$\begin{aligned}
\mathbf{R}(x) \;=\; & \frac{1}{2 \sinh(x - \eta)} \sum_{a=1}^{N+M} (e^{x+\eta} q^{-2\mathsf{p}(a)} - e^{-x-\eta} q^{2\mathsf{p}(a)}) e_{aa} \otimes e_{aa} \\
& + \frac{\sinh x}{\sinh(x - \eta)} \sum_{a \neq b}^{N+M} e_{aa} \otimes e_{bb} \\
& + \frac{\sinh \eta}{\sinh(x - \eta)} \sum_{a < b}^{N+M} \left(e^{x} (-1)^{\mathsf{p}(b)} e_{ab} \otimes e_{ba} + e^{-x} (-1)^{\mathsf{p}(a)} e_{ba} \otimes e_{ab} \right) .
\end{aligned} \tag{4.24}$$

It is an analog of (3.14) in the rational case. Expression (4.24) can be rewritten in the form

$$\mathbf{R}_{12}(x) = -\mathbf{P}_{12} + \frac{\sinh x}{\sinh(x-\eta)}\Big(\mathbf{I} + \mathbf{P}^q_{12}\Big) + \mathbf{G}^-_{12}\,, \tag{4.25}$$

where

$$\begin{aligned}\mathbf{G}^-_{12} &= \sum_{a=1}^{N+M}\Big(\frac{\sinh(x+\eta-2\eta\mathsf{p}(a))}{\sinh(x-\eta)} + (-1)^{\mathsf{p}(a)} - \frac{\sinh(x)}{\sinh(x-\eta)}((-1)^{\mathsf{p}(a)}+1)\Big)e_{aa}\otimes e_{aa} \\ &= 2\sum_{a\in\mathfrak{B}}\frac{(\cosh\eta-1)\sinh(x)}{\sinh(x-\eta)}\,e_{aa}\otimes e_{aa} = \sum_{a=1}^{N+M}\mathbf{G}^-_a\, e_{aa}\otimes e_{aa}\,.\end{aligned} \tag{4.26}$$

Similarly to the case of symmetric vector (and also similarly to (2.14)) it is easy to see that the vector $\Big\langle\Omega_q\Big|$ with the property

$$\Big\langle\Omega_q\Big|\mathbf{P}^q_{i,i-1} = -\Big\langle\Omega_q\Big| \tag{4.27}$$

can not contain two or more identical bosonic vectors. On the other hand, $\mathbf{G}^-_{12}$ acts by zero on the pair of identical fermions. Thus

$$\Big\langle\Omega_q\Big|\mathbf{R}_{i,i-1} = -\Big\langle\Omega_q\Big|\mathbf{P}_{i,i-1}\,. \tag{4.28}$$

Repeating the steps from the previous paragraphs we obtain the following expressions for the asymptotics of the R-matrix at infinity:

$$\begin{aligned}\widetilde{\mathbf{R}}(\infty) &= \mathbf{I} + (q-q^{-1})\sum_{a>b}^{N+M}(-1)^{\mathsf{p}(b)}e_{ab}\otimes e_{ba} + \sum_{a=1}^{N+M}\Big(q^{1-2\mathsf{p}(a)}-1\Big)e_{aa}\otimes e_{aa}\,, \\ \widetilde{\mathbf{R}}(-\infty) &= \mathbf{I} + (q^{-1}-q)\sum_{a<b}^{N+M}(-1)^{\mathsf{p}(b)}e_{ab}\otimes e_{ba} + \sum_{a=1}^{N+M}\Big(q^{-1+2\mathsf{p}(a)}-1\Big)e_{aa}\otimes e_{aa}\,,\end{aligned} \tag{4.29}$$

where

$$\widetilde{\mathbf{R}}(x) = \frac{\sinh(x-\eta)}{\sinh x}\mathbf{R}(x)\,. \tag{4.30}$$

It is easy to see that these asymptotics differ from the corresponding asymptotics in the q-symmetric case by non-diagonal part only, but the latter does not contribute to the trace in the transfer matrix. Therefore, the Hamiltonian with the opposite sign of η has the same eigenvalue:

$$\sum_{i=1}^{n}\left(\prod_{j\neq i}^{n}\frac{\sinh(x_i-x_j-\eta)}{\sinh(x_i-x_j)}\right)e^{\eta\hbar\partial_{x_i}}\Psi = \left(\sum_{a=1}^{N+M}g_a\frac{\sinh(\eta M_a)}{\sinh\eta}\right)\Psi\,. \tag{4.31}$$

4.4. Symmetry between q-(anti)symmetric vectors

In this paragraph we will show that the usage of q-antisymmetric vectors do not actually lead to any new wave functions of the Ruijsenaars–Schneider system. For this paragraph let us introduce more refined notations:

$$
\begin{aligned}
\widetilde{\mathbf{R}}^{\mathsf{p}}(x|\eta) = {} & \frac{1}{2\sinh x}\sum_{a=1}^{N+M}\left(e^{x+\eta}q^{-2\mathsf{p}(a)} - e^{-x-\eta}q^{2\mathsf{p}(a)}\right)e_{aa}\otimes e_{aa} + \sum_{a\neq b}^{N+M} e_{aa}\otimes e_{bb} \\
& + \frac{\sinh\eta}{\sinh x}\sum_{a<b}^{N+M}\left(e^{x}(-1)^{\mathsf{p}(b)}e_{ab}\otimes e_{ba} + e^{-x}(-1)^{\mathsf{p}(a)}e_{ba}\otimes e_{ab}\right)
\end{aligned}
\tag{4.32}
$$

and

$$
\mathbf{R}^{\mathsf{p}}_{\pm}(x|\eta) = \frac{\sinh x}{\sinh(x\pm\eta)}\,\widetilde{\mathbf{R}}^{\mathsf{p}}(x|\eta)\,, \tag{4.33}
$$

where the index p stands for a fixed choice of grading.

Let us introduce the operator Q of the grading change:

$$
\mathsf{p}(Qe_a) = \mathsf{p}(e_a) + 1 \pmod 2). \tag{4.34}
$$

This operator simply changes all basis bosonic vectors e_a to fermionic ones and vice versa. It is easy to see from this definition that the R-matrix has a symmetry

$$
Q\widetilde{\mathbf{R}}^{\mathsf{p}}(x|\eta)Q^{-1} = \widetilde{\mathbf{R}}^{\mathsf{p}+1}(x|-\eta)\,, \tag{4.35}
$$

where the index $\mathsf{p}+1$ means simultaneous shift of all grading parameters by 1 modulo 2 in (4.32). Therefore,

$$
Q\mathbf{R}^{\mathsf{p}}_{-}(x|\eta)Q^{-1} = \mathbf{R}^{\mathsf{p}+1}_{+}(x|-\eta)\,. \tag{4.36}
$$

For the special vectors (on which we project the solutions) we also reserve the following notation:

$$
\left\langle\Omega^{\mathsf{p}}_{q+}\right|\mathbf{P}^{q,\mathsf{p}}_{i,i-1} = \left\langle\Omega^{\mathsf{p}}_{q+}\right|, \quad \left\langle\Omega^{\mathsf{p}}_{q-}\right|\mathbf{P}^{q,\mathsf{p}}_{i,i-1} = -\left\langle\Omega^{\mathsf{p}}_{q-}\right|. \tag{4.37}
$$

By changing all bosons to fermions in these equations and vice versa, and taking into account that

$$
Q\mathbf{P}^{q,\mathsf{p}}_{i,i-1}Q^{-1} = -\mathbf{P}^{q,\mathsf{p}+1}_{i,i-1}\,, \tag{4.38}
$$

we get

$$
\left\langle\Omega^{\mathsf{p}}_{q+}\right|Q = \left\langle\Omega^{\mathsf{p}+1}_{q-}\right|. \tag{4.39}
$$

As a first step towards the explanation of the origin of the wavefunctions for Hamiltonians with signs of η and $\hbar$ changed we will prove the following

Proposition 4.2. *For any solution $\left|\Phi^{\mathsf{p}}_{-}(x|\eta,\hbar)\right\rangle$ of the qKZ equations with the R-matrix $\mathbf{R}^{\mathsf{p}}_{-}(x|\eta)$ suitable for projecting on the q-antisymmetric vector $\left\langle\Omega^{\mathsf{p}}_{q-}\right|$, we can construct the solution $\left|\Phi^{\mathsf{p}+1}_{+}(x|\eta,\hbar)\right\rangle$ of the qKZ equations, with the R-matrix $\mathbf{R}^{\mathsf{p}+1}_{+}(x|\eta)$ suitable for projecting on the q-symmetric vector $\left\langle\Omega^{\mathsf{p}+1}_{q+}\right|$.*

Proof. Consider the qKZ-equations:

$$
\begin{aligned}
e^{\eta\hbar\partial_{x_i}}\left|\Phi^{\mathsf{p}}_{-}(x|\eta,\hbar)\right\rangle = {} & \mathbf{R}^{\mathsf{p}}_{-,i\,i-1}(x_i - x_{i-1} + \eta\hbar|\eta)\dots\mathbf{R}^{\mathsf{p}}_{-,i1}(x_i - x_1 + \eta\hbar|\eta)\mathbf{G}^{(i)} \\
& \times\mathbf{R}^{\mathsf{p}}_{-,in}(x_i - x_n|\eta)\dots\mathbf{R}^{\mathsf{p}}_{-,i\,i+1}(x_i - x_{i+1}|\eta)\left|\Phi^{\mathsf{p}}_{-}(x|\eta,\hbar)\right\rangle, \qquad i = 1,\dots,n\,.
\end{aligned}
$$

Changing signs of η and $\hbar$ yields

$$e^{\eta\hbar\partial_{x_i}}\left|\Phi^{\mathrm{p}}_{-}(x|-\eta,-\hbar)\right\rangle = \mathbf{R}^{\mathrm{p}}_{-,i\,i-1}(x_i-x_{i-1}+\eta\hbar|-\eta)\dots\mathbf{R}^{\mathrm{p}}_{-,i1}(x_i-x_1+\eta\hbar|-\eta)\mathbf{G}^{(i)}$$
$$\times\mathbf{R}^{\mathrm{p}}_{-,in}(x_i-x_n|-\eta)\dots\mathbf{R}^{\mathrm{p}}_{-,i\,i+1}(x_i-x_{i+1}|-\eta)\left|\Phi^{\mathrm{p}}_{-}(x|-\eta,-\hbar)\right\rangle,\qquad i=1,\dots,n\,.$$

Using the symmetry (4.35) this could be rewritten in the form:

$$e^{\eta\hbar\partial_{x_i}}\,Q\left|\Phi^{\mathrm{p}}_{-}(x|-\eta,-\hbar)\right\rangle = \mathbf{R}^{\mathrm{p}+1}_{+,i\,i-1}(x_i-x_{i-1}+\eta\hbar|\eta)\dots\mathbf{R}^{\mathrm{p}+1}_{+,i1}(x_i-x_1+\eta\hbar|\eta)\mathbf{G}^{(i)}$$
$$\times\mathbf{R}^{\mathrm{p}+1}_{+,in}(x_i-x_n|\eta)\dots\mathbf{R}^{\mathrm{p}+1}_{+,i\,i+1}(x_i-x_{i+1}|\eta)\,Q\left|\Phi^{\mathrm{p}}_{-}(x|-\eta,-\hbar)\right\rangle,\qquad i=1,\dots,n\,.$$

It can be seen from here that the desired solution $\left|\Phi^{\mathrm{p}+1}_{+}(x|\eta,\hbar)\right\rangle$ is the following:

$$\left|\Phi^{\mathrm{p}+1}_{+}(x|\eta,\hbar)\right\rangle = Q\left|\Phi^{\mathrm{p}}_{-}(x|-\eta,-\hbar)\right\rangle. \quad\square \tag{4.40}$$

Consider the space of all wavefunctions $\Psi_{-}(x|\eta,\hbar)$ of the Ruijsenaars Hamiltonian with signs of η and $\hbar$ changed:

$$\sum_{i=1}^{n}\left(\prod_{j\neq i}^{n}\frac{\sinh(x_i-x_j-\eta)}{\sinh(x_i-x_j)}\right)e^{\eta\hbar\partial_{x_i}}\,\Psi_{-}(x|\eta,\hbar) = \left(\sum_{a=1}^{N+M}g_a\frac{\sinh(\eta M_a)}{\sinh\eta}\right)\Psi_{-}(x|\eta,\hbar)\,, \tag{4.41}$$

which could be obtained with our construction, i.e. they have the form

$$\Psi_{-}(x|\eta,\hbar) = \left\langle\Omega^{\mathrm{p}}_{q-}\middle|\Phi^{\mathrm{p}}_{-}(x|\eta,\hbar)\right\rangle. \tag{4.42}$$

For any such $\Psi_{-}(x|\eta,\hbar)$ the function $\Psi_{+}(x|\eta,\hbar)=\Psi_{-}(x|-\eta,-\hbar)$ is automatically satisfies the equation

$$\sum_{i=1}^{n}\left(\prod_{j\neq i}^{n}\frac{\sinh(x_i-x_j+\eta)}{\sinh(x_i-x_j)}\right)e^{\eta\hbar\partial_{x_i}}\,\Psi_{+}(x|\eta,\hbar) = \left(\sum_{a=1}^{N+M}g_a\frac{\sinh(\eta M_a)}{\sinh\eta}\right)\Psi_{+}(x|\eta,\hbar)\,. \tag{4.43}$$

Now we are ready to prove the main statement of this section.

Proposition 4.3. *For any wavefunction of the form (4.42) the corresponding $\Psi_{+}(x|\eta,\hbar)=\Psi_{-}(x|-\eta,-\hbar)$ can be also obtained from our construction, i.e., it has the form*

$$\Psi_{+}(x|\eta,\hbar) = \left\langle\Omega^{\mathrm{p}+1}_{q}\middle|\Phi^{\mathrm{p}+1}_{+}(x|\eta,\hbar)\right\rangle. \tag{4.44}$$

The proof follows from the previous proposition with $\left|\Phi^{\mathrm{p}+1}_{+}(x|\eta,\hbar)\right\rangle$ defined as in (4.40) and the remark (4.39).

This proposition actually means that for any wavefunction constructed with the help of the q-antisymmetric vector the existence of the corresponding solution of the qKZ equation is a simple consequence of the existence of such solution for the wavefunction with signs of η and $\hbar$ changed, constructed with the help of the q-symmetric vector.

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Appendix A

Here we give a short summary of notations and definitions related to the Lie superalgebra $gl(N|M)$.

Let $\mathfrak{B}$ be any one of the subsets of $\{1, 2, \dots, N+M\}$ with $\mathrm{Card}(\mathfrak{B}) = N$, and $\mathfrak{F}$ be the complement set $\mathfrak{F} = \{1, 2, \dots, N+M\} \setminus \mathfrak{B}$. The vector space $\mathbb{C}^{N|M}$ is endowed with the $\mathbb{Z}_2$-grading. The grading parameter is defined as

$$\mathsf{p}(a) = \begin{cases} 0\,, & a \in \mathfrak{B} \quad \text{(bosons)}\,, \\ 1\,, & a \in \mathfrak{F} \quad \text{(fermions)}\,. \end{cases} \tag{A.1}$$

The Lie superalgebra $gl(N|M)$ is defined by the following relations for the generators $\mathbf{e}_{ab}$:

$$\mathbf{e}_{ab}\mathbf{e}_{cd} - (-1)^{\mathsf{p}(\mathbf{e}_{ab})\mathsf{p}(\mathbf{e}_{cd})}\mathbf{e}_{cd}\mathbf{e}_{ab} = \delta_{bc}\mathbf{e}_{ad} - (-1)^{\mathsf{p}(\mathbf{e}_{ab})\mathsf{p}(\mathbf{e}_{cd})}\delta_{ad}\mathbf{e}_{cb}\,, \tag{A.2}$$

where

$$\mathsf{p}(\mathbf{e}_{ab}) = \mathsf{p}(a) + \mathsf{p}(b) \bmod 2\,. \tag{A.3}$$

In the fundamental representation the set of generators $\{\mathbf{e}_{ab}\}$ forms the standard basis in matrices $\mathrm{End}(\mathbb{C}^{N|M})$: $(e_{ab})_{ij} = \delta_{ia}\delta_{jb}$, so that for the orthonormal basis vectors e_a, $a = 1, ..., N+M$ in $\mathbb{C}^{N|M}$ (i.e. $(e_a)_k = \delta_{ak}$) we have

$$e_{ab}\, e_c = \delta_{bc}\, e_a\,. \tag{A.4}$$

For any homogeneous (with a definite grading) operators $\{\mathbf{A}_i \in \mathrm{End}(\mathbb{C}^{N|M})\}_{i=1}^4$ and homogeneous vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{N|M}$ we have:

$$(\mathbf{A}_1 \otimes \mathbf{A}_2)(\mathbf{x} \otimes \mathbf{y}) = (-1)^{\mathsf{p}(\mathbf{A}_2)\mathsf{p}(\mathbf{x})}(\mathbf{A}_1\mathbf{x} \otimes \mathbf{A}_2\mathbf{y}) \tag{A.5}$$

and

$$(\mathbf{A}_1 \otimes \mathbf{A}_2)(\mathbf{A}_3 \otimes \mathbf{A}_4) = (-1)^{\mathsf{p}(\mathbf{A}_2)\mathsf{p}(\mathbf{A}_3)}(\mathbf{A}_1\mathbf{A}_3 \otimes \mathbf{A}_2\mathbf{A}_4)\,. \tag{A.6}$$

The graded permutation operator $\mathbf{P}_{12} \in \mathrm{End}\big(\mathbb{C}^{N|M} \otimes \mathbb{C}^{N|M}\big)$ is of the form:

$$\mathbf{P}_{12} = \sum_{a,b=1}^{M+N} (-1)^{\mathsf{p}(b)} e_{ab} \otimes e_{ba}\,. \tag{A.7}$$

Due to (A.5) it permutes any pair of homogeneous vectors $\mathbf{x}$ and $\mathbf{y}$ according to the rule

$$\mathbf{P}_{12}\, \mathbf{x} \otimes \mathbf{y} = (-1)^{\mathsf{p}(\mathbf{x})\mathsf{p}(\mathbf{y})}\mathbf{y} \otimes \mathbf{x}\,. \tag{A.8}$$

In particular,

$$\mathbf{P}_{12}\, e_a \otimes e_a = (-1)^{\mathsf{p}(a)}\, e_a \otimes e_a\,. \tag{A.9}$$

The supertrace and the superdeterminant of $\mathcal{M} \in \mathrm{End}(\mathbb{C}^{N|M})$ are given by

$$\mathrm{str}\,\mathcal{M} = \sum_{a=1}^{N+M} (-1)^{\mathsf{p}(a)} \mathcal{M}_{aa} \tag{A.10}$$

and $\mathrm{sdet}\,\mathcal{M} = \exp(\mathrm{str}\log\mathcal{M})$. For an operator $\mathcal{M}^{(i)}$ acting as $\mathcal{M}$ on the i-th component of $(\mathbb{C}^{N|M})^{\otimes n}$ we have

$$\mathbf{P}_{ij}\, \mathcal{M}^{(j)} = \mathcal{M}^{(i)}\, \mathbf{P}_{ij}\,, \tag{A.11}$$

$$\mathrm{str}_0(\mathbf{P}_{0i}\, \mathcal{M}^{(0)}) = \mathcal{M}^{(i)}\,. \tag{A.12}$$

References

[1] A. Alexandrov, V. Kazakov, S. Leurent, Z. Tsuboi, A. Zabrodin, J. High Energy Phys. 09 (2013) 064, arXiv:1112.3310.
[2] M. Beketov, A. Liashyk, A. Zabrodin, A. Zotov, Nucl. Phys. B 903 (2016) 150–163, arXiv:1510.07509.
[3] V.V. Bazhanov, A.G. Shadrikov, Theor. Math. Phys. 73 (3) (1987) 1302–1312;
S. Belliard, E. Ragoucy, J. Phys. A 41 (2008) 295202, arXiv:0804.2822.
[4] F. Calogero, J. Math. Phys. 12 (1971) 419–436;
B. Sutherland, Phys. Rev. A 4 (5) (1971) 2019–2021;
B. Sutherland, Phys. Rev. A 5 (3) (1972) 1372–1376.
[5] D. Gaiotto, P. Koroteev, J. High Energy Phys. 05 (2013) 126, arXiv:1304.0779;
P. Koroteev, A.M. Zeitlin, arXiv:1802.04463.
[6] M. Gaudin, J. Physiol. 37 (10) (1976) 1087–1098.
[7] A. Gorsky, A. Zabrodin, A. Zotov, J. High Energy Phys. 01 (2014) 070, arXiv:1310.6958.
[8] V. Knizhnik, A. Zamolodchikov, Nucl. Phys. B 247 (1984) 83–103.
[9] P. Kulish, E. Sklyanin, Zap. Nauč. Semin. LOMI 95 (1980) 129–160; Engl. transl.: J. Sov. Math. 19 (1982) 1956;
P. Kulish, Zap. Nauč. Semin. LOMI 145 (1985) 140–163;
J.H.H. Perk, C.L. Schultz, Phys. Lett. A 84 (1981) 407–410.
[10] G. Felder, A. Veselov, Commun. Math. Phys. 160 (1994) 259–273;
A. Veselov, Theor. Math. Phys. 98 (3) (1994) 368–376.
[11] I. Frenkel, N. Reshetikhin, Commun. Math. Phys. 146 (1992) 1–60.
[12] A. Matsuo, Invent. Math. 110 (1992) 95–121;
I. Cherednik, Adv. Math. 106 (1994) 65–95.
[13] E. Mukhin, V. Tarasov, A. Varchenko, SIGMA 8 (2012) 072, arXiv:1201.3990.
[14] S.N.M. Ruijsenaars, H. Schneider, Ann. Phys. 146 (1986) 1–34;
S.N.M. Ruijsenaars, Commun. Math. Phys. 110 (1987) 191–213.
[15] V. Tarasov, A. Varchenko, St. Petersburg Math. J. 6 (1994) 275–313, arXiv:hep-th/9311040;
V. Tarasov, A. Varchenko, Amer. Math. Soc. Transl. (2) 174 (1996) 235–273, arXiv:hep-th/9406060.
[16] Z. Tsuboi, A. Zabrodin, A. Zotov, J. High Energy Phys. 05 (2015) 086, arXiv:1412.2586.
[17] C.N. Yang, Phys. Rev. Lett. 19 (1967) 1312–1315.
[18] A. Zabrodin, A. Zotov, J. Phys. A, Math. Theor. 50 (2017) 205202, arXiv:1701.06074.
[19] A. Zabrodin, A. Zotov, Nucl. Phys. B 922 (2017) 113–125, arXiv:1704.04527.