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MORSE–SMALE SYSTEMS AND TOPOLOGICAL STRUCTURE OF CARRIER MANIFOLDS

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Abstract. We review the results describing the connection between the global dynamics of Morse–Smale systems on closed manifolds and the topology of carrier manifolds. Also we consider the results related to topological classification of Morse–Smale systems.

CONTENTS

Introduction

In [3], *rough* dynamical systems with a continuous (flow) time are introduced on compact plane domains diffeomorphic to disks and bounded by cycles without contacts. The sense of the introduced notion is as follows: sufficiently small C^1-perturbations of the system do not change the qualitative behavior of the system. This notion is fruitful: actually, in [3], it is proved that rough systems are typical, i. e., they form an open everywhere dense set in the space of the considered systems, equipped with the C^1-topology. Moreover, it is shown by Andronov and Pontryagin that the dynamics of rough systems is sufficiently clear: according to [3], rough flows in a bounded part of a plane have finite numbers of equilibrium states and periodic trajectories such that they are hyperbolic and their union contains the limit set of any trajectory. Moreover, there are no separatrices from a saddle to a saddle (including the case where those two saddles coincide with each other). Note that the notion of plane rough flows is naturally extended to the notion of flows on two-dimensional spheres; thus, in the sequel, we mainly deal with manifolds without boundaries (for simplicity).

The paper [3] influenced investigations of the so-called Gor'kiy school (Andronov himself and his disciples and colleagues) a lot. In [62], the *roughness* notion is introduced for discrete-time dynamical systems (cascades) on circles and, in fact, the notion of rough flows without equilibrium states on toruses. From [62], it follows that rough cascades on a circle are typical and their dynamics is

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sufficiently clear: for any rough cascade, its number of periodic points is finite and each such point is hyperbolic.

In [74], Andronov–Pontryagin results are extended to arbitrary orientable closed surfaces, the roughness notion is modified, and the *structural stability* notion is introduced. In fact, the Peixoto result coincides with the Andronov–Pontryagin result on the dynamics of structurally stable flows, but the statement that such flows appears explicitly (formally, Andronov and Pontryagin do not separate that statement). Note, that Peixoto, being unaware of [62], has reproved several of Mayer's results.

In [87], Smale influenced by [3, 74] introduces a class of dynamical systems on manifolds that are called *Morse–Smale systems* nowadays; at that time, they were claimed to be typical dynamical systems with sufficiently clear dynamics. Actually, the Smale definition just lists the properties obtained in [3, 74], but the Andronov–Pontryagin condition about the absence of separatrix interfaces is changed for the following many-dimensional analog: any intersection of separatrices is transversal. Earlier, Smale proved the Poincaré conjecture for manifolds of dimensions greater than or equal to 5 (see [88, 90]), substantially using the Morse theory and vector fields generated by gradients of Morse functions. In [89], a class of gradient-like vector fields (i. e., Morse–Smale flows without periodic trajectories) is selected and it is proved that it is open and dense in the set of gradient vector fields.

It is reasonable to study the existence of Morse–Smale systems on closed manifolds. In [89], it is proved that any Morse function defined on a manifold can be approximated by a Morse function such that its gradient is a Morse–Smale vector field without periodic orbits. Then the translation to the time $t = 1$ along trajectories of such a field is a Morse–Smale diffeomorphism. Since Morse functions exist on any closed manifold, it follows that Morse–Smale systems (both flows and diffeomorphisms) exist on any closed manifold. In [72, 73], the structural stability of Morse–Smale systems is proved. Therefore, those systems form an open set in the space of C^1-smooth dynamical systems. From the contemporary viewpoint, Morse–Smale systems on closed manifolds are nothing but structurally stable dynamical systems with zero topological entropy. From this viewpoint, they are the simplest structurally stable systems (in [4, 5, 91, 92], the existence of broad classes of structurally stable dynamical systems with positive topological entropies is proved).

A close relation between dynamical characteristics of a Morse–Smale system and the topology of the carrier manifold is found already in the pioneering work [87]. This is why Morse–Smale systems have been under close attention up to now, and a lot of new publications in this direction still appear. Results of various areas of topology, originally not related to dynamical systems, are applied to investigate Morse–Smale systems. For example, for Morse–Smale flows without equilibrium states, it is found that periodic trajectories form a special collection of knots and links. Rather recently, it is found that invariant manifolds of saddle points might have wild embeddings.

The present review provides results about relations between the global dynamics of Morse–Smale systems on closed manifolds and the topology of carrier manifolds. Also, we provide results related to the topological classification of Morse–Smale systems considered on closed smooth n-dimensional connected manifolds M^n ($n \geq 1$).

The structure of this paper is as follows. In Sec. 1, the earliest definition of Morse–Smale systems, i. e., the Smale definition, and basic corollaries following intermediately from it are provided. In Sec. 2, we provide a filtration construction for the carrier manifold, related to the dynamics of the Morse–Smale system; it was introduced by Smale to deduce a system of inequalities called (by Smale) *Morse inequalities*. Those inequalities establish relations between Betti numbers of the carrier manifold and dynamical characteristics of the Morse–Smale system. In Sec. 3, Morse–Smale systems on one-dimensional and two-dimensional manifolds are considered. Further, Morse–Smale systems on manifolds of dimension exceeding two are considered. In Sec. 4, we consider Morse–Smale systems without equilibrium states (the nonwandering set of such a system consists of periodic trajectories) and provide the classical Morgan–Azimov results about the structure of the carrier manifold. In Sec. 5, Morse–Smale flows are considered on 3-manifolds such that their nonwandering sets include equilibrium points. We provide theorems on the structure of the carrier manifold and existence conditions for periodic trajectories (usually, this is important for applications). In Sec. 6, we describe flows with three equilibrium states and study the topology of the manifold admitting such systems. In Sec. 7, we consider Morse–Smale systems (both flows and diffeomorphisms) with restrictions; mainly, the restrictions refer to the absence of heteroclinic intersections of various kinds. Then we obtain sufficient existence conditions for heteroclinic curves and heteroclinic points. Existence theorems for heteroclinic curves are important for the study of magnetic fields in electrically conducting media (see, e. g., [43]). In Sec. 8, an arbitrary Morse–Smale diffeomorphism is represented as a diffeomorphism between a source and a sink; in Sec. 9, this representation is used to obtain classification results.

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1. Main Definition

Recall notions and introduce notation needed to provide the first definition of dynamical Morse–Smale systems (in the contemporary terminology). The space of C^r-diffeomorphisms (vector fields of smoothness C^r), endowed the uniformed C^r-topology, is defined by $Diff^r\ (M^n)$ ($\chi^r\ (M^n)$ respectively). For $r = 1$, the notation $Diff\ (M^n)$ ($\chi\ (M^n)$ respectively) is usually used. For brevity, we provide definitions mainly for diffeomorphisms; the corresponding definitions for vector fields (and flows) are similar.

Fix f from $Diff\ (M^n)$. Recall that a point x from M^n is called *nonwandering* if for any its neighborhood U and any positive integer N there exists n_0 from $\mathbb{Z}$ such that $|n_0| \geq N$ and $f^{n_0}(U) \cap U \neq \varnothing$. For any diffeomorphism f, the set of its nonwandering points is denoted by $NW(f)$. Obviously, each periodic point is nonwandering. A periodic point x_0 from $Per\ (f)$, $f^q(x_0) = x_0$, is called *hyperbolic* if the derivative $Df^q(x_0) : T_{x_0}M^n \to T_{x_0}M^n$ (treated as linear mapping of the tangential space into itself) has no eigenvalues with the absolute value equal to one. Due to the Grobman–Hartman theorem, if x_0 is a hyperbolic fixed point of a diffeomorphism f, then f is adjoint to the linear diffeomorphism defined by the Jacobi matrix $\left(\dfrac{\partial f}{\partial x}\right)\Big|_{x_0}$ in a neighborhood of x_0 (see [49, 50, 56]). Recall that diffeomorphisms $f : M \to M$ and $f' : M' \to M'$ are called *topologically adjoint* if there exists a homeomorphism $h : M \to M'$ such that $hf = f'h$.

This means that any hyperbolic point x_0 has a so-called *stable manifold* $W^s(x_0)$ and *unstable manifold* $W^u(x_0)$ defined as the set of points y from M^n such that $\varrho_M(f^{qk}x_0, f^{qk}y) \to 0$ as $k \to +\infty$ ($k \to -\infty$ respectively), where ϱ_M is the metric in M^n. Note that the unstable manifold $W^u(x_0)$ is a stable manifold with respect to f^{-1}. It is known that $W^s(x_0)$ and $W^u(x_0)$ are homeomorphic (with respect to the inner topology) to the Euclidean spaces $\mathbb{R}^{\dim W^s(x_0)}$ and $\mathbb{R}^{\dim W^u(x_0)}$ respectively and are smooth injective immersions of those spaces into M^n (see [57, 92]). A periodic hyperbolic point p from $NW(f)$ is called a *node* if either $\dim W^s(p) = n$ (in this case, p is called a *sink point*, see Fig. 1.1 (c)) or $\dim W^u(p) = n$ (in this case, p is called a *source point*, see Fig. 1.1 (b)). In the special case where p is a fixed point, it is called a *knot* (*sink* or *source* respectively). A hyperbolic periodic point σ from $NW(f)$ is called a *saddle point* if the topological dimensions of its stable and unstable manifolds are different from zero (see Fig. 1.1 (a)).

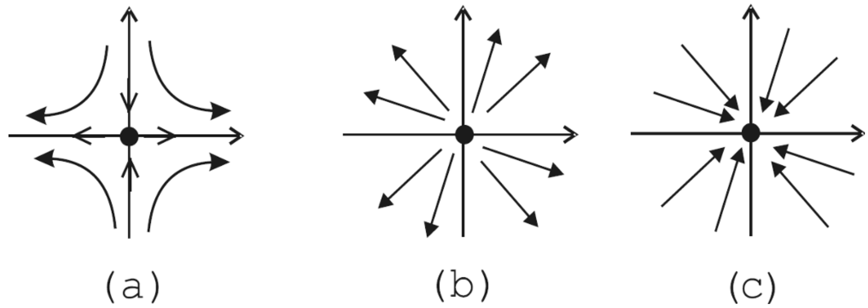


Fig. 1.1

The definitions of hyperbolic equilibrium states of flows, hyperbolic periodic movements, and their stable and unstable manifolds are introduced in the same way. However, unstable manifolds of equilibrium states of flows are homeomorphic to Euclidean spaces, while unstable manifolds of equilibrium states of one-dimensional periodic trajectories are homeomorphic to cylinders of the corresponding dimension.

A diffeomorphism f is called a *Morse–Smale diffeomorphism* if $NW(f)$ is a finite set of periodic points (and, therefore, $NW(f) = Per\ (f)$), all periodic points are hyperbolic, and the intersection of invariant manifolds $W^s(x)$ and $W^u(y)$ is transversal (if that intersection is not empty) provided that x and y belong to $NW(f)$. For flows, the definition is similar. Let $\mathcal{MS}^r\ (M^n)$ $(\Sigma^r(M^n))$ denote the set of Morse–Smale C^r-diffeomorphisms (vector fields respectively) on M^n. For $r = 1$, we use the notation $\mathcal{MS}\ (M^n)$ and $\Sigma\ (M^n)$.

Let f belong to $\mathcal{MS}\ (M^n)$. If $\dim W^u(\sigma) = i$, then each component of the set $W^u(\sigma) \setminus \sigma$ is called an *i-dimensional unstable separatrix* and each component of the set $W^s(\sigma) \setminus \sigma$ is called an $(n - i)$-*dimensional stable separatrix*. Since a point decomposes the one-dimensional Euclidean space, but decomposes no Euclidean space of a higher dimension, it follows that any one-dimensional (stable or unstable) manifold of a saddle periodic point consists of that saddle point and two periodic separatrices, while any i-dimensional manifold, $i \geq 2$, consists of that saddle point and one i-dimensional separatrix.

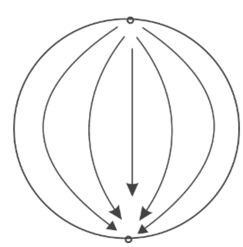


Fig. 1.2

The simplest Morse–Smale diffeomorphism is the diffeomorphism of a closed manifold with two fixed points such that one of them is an attracting point (sink), another one is a repelling point (source), and there are no other periodic points (see Fig. 1.2). In this case, the manifold is the sphere $\mathbb{S}^n$ and the dynamics of such a diffeomorphism is simple: all points different from fixed ones are wandering points moving from the source to the sink under the action of the diffeomorphism. All such diffeomorphisms are pairwise topologically adjoint to each other (see, e. g., [44, Th. 2.2.1]).

If $\psi : M^n \to \mathbb{R}$ is a Morse function, then the vector field $\nabla\psi = grad\ \psi$ on M^n defines a flow ψ^t without periodic trajectories and such that the number of its equilibrium states is finite and each equilibrium state is hyperbolic. The flow ψ^t is called a *gradient flow*; it is not guaranteed that it is a

Morse–Smale flow because the transversality of the intersection of separatrices of different its saddles can be broken. However, it is proved in [89] that the space of Morse functions contains an open everywhere dense set of Morse functions such that their gradient determines a Morse–Smale flow.

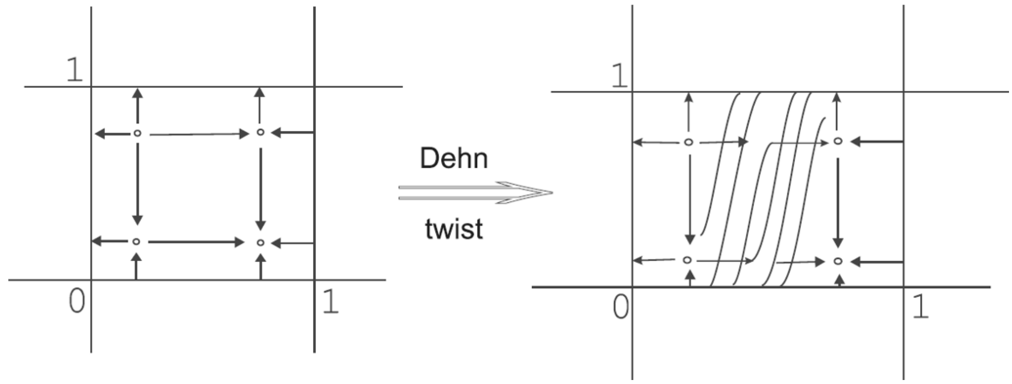


Fig. 1.3

Obviously, any Morse–Smale diffeomorphism that is a translation to the time unit of a flow induces the identity map in the group of homologies. There exist Morse–Smale diffeomorphisms inducing nontrivial isomorphisms in the group of homologies. For example, a Morse–Smale diffeomorphism of the two-dimensional torus such that it induces a nontrivial isomorphism in the one-dimensional group of homologies, but it cannot be embedded in a flow, can be obtained as a composition of the translation along the trajectories of the gradient Morse–Smale vector field and the corresponding Dehn twist along the closed transversal. The left-hand part of Fig. 1.3 displays the phase portrait of a gradient vector field with two saddles and two knots on the square that is the fundamental region of the universal covering of the torus. The Dehn twist acts along the curve such that the line $x = \dfrac{1}{2}$ is projected to that curve. The result of the composition of the translation along the trajectories and the Dehn twist is a diffeomorphism such that the stable manifold of one saddle intersects the unstable manifold of another saddle (see the right-hand part of Fig. 1.3). The Dehn twist yields a nontrivial action in the group of homologies the arising of so-called heteroclinic points preventing the inclusion of such a diffeomorphism to a flow. In [72], it is proved that any neighborhood of the identity map of a surface contains Morse–Smale diffeomorphisms that cannot by embedded in a flow.

The following theorem is announced in [83] and is proved in [84].

Theorem 1.1. *Let $f : M^d \to M^d$ be a Morse–Smale diffeomorphism. Then eigenvalues of the induced map $f_* : H_*(M^d, \mathbb{R}) \to H_*(M^d, \mathbb{R})$ are roots of unity.*

Recall that f_* denotes the family of all maps $f_{*,k} : H_k(M^d, \mathbb{R}) \to H_k(M^d, \mathbb{R})$, $k \in \{0, \dots, d\}$.

Let $f : M^n \to M^n$ be a Morse–Smale diffeomorphism and σ_1 and σ_2 be different saddle points from $NW(f)$. If $W^s(\sigma_1)$ and $W^u(\sigma_2)$ are their invariant manifolds and $W^s(\sigma_1) \cap W^u(\sigma_2) \neq \varnothing$, then we say that the said intersection is *heteroclinic*. Since $W^s(\sigma_1)$ and $W^u(\sigma_2)$ are locally embedded submanifolds, it follows that any connected component of the heteroclinic intersection $W^s(\sigma_1) \cap W^u(\sigma_2)$ is a locally embedded submanifold as well. If $\dim (W^s(\sigma_1) \cap W^u(\sigma_2)) \geq 1$, that any connected component of the intersection $W^s(\sigma_1) \cap W^u(\sigma_2)$ is called a *heteroclinic manifold*. In particular, if $\dim (W^s(\sigma_1) \cap W^u(\sigma_2)) = 1$, then the heteroclinic manifold is a *heteroclinic curve* (see Fig. 1.4).

If $\dim (W^s(\sigma_1) \cap W^u(\sigma_2)) = 0$, then the intersection $W^s(\sigma_1) \cap W^u(\sigma_2)$ is an enumerable set of points; they are called *heteroclinic points*. The set of heteroclinic points is invariant and is a union of heteroclinic orbits. Note that Morse–Smale flows do not have heteroclinic points and their heteroclinic manifolds consist of one-dimensional trajectories.

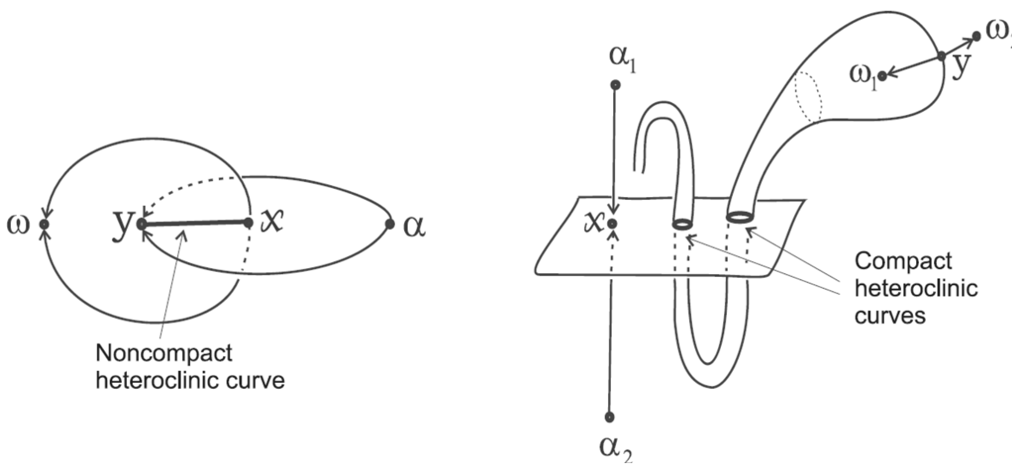


Fig. 1.4

2. Smale Filtration and Morse Inequalities

The definition of Morse–Smale systems imply the following results proved in [87].

Proposition 2.1. *Let $f : M^n \to M^n$ be a Morse–Smale diffeomorphism. Then*

$$M^n = \bigcup_{p \in NW(f)} W^s(p) = \bigcup_{p \in NW(f)} W^u(p).$$

A similar assertion is valid for Morse–Smale flows as well.

Theorem 2.1. *Let $f : M^n \to M^n$ be a Morse–Smale diffeomorphism and p and q belong to $NW(f)$. If $W^u(p) \cap W^s(q) \neq \varnothing$, then*

(1) $W^u(q) \subset \operatorname{clos} W^u(p)$ *and* $\dim W^u(p) \geq \dim W^u(q)$;
(2) $W^s(p) \subset \operatorname{clos} W^s(q)$ *and* $\dim W^s(p) \leq \dim W^s(q)$.

The inequalities for the dimensions are proved as follows. Really, let p and q belong to $NW(f)$ and $W^u(p) \cap W^s(q) \neq \varnothing$. Since the intersection of the invariant manifolds $W^u(p)$ and $W^s(q)$ is transversal, it follows that $\dim\ W^s(q) + \dim\ W^u(p) \geq n$. Hence, $\dim\ W^s(q) \geq \dim\ W^s(p)$ because $\dim\ W^s(p) = n - \dim\ W^u(p)$. The inequality $\dim W^u(p) \geq \dim W^u(q)$ is proved in the same way.

The condition $W^u(p) \cap W^s(q) \neq \varnothing$ is denoted by $p \succ q$. The finiteness of the number of nonwandering orbits and Theorem 2.1 imply the following assertion.

Proposition 2.2. *For any Morse–Smale diffeomorphism, the relation $\succ$ is a partial order relation on the set of periodic orbits.*

The relation $\succ$ is naturally extended to the set of periodic orbits. Let O_r denote the orbit of a point r from $NW(f)$. Assign $O_p \succ O_q$ if there exist points r from O_p and s from O_q such that $r \succ s$. It follows from Proposition 2.2 that the relation $\succ$ is a partial order relation on the set of periodic orbits.

In [92], it is proposed to represent the dynamics of a Morse–Smale system by means of the graph constructed as follows. To the periodic orbit O_p of a point p, we put in correspondence the node $v(O_p)$ of the graph G. Nodes $v(O_p)$ and $v(O_q)$ of the graph G are joined by the rib from $v(O_p)$ to $v(O_q)$ if and only if $W^u(p) \cap W^s(q) \neq \varnothing$ and there are no r from $NW(f)$ such that $W^u(p) \cap W^s(r) \neq \varnothing$ and $W^u(r) \cap W^s(q) \neq \varnothing$ (i. e., $p \succ r \succ q$). The graph constructed this way (its nodes can be endowed by additional data about the dimension of stable and unstable manifolds) is called the *Smale graph* of the diffeomorphism f.

For Morse–Smale flows, similar definitions of the order and Smale graph can be introduced. It is clear that Smale graphs of topologically equivalent Morse–Smale systems are isomorphic to each other. The inverse is not true even if we consider only Morse–Smale flows without periodic trajectories on surfaces. In [77] (also, see [70]), Morse–Smale flows on a sphere with isomorphic Smale graphs are constructed such that they are not equivalent topologically.

Recall that a sequence of subsets $M_0 \subset M_1 \subset \dots$ of a topological space M^n is called a *filtration* if the set family $M_0, M_1, \dots$ is a fundamental covering of the space M^n. In [87], a finite filtration related to the dynamics of a Morse–Smale system is constructed by means of Theorem 2.1. It is found that the subsets $K^d = \bigcup\limits_{\dim W_i^u \le d} W_i^u$ form a filtration, but the inclusion $W_i^u \subset K^d$ does not imply the inclusion $\partial W_i^u \subset K^{d-1}$ because it is possible that unstable manifolds lay in the limit set of other unstable manifolds of the same dimension. For a Morse–Smale diffeomorphism $f : M^n \to M^n$, the following filtration is used. Let $M_0 = K^0$ be the set of sink periodic points. Let M_1 be the union of M_0 and one-dimensional unstable manifolds such that their boundaries belong to M_0. Note that M_1 is a subset of K^1, but M_1 does not include one-dimensional unstable manifolds such that their limit set contains other one-dimensional unstable manifolds (see Fig. 1.3). If M_{i-1} is constructed, then the union of M_{i-1} and all unstable manifolds such that their boundaries belong to M_{i-1} is M_i. From Proposition 2.1, it follows that there exists k such that $M_k = M^n$. It is clear that all M_i are closed and invariant with respect to f and they form a finite filtration of the manifold M^n.

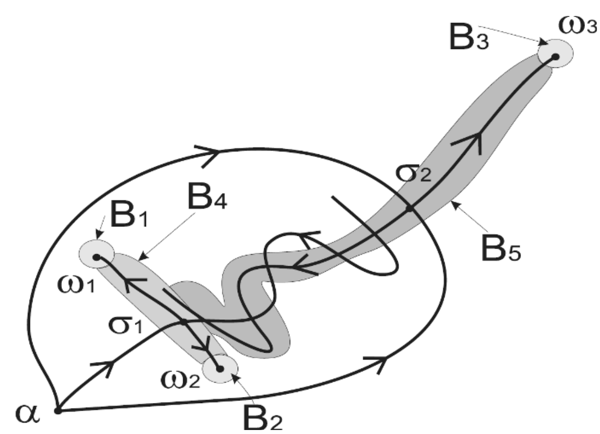


Fig. 2.1

Figure 2.1 displays the phase portrait of a diffeomorphism of the sphere S^2 such that its nonwandering set is hyperbolic and consists of three fixed sink points ω_1, ω_2, and ω_3, two fixed saddle points σ_1 and σ_2, and one fixed source α. The filtration elements for this diffeomorphism are $M_i = \bigcup\limits_{j=1}^{i} B_j$, $i = \overline{1,5}$, and $M_6 = S^2$, where B_i are appropriate disks.

Let C_j be the number of periodic points p of a diffeomorphism f such that the dimension of the unstable manifold of any such point is equal to $j = \dim W^s(p)$, $p \in Per\ (f)$. Let $\beta_i(M^n) = \beta_i$ be the ith Betti number of the manifold M^n, i.e., $\beta_i(M^n) = rank\ H_i(M^n, \mathbb{Z})$. In [87], it is shown that the following relations hold:

$$C_0 \ge \beta_0, \qquad C_1 - C_0 \ge \beta_1 - \beta_0, \qquad C_2 - C_1 + C_0 \ge \beta_2 - \beta_1 + \beta_0,$$

$$\dots\dots\dots\dots$$

$$\sum_{i=0}^{n} (-1)^i C_i = \sum_{i=0}^{n} (-1)^i \beta_i.$$

These relations hold for Morse–Smale flows as well, but the Betti numbers are computed in the ring $\mathbb{Z}_2$, $\beta_i(M^n) = rank\ H_i(M^n, \mathbb{Z}_2)$, and C_j is the sum of the number of equilibrium states with j-dimensional stable separatrices and the number of one-dimensional periodic trajectories with $(j+1)$-dimensional stable separatrices. In [36] (also, see [37]), the Morse–Smale inequalities are strengthened for Morse–Smale flows without equilibrium states.

The term "Morse inequalities" comes from [67] investigating relations between the number of critical points of the Morse function and the topological structure of the manifold.

Since $\beta_0 = 1$ for any connected manifold, it follows from the first inequality of the above system of inequalities that any Morse–Smale system has at least one sink and one source periodic orbit. This confirms the above mentioned fact that the simplest Morse–Smale diffeomorphism is the diffeomorphism of a closed manifold with two fixed points such that one of them is an attracting point (sink), another one is a repelling point (source), and there are no other periodic points (see Fig. 1.2).

3. One-Dimensional and Two-Dimensional Systems

The circle S^1 is the only closed one-dimensional manifold. Morse–Smale systems (both flows and diffeomorphisms) are everywhere dense in S^1; their combinatorial description is rather simple and it is omitted here because the number of trajectories of any such flow is finite. To prove the density of the diffeomorphisms, the closure lemma is to be proved. For the first time, it was done in [62]. Other proofs can be found in [75, 76, 79]. For the circle, the closure lemma is proved in any class of smoothness (even in the analytic one). Therefore, Morse–Smale diffeomorphisms $\mathcal{MS}^k(S^1)$ are everywhere dense in $Diff^r\ (S^1)$ provided that $1 \le k \le r \le \omega$. Taking into account this result, it is easy to prove that the set of C^r-rough or, which is the same, of C^r-structurally stable diffeomorphisms of the circle coincides with $\mathcal{MS}^r(S^1)$.

The description and topological classification of Morse–Smale diffeomorphisms of the circle, preserving the orientation, is rather simple. Decompose $\mathcal{MS}^r(S^1)$ into the subclass $\mathcal{MS}^r_+(S^1)$ of diffeomorphisms preserving the orientation and the subclass $\mathcal{MS}^r_-(S^1)$ of diffeomorphisms changing the orientation. From [62], the following fact is known.

Proposition 3.1.

(1) *For any diffeomorphism φ from $\mathcal{MS}^r_+(S^1)$, the nonwandering set $NW(\varphi)$ consists of* $2n$, $n \in \mathbb{N}$, *periodic orbits such that the period of each one is equal to* k.

(2) *For any diffeomorphism φ from $\mathcal{MS}^r_-(S^1)$, the nonwandering set $NW(\varphi)$ consists of* $2q$, $q \in \mathbb{N}$, *periodic points such that two of them are fixed and the period of each other one is equal to* 2.

Let φ belong to $\mathcal{MS}^r_+(S^1)$. Renumber periodic points from $NW(\varphi)$ as follows: $p_0, p_1, \dots, p_{2nk-1}$, $p_{2nk}{=}p_0$ clockwise, beginning from an arbitrary periodic point p_0; then there exists an integer l such that $\varphi(p_0) = p_{2nl}$, $l = 0$ for $k = 1$, $l \in \{1, \dots, k-1\}$ for k exceeding one, and the numbers (k, l) are relatively prime.[1] Note that l does not depend on the choice of the point p_0 (see Fig. 3.1 (A)). For φ from $\mathcal{MS}^r_-(S^1)$, assign $\nu = -1$, $\nu = 0$, and $\nu = +1$ if its fixed points are sources, a source and a sink, and sinks, respectively. Note that $\nu = 0$ if q is odd, while $\nu = \pm 1$ if q even (see Fig. 3.1 (B)).

The following fact is known from [62] as well.

Theorem 3.1.

(1) *Diffeomorphisms φ and φ' from $\mathcal{MS}^r_+(S^1)$ with parameters n, k, and l and n', k', and l' (respectively) are topologically adjoint if and only if $n = n'$, $k = k'$, and one of the following assertions is valid*:
   - $l = l'$ (*in this case, if* $l \neq 0$, *then the conjugating homeomorphism preserves the orientation*),
   - $l = k' - l'$ (*in this case, the conjugating homeomorphism changes the orientation*).

[1] In [62], the *order number* r_1 is used instead of l such that $l \cdot r_1 \equiv 1 (mod\ k)$.

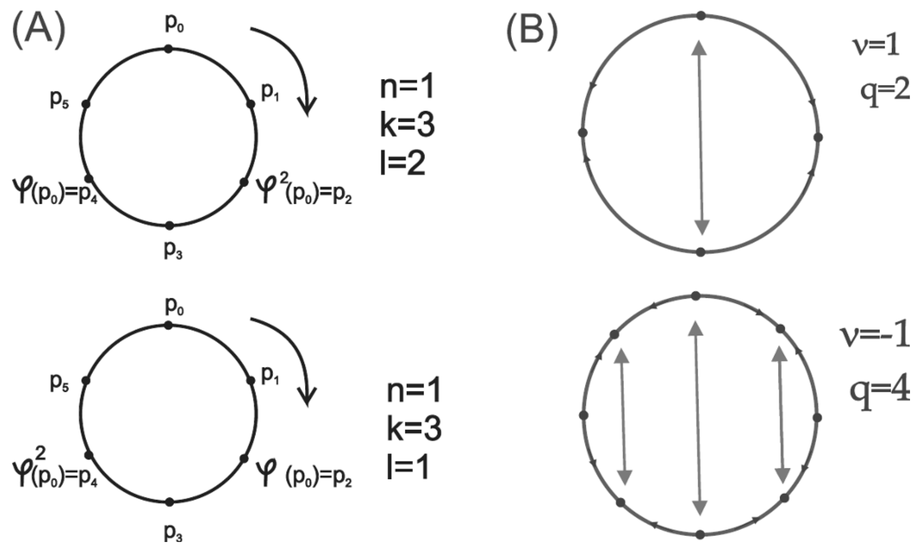


Fig. 3.1

(2) *Diffeomorphisms* φ *and* φ' *from* $\mathcal{MS}^r_-(S^1)$ *with parameters* q *and* q' (*respectively*) *are topologically adjoint if and only if* $q = q'$ *and* $\nu = \nu'$.

Pass to Morse–Smale systems on two-dimensional manifolds. First, we consider flows. Then we consider diffeomorphisms.

For a Morse–Smale flow with a predefined collection of equilibrium states, the only necessary restriction for the existence on a given closed surface is the Euler–Poincaré relation: the sum of indices of the equilibrium states is equal to the Euler characteristic (see, e. g., [7]). Regarding the typicalness of Morse–Smale flows, the following theorem is valid.

Theorem 3.2. *The set* $\Sigma\ (M^2)$ *of Morse–Smale vector fields coincides with the space of structurally stable vector fields on the closed surface* M^2. *The set* $\Sigma\ (M^2)$ *is open and dense in the space* $\chi\ (M^2)$ *of all vector fields on* M^2. *If* M^2 *is an orientable surface or a nonorientable surface of kind* $g, 1 \le g \le 3$, *then* $\Sigma^r(M^2)$ *coincides with the space of* C^r*-structurally stable vector fields and is open and everywhere dense in* $\chi^r\ (M^2)$ *provided that* $r \ge 1$.

In fact, Theorem 3.2 is proved for the sphere $M^2 = S^2$ in [3]. In [75, 76], this theorem is proved for all other surfaces (for the case where $r = 1$) and for all orientable surfaces and for the projective plane (for the case where $r \ge 1$). According to the Aranson–Markley theorem (see [6, 60]), any flow on the Klein bottle has no nontrivially recurrent trajectories. This implies the validity of Theorem 3.2 for the Klein bottle provided that $r \ge 1$. Finally, for nonorientable surfaces of kind 3, Theorem 3.2 for $r \ge 1$ follows from the following result of [53]: no flow on a nonorientable surface of kind 3 has a so-called nonorientable nontrivially recurrent trajectory.

Recall definitions to pass to the classification. We say that flows f_1^t and f_2^t on M^n are *topologically equivalent* if there exists a homeomorphism $h : M^n \to M^n$ mapping trajectories of one of those flows to trajectories of another one. If this is satisfied and h preserves the direction with respect to time, then we say that f_1^t and f_2^t are *topologically trajectory* (*orbitally*) *equivalent*. We say that this invariant is *complete* if its coincidence for two flows is a necessary and sufficient condition of their equivalence. Usually, the invariant of the topological equivalence is closely related to the invariant of the trajectory equivalence: once one of them is constructed, it is easy to construct another one. Therefore, it is not always specified what equivalence is meant.

In [77], the following complete topological invariant is introduced for Morse–Smale flows: it is an equipped graph (see Fig. 3.2) including the information about the mutual location of nonwandering trajectories and their invariant manifolds (in particular, separatrices). In fact, Peixoto equipped the Smale graph by additional information to transform it to a complete topological invariant (the so-called distinguishing graph). Note that the scheme of the flow introduced in [58, 59] (also, see [2]) is a complete topological invariant for flows on the sphere such that their numbers of special trajectories are finite; fundamentally, it coincides with the Peixoto graph. For such flows, there exist other forms of complete topological invariants in the spirit of the Peixoto equipped graph (see, e.g., [69, 95]).

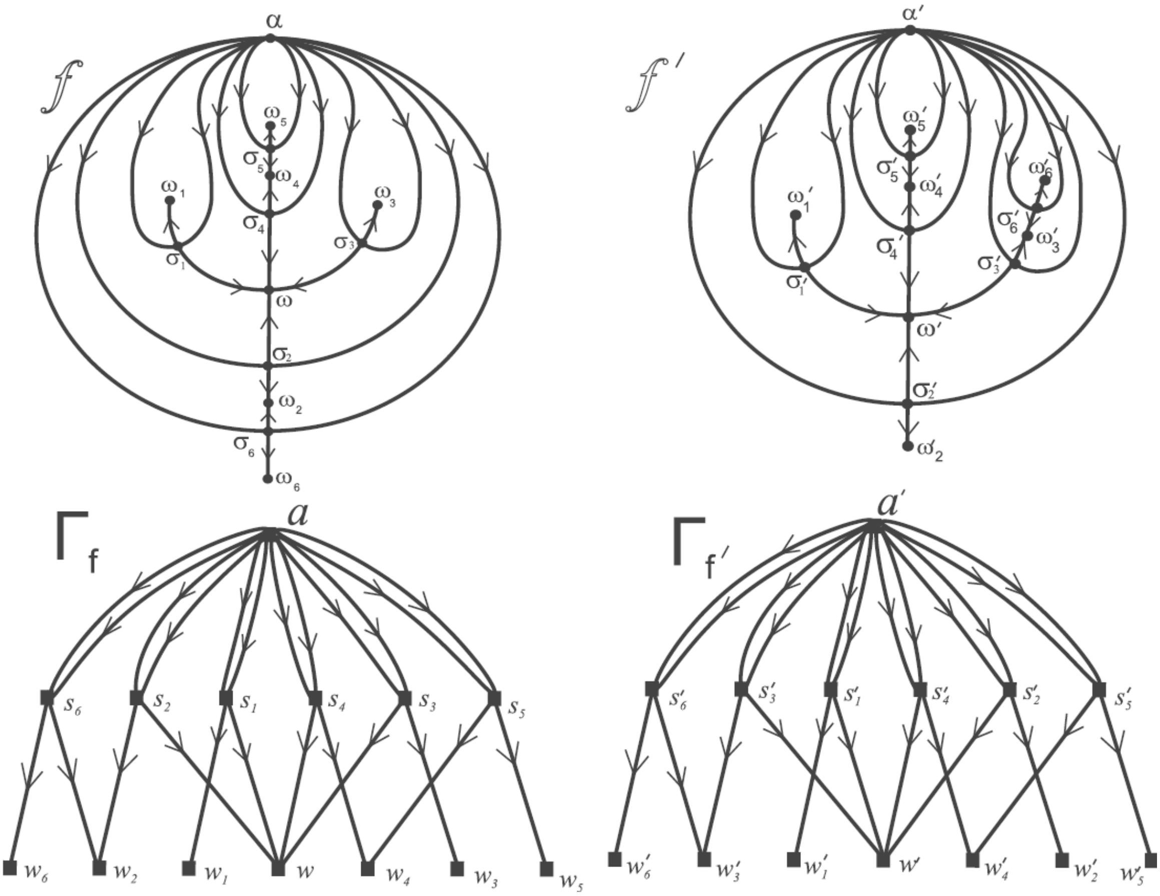


Fig. 3.2

In Fig. 3.2, two gradient-like flows and their graphs are displayed. This illustration shows that if a graph is not equipped by additional information, then it is not a complete topological invariant: the displayed flows are not equivalent to each other, but their graphs coincide with each other.

In [71], a complete topological invariant is constructed for arbitrary Morse–Smale flows on closed surfaces. Let us describe this invariant schematically.

Consider a vector field $\vec{v}$ on a compact surface N such that it is transversal to the boundary ∂N. We say that N is an *elementary* surface if either it contains one and only one nonwandering trajectory (a knot or a limit cycle) or all nonwandering trajectories are saddles. We say that an elementary surface is *nodal* and assign a $\vec{v}$-atom of type K to it if it is homeomorphic to a disk and contains one and only one knot (a sink or a source). If it is homeomorphic either to a ring or to a Möbius band and contains one and only one limit cycle, then we assign a $\vec{v}$-atom of type R-Möb to it. If it contains only

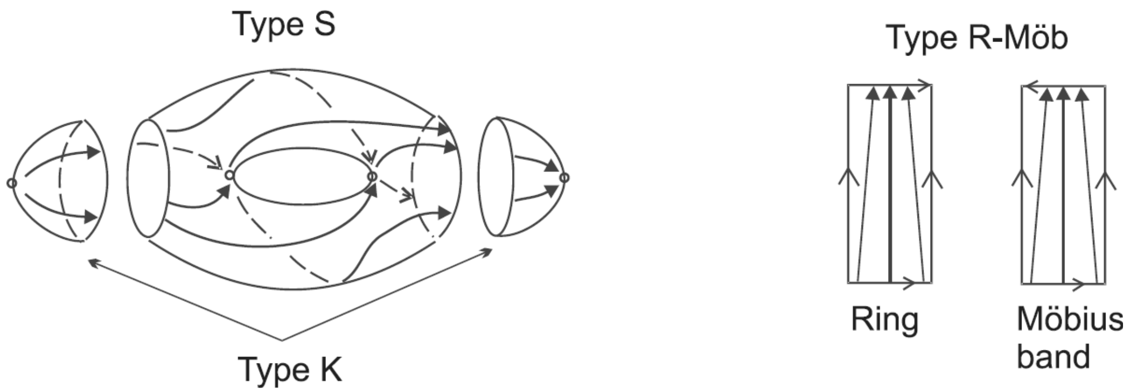


Fig. 3.3

saddles, then a $\vec{v}$-atom of type S is assigned to it (see Fig. 3.3). It is possible to show that for any Morse–Smale vector field $\vec{v}$ there exists a family of closed pairwise disjoint transversals, decomposing the carrier surface M^2 into elementary surfaces. Therefore, a graph $\Gamma(\vec{v})$ such that its nodes are $\vec{v}$-atoms with the specified types corresponds to the field $\vec{v}$. Two nodes are connected by a rib if the corresponding elementary surfaces have a common boundary component. Each rib is endowed with a direction according to the direction of the vector field at the common boundary component. If a rib connects nodes such that no one has type K, then either the number -1 or the number $+1$ is assigned to that rib: if the homeomorphism gluing the two boundary components changes the orientation, then we assign the number -1; the number $+1$ is assigned otherwise.

The structure of a flow on elementary surfaces corresponding to atoms of type K and R-Möb is clear because it contains only one nonwandering attracting or repelling trajectory. For atoms of type S, an invariant similar to the distinguishing Peixoto graph is constructed. Let a $\vec{v}$-atom of type C be assigned to an elementary surface N. Let σ be a saddle from N. For any Morse–Smale vector field, any separatrix of the saddle σ intersects ∂N. Any arc of an unstable (stable) separatrix from σ to ∂N is called a $u(s)$-arc. The closures of all $u(s)$-arcs decompose N into open domains such that each one has one and only one boundary component such that the field is directed out and one and only one boundary component such that the field is directed in (see Fig. 3.4 (a) and (b)). Any arc of a trajectory from one boundary component to another one is called a *t-arc*. In each domain, select one t-arc arbitrarily. Then the family of all $u(s)$-arcs and selected t-arcs decompose N into curvilinear polygons (see the qualitative picture at Fig. 3.4 (c)). The obtained curvilinear polygons are called *canonical*.

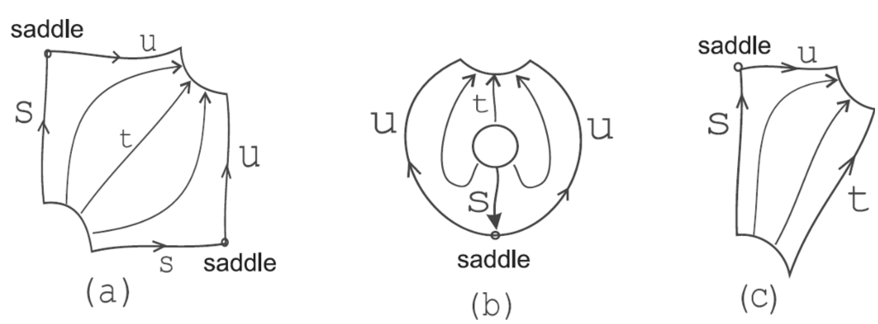


Fig. 3.4

In [71], for a given partition of an elementary surface N of type S, the following graph $\Gamma(N)$ (the so-called *three-color* graph) is constructed (see Fig. 3.5):

(1) nodes of the graph $\Gamma(N)$ are in a one-to-one correspondence to canonical curvilinear polygons;
(2) if two canonical polygons have a common side formed by a $u(s,t)$-arc, then the rib connecting the corresponding nodes is provided with the label u (s, t respectively).

The constructed graph does not depend on the choice of t-arcs.

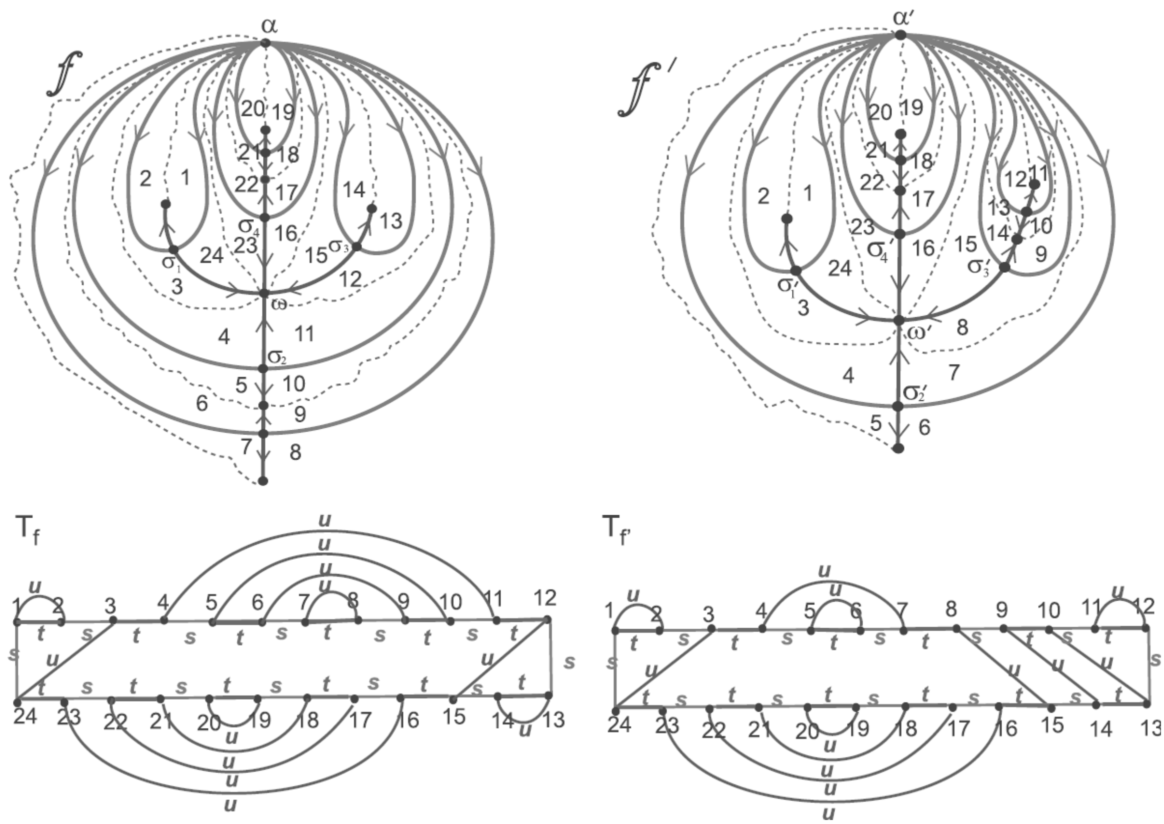


Fig. 3.5

Using the information about the atoms, curves connecting them, and three-color graphs, one constructs the graph of the flow (in [71], it is called the *molecule*) and proves the following theorem.

Theorem 3.3. *Morse–Smale vector fields $\vec{v}$ and $\vec{v}'$ on a closed surface are topologically trajectory equivalent if and only if the corresponding molecules $\Gamma(\vec{v})$ and $\Gamma(\vec{v}')$ are isomorphic.*

Not any molecule (in the sense of the above abstract definition) can be treated as a molecule of a Morse–Smale vector field: it and the three-color graphs should satisfy certain restrictions (see [71, Th. 3.24]). In [71], algorithms to compare molecules and to enumerate them with respect to a complexity criterion are proposed. In [42], it is shown that the specified algorithm is not efficient. Recall that a resolving algorithm for the problem to distinguish graphs is called *efficient* if the time to execute it polynomially depends on the number of nodes of the graph. In [31], a notion of efficiently resolved problems is proposed: a computational problem can be really resolved by a device if the resolving time is bounded by a polynomial of the parameter representing the length of the input data. At the moment, it is not known whether an efficient algorithm to distinguish arbitrary graphs exist. In [42], the efficiency of distinguishing algorithms for three-color graphs and Peixoto graphs is achieved by means of the following facts: three-color graphs are three-valent, while Peixoto graphs are embeddable to the 2-sphere. Also, it is proved that the problem to establish the orientability of the surface and to find its kind is polynomial as well.

Apply Theorem 3.3, which is a classification theorem, to illustrate the relation between the dynamics of the Morse–Smale flow and the carrier surface.

Proposition 3.2. *Let f^t be a Morse–Smale flow on a closed surface M^2 such that the nonwandering set of the flow f^t consists of three equilibrium states. Then $M^2 = \mathbb{P}^2$ is a projective plane. All Morse–Smale flows on $\mathbb{P}^2$ such that the nonwandering set of the flow consists of three equilibrium states are topologically equivalent to each other.*

Consider diffeomorphisms on surfaces.

The next theorem establishes a relation between the kind of the carrier surface and dynamical characteristics of Morse–Smale diffeomorphisms of closed (orientable and nonorientable) surfaces. It follows from the last relation of the system of Morse inequalities.

Theorem 3.4. *Let $f : M_g^2 \to M_g^2$ be a Morse–Smale diffeomorphism of a closed surface M_g^2 of kind g ($g \geq 0$ if M_g^2 is an orientable surface and $g \geq 1$ if M_g^2 is a nonorientable surface). Suppose that f has $\nu(f)$ saddle periodic points and $\mu(f)$ nodal periodic points. Then*

$$g = \frac{\nu(f) - \mu(f) + 2}{2} \text{ if } M_g^2 \text{ is an orientable surface and}$$

$$g = \nu(f) - \mu(f) + 2 \text{ if } M_g^2 \text{ is an nonorientable surface.}$$

Note that Theorem 3.4 contains no assumptions about possible intersections of invariant manifolds of saddle periodic points or about the character of the embedding of those manifolds to M_g^2.

In [12–14, 20, 54, 68], more refined results regarding relations between periodic data of Morse–Smale diffeomorphisms, their homotopic classes, and topological characteristics of the carrier surfaces are obtained; in many of the above papers, wider classes of diffeomorphisms, including Morse–Smale diffeomorphisms, are considered. In [35], the zeta-function of the Morse–Smale diffeomorphism is expressed via its periodic data.

To solve the classification problem (up to the adjointness) for diffeomorphisms of surfaces, it is natural to start from the investigation of the topological structure of domains formed by the partition of the wandering set by separatrices of saddle periodic points. Let f belong to $\mathcal{MS}\ (M^2)$. Take the wandering set $M^2 \setminus NW(f)$ and remove the separatrices of all saddle periodic points of the diffeomorphism f. Then only three cases are possible for the connected component of the remaining set:

(1) it is a 1-connected wandering-type component;
(2) it is a 1-connected periodic-type component;
(3) it is a 2-connected periodic-type component.

In the last case, M^2 is S^2 and the boundary of the component consists of two and only two points: the sink and the source (see [8]).

Morse–Smale diffeomorphisms without heteroclinic points (*gradient-like diffeomorphisms*) on orientable closed surfaces are classified in [15–17]. Roughly speaking, such diffeomorphisms are obtained as follows: Morse–Smale flows without closed trajectories and periodic transformations of surfaces undergo superpositions of translations to the time unit. The complete invariant for such diffeomorphisms is an analog of the distinguishing Peixoto graph equipped with periodic automorphisms. Also, an exhausting classification (including the realization) of gradient-like diffeomorphisms on surfaces is obtained in [41] in terms of three-color graphs equipped with periodic automorphisms.

If Morse–Smale diffeomorphisms have heteroclinic points, then their classification becomes much more complicated (see Fig. 3.6). Heteroclinic points lead to the appearance of so-called heteroclinic chains of saddle orbits (i. e., orbits generated by saddle periodic points). Recall that a sequence of saddle orbits $O_1, \ldots, O_h$ forms a *heteroclinic chain* if $W^u(O_i) \cap W^s(O_{i+1}) \neq \varnothing$ for any i such that $1 \leq i < h$. Since Morse–Smale diffeomorphisms have no homoclinic points, it follows that each heteroclinic chain consists of a finite number of pairwise different saddle orbits. Denote that number by h, $h \geq 2$, and call the number $h - 1$ the *length* of the chain $O_1, \ldots, O_h$. The greatest length of the chains connecting orbits O and O' such that $O \prec O'$ is denoted by $beh(O'|O)$. Any heteroclinic chain of the greatest length corresponds to a simple nonclosed path in the Smale graph of the given diffeomorphism. Figures 3.6 (a)-(b) display heteroclinic chains of length 1 and length 2.

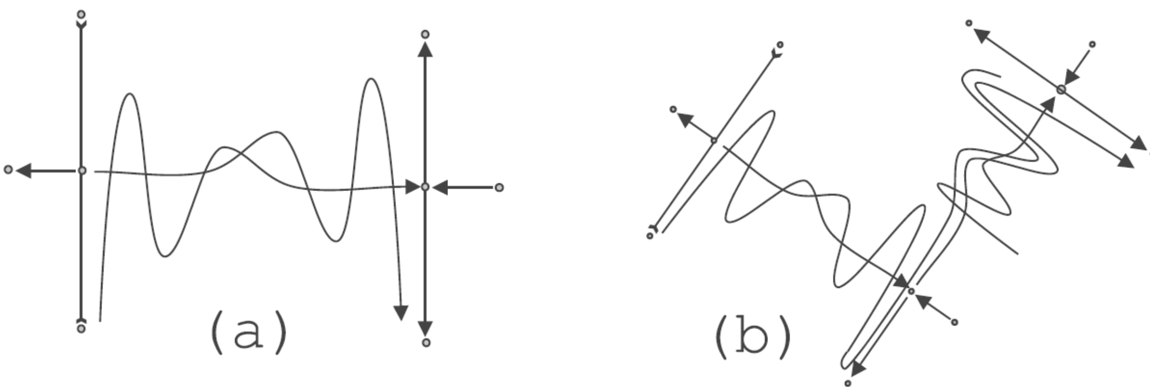
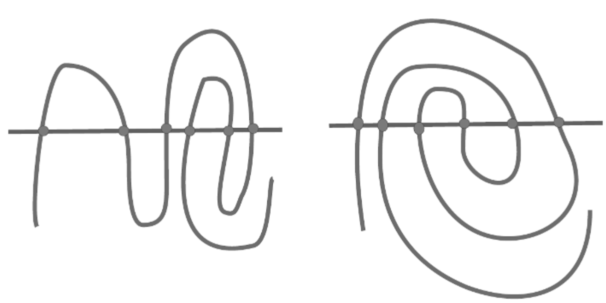


Fig. 3.6

Since heteroclinic points are formed by an intersection of invariant manifolds of saddle orbits, it follows that the set of heteroclinic points is invariant and, therefore, is a union of orbits. First, we consider the class of diffeomorphisms with finite numbers of heteroclinic orbits. In this case, the greatest heteroclinic chains have length 1. Morse–Smale diffeomorphisms with finite numbers of heteroclinic orbits are classified in [38]: the so-called signature carrying the information about heteroclinic points is added to the Bezdenezhnykh–Grines graph (see Fig. 3.7).



$$\begin{pmatrix} 123456 \\ 123654 \end{pmatrix} \qquad \begin{pmatrix} 123456 \\ 163452 \end{pmatrix}$$

Fig. 3.7

Also, an exhausting classification (including the realization) of Morse–Smale diffeomorphisms with finite numbers of heteroclinic orbits on surfaces is obtained in [65], where a wider class of diffeomorphisms is considered, namely, diffeomorphisms with finite numbers of stability modules on surfaces.

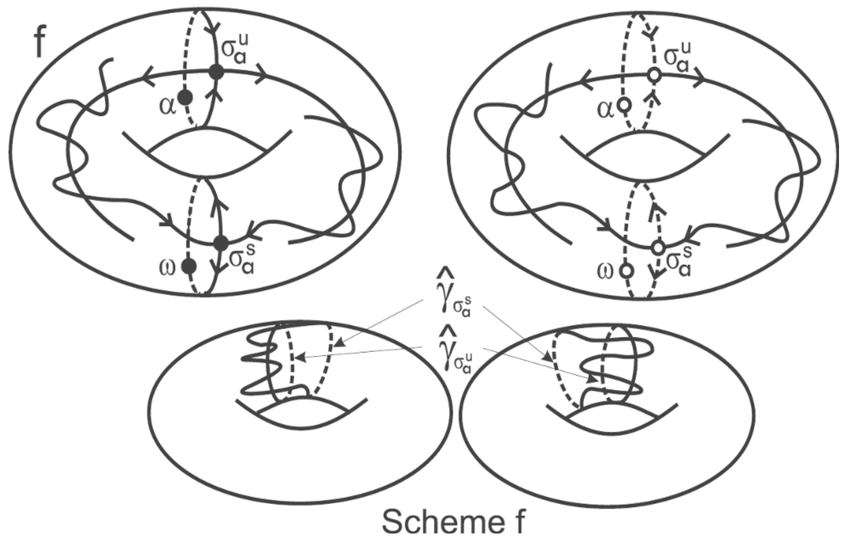


Fig. 3.8

In [65], a complete topological invariant called the *scheme* is introduced: up to a homeomorphism, this is a finite number of two-dimensional toruses that are spaces of wandering orbits with a collection of simple closed curves or spaces of orbits of separatrices (see Fig. 3.8).

Arbitrary Morse–Smale diffeomorphisms on orientable surfaces are classified in [28], where necessary and sufficient conditions for the adjointness of arbitrary structurally stable diffeomorphisms of orientable closed surfaces are obtained in terms of Markov partitions. The main result of [28] is valid for all structurally stable diffeomorphisms. We present its simplified version for Morse–Smale diffeomorphisms.

Let $O_1, \ldots, O_h$ be the greatest heteroclinic chain of saddle orbits of a Morse–Smale diffeomorphism f. The union $\bigcup\limits_{i=1}^{h} O_i \cup K_{1h}$, where $K_{1h} = \left(\bigcup\limits_{i=1}^{h} W^u(O_i)\right) \bigcap \left(\bigcup\limits_{i=1}^{h} W^s(O_i)\right)$, is called the *saturation* of the given chain. In other words, the saturation of a chain is the union of saddle orbits and all possible intersections of stable and unstable separatrices of saddle periodic points included to the chain. Note that the saturation of the greatest heteroclinic chain is an invariant set. By virtue of the structural stability of Morse–Smale diffeomorphisms, separatrices of saddle periodic points intersect each other transversally. This implies (see [28]) that the saturation of greatest heteroclinic chains can be provided with a uniform hyperbolic structure, and a coordinated hyperbolic structure can be introduced on the intersection of the saturations. Therefore, a saddle-type hyperbolic set K_f called a *saturated hyperbolic set* corresponds to any Morse–Smale diffeomorphism f. In [28], it is shown that any saturated hyperbolic set has an invariant neighborhood of the finite topological type, i.e., a closed surface with a finite number of holes.

The notion of saturated hyperbolic sets can be treated as a generalization of the notion of saddle-type base sets (for those sets, we have the transitivity apart from the hyperbolic structure). Therefore, Markov partitions of such sets can be constructed by means of the Bowen–Sinay technique (see [29, 85, 86]). In other words, the saturated hyperbolic set is covered by a special family of curvilinear quadrangles formed by segments of stable and unstable separatrices. In [28], for so-called good Markov partitions, the notion of the geometric type is introduced. A geometric type includes the description of the reciprocal location, orientation, and enumeration of the curvilinear quadrangles and their images under the action of the diffeomorphism. Two geometric types are *equivalent* if they are geometric types of good Markov partitions of the same hyperbolically saturated set (e. g., changing the enumeration of the curvilinear quadrangles, we obtain equivalent geometric types). The main result for Morse–Smale systems is as follows (see [28, Th. 1.0.3]).

Theorem 3.5. *Let K_{f_1} and K_{f_2} be saturated hyperbolic sets of Morse–Smale diffeomorphisms f_1 and f_2 of closed orientable surfaces M_1 and M_2 respectively. Suppose that K_{f_1} and K_{f_2} have good Markov partitions with equivalent geometric types. Then f_1 and f_2 are adjoint on invariant neighborhoods of the sets K_{f_1} and K_{f_2}, i. e., there exists a homeomorphism h of a finite topological type, mapping an invariant neighborhood U_1 of the set K_{f_1} to an invariant neighborhood U_2 of the set K_{f_2} such that U_2 has a finite topological type and h conjugates the restrictions $f_1|_{U_1}$ and $f_2|_{U_2}$.*

In [18], a finite algorithm is presented; using it, one can check whether two geometric types are equivalent.

4. Morse–Smale Flows without Equilibrium States

To describe carrier manifolds for Morse–Smale flows without equilibrium states, we use partitions into (standard) handles, partitions into circular handles, the Seifert space, and other notions of topology of manifolds.

The *Seifert space* (manifold or fibering) is any 3-manifold M^3 representable as a union of pairwise disjoint closed simple curves C_α such that each curve C_α has a closed neighborhood homeomorphic to the solid torus $D^2 \times S^1$, arising as follows: the disk D^2 is multiplied by the segment $[0;1]$ such that any point $(x;0)$ is glued with the point $(d(x);1)$, where $d : D^2 \to D^2$ is the rotation of the disk D^2 to the angle $2\pi\dfrac{m}{n}$ (m and n are relatively prime integers such that $0 \leq m < n$).

A three-dimensional manifold M^3 is *big* in the Waldhausen sense if M^3 contains an embedded surface such that its fundamental group is infinite and is a subgroup of the fundamental group of the manifold M^3. In particular, any 3-manifold such that its first group of homologies is infinite and any 3-manifold such that its edge is incompressible (e. g., the manifold $T^2 \times [0;1]$) are big. Any complement to a knot embedded to a 3-sphere is a manifold big in the Waldhausen sense (see [61]).

The next theorem on the topological structure of the carrier 3-manifold for Morse–Smale flows without equilibrium states is proved in [66].

Theorem 4.1. *Let f^t be a Morse–Smale flow without equilibrium states, defined on a closed three-dimensional manifold M^3. Then, if M^3 is not big in the Waldhausen sense, then M^3 is a Seifert space. If M^3 is big in the Waldhausen sense, then M^3 is a union of Seifert spaces and direct products $T^2 \times [0;1]$.*

In [36], necessary and sufficient existence conditions are obtained for Morse–Smale S^3-flows (S^3 is the three-dimensional sphere) without equilibrium states, but with a prescribed collection of periodic trajectories. Namely, denote the number of one-dimensional non-twisted[2] periodic trajectories of index k (the index of a periodic trajectory is equal to the dimension of its unstable manifold minus one) by A_k. Then a Morse–Smale flow on S^3 such that it has no equilibrium states and its prescribed collection is (A_0, A_1, A_2) exists if and only if the following conditions are satisfied:

(1) $A_0 \geq 1$ and $A_2 \geq 1$;
(2) $A_1 \geq A_0 - 1$ and $A_1 \geq A_2 - 1$.

If the above conditions are satisfied, then the number of twisted periodic trajectories of index 1 can be arbitrary (see [36]).

Periodic trajectories of a Morse–Smale flow on the three-dimensional sphere S^3 form indexed links (this means that the index of the corresponding periodic trajectory is assigned to any knot). The

[2]We say that a periodic trajectory is *twisted* if the Poincaré map on its unstable manifold is adjoint to the involution $\vec{x} \to -\vec{x}$. Otherwise, we say that the trajectory is *non-twisted*.

simplest Morse–Smale S^3-flow without equilibrium states is a flow with two and only two periodic trajectories such that the index of one of them (the attracting trajectory) is equal to 0, while the index of another one (the repelling trajectory) is equal to 2 (see Fig. 4.1). Those trajectories form the known Hopf link. Taking into account the indices, we call it the Hopf $(0, 2)$-link.

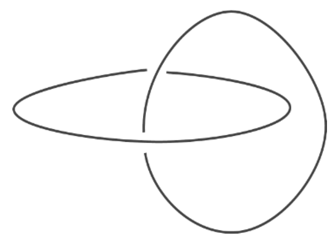


Fig. 4.1. The Hopf link

In [94], six operations are introduced on the set of indexed links (we call them Wada operations) and it is proved that any indexed link formed by periodic trajectories of a Morse–Smale S^3-flow without equilibrium states can be obtained from the Hopf $(0, 2)$-link by means of Wada operations. Conversely, any indexed link on S^3, obtained from the Hopf $(0, 2)$-link by means of Wada operations can be realized by means of periodic trajectories of a Morse–Smale flow without equilibrium states (also, see [82, 96]). In [19], indexed links of nonsingular S^3-flows without heteroclinic intersections are described in terms of the *Lyapunov graph*, i. e., the oriented graph such that its nodes and ribs correspond to filtration elements and regular level curves (respectively) of the Lyapunov function for the flow.

Recall that the *k-handle* (*handle of index k*) of dimension n, $0 \leq k \leq n$, is the product $D^k \times D^{n-k}$ or its homeomorphic image, where D^j is the j-dimensional closed ball. The part $S^{k-1} \times D^{n-k}$ of the boundary $\partial\left(D^k \times D^{n-k}\right)$ is called the *base* of the handle $D^k \times D^{n-k}$. The gluing of the k-handle $D^k \times D^{n-k}$ to an n-dimensional manifold M^n with a nonempty boundary is the identifying of the base

$$S^{k-1} \times D^{n-k} \subset S^{k-1} \times D^{n-k} \bigcup D^k \times S^{n-k-1} = \partial\left(D^k \times D^{n-k}\right)$$

and a part of the boundary ∂M^n by means of a homeomorphism $S^{k-1} \times D^{n-k} \to \partial M^n$. To glue the handle of index 0 is to add an n-dimensional ball to M^n. To glue the handle of index n is to paste an n-dimensional ball into a component of ∂M^n.

The *circular k-handle* of dimension n, $0 \leq k \leq n$, is the product $S^1 \times D^k \times D^{n-k-1}$. The gluing of the circular k-handle $S^1 \times D^k \times D^{n-k-1}$ to M^n is the identifying of the set

$$S^1 \times S^{k-1} \times D^{n-k-1} \subset \partial\left(S^1 \times D^k \times D^{n-k-1}\right)$$

and a part of the boundary ∂M^n by means of a diffeomorphism $S^1 \times S^{k-1} \times D^{n-k-1} \to \partial M^n$. If M^n can be obtained from $M^{n-1} \times [0; 1]$, where M^{n-1} is an $(n-1)$-manifold, by means of an iterated gluing of circular handles (circular handles of different indices are allowed), then we say that M^n *admits an expansion into circular handles.*

For the case where $n \geq 4$, the following theorem is proved in [10].

Theorem 4.2. *A Morse–Smale flow without equilibrium states exists on a manifold M^n $(n \geq 4)$ if and only if M^n admits an expansion into circular handles.*

In [11], it is proved that if the dimension of the manifold is greater than or equal to four, then any vector field without singularities is homotopic (in the space of vector fields without singularities) to the Morse–Smale vector field. Therefore, by virtue of Theorem 4.2, the description of the carrier manifold can be applied to a wider class of flows without equilibrium states.

5. Morse–Smale Flows with Equilibrium States, Defined on 3-manifolds

To describe carrier manifolds for Morse–Smale flows with equilibrium states, the Heegaard expansion, the Heegaard kind, and other notions of topology of manifolds are used.

A three-dimensional manifold D_g^3 is called a *3-ball with g handles* if D_g^3 is obtained from a three-dimensional disk D^3 by means of the gluing of g handles of index 1, $g \geq 0$. A *Heegaard partition* of kind g, $g \geq 0$, of a closed three-dimensional manifold M^3 is a representation of M^3 as a glueing of 3-balls with g handles by means of a homeomorphism identifying their boundaries. We say that a manifold M^3 has the *Heegaard kind* $h(M^3)$ if $h(M^3)$ is equal to the least g such that a Heegaard partition of kind g exists for the manifold M^3.

In [45, 46], we consider the case where the Morse–Smale flow has equilibrium states ultimately; in a way, this situation is alternative to [10]. The following theorem is proved in [46].

Theorem 5.1. *Let a Morse–Smale flow f^t be defined on a closed three-dimensional manifold M^3 and let the nonwandering set of f^t consist of $\nu(f^t)$ saddle equilibrium states and $\mu(f^t)$ nodal equilibrium states. Then the manifold M^3 can be represented by a Heegaard partition of kind*

$$h_D = \frac{\nu(f^t) - \mu(f^t) + 2}{2}.$$

Let M^3 be a closed manifold. Then there exists a Morse–Smale flow on M^3 such that it has no periodic trajectories and $2h(M^3) = \nu(f^t)$.

Corollary 5.1. *Let a Morse–Smale flow f^t be defined on a closed 3-manifold M^3 such that the nonwandering set of f^t consists of $\nu(f^t)$ saddle equilibrium states and $\mu(f^t)$ nodal equilibrium states. Then*

$$h(M^3) \leq \frac{\nu(f^t) - \mu(f^t) + 2}{2}.$$

From Theorem 5.1, one can deduce the following sufficient existence conditions for periodic trajectories.

Proposition 5.1. *Let a Morse–Smale flow f^t be defined on a closed 3-manifold M^3 such that the set of equilibrium states of f^t consists of $\nu(f^t)$ saddles and $\mu(f^t)$ knots. Then the flow f^t has periodic trajectories if*

$$h(M^3) > \frac{\nu(f^t) - \mu(f^t) + 2}{2}.$$

Theorem 5.2. *Let a Morse–Smale flow f^t be defined on a closed 3-manifold M^3 such that the nonwandering set of f^t consists of $\nu(f^t)$ saddle equilibrium states, $\nu(f^t) \geq 0$, $\mu(f^t)$ nodal equilibrium states, $\mu(f^t) \geq 0$, $s(f^t)$ saddle periodic trajectories, $s(f^t) \geq 0$, and $r(f^t)$ nodal periodic trajectories, $r(f^t) \geq 0$. Then the manifold partition M^3 can be represented as a Heegaard partition of kind*

$$h_D = \frac{\nu(f^t) - \mu(f^t) + 2}{2} + s(f^t)$$

and

$$h(M^3) \leq \frac{\nu(f^t) + r(f^t)}{2} + s(f^t).$$

Several classification results are obtained for special classes of Morse–Smale flows on closed 3-manifolds. In [33], *polar flows*, i. e., gradient-like flows such that each one has one and only one stable knot and one and only one unstable knot, are classified. To construct a complete classification invariant of such flows (in [33], it is called the *Heegaard diagram*), intersections of separatrices of the saddles with spheres around the knots are considered. In [93], a complete topological invariant is constructed for Morse–Smale flows such that each one has a finite number of trajectories belonging to intersections

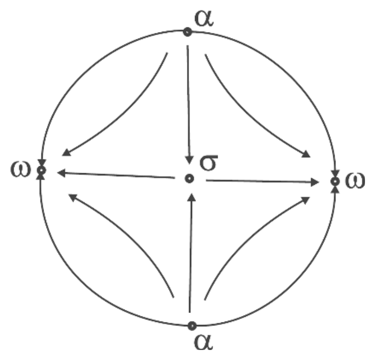


Fig. 6.1. A flow on a projective plane (diametrically opposite points of the circle are identified)

of two-dimensional separatrices of saddle nonwandering trajectories. This invariant is a combinatorial description of the cells of the flow and a character description for the joining of the cells to the sinks and the sources. In [81], a complete invariant is constructed for Morse–Smale flows without periodic trajectories. Up to a homeomorphism, this invariant is a surface such that it is transversal to flow trajectories located outside the closure of the set of one-dimensional separatrices of saddle points and it contains traces of the intersections of that surface with two-dimensional separatrices of saddle points.

6. Flows with Three Equilibrium States

As we note above, if the nonwandering set of a Morse–Smale flow on a closed manifold M^n consists of two points, then the flow has one and only one stable knot and one and only one unstable knot, the manifold M^n is the n-dimensional sphere, and the flow is topologically equivalent to the standard flow of the type "north–south." Consider the dynamics of Morse–Smale flows with nonwandering sets consisting of three points each. Due to Proposition 3.2, the carrier two-dimensional manifold of any such flow is a projective plane $M^2 = \mathbb{P}^2$, and all Morse–Smale flows on $\mathbb{P}^2$ such that the nonwandering set of the flow consists of three equilibrium states are topologically equivalent. The phase portrait of such a flow is displayed on Fig. 6.1. It is easy to see that the sink ω and the unstable manifold $W^u(\sigma)$ of the saddle σ form a locally plainly embedded circle $S^1 \subset \mathbb{P}^2$ such that $\mathbb{P}^2 \setminus S^1$ is homeomorphic to an open ball. Thus, the following definition is reasonable.

Recall that we say that a k-dimensional sphere S^k, $1 \leq k \leq n-1$, topologically embedded in M^n is *locally plainly embedded* if for any point z from S^k there exist a neighborhood $U(z) = U \subset M^n$ and a homeomorphism $\varphi_z : U \to \mathbb{R}^n$ such that $\varphi_z(S^k \cap U) = \mathbb{R}^k \subset \mathbb{R}^n$. We say that M^n is a *projective-like* manifold if

(1) n belongs to the set $\{2, 4, 8, 16\}$;

(2) M^n is the disjoint union[3] of a $\frac{n}{2}$-dimensional sphere $S^{\frac{n}{2}}$ locally plainly embedded in M^n and an open n-dimensional ball B^n, i. e.,

$$M^n = S^{\frac{n}{2}} \cup B^n \quad \text{and} \quad S^{\frac{n}{2}} \cap B^n = \varnothing.$$

In [63] (also, see [97]), the following theorem is proved.

Theorem 6.1. *Let f^t be a Morse–Smale flow on a closed n-dimensional manifold M^n, $n \geq 2$, such that the nonwandering set of f^t consists of three equilibrium states. Then M^n is a projective-like manifold. For $n = 2$, that manifold is a projective plane $M^2 = \mathbb{P}^2$ (a nonorientable surface of kind* 1

[3]A *disjoint union* is any union of pairwise disjoint sets.

with the fundamental group $\pi_1(M^2) = \mathbb{Z}_2$). For $n \geq 4$, we have

$$\pi_1(M^n) = \cdots = \pi_{\frac{n}{2}-1}(M^n) = 0, \quad \text{and, therefore, } M^n \text{ is orientable.}$$

For any projective-like manifold, there exists a Morse–Smale flow on that manifold such that the nonwandering set of that flow consists of three equilibrium states.

Note that any Morse–Smale flow on M^n such that its nonwandering set consists of three equilibrium states has one and only one saddle and closures of invariant (stable and unstable) manifolds of that saddle are topologically embedded spheres of dimension $n/2$. Now, to present topological equivalence results for flows with three equilibrium states, obtained in [64], introduce the key notion of submanifolds locally equivalently embedded in the carrier manifold.

Let M_1^k and M_2^k be k-dimensional submanifolds topologically embedded in M^n, $1 \leq k \leq n-1$. We say that M_1^k and M_2^k are *locally equivalently embedded* if there exist neighborhoods $U(\operatorname{clos} M_1^k)$ and $U(\operatorname{clos} M_2^k)$ of the topological closures $\operatorname{clos} M_1^k$ and $\operatorname{clos} M_2^k$ of the submanifolds M_1^k and M_2^k (respectively) and a homeomorphism $h : U(\operatorname{clos} M_1^k) \to U(\operatorname{clos} M_2^k)$ such that $h(M_1^k) = M_2^k$. For carrier manifolds of dimensions $n = 8$ and $n = 16$, the following theorem is valid.

Theorem 6.2. *Let f_i^t be a Morse–Smale flow ($i = 1, 2$) on a closed n-dimensional topological manifold M_i^n, where $n = 8, 16$, such that the nonwandering set of f_i^t consists of three equilibrium states. Then the flows f_1^t and f_2^t are topologically equivalent if and only if the stable (equivalently unstable) manifolds of the saddles of the flows f_1^t and f_2^t are locally equivalently embedded.*

For carrier manifolds of dimensions $n = 2$ and $n = 4$, the following theorem is valid.

Theorem 6.3. *Let f_1^t and f_2^t be Morse–Smale flows on closed topological manifolds M_1^n and M_2^n respectively, where $n = 2, 4$, such that the nonwandering set of each flow consists of three equilibrium states. Then f_1^t and f_2^t are topologically equivalent. In particular, the manifolds M_1^4 and M_2^4 are homeomorphic to each other, while the manifolds M_1^2 and M_2^2 are homeomorphic to the two-dimensional projective plane $\mathbb{P}^2$.*

7. Nonwandering Sets of Morse–Smale Systems and Topology of Carrier Manifolds: Relations between Numeric Characteristics

In this section, Morse–Smale diffeomorphisms satisfying special conditions are considered.

As we note above, a Morse–Smale diffeomorphism can be defined on any three-dimensional manifold. However, this is not valid for Morse–Smale diffeomorphisms without heteroclinic curves. The following result about the topological structure of closed three-dimensional manifolds possessing the said property is obtained in [23, 24].

Theorem 7.1. *Let $f : M^3 \to M^3$ be a Morse–Smale diffeomorphism preserving the orientation and M^3 be a closed orientable 3-manifold without heteroclinic curves. Then M^3 is either the sphere S^3 ($\nu(f) = \mu(f) - 2$ in that case) or the connected sum $(S^2 \times S^1)\sharp \cdots \sharp(S^2 \times S^1)$ such that the number of its terms is equal to*

$$\frac{\nu(f) - \mu(f) + 2}{2},$$

where $\nu(f)$ is the number of saddle periodic points and $\mu(f)$ is the number of nodal periodic points.

By $G^*(M^n)$ denote the set of Morse–Smale diffeomorphisms of an orientable n-dimensional ($n \geq 4$) closed manifold M^n such that each diffeomorphism preserves the orientation, each its saddle point has a one-dimensional (stable or unstable) manifold, and invariant manifolds of different saddle points do not intersect each other. In [40, 51, 52], Theorem 7.1 is generalized as follows.

Theorem 7.2. *A closed orientable n-manifold M^n ($n \geq 4$) admits a Morse–Smale diffeomorphism $f : M^n \to M^n$ from the class $G^*(M^n)$ such that it preserves the orientation and has $\nu(f)$ saddle periodic points, $\nu(f) \geq 0$, and $\mu(f)$ nodal periodic points, $\mu(f) \geq 2$, if and only if M^n is either a sphere ($\nu(f) = \mu(f) - 2$ in that case) or is the connected sum $(S^{n-1} \times S^1)\sharp \cdots \sharp(S^{n-1} \times S^1)$. In the last case, the number of terms of the connected sum is equal to*

$$\frac{\nu(f) - \mu(f) + 2}{2}.$$

From Theorem 7.1, it follows that no Morse–Smale diffeomorphism with three and only three periodic points exists on 3-manifolds. Omitting the trivial case of a diffeomorphism with two and only two periodic points, we see that the Morse–Smale diffeomorphism with four and only four periodic points is the simplest one. For such diffeomorphisms, the embedding of invariant manifolds of saddle periodic points can be rather complicated; therefore, nontrivial examples exist even on the 3-sphere S^3. In [78], it is found that the tame (not wild) embedding of invariant manifolds of saddle periodic points is possible for the Morse–Smale diffeomorphism $f : S^3 \to S^3$ with three knots (two sinks and one source) and one saddle with a one-dimensional (unstable) separatrix and a two-dimensional (stable) separatrix (in the sequel, the class of such diffeomorphisms is called the *Pixton* class). To understand the arising of a wild embedding of separatrices for a saddle point of a diffeomorphism f, denote the sinks by ω_1 and ω_2 and denote the separatrix of a saddle σ, going to the sink ω_2, by $W^{u+}(\sigma)$ (see Fig. 7.1 (a)). Now, represent the union $\alpha \cup W^s(\sigma) \cup W^{u+}(\sigma) \cup \omega_2$ as an infinite "curve" with a slightly inflated half (this half corresponds to $\alpha \cup W^s(\sigma)$), where $W^s(\sigma)$ is the stable manifold of the saddle σ. Then embed that curve and its neighborhood in S^3, treating it as a curve with two wildness endpoints, e. g., as the well-known (see [9]) Artin–Fox arc (see Fig. 7.1 (b)).

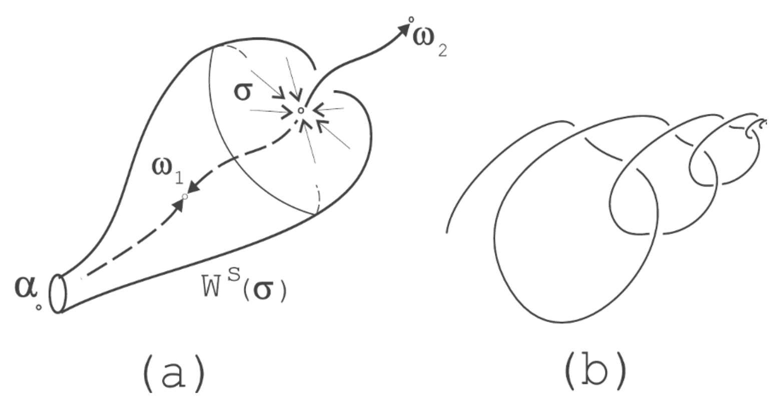


Fig. 7.1

The strict definition is as follows. Let f belong to $\mathcal{MS}(M^3)$, ω be a sink point, and $\ell^u(\sigma)$ be a one-dimensional separatrix of a saddle point σ such that $\ell^u(\sigma)$ is a subset of $W^s(\omega)$. We say that the separatrix $\ell^u(\sigma)$ is *tamely embedded* in M^3 if there exists a homeomorphism $h : W^s(\omega) \to \mathbb{R}^3$ such that $h(\ell^u(\sigma))$ is a ray in $\mathbb{R}^3$. In the same way, a bundle L_ω of one-dimensional separatrices such that its closure contains ω is called a *tame* bundle if there exists a homeomorphism $h : W^s(\omega) \to \mathbb{R}^3$ unbending separatrices.

From [55], the following tame embedding criterion for one-dimensional separatrices is deduced.

Proposition 7.1. *Let f belong to $\mathcal{MS}(M^3)$, ω be a sink point, and $\ell^u(\sigma)$ be a one-dimensional separatrix of a saddle point σ such that $\ell^u(\sigma)$ is a subset of $W^s(\omega)$. Then the separatrix $\ell^u(\sigma)$ is tamely embedded in M^3 if and only if there exists a smooth 3-ball $B_\omega \subset W^s(\omega)$ such that ω belongs to it interior and the intersection of $\ell^u(\sigma)$ an ∂B_ω consists of one point.*

For bundles of one-dimensional separatrices, no similar criterion is valid. Really, in [32], a bundle of arcs is constructed in $\mathbb{R}^3$ such that the intersection with the boundary of a 3-ball consists of one point, but the bundle is tame (see Fig. 7.2, where the Debrunner–Fox bundle of arcs is realized as a bundle of separatrices of a Morse–Smale diffeomorphism on S^3). We call such a bundle *moderately wild* because, extracting any arc from such a bundle, we obtain that the remaining union is tame.

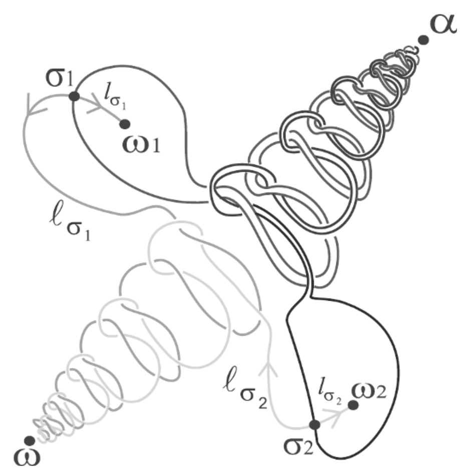


Fig. 7.2

Let $f : M^3 \to M^3$ be a Morse–Smale diffeomorphism on a closed 3-manifold M^3 such that f has no heteroclinic points. We say that one-dimensional separatrices of the diffeomorphism f are *trivially embedded* if all bundles of one-dimensional separatrices are tame.

The next theorem and corollaries from it refer to relations between dynamical characteristics of a Morse–Smale diffeomorphism with trivially embedded separatrices, but without heteroclinic points, and the kind of the Heegaard partition of the carrier 3-manifold; they are obtained in [45, 46].

Theorem 7.3. *Let $f : M^3 \to M^3$ be a gradient-like diffeomorphism of a 3-manifold M^3 such that it has $\nu(f)$ saddle periodic points and $\mu(f)$ nodal periodic points. If one-dimensional separatrices of the diffeomorphism f are trivially embedded, then the manifold M^3 can be represented as a Heegaard partition of kind*

$$h = \frac{\nu(f) - \mu(f) + 2}{2}.$$

Corollary 7.1. *Let $f : M^3 \to M^3$ be a gradient-like diffeomorphism of a closed 3-manifold M^3. If one-dimensional separatrices of the diffeomorphism f are trivially embedded, then the number of saddle periodic points of the diffeomorphism f is not less than the doubled Heegaard kind of the manifold M^3. For any closed manifold M^3 of a Heegaard kind $h(M^3)$, there exists a Morse–Smale diffeomorphism f on that manifold such that it has no heteroclinic points and the number of its saddle periodic points is equal to $2h(M^3)$.*

Corollary 7.1 implies the following sufficient condition for the existence of heteroclinic points of a Morse–Smale diffeomorphism with trivially embedded one-dimensional separatrices of saddle periodic points.

Proposition 7.2. *Let $f : M^3 \to M^3$ be a Morse–Smale diffeomorphism of a closed 3-manifold M^3 such that it has $\nu(f)$ saddle periodic points and $\mu(f)$ nodal periodic points. Suppose that the one-dimensional separatrices of saddle periodic points of the diffeomorphism f are trivially embedded. Then,* *if*

$$h(M^3) > \frac{\nu(f) - \mu(f) + 2}{2},$$

then f has heteroclinic points. In particular, f is embedded in no flow.

From [61], it is known that the Heegaard kind is additive with respect to connected sums of 3-manifolds. Since $h(S^2 \times S^1) = 1$, it follows that the Heegaard kind of a connected sum $h\left((S^2 \times S^1)\sharp \cdots \sharp(S^2 \times S^1)\right)$ is equal to the number of its terms. This, Theorem 7.1, and the Seifert–van Campen theorem imply the following assertion.

Proposition 7.3. *Let $f : M^3 \to M^3$ be a Morse–Smale diffeomorphism of a closed 3-manifold such that it preserves the orientation and has $\nu(f)$ saddle periodic points and $\mu(f)$ nodal periodic points. Suppose that one of the following conditions is satisfied:*

(1) $h(M^3) > \dfrac{\nu(f) - \mu(f) + 2}{2}$;
(2) *the fundamental group $\pi_1(M^3)$ is not equal to the free product $\mathbb{Z} * \cdots * \mathbb{Z}$ of g samples of the group $\mathbb{Z}$ of integers.*

Then f has heteroclinic curves.

Frequently, closed heteroclinic curves have an artificial genesis and their presence is not caused by the topological structure of the carrier manifold or by dynamical restrictions. Regarding nonclosed heteroclinic curves, we present the next theorem proved in [47].

Theorem 7.4. *Let $f : M^3 \to M^3$ be a Morse–Smale diffeomorphism of a closed orientable three-dimensional manifold such that the three-dimensional sphere S^3 is its universal covering and the nonwandering set of f consists of two saddle periodic points and two nodal periodic points. Then there exists at least one heteroclinic nonclosed curve such that its boundary consists of saddle points (see Fig. 7.3). Any such heteroclinic curve is invariant with respect to an iteration of the diffeomorphism f. If the intersection of the two-dimensional invariant manifolds of saddle points is not exhausted by such heteroclinic curves, then the remaining part of the intersection contains a denumerable family of closed heteroclinic curves that are a union of orbits of a finite set of closed heteroclinic curves.*

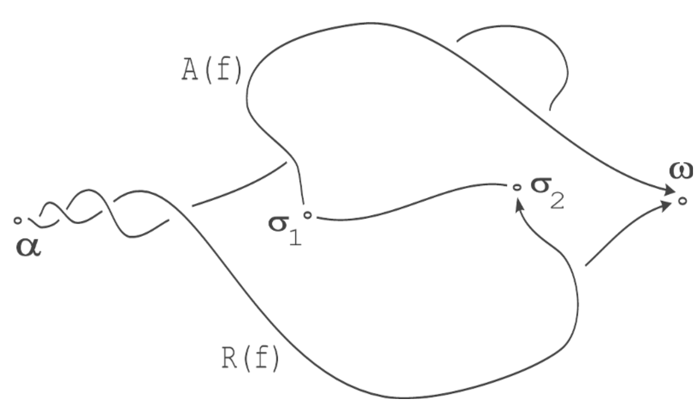


Fig. 7.3

The next theorem provides an estimate from below for the number of nonclosed heteroclinic curves for Morse–Smale diffeomorphisms on lenses $L_{p,q}$ under the assumption that the nonwandering set of the diffeomorphism contains four and only four periodic points (see [47]).

Theorem 7.5. *Let $f : L_{p,q} \to L_{p,q}$ be a Morse–Smale diffeomorphism such that its nonwandering set of the diffeomorphism consists four and only four periodic points. Then*

(1) *The diffeomorphism f has two saddle periodic points and two nodal periodic points and the Morse indices of the saddle points are different from each other.*
(2) *If the one-dimensional separatrices of the diffeomorphism f are trivially embedded, then there exist at least p heteroclinic nonclosed curves such that the boundary of each one consists of saddle points. Any such heteroclinic curve is invariant with respect to an iteration of the diffeomorphism f.*

It follows from Theorem 7.1 that any gradient-like diffeomorphism has heteroclinic (compact or noncompact) curves on any irreducible manifold provided that at least one saddle point exists. The next theorem proved in [43] refines that result for polar systems.

Theorem 7.6. *The two-dimensional manifold of any saddle point of any polar gradient-like diffeomorphism on an irreducible manifold contains a noncompact heteroclinic curve.*

8. Global Dynamics of Morse–Smale Systems

The complexity of the dynamics of arbitrary Morse–Smale systems is illustrated by the following result from [1]. Let f^t be a flow on a compact manifold M^n. Let $G(f^t)$ be the group of homeomorphisms of the flow f^t on itself, i. e., any homeomorphism from $G(f^t)$ maps any trajectory of the flow f^t to a trajectory of the flow f^t, preserving the orientation with respect to time. The metric on M^n induces a natural metric on $G(f^t)$. Following [1], we say that a trajectory l of the flow f^t is *special* if there exists a positive ε such that if g belongs to $G(f^t)$ and is ε-close to the identity homeomorphism, then the condition $g(l) = l$ is satisfied.

All equilibrium states and periodic trajectories of the Morse–Smale flow are special because they are isolated in the nonwandering set. However, a wandering trajectory l belonging to the intersection of the stable manifold W^s and unstable manifold W^u of nonwandering trajectories can be special as well. In such a case, the relation $\dim W^s + \dim W^u = \dim M^n + 1$ is valid. The type $(\dim W^s, \dim W^u)$ is assigned to the trajectory l. The set of all such special trajectories of type $(m+1, \dim M^n - m)$ and their limit nonwandering trajectories is defined by $\mathfrak{M}_{m+1}$. In general, this set is disconnected and consists of a finite number of components denoted by $\mathfrak{M}^{(1)}_{m+1}$, ..., $\mathfrak{M}^{(\kappa)}_{m+1}$. In this notation, the following theorem is valid (see [1]).

Theorem 8.1. *Let f^t be a Morse–Smale flow and $\mathfrak{M}^{(i)}_{m+1}$ be a component of the set $\mathfrak{M}_{m+1}$. Then the restriction of the Morse–Smale flow to $\mathfrak{M}^{(i)}_{m+1}$ is topologically orbitally equivalent to a special suspension (Σ_A, f) over a topological Markov chain (Σ_A, σ) with a finite number of states, where Σ_A is the space of two-side binary sequences determined by a matrix A and the nonnegative function $f : \Sigma_A \to [0;1]$ vanishes only at points corresponding to constant sequences (σ is the translation to the left by* 1).

The set of special trajectories has a complex structure, but any Morse–Smale system can be represented as a “source–sink” pair, where there are (appropriately simple) invariant closed sets such that one of them is the attractor and another one is the repeller (see Fig. 1.2). Let us describe the construction of the attractor A_f and repeller R_f for the Morse–Smale diffeomorphism $f : M^n \to M^n$. If the space $\hat{V}_f$ of orbits of the nonwandering set $V_f = M^n \setminus (A_f \cup R_f)$ (together with the images embedded to it under the factorization of invariant manifolds of saddle periodic points) can be described, then new topological invariants describing the embedding (it might be wild) of stable and unstable manifolds of saddle periodic points to the carrier manifold are found. This creates a premise for the topological classification in the framework of that class of diffeomorphisms (see Sec. 9).

Let $f : M^n \to M^n$ be a Morse–Smale diffeomorphism preserving the orientation. Let Σ_f, Δ^s_f, and Δ^u_f denote the sets of all saddles, sinks, and sources (respectively) of the diffeomorphism f. Divide the set of saddle points Σ_f into disjoint subsets Σ^A_f and Σ^R_f such that the sets

$$A_f = \Delta^s_f \cup W^u_{\Sigma^A_f} \quad \text{and} \quad R_f = \Delta^u_f \cup W^s_{\Sigma^R_f}$$

are closed and invariant. Note that if at least one of the sets A_f or R_f is closed and invariant, then another one is closed and invariant as well. The sets A_f and R_f contain all periodic points of the diffeomorphism f and do not intersect each other. The greatest dimension of an unstable (stable) manifold of points from Σ^A_f (Σ^R_f) is called the *dimension* of A_f (R_f respectively).

Theorem 8.2. *Let $f : M^n \to M^n$ be a Morse–Smale diffeomorphism. Then the set A_f (R_f) is an attractor (repeller respectively) of the diffeomorphism f. If the dimension of the attractor A_f (repeller R_f) does not exceed $n-2$, then the repeller R_f (attractor A_f respectively) is connected.*

We say that A_f and R_f are the *global attractor and repeller* (respectively) of the Morse–Smale diffeomorphism $f : M^n \to M^n$.

The next theorem describes the topological structure of the space of orbits of the set

$$V_f = M^n \setminus (A_f \cup R_f)\,.$$

Let

$$\hat{V}_f = V_f / f$$

denote the set of orbits of the action of f on the manifold V_f coinciding with the set of orbits of the diffeomorphism f on V_f. Let

$$p_f : V_f \to \hat{V}_f$$

be the natural projection that maps any point x from V_f to its orbit by virtue of the diffeomorphism f and endows the set $\hat{V}_f$ with the quotient topology.

Recall that a sphere S^{n-1} in M^n is said to be *cylindrically embedded in M^n* if there exists a topological embedding $h : \mathbb{S}^{n-1} \times [-1;+1] \to M^n$ such that $h(\mathbb{S}^{n-1} \times \{0\}) = S^{n-1}$. Similarly to the three-dimensional case, we say that M^n is *irreducible* if any $(n-1)$-sphere cylindrically embedded in M^n bounds an n-ball in M^n.

Theorem 8.3. *The space V_f is a smooth orientable n-manifold. If the dimension of the attractor A_f and repeller R_f does not exceed $n-2$, then V_f is connected and $\hat{V}_f$ either is irreducible or is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$.*

Several corollaries from Theorems 8.2-8.3 are provided below.

Corollary 8.1. *If a Morse–Smale diffeomorphism $f : M^n \to M^n$ has no saddle points with one-dimensional unstable (stable) manifolds, then the nonwandering set of f contains one and only one sink (source).*

If a Morse–Smale diffeomorphism has one and only one sink and one and only one source, then it is called a *polar diffeomorphism.* Polar diffeomorphisms generalize "source–sink" diffeomorphisms in a natural way. The next assertion follows directly from Corollary 8.1.

Corollary 8.2. *If a Morse–Smale diffeomorphism $f : M^n \to M^n$ has no saddle points with one-dimensional invariant manifolds, then f is a polar diffeomorphism.*

Recall that the direct product $H^n_q = \mathbb{B}^q \times \mathbb{B}^{n-q}$ of two disks is called a *handle of index q of dimension n* ($0 \le q \le n$) and the disk $\mathbb{B}^q$ is called the *axis of the handle.* The handle H^n_q is a smooth manifold with boundary $\partial H^n_q = \partial(\mathbb{B}^q \times \mathbb{B}^{n-q}) = (\partial\mathbb{B}^q \times \mathbb{B}^{n-q}) \cup (\mathbb{B}^q \times \partial\mathbb{B}^{n-q}) = (\mathbb{S}^{q-1} \times \mathbb{B}^{n-q}) \cup (\mathbb{B}^q \times \mathbb{S}^{n-q-1})$.

Recall the operation to glue a handle H^n_q to an n-manifold A^n with boundary $B^{n-1} = \partial A^n$. Let $\mathbb{S}^{q-1} \subset B^{n-1}$ be a smoothly embedded sphere and $N(\mathbb{S}^{q-1})$ is its tubular neighborhood diffeomorphic to the direct product $\mathbb{S}^{q-1} \times \mathbb{B}^{n-q}$. Glue the manifold A^n with the handle H^n_q along the map $g : \mathbb{S}^{q-1} \times \mathbb{B}^{n-q} \to N(\mathbb{S}^{q-1})$, which is a diffeomorphism between the tubular neighborhood $N(\mathbb{S}^{q-1})$ and the manifold $\mathbb{S}^{q-1} \times \mathbb{B}^{n-q}$, which is a part of the boundary ∂H^n_q. Then, smoothing "angles" arising at the points $\partial N(\mathbb{S}^{q-1}) = \mathbb{S}^{q-1} \times \mathbb{S}^{n-q-1}$, we obtain a smooth manifold $\tilde{A}^n$ with a smooth boundary $\tilde{B}^{n-1}$.

Any triple (K, L_0, L_1), where L_0 and L_1 are closed manifolds of dimension $n-1$ and K is a compact n-dimensional manifold such that $\partial K = L_0 \sqcup L_1$, is called a compact n-dimensional *cobordism*.

Corollary 8.3. *Let $f : M^n \to M^n$ be a Morse–Smale diffeomorphism, Σ^A_f be the set of saddle points with one-dimensional unstable manifolds, and $\Sigma^R_f = \Sigma_f \setminus \Sigma^A_f$. Then the corresponding global attractor A_f and global repeller R_f are connected and there exists an n-dimensional cobordism (K, L_0, L_1), where $K \subset V_f$ and L_1 and L_2 are homeomorphic to the boundary of the n-ball with g glued n-dimensional handles of index* 1, $g \geq 0$, *such that $\hat{V}_f$ is obtained from (K, L_0, L_1) by means of the identifying of its boundaries by virtue of the diffeomorphism f.*

The next theorem describes the structure of the global attractor and the global repeller for the case where they do not contain heteroclinic intersections.

The image of any open n-disk $int\ \mathbb{B}^q$ in a Hausdorff space X under a continuous mapping $g^q : \mathbb{B}^q \to X$ such that its restriction $g^q|_{int\ \mathbb{B}^q} : int\ \mathbb{B}^q \to g^q(int\ \mathbb{B}^q)$ is a homeomorphism is called a *q-block* e^q ($q \geq 0$). Note that $\partial \mathbb{B}^q = \mathbb{S}^{q-1}$ for any nonnegative q. If $q = 1$, then the boundary $\mathbb{S}^0$ of the disk $\mathbb{B}^1$ consists of two points. If $q = 0$, then the disk $\mathbb{B}^0$ is a point and its boundary $\mathbb{S}^{-1}$ is the empty set. A *finite block complex* is any Hausdorff space X representable as a union of pairwise disjoint cells (block decomposition) $X = \bigcup\limits_{q=0}^{n} \Big(\bigcup\limits_{j=1}^{c_q(X)} e^q_j \Big)$ such that the boundary $Fr\ e^q_j$ of any block e^q_j is contained in a union of blocks of lower dimensions. The dimension of the greatest block of a block complex is called the *dimension* of that block complex.

A special case of block complexes is a *union of spheres*: to obtain it, we take spheres $X_1, \ldots, X_m$ of positive dimensions (different dimensions for different spheres are allowed), mark a point of each sphere, and identify all marked points with one point (see [34]). Any isolated point can be treated as a trivial union of spheres.

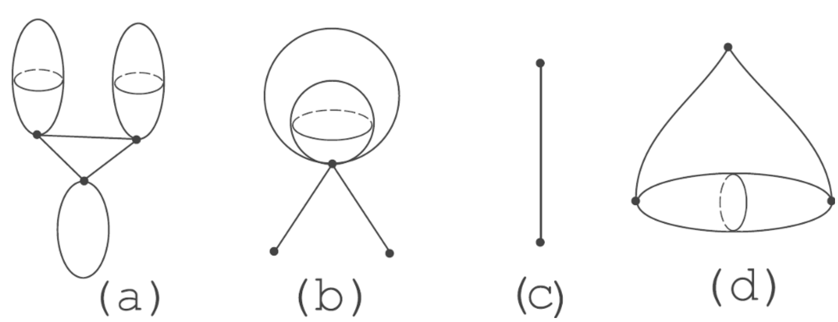


Fig. 8.1

We say that two unions of spheres, embedded to a same space, *are joined by an arc* if an arc (a topological embedding of a segment) is added to the unions such that it joins the marked points of the unions and its interior does not intersect the unions. That arc is called the *binding arc.* Any connected set consisting of a finite number of unions of spheres and binding arcs is called a *sheaf of* *unions of spheres*. For example, Figure 8.1 (a)–(c) displays sheafs of unions of spheres, while the set displayed at Fig. 8.1 (d) is not a sheaf of unions of spheres because the sphere of a union has the only marked point.

Theorem 8.4. *Let A_f (R_f) be a global attractor (global repeller respectively) of a Morse–Smale diffeomorphism $f : M^n \to M^n$ such that A_f (R_f respectively) does not contain heteroclinic intersections. Then A_f (R_f respectively) is a finite family of sheafs of unions of spheres.*

9. Morse–Smale Diffeomorphisms on n-Manifolds: Topological Classification for $n \geq 3$

Consider the problem to classify Morse–Smale diffeomorphisms on closed 3-manifolds.

For diffeomorphisms from the Pixton class, the topological classification is reduced to the isotopic classification of knots in $S^2 \times S^1$, homotopic to knots $\{\cdot\} \times S^1$ (for details, see [21]). In particular, this implies the existence of a denumerable set of topological adjointness classes for Morse–Smale S^3-diffeomorphisms with three knots and one saddle for each.

In [21], a qualitatively new adjointness invariant is found and the main stages of the proof of necessary and sufficient adjointness conditions are developed (for the case of a basic class) for two diffeomorphisms from wider classes. This work inspired a flow of publications devoted to the classification of various classes of Morse–Smale diffeomorphisms on closed 3-dimensional manifolds (see [22, 23, 25–27] etc.) Finally, in [80], a complete topological classification is obtained for diffeomorphisms from the set $\mathcal{MS}\ (M^3)$, preserving the orientation. We describe the scheme to construct a complete topological invariant for such diffeomorphisms.

Let f be a diffeomorphism from the class $\mathcal{MS}\ (M^3)$, preserving the orientation. By A_f (R_f) denote the union of all sink (source respectively) periodic points and all one-dimensional unstable (stable respectively) manifolds of saddle periodic points of the diffeomorphism f. Then A_f is an attracting set and is a union of a finite number of arcs and circles (points of their wild knotting at sink periodic points are possible). The repelling set R_f is described in the same way. Also, the sets A_f and R_f can contain arcs such that one-dimensional separatrices of saddle periodic points tend to those arcs, "oscillating."

Assign $V_f = M^3 \setminus (A_f \bigcup R_f)$. The set V_f is an invariant open connected subset belonging to the wandering set of the diffeomorphism f. Since it is invariant, one can consider the space of orbits $\hat{V}_f$ lying in V_f. Formally, $\hat{V}_f$ is a quotient space with respect to the following equivalence relation: two points are equivalent to each other if they belong to a same orbit. Let $p_f : V_f \to \hat{V}_f$ denote the natural projection. By virtue of Theorem 8.3, $\hat{V}_f$ is a connected closed oriented three-dimensional manifold and the projection p_f is a covering with a group of covering transformations isomorphic to $\mathbb{Z}$ (see, e. g., [48]). Therefore, p_f determines an epimorphism $\alpha_f : \pi_1(\hat{V}_f) \to \mathbb{Z}$. Let $\hat{\Gamma}^u_f$ and $\hat{\Gamma}^s_f$ denote the images with respect to p_f for all two-dimensional stable and unstable separatrices, respectively. These sets are compact and any component of the arcwise connectedness of the sets $\hat{\Gamma}^u_f$ and $\hat{\Gamma}^s_f$ is the two-dimensional torus or the Klein bottle with an empty, finite, or denumerable set of punctured points (see Fig. 9.1); it is possible that the sets $\hat{\Gamma}^u_f$ and $\hat{\Gamma}^s_f$ transversally intersect each other.

Any collection $S_f = (\hat{V}_f, \alpha_f, \hat{\Gamma}^u_f, \hat{\Gamma}^s_f)$ is called a *scheme*. We say that schemes $S_f = (\hat{V}_f, \alpha_f, \hat{\Gamma}^u_f, \hat{\Gamma}^u_f)$ and $S_{f'} = (\hat{V}_{f'}, \alpha_{f'}, \hat{\Gamma}^u_{f'}, \hat{\Gamma}^s_{f'})$ are *equivalent* to each other if there exists a homeomorphism $h : \hat{V}_f \to \hat{V}_{f'}$ such that

(1) $h(\hat{\Gamma}^u_f) = \hat{\Gamma}^u_{f'}$, $h(\hat{\Gamma}^u_f) = \hat{\Gamma}^u_{f'}$;

(2) $h_*(\alpha_{f'}) = \alpha_f$, where the isomorphism $h_* : \pi_1(\hat{V}_{f'}, \mathbb{Z}) \to \pi_1(\hat{V}_f, \mathbb{Z})$ is induced by the homeomorphism h.

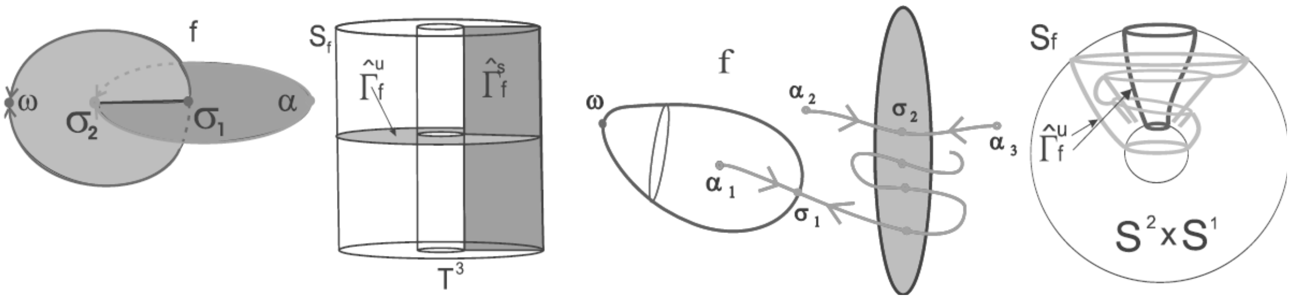


Fig. 9.1

Theorem 9.1. *Up to an equivalence, the scheme is a complete topological invariant in the class of diffeomorphisms from* $\mathcal{MS}$ (M^3) *preserving the orientation.*

In [40], Theorem 9.1 is generalized to the class G^* (M^n): a scheme similar to the scheme of the Morse–Smale 3-diffeomorphism is introduced for diffeomorphisms from G^* (M^n).

Theorem 9.2. *Up to an equivalence, the scheme is a complete topological invariant in the class of diffeomorphisms from* G^* (M^n) *preserving the orientation.*

In [39], for a diffeomorphism f from the class $G(S^n)$ of Morse–Smale diffeomorphisms with a one-dimensional set of unstable separatrices on the n-sphere, we introduce a graph $G(f)$ similar to the distinguishing Bezdenezhnykh–Grines graph (recall that the latter is an analog of the distinguishing Peixoto graph) and define an automorphism on it.

In [47], the following assertion is proved.

Theorem 9.3. *Let* f *and* g *be Morse–Smale diffeomorphisms on the* n*-dimensional sphere* S^n $(n \geq 4)$ *such that they preserve the orientation, and the nonwandering set of each one consists of four fixed points*: *one saddle of codimension one and three knots. Then* f *and* g *are adjoint if and only if the Morse indices of their saddles are equal to each other* (*it is equal either to* 1 *or to* $n-1$).

Comparing this result with fact that there is a denumerable family of Morse–Smale diffeomorphisms on the 3-dimensional sphere such that they are pairwise and each one has one saddle and three knots (see [21]), we see a contrast: the multidimensional case is simpler. The reason is that no wild embedding of a one-dimensional separatrix and its topological closure (and one knot) is possible. This follows from [30], where it is proved that if $n \geq 4$, then any wildly embedded arc has a continual set of points of wildness. Therefore, this case does not require knotting points (unlike the case where $n = 3$.)

Theorem 9.4. *Let* f *and* f' *be diffeomorphisms from the class* $G(S^n)$, *preserving the orientation. They are topologically adjoint if and only if there exists an isomorphism of the graphs* $G(f)$ *and* $G(f')$, *conjugating the automorphisms.*

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