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König Graphs with Respect to the 4-Path and Its Spanning Supergraphs

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Abstract—We describe the class of graphs whose every subgraph has the next property: The maximal number of disjoint 4-paths is equal to the minimal cardinality of sets of vertices such that every 4-path in the subgraph contains at least one of these vertices. We completely describe the set of minimal forbidden subgraphs for this class. Moreover, we present an alternative description of the class based on the operations of edge subdivision applied to bipartite multigraphs and the addition of the so-called pendant subgraphs, isomorphic to triangles and stars.

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INTRODUCTION

Let $\mathcal{X}$ be a set of graphs. An arbitrary set G of pairwise vertex disjoint induced subgraphs in G isomorphic to graphs in $\mathcal{X}$ is called an *$\mathcal{X}$-packing* of G. An arbitrary subset of vertices in a graph G covering all induced subgraphs of G isomorphic to graphs in $\mathcal{X}$ is called a *vertex cover G with respect to $\mathcal{X}$* or simply *$\mathcal{X}$-cover*. In other words, each induced subgraph in G isomorphic to a graph in $\mathcal{X}$ contains a vertex of an $\mathcal{X}$-cover. A graph is called *König with respect to $\mathcal{X}$* if, in its every induced subgraph, the greatest size of an $\mathcal{X}$-packing is equal to the least size of an $\mathcal{X}$-cover (see [1]). The class of all König graphs with respect to $\mathcal{X}$ is denoted by $\mathcal{K}(\mathcal{X})$. If the set $\mathcal{X}$ consists of a single graph H then we speak of H-packings, H-covers, and König graphs with respect to H.

Note that, in the literature, a $\mathcal{X}$-cover is often understood as a set of vertices of a graph G covering all (not necessarily induced) subgraphs of G isomorphic to graphs in $\mathcal{X}$ (see, for example, [2, 3]). We will call such $\mathcal{X}$-covers *weak*. By analogy, by *weak $\mathcal{X}$-packings* we mean the sets of all (not necessarily induced) pairwise vertex disjoint subgraphs isomorphic to graphs in $\mathcal{X}$.

The $\mathcal{X}$-packing and $\mathcal{X}$-cover problems of a graph are the topics of many works, especially as regards their algorithmic aspects (for example, see [4–6]). In particular, some studies are devoted to the packing and cover problems for paths of different lengths. It is known that the weak P_k-cover problem is polynomially solvable in the class of trees and NP-hard in general for $k \geq 2$ [7, 8], and the weak P_k-packing problem is polynomially solvable for $k = 2$ [9] and NP-complete for $k \geq 3$ in the general case (see [10]) and in 3-regular graphs [11]. Moreover, for $k \geq 4$ the weak P_k-packing problem is APX-complete; i.e., under the assumption that P $\neq$ NP, there is no approximate algorithm for it with arbitrary constant approximation coefficient (see [12]).

Nevertheless, it is known that the P_k-packing problem (both in the weak and "induced" cases) is solved in linear time in the class of forests for every k [11]. Moreover, a description is known of several graph classes on which the P_k-packing problem and the P_k-cover problem are solved in polynomial time for various k (for example, see [13–16] for the weak case and [13, 17] for the "induced" case).

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Let $\langle H\rangle$ denote the set of all spanning supergraphs of a graph H; i.e., the set of graphs obtained from H by addition of edges. In particular,

$$\langle P_4\rangle = \{P_4, C_4, \text{3-pan}, K_4 - e, K_4\},$$

where the 3-pan is the graph obtained from P_4 by adding an edge joining it to a vertex of degree 2 not adjacent to it and $K_4 - e$ is the graph obtained from K_4 by removing an edge.

It is not hard to see that if H is a subgraph in a graph G then the subgraph induced by its vertices is isomorphic to one of the graphs in $\langle H\rangle$. Therefore, it is true that every weak H-packing of the graph G is its $\langle H\rangle$-packing, and every weak H-cover is its $\langle H\rangle$-cover. Thus, for an arbitrary H, the class $\mathcal{K}(\langle H\rangle)$ can be characterized as the set of graphs in whose each induced subgraph the greatest size of a weak H-packing is equal to the least size of a weak H-cover.

For an arbitrary H, the class $\mathcal{K}(\langle H\rangle)$ is *hereditary*; i.e., it is closed under vertex removal. It is known that a hereditary class can be characterized by the set of forbidden induced subgraphs; i.e., by the graphs minimal with respect to inclusion of the vertices not lying in the class. A detailed description is given, for example, in [18] for the class $\mathcal{K}(\mathcal{C})$, where $\mathcal{C}$ is the class of all simple cycles, and in [13], for the class $\mathcal{K}(\langle P_3\rangle)$.

This article is devoted to describing the class of König graphs with respect to $\langle P_4\rangle$. We show that this class is *monotone*, i.e., closed under the removal not only of vertices but also edges. It is known that every monotone class can be characterized by the set of minimal (not necessarily induced) forbidden subgraphs; i.e., inclusion minimal vertices and edges of the graphs not belonging to this class. All such graphs for the class $\mathcal{K}(\langle P_4\rangle)$ are described in Section 2. Moreover, in Sections 3 and 4, we describe the class of *ST*-graphs obtained from bipartite multigraphs by applying the procedure of edge subdivision and the addition of pendant subgraphs, and prove that it coincides with the class $\mathcal{K}(\langle P_4\rangle)$.

1. DEFINITIONS AND NOTATIONS

In the article, we use the standard notations K_n, P_n, and C_n for complete graphs, simple paths, and simple cycles on n vertices respectively.

Denote by S_n the tree on $n+1$ vertices n of which are leaves. Denote the greatest number of elements in a $\langle P_4\rangle$-packing of a graph G by $\mu_{\langle P_4\rangle}(G)$ and designate the minimal number of vertices in its $\langle P_4\rangle$-cover by $\beta_{\langle P_4\rangle}(G)$. Refer to a subgraph isomorphic to one of the graphs in $\langle P_4\rangle$ as a *quartet*. Denote by (v_1, v_2, v_3, v_4) the quartet consisting of vertices v_1, v_2, v_3, and v_4. Denote by $V(G)$ the vertex set of a graph G. The set of vertices adjacent to the vertex v will be denoted by $N(v)$.

Let G be a graph and let $A \subseteq V(G)$. Denote by $G \setminus A$ the graph obtained from G by removing all vertices of A.

Considering a cycle C_n, we assume that the vertices of C_n are enumerated along the cycle with the numbers $0, 1, \ldots, n-1$. In addition, the arithmetic operations with the numbers of vertices are carried out modulo n. Refer to each of the sets of its vertices with numbers in a given residue class modulo 4 as a 4-*class*.

Note that for every graph G we have

$$\mu_{\langle P_4\rangle}(G) \le \beta_{\langle P_4\rangle}(G).$$

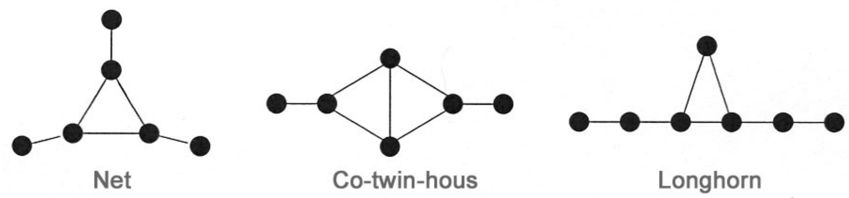
Therefore, for proving that these quantities are equal, it suffices to present for G a $\langle P_4\rangle$-packing and a $\langle P_4\rangle$-cover of the same size.

2. FORBIDDEN SUBGRAPHS

If F is a minimal graph as regards the inclusion of vertices and edges not contained in $\mathcal{K}(\langle P_4\rangle)$ then we call it a *minimal forbidden subgraph* of this class.

It is easy to show by a straightforward check that the graphs net, co-twin-house, and longhorn depicted in the figure do not belong to the class $\mathcal{K}(\langle P_4\rangle)$. For each of them,

$$\mu_{\langle P_4\rangle}(G) = 1, \qquad \beta_{\langle P_4\rangle}(G) = 2,$$



The three forbidden graphs for the class $\mathcal{K}(\langle P_4\rangle)$.

and each of them consists of at most 7 vertices. Moreover, each proper subgraph in each of these graphs is König with respect to $\langle P_4\rangle$. Thus, we have

Lemma 1. *The* net, co-twin-house, *and* longhorn *are the minimal forbidden graphs for* $\mathcal{K}(\langle P_4\rangle)$.

It was proved in [17] that each forest is a König graph with respect to P_4. Since a forest contains no other subgraphs isomorphic to graphs in $\langle P_4\rangle$ but P_4, we have

Lemma 2. *Each forest is a König graph with respect to* $\langle P_4\rangle$.

Consider the infinite series of minimal forbidden subgraphs for the class $\mathcal{K}(\langle P_4\rangle)$. Obviously,

$$\mu_{\langle P_4\rangle}(C_{4k}) = \mu_{\langle P_4\rangle}(C_{4k+1}) = \mu_{\langle P_4\rangle}(C_{4k+2}) = \mu_{\langle P_4\rangle}(C_{4k+3}) = k,$$
$$\beta_{\langle P_4\rangle}(C_{4k}) = \beta_{\langle P_4\rangle}(C_{4k-1}) = \beta_{\langle P_4\rangle}(C_{4k-2}) = \beta_{\langle P_4\rangle}(C_{4k-3}) = k.$$

Therefore, by Lemma 2, we have

Lemma 3. *A cycle C_n belongs to the class $\mathcal{K}(\langle P_4\rangle)$ if $n = 3$ or n divides by 4, and C_n is a minimal forbidden graph for $\mathcal{K}(\langle P_4\rangle)$ if $n > 3$ and n does not divide by 4.*

Consider the graph obtained from C_n by adding two vertices not adjacent to each other and each of which is joined with a single vertex in the cycle. Denote this graph by $A(n, k)$, where k is the distance between the vertices of the cycle adjacent to the added vertices.

Lemma 4. *If n divides by 4 and k does not divide by 2 then $A(n, k)$ is a minimal forbidden graph in $\mathcal{K}(\langle P_4\rangle)$.*

Proof. Let $n = 4t$. Then $|V(A(n, k))| = 4t + 2$. Obviously, $\mu_{\langle P_4\rangle}(A(n, k)) = t$. Suppose that there exists a $\langle P_4\rangle$-cover C of $A(n, k)$ of size t. Then it is included in a cycle C_n of this graph and is its least $\langle P_4\rangle$-cover, and hence its 4-class. Since k is odd, one of the vertices adjacent to the vertices of degree 1 in $A(n, k)$ is adjacent also to a vertex in C. But then the distance to the other closest vertex of C is equal to 3. Hence, this vertex, the adjacent vertex of degree 1, and two more vertices of the cycle induce a quartet not covered by C. Consequently,

$$\beta_{\langle P_4\rangle}(A(n, k)) > t \quad \text{and} \quad A(n, k) \notin \mathcal{K}(\langle P_4\rangle).$$

Considering a subgraph in $A(n, k)$ obtained by removing a vertex of degree 1 or an edge incident to it. Its connected component, which is not an isolated vertex, consists of a cycle with an added vertex of degree 1. For this graph, obviously,

$$\mu_{\langle P_4\rangle}(G) = \beta_{\langle P_4\rangle}(G) = t;$$

therefore, it is König with respect to $\langle P_4\rangle$.

The remaining subgraphs in $A(n, k)$ are forests or are isomorphic to the cycle C_{4t}, and, by Lemmas 2 and 3, they are all König with respect to $\langle P_4\rangle$. Lemma 4 is proved. □

Let $\mathcal{F}$ stand for the set of all forbidden graphs in Lemmas 1, 3, and 4:

$$\mathcal{F} = \{\text{net, co-twin-house, longhorn}\} \cup \{C_n \mid n > 3 \text{ does not divide by } 4\} \cup \{A(n,k) \mid n \text{ divides by } 4,\ k \text{ is odd}\}.$$

Since every quartet consists of 4 vertices, each graph G satisfies the inequality

$$4\mu_{\langle P_4\rangle}(G) \leq |V(G)|.$$

Note also that every graph $H \in \mathcal{F}$ enjoys the inequality

$$|V(H)| \leq 4\mu_{\langle P_4\rangle}(H) + 3;$$

i.e., the value of the parameter $\mu_{\langle P_4\rangle}$ is maximal in H for the given number of vertices. Let H' be a spaning supergraph of H. Obviously, every set of vertices in H that induces a quartet also generates a quartet in H'. Then

$$\beta_{\langle P_4\rangle}(H) \leq \beta_{\langle P_4\rangle}(H'), \qquad \mu_{\langle P_4\rangle}(H) \leq \mu_{\langle P_4\rangle}(H').$$

Since $|V(H')| = |V(H)|$ and $\mu_{\langle P_4\rangle}(H)$ is maximal for the given number of vertices, we have

$$\mu_{\langle P_4\rangle}(H') = \mu_{\langle P_4\rangle}(H) < \beta_{\langle P_4\rangle}(H) \leq \beta_{\langle P_4\rangle}(H').$$

Hence, $H' \notin \mathcal{K}(\langle P_4\rangle)$.

Suppose that a graph G has a subgraph H isomorphic to one of the graphs of $\mathcal{F}$. Then G has H or one of its spanning supergraphs as an induced subgraph, i.e., is not König with respect to $\langle P_4\rangle$. Thus, we can formulate a corollary to Lemmas 1, 3, and 4:

Corollary 2. *No graph of class $\mathcal{K}(\langle P_4\rangle)$ has subgraphs isomorphic to graphs in $\mathcal{F}$.*

3. *ST*-GRAPHS

Describe the procedure of *ST*-extension and the class of *ST*-graphs and prove that the *ST*-extension of bipartite multigraphs always gives König graphs with respect to $\langle P_4\rangle$.

Definition 1. Refer to a connecting subgraph H in a graph G as *pendant* if there is a vertex $c \in V(G \setminus H)$ such that H is not a connected component of the graph $G \setminus \{c\}$. Refer to c as the *contact vertex* of the pendant subgraph H.

Definition 2. Let H be a bipartite multigraph. The procedure of the *ST-extension* of H is as follows:

(1) Subdivide each cyclic edge (each edge belonging to a cycle including of length 2) of H by a vertex. All vertices except for those added in subdividing will be called *old*.

(2) Add to graphs several pendant subgraphs isomorphic to C_3 or S_k, where k are arbitrary nonnegative integers so that the contact vertex of each of them is old.

Refer to the obtained graph as an *ST-extension* of the multigraph H. Refer as an *ST-graph* to a graph presenting an *ST*-extension of an arbitrary bipartite multigraph.

Observe that if we allow subdivisions of noncyclic edges of H in item (1) of the definition of an *ST*-extension then the graph class remains unchanged since as a result we obtain an *ST*-extension of the graph obtained from H by subdividing the corresponding edges.

Theorem 1. *Every ST-graph is a König graph with respect to $\langle P_4\rangle$.*

Proof. Let G be the graph obtained by an *ST*-extension of an arbitrary bipartite multigraph H. The above remark implies that each subgraph in G is also an *ST*-graph. Thus, for proving the theorem, it suffices to show that $\mu_{\langle P_4\rangle}(G) = \beta_{\langle P_4\rangle}(G)$.

It is not hard to see that a graph is König with respect to $\langle P_4\rangle$ if and only if each of its connected components is a König graph with respect to $\langle P_4\rangle$. Thus, we can assume without loss of generality that G is connected.

Proceed by induction on the number of vertices in G. If G is isomorphic to one of the graphs K_1, C_3, and S_k, where $k \in \mathbb{N}$, then G contains no quartets and $\mu_{\langle P_4\rangle}(G) = \beta_{\langle P_4\rangle}(G) = 0$.

Suppose that G contains at least one quartet and $\mu_{\langle P_4\rangle}(G') = \beta_{\langle P_4\rangle}(G')$ for its every induced subgraphs G'.

The following cases are possible:

1. The graph G has a pendant subgraph T isomorphic to C_3 with contact vertex y. Then the vertices $V(T)\cup\{y\}$ form a quartet. Consider the graph G' obtained from G by removing the vertices of this quartet. Let M be a greatest $\langle P_4\rangle$-packing and let C be a least $\langle P_4\rangle$-cover of G'. By the induction assumption, $|M| = |C|$. Adding a quartet composed of vertices from $V(T)\cup\{y\}$ to M, we obtain a $\langle P_4\rangle$-packing of size $|M|+1$. Adding the vertex y to C, we obtain a $\langle P_4\rangle$-cover of G of the same size.

2. The graph G has a pendant subgraph T isomorphic to S_k, where $k \ge 2$, with contact vertex y; moreover, y is adjacent to at least one leaf of T. Then all quartets containing vertices from $V(T)\cup\{y\}$ pass through y. Let x_1 be a leaf of T adjacent to y, let x_2 be its central vertex, and let x_3 be its another leaf. Then (x_1, x_2, x_3, y) is a quartet. Consider the graph G' obtained from G by removing the vertices of this quartet. Let M be a greatest $\langle P_4\rangle$-packing and let C be a least $\langle P_4\rangle$-cover of G'. By the induction assumption, $|M| = |C|$. Now, $M\cup\{(x_1,\ x_2,\ x_3,\ y)\}$ is a $\langle P_4\rangle$-packing while $C\cup\{y\}$ is $\langle P_4\rangle$-cover of G of the same size.

3. The graph G has a pendant subgraph T_1 isomorphic to S_k with $k\ge 1$ and a pendant subgraph T_2 isomorphic to S_l with $l \ge 1$, or to the graph K_1 with common contact vertex y. Then all quartets containing vertices from $V(T_1)\cup V(T_2)\cup\{y\}$ pass through y. Let x_1 and x_2 be vertices of the graphs T_1 and T_2 respectively adjacent to y. Let x_3 be a vertex of T_1 different from x_1. Then (x_1, x_2, x_3, y) is a quartet. Consider the graph G' obtained by all vertices of this quartet from G. Let M be a greatest $\langle P_4\rangle$-packing and let C be a least $\langle P_4\rangle$-cover of G'. By the induction assumption, $|M| = |C|$. Now, $M\cup\{(x_1,\ x_2,\ x_3,\ y)\}$ is a $\langle P_4\rangle$-packing and $C\cup\{y\}$ is a $\langle P_4\rangle$-cover of G of the same size.

4. The graph G satisfies none of the conditions of Cases 1–3 but has a pendant subgraph T isomorphic to S_k, where $k\ge 1$, whose contact vertex y is cyclic in the multigraph H; moreover, if $k\ge 2$ then y is adjacent to only one of its central vertices. Then all quartets containing vertices from $V(T)\cup\{y\}$ pass through y. Let x_1 be a central vertex of T and let x_2 be its leaf. Let z be a cyclic vertex of $G\setminus T$ adjacent to y and added in subdividing a cyclic edge of the multigraph H. Then (x_2, x_1, y, z) is a quartet. Since y is a cyclic vertex in H, it is also a cyclic vertex in G. Consider one of the cycles containing y and z and enumerate its vertices along the cycle with $0, 1, \dots, 4n-1$ so that y have number 0 and z have number 1. Note that all vertices with odd numbers have degree 2.

Consider the graph G' obtained from G by removing the vertices x_1, x_2, and y. Let M be a greatest $\langle P_4\rangle$-packing and let C be a least $\langle P_4\rangle$-cover of G'. Note that if a quartet of G' contains an enumerated vertex with odd number then it also contains one of its adjacent vertices with even number. Moreover, if the quartet contains vertices $2i-1$ and $2i+1$ then it also contains $2i$ and one of the vertices $2i-2$ and $2i+2$. Thus, the number of enumerated vertices belonging to the quartets of the $\langle P_4\rangle$-packing with even numbers is not less that those with odd numbers.

Suppose that M includes a quartet containing z. Then the enumerated vertices contain the vertex with number $2i+1$, where $1\le i\le 2n-1$, belonging to no quartet of M. Take i such that all vertices with odd (and hence with even) numbers less than $2i+1$ are contained in quartets of M. Then there exists a quartet $q\in M$ containing vertices $2i-1$ and $2i$. Suppose that q does not contain vertex $2i-2$; i.e., $q = (2i-1, 2i, v_1, v_2)$, where v_1 is a nonenumerated vertex. Then

$$M\setminus\{(2i-1, 2i, v_1, v_2)\}\cup\{(2i, 2i+1, v_1, v_2)\}$$

is also a greatest $\langle P_4\rangle$-packing of G' and the vertex with number $2i-1$ belongs to none of its quartets. Otherwise, $q = (2i-3, 2i-2, 2i-1, 2i)$. Then

$$M\setminus\{(2i-3, 2i-2, 2i-1, 2i)\}\cup\{(2i-2, 2i-1, 2i, 2i+1)\}$$

is also a greatest $\langle P_4\rangle$-packing of G' and the vertex with number $2i-3$ belongs to none of its quartets. Observe that the minimal number of a vertex belonging to no quartet of a greatest $\langle P_4\rangle$-packing has decreased. Using several such “shifts,” we can obtain a $\langle P_4\rangle$-packing none of whose quartets contains the vertex z with number 1.

Thus, there exists a greatest $\langle P_4\rangle$-packing such that the vertex z belongs to none of its quartets. Assume without loss of generality that M is such a $\langle P_4\rangle$-packing.

By the induction assumption, $|M| = |C|$. Now, $M \cup \{(x_1, x_2, y, z)\}$ is a $\langle P_4 \rangle$-packing and $C \cup \{y\}$ is a $\langle P_4 \rangle$-cover of the same size of G.

5. The graph G satisfies none of the conditions of Cases 1–4; then every quartet in G contains at least one vertex that is not a vertex in a pendant subgraph isomorphic to C_3 and S_k or the contact vertex of such pendant subgraphs; moreover, every cyclic vertex can be the contact vertex only for pendant subgraphs isomorphic to K_1.

Consider a block X in G such that $|X| \geq 4$ and at most one junction in it has a neighbor of degree 1 in $G \setminus X$ (if we consecutively remove all pendant vertices from G then X corresponds to one of the leaves of the block and junction tree of the obtained graph). Denote such a junction, if it exists, by v_0. Otherwise, denote by v_0 an arbitrary vertex of the block X not added in subdividing. Each of the vertices in X is cyclic.

Denote by X_0 the subgraph in the multigraph H from which the graph X is obtained by subdivision and construct in it a greatest matching M_0 and a least vertex cover C_0. Since X_0 contains only cyclic edges; therefore, fixing one of the parts of the multigraph X_0, with each edge $m_i \in M_0$ we can associate an edge m'_i having a common vertex with m_i in the chosen part. Obviously, $m'_i \notin M_0$ and $m'_i \neq m'_j$ for $i \neq j$. During the procedure of *ST*-extension, each edge of the subgraph X_0 is subdivided by a single vertex. For each $m_i \in M_0$, take a quartet consisting of the vertices incident to m_i and the vertices added in subdividing the edges m_i and m'_i. The so-obtained set of quartets M_1 constitutes a $\langle P_4 \rangle$-packing of X.

It is easy that C_0 is a $\langle P_4 \rangle$-cover of X; moreover, $|C_0| = |M_1|$. Note that C_0 is also a $\langle P_4 \rangle$-cover of the subgraph in G constituted by the vertices of X and of the pendant subgraphs having their contact vertices in X.

If v_0 belongs to a least vertex cover of X_0 then as C_0 we take such a $\langle P_4 \rangle$-cover.

Suppose that v_0 belongs to no least vertex cover of X_0 but one of the edges in M_0 is incident to v_0 (note that such a matching always exists).

Consider the set A of the vertices of the same part of X_0 as v_0 incident to no edges in M_0. Note that A is not empty; otherwise, the entire part containing v_0 would be the least vertex cover containing v_0. Since v_0 belongs to no least vertex cover of X_0, there exists an alternating path to v_0 from at least one vertex $a \in A$ (here by an *alternating* path we mean a path in which the edges belonging and not belonging to M_0 alternate). This follows from the proof of König's Theorem (see, for example, [19]). Replacing the edges of the path not belonging to M_0 with the edges of the same path not belonging to M_0, we obtain a greatest matching M'_0 in X_0. With each new edge $m_i \in M'_0$, associate an edge m'_i of this alternating path directed from the vertex v_0, and with an edge incident to a, associate another edge incident to the same vertex. Thus, v_0 is not covered by M'_0 and one of the edges incident to v_0 is associated with no edge of the matching. Denote by v_1 the vertex subdividing it in the process of *ST*-extension.

Denote by X' the subgraph in G consisting of X and all pendant vertices of G added to X. Note that since G contains no other pendant subgraphs with contact vertices in X but K_1, C_0 is a $\langle P_4 \rangle$-cover also for X'.

Consider the graph G' obtained from G by removing the vertices of the subgraph X' if v_0 belongs to some least vertex cover of X_0 and by removing the vertices of the subgraph $X' \setminus \{v_0, v_1\}$ otherwise. Let M be a greatest $\langle P_4 \rangle$-packing and let C be a least $\langle P_4 \rangle$-cover of G'. By the induction assumption, $|M| = |C|$. The set $M \cup M_1$ is a $\langle P_4 \rangle$-packing, and $C \cup C_0$ is a $\langle P_4 \rangle$-cover of G. Since $|C_0| = |M_1|$, we have $\mu_{\langle P_4 \rangle}(G) = \beta_{\langle P_4 \rangle}(G)$.

Theorem 1 is proved. □

4. COMPLETE DESCRIPTION OF THE GRAPHS OF CLASS $\mathcal{K}(\langle P_4 \rangle)$

Let us show that every König graph with respect to $\langle P_4 \rangle$ is an ST-extension of a bipartite multigraph and prove that the forbidden subgraphs described in Section 2 completely describe this graph class.

Theorem 2. *The following are equivalent for a graph G :*

(1) *G is a König graph with respect to $\langle P_4 \rangle$;*

(2) *G contains no subgraphs from $\mathcal{F}$;*

(3) *G is an ST-extension of a bipartite multigraph.*

Proof. By Corollary 2, (1) $\Rightarrow$ (2). It follows from Theorem 1 that (3) $\Rightarrow$ (1). Show that (2) $\Rightarrow$ (3).

Let G be a connected graph not containing subgraphs from $\mathcal{F}$. Let G_0 denote the subgraph of G obtained by removing all its pendant subgraphs isomorphic to C_3 and also those pendant subgraphs isomorphic to S_k, where $k \geq 1$, in which the contact vertex is adjacent with at least two vertices one of which is a central vertex of S_k.

If two such pendant subgraphs intersect but contained in one another then it is not hard to see G is either a subgraph in K_4 or a graph C_3 to one or two vertices of which graphs S_0 are attached. In each of the indicated cases, G is an ST-extension of a bipartite multigraph. Therefore, we may assume that all pendant graphs removed from G are pairwise vertex disjoint.

Show that G_0 has no triangles. Suppose that G_0 contains a triangle. Let x, y, and z denote the vertices of a triangle in this graph. Consider the following cases:

1. There exists a vertex u in G adjacent to all vertices of the triangle. Since none of the triangles consisting of vertices of $\{x, y, z, u\}$ is a pendant subgraph in G, at least two of these vertices are adjacent to other vertices of this graph. Suppose that v is adjacent to x and w is adjacent to y in G. If $v = w$ then G has a subgraph C_5. Otherwise, G has a subgraph $A(4, 1)$.

2. There exist pairwise distinct vertices u, v, and w such that (u, x), (v, y), and (w, z) are edges of G. Then G contains a subgraph net.

3. There are distinct vertices u and v such that (u, x), (u, y), and (v, z) are the edges of G. Since the triangle u, x, y is not pendant in G, there exists a vertex $w \notin \{u, x, y, z\}$ adjacent to one of the vertices of this triangle. If $w = v$ then G contains a subgraph C_5. If w does not coincide with v and is adjacent to u then G contains a subgraph co-twin-house. Otherwise, G contains a subgraph net.

4. The vertex z has no other neighbors but x and y. Since none of the subgraphs induced by the sets of vertices $\{x, y\}$, $\{x, z\}$, $\{y, z\}$, or $\{x, y, z\}$ is pendant in G, there are distinct vertices u and v such that (u, x) and (v, y) are edges of G. It is not hard to see that the vertices u, x, and z induce a subgraph isomorphic to S_2 with central vertex x. The star with central vertex x is not a pendant subgraph in G. Hence, there exists a vertex $s \neq y$ adjacent to one of the leaves of this star. The case when this leaf is a neighbor of y was already examined in Case 3. Assume without loss of generality that the vertex s is adjacent to u and $s \neq v$; otherwise, G contains a subgraph C_5. By analogy, there exists a vertex $t \notin \{y, u\}$ adjacent to v. If $s = t$ then G contains a subgraph C_5; otherwise, G contains a subgraph longhorn.

Thus, G_0 has no triangles. Let Z be a block of size greater than 2 in G_0. Each of its vertices is cyclic. Lemma 3 implies that the length of each cycle in Z divides by 4. Let x be a vertex in Z having degree greater than 2 in G if such a vertex exists and its arbitrary vertex otherwise. Then all vertices in Z whose distance from x is odd have degree 2 in G; otherwise, G contains a forbidden subgraph of type $A(n, k)$. Call such vertices *passable* in G_0.

Construct a multigraph H by replacing each passable vertex in all cycles of G_0 by a single edge joining its neighboring vertices. Obviously, G_0 is obtained from H by subdividing each cyclic edge and G is an ST-extension of H. Since all cycles in G_0 have length divisible by 4, all cycles in H have even length, and hence the Multigraph H is bipartite. Theorem 2 is proved. □

Corollary 3. *The class $\mathcal{K}(\langle P_4 \rangle)$ is monotone and completely defined by the set of its minimal forbidden subgraphs $\mathcal{F}$.*

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