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Invariant subalgebras and affine embeddings of homogeneous spaces

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1. Invariant subalgebras.

Let G be a connected semisimple complex Lie group, and let $\mathbb{C}[G]$ be the algebra of polynomial functions on G. For any integral commutative complex algebra A denote by $Q(A)$ its quotient field and by tr.deg $Q(A)$ the transcendency degree of this field over $\mathbb{C}$. If M is a set with a G-action, then M^G denotes the subset of G-fixed points.

Theorem 1.1. *Let $A \subset \mathbb{C}[G]$ be a G-invariant finitely generated subalgebra and $I \lhd A$ be a G-invariant prime ideal. Then*

$$\text{tr.deg}\,(Q(A/I))^G \leq \frac{1}{2}(\dim G - \text{rk}\, G) - 1. \quad (1)$$

Moreover, there exist a subalgebra A and an ideal I such that (1) *is an equality.*

The proof of Theorem 1.1 is based on some properties of affine embeddings of homogeneous spaces.

2. Embeddings of homogeneous spaces.

Let G be a connected reductive algebraic group over an algebraically closed field $\Bbbk$ of characteristic zero, and let H be an algebraic subgroup of G. A pointed irreducible algebraic G-variety X is said to be an *embedding* of the homogeneous space G/H if the base point of X has the dense orbit and stabilizer H. We shall denote this by $G/H \hookrightarrow X$. For an algebraic G-variety Z the closure of a G-orbit on Z is an embedding of this orbit. Thus the study of embeddings can be considered as a starting point for the theory of algebraic transformation groups.

The general theory of embeddings of homogeneous spaces was developed in the famous work of D. Luna and Th. Vust [8]. The notion of *complexity* plays here the key role. Let B be a Borel subgroup of G. By definition, the complexity $c(X)$ of a G-variety X is the codimension of a generic B-orbit for the restricted action $B : X$, see [12] and [8]. By Rosenlicht's theorem, $c(X) = \text{tr.deg}\, \Bbbk(X)^B$. The classification of embeddings of a given homogeneous space G/H is known if $c(G/H) \leq 1$, see [4] and [11].

A normal G-variety X is called *spherical* if $c(X) = 0$, or, equivalently, $\Bbbk(X)^B = \Bbbk$. A homogeneous space G/H and a subgroup $H \subset G$ are said to

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be spherical if G/H is a spherical G-variety. It was proved by F. J. Servedio, D. Luna, Th. Vust and D. N. Akhiezer that a space G/H is spherical if and only if each embedding of G/H has finitely many G-orbits.

More generally, the *modality* of an action $G : X$ is the integer

$$\mathrm{mod}(X) = \max_{Y \subseteq X} \ \mathrm{tr.deg}\ \Bbbk(Y)^G,$$

where Y runs through G-stable irreducible subvarieties of X. (This notion appeared in works of V. I. Arnold on singularities). The modality is equal to the maximal number of parameters in a family of G-orbits of the same dimension on X. In particular, $\mathrm{mod}(X) = 0$ iff the number of G-orbits on X is finite. It was shown by E. B. Vinberg [12] that $\mathrm{mod}(X) \le c(X)$.

Denote by $m(G/H)$ the maximum of $\mathrm{mod}(X)$, where X runs through all embeddings $G/H \hookrightarrow X$. D. N. Akhiezer [1] proved that $m(G/H) = c(G/H)$.

3. Affine embeddings.

An embedding $G/H \hookrightarrow X$ is called *affine* if the variety X is affine. In many problems of invariant theory, representation theory and other branches of mathematics, only affine embeddings appear. Hence they deserve a special consideration. On the other hand, there are some interesting properties that hold for affine embeddings only. Some examples will be considered in this section.

Note that a given homogeneous space G/H admits an affine embedding iff G/H is quasiaffine (as an algebraic variety), see [9, Th. 1.6]. In this situation, the subgroup H is said to be *observable* in G. For a description of observable subgroups, see [10], [9, Th. 4.18]. By Matsushima's criterion, G/H is affine iff H is reductive. In particular, any reductive subgroup is observable.

Let us say that an embedding $G/H \hookrightarrow X$ is *trivial* if $X = G/H$. It is well-known that any embedding of G/H is trivial iff H is a parabolic subgroup of G. The following result due to D. Luna is an affine version of this fact. Denote by $N_G(H)$ the normalizer of H in G and by $W(H)$ the quotient group $N_G(H)/H$. (The group $W(H)$ can be identified with the group $Aut_G(G/H)$ of G-equivariant automorphisms of G/H). Then any affine embedding of G/H is trivial iff H is reductive and $W(H)$ is finite [7]. For example, this is the case if H is a reductive subgroup containing a maximal torus of G.

By analogy with the previous section, we associate with any quasiaffine homogeneous space G/H the integer

$$a(G/H) = \max_X \ \mathrm{mod}(X),$$

where X runs through all affine embeddings of G/H. It is clear that $a(G/H) \le m(G/H)$.

Theorem 3.1. *Let H be a reductive subgroup of G.*

(1) If the group $W(H)$ is finite, then $a(G/H) = 0$.

(2) If $W(H)$ is infinite, then

$$a(G/H) = \max_{H_1} \ c(G/H_1),$$

where H_1 runs through all non-trivial extensions of H by a one-dimensional subtorus of $N_G(H)$. In particular, $a(G/H) = c(G/H)$ or $c(G/H) - 1$.

In the case $a(G/H) = 0$, we obtain

Corollary 3.2. [2, Theorem 3] *For a reductive subgroup H of G, following conditions are equivalent:*

(1) for any affine embedding $G/H \hookrightarrow X$, the number of G-orbits in X is finite;

(2) either $W(H)$ is finite or any non-trivial extension of H by a one-dimensional torus of $N_G(H)$ is spherical in G.

4. Proof of Theorem 3.1.

Here we follow the scheme of the proof of Theorem 3 from [2].

Proposition 4.1. *Let H be an observable subgroup of G. Suppose that there is a non-trivial one-parameter subgroup $\lambda : \mathbb{k}^* \to W(H)$ and let H_1 be the preimage of $\lambda(\mathbb{k}^*)$ in $N_G(H)$. Then there exists an affine embedding $G/H \hookrightarrow X$ with* $\mathrm{mod}(X) \geq c(G/H_1)$.

The idea of the proof is to apply Akhiezer's construction [1] to the homogeneous space G/H_1 and to consider the affine cone over a projective embedding X' of G/H_1 with $\mathrm{mod}(X') = c(G/H_1)$.

Lemma 4.2. *Let $H \subseteq G$ be an observable subgroup and H_1 be the extension of H by a one-dimensional torus $\lambda(\mathbb{k}^*) \subseteq W(H)$. Then there exists a finite-dimensional G-module V and an H_1-eigenvector $v \in V$ such that*

(1) *the orbit $G\langle v\rangle$ of the line $\langle v\rangle$ in the projective space $\mathbb{P}(V)$ is isomorphic to G/H_1;*

(2) *H fixes v;*

(3) *H_1 acts transitively on $\mathbb{k}^* v$;*

(4) $\mathrm{mod}(G\langle v\rangle) = c(G/H_1)$.

Proof. (1)-(3) By Chevalley's theorem, there exists a G-module V' and a vector $v' \in V'$ having property (1). Let us denote by χ the character of H at v'. Since H is observable in G, every finite-dimensional H-module can be embedded in a finite-dimensional G-module [3]. In particular, there exists a finite-dimensional G-module V'' containing H-eigenvectors of character $-\chi$. Choose among them a H_1-eigenvector v'' and put $V = V' \otimes V''$ and $v = v' \otimes v''$. Properties (1) and (2) are satisfied.

If condition (3) also holds, then we are done. Otherwise, consider any G-module W having a vector with stabilizer H. Take an H_1-eigenvector $w \in W^H$ with nontrivial character, and replace V by $V \otimes W$ and v by $v \otimes w$. Now properties (1)-(3) are satisfied.

(4) By a result due to Akhiezer [1], we may choose (V', v') so that properties (1) and (4) are satisfied. Then we proceed as in (1)-(3) to obtain the couple (V, v). The closure $\overline{G\langle v\rangle} \subseteq \mathbb{P}(V)$ is contained in the image of the Segre embedding

$$\mathbb{P}(V') \times \mathbb{P}(V'') \hookrightarrow \mathbb{P}(V), \qquad \text{or} \quad \mathbb{P}(V') \times \mathbb{P}(V'') \times \mathbb{P}(W) \hookrightarrow \mathbb{P}(V),$$

and projects G-equivariantly onto $\overline{G\langle v'\rangle} \subseteq \mathbb{P}(V')$. This implies (4) for (V, v). ∎

Proof. (Proposition 4.1) Let $(V,\ v)$ be the couple from Lemma 4.2. Denote by H' the stabilizer G_v of the vector v and set $\tilde{X} = \overline{Gv}$. By (1)–(3) and since H_1/H is isomorphic to $\mathbb{k}^*$, H' is a finite extension of H. By (3), the closure of Gv in V is a cone, so by (4) we have $\operatorname{mod}(\tilde{X}) \geq c(G/H_1)$.

Consider the morphism $G/H \to G/H'$. It determines an embedding $\mathbb{k}[\tilde{X}] \subseteq \mathbb{k}[G/H'] \subseteq \mathbb{k}[G/H]$. Let A be the integral closure of the subalgebra $\mathbb{k}[\tilde{X}] \subseteq \mathbb{k}[G/H']$ in the field $\mathbb{k}(G/H)$. We have the following commutative diagrams:

$$\begin{array}{ccccc} A & \hookrightarrow & \mathbb{k}[G/H] & \hookrightarrow & \mathbb{k}(G/H) \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{k}[\tilde{X}] & \hookrightarrow & \mathbb{k}[G/H'] & \hookrightarrow & \mathbb{k}(G/H') \end{array} \qquad \begin{array}{ccc} \operatorname{Spec} A & \hookleftarrow & G/H \\ \downarrow & & \downarrow \\ \tilde{X} & \hookleftarrow & G/H' \end{array}$$

The affine variety $X = \operatorname{Spec} A$ with a natural G-action can be considered as an affine embedding of G/H. The embedding $\mathbb{k}[\tilde{X}] \subseteq A$ defines a finite (surjective) morphism $X \to \tilde{X}$ and therefore, $\operatorname{mod}(X) = \operatorname{mod}(\tilde{X}) \geq c(G/H_1)$. ∎

We shall use some results due to F. Knop.

Lemma 4.3. ([5, 7.3.1], see also [2, Lemma 3]) *Let X be an irreducible G-variety, and v be a G-invariant valuation of $\mathbb{k}(X)/\mathbb{k}$ with residue field $\mathbb{k}(v)$. Then $\mathbb{k}(v)^B$ is the residue field of the restriction of v to $\mathbb{k}(X)^B$.*

Definition 4.4. [6, §7] Let X be a normal G-variety. A discrete $\mathbb{Q}$-valued G-invariant valuation of $\mathbb{k}(X)$ is called *central* if it vanishes on $\mathbb{k}(X)^B \setminus \{0\}$. A *source* of X is a non-empty G-stable subvariety $Y \subseteq X$ which is the center of a central valuation of $\mathbb{k}(X)$.

The following lemma is an easy consequence of results from [6], for more details see [2, Lemma 4].

Lemma 4.5. *If X is a normal affine G-variety containing a proper source, then there exists a one-dimensional torus $S \subseteq \operatorname{Aut}_G(X)$ such that $\mathbb{k}(X)^B \subseteq \mathbb{k}(X)^S$.*

Now we are able to prove Theorem 3.1. Statement (1) follows from Luna's Theorem. To prove (2), one can use Proposition 4.1. Since H is reductive, the group $W(H)$ is reductive and contains a one-dimensional subtorus $\lambda(\mathbb{k}^*)$. Hence $a(G/H) \geq c(G/H_1) \geq c(G/H) - 1$ for the extended subgroup H_1. If there is a one-dimensional torus in $W(H)$ such that $c(G/H) = c(G/H_1)$, then there exists an affine embedding of G/H of modality $c(G/H)$.

Conversely, suppose that $G/H \hookrightarrow X$ is an affine embedding of modality $c(G/H)$. We need to find a one-dimensional subtorus $\lambda(\mathbb{k}^*) \subseteq W(H)$ such that

$c(G/H_1) = c(G/H)$. By the definition of modality, there exists a proper G-invariant subvariety $Y \subset X$, such that the codimension of generic G-orbit in Y is $c(G/H)$. Therefore, $c(Y) = c(G/H)$. Consider a G-invariant valuation v of $\Bbbk(X)$ with the center Y. For the residue field $\Bbbk(v)$ we have $\operatorname{tr.deg} \Bbbk(v)^B \geq \operatorname{tr.deg} \Bbbk(Y)^B$, hence $\operatorname{tr.deg} \Bbbk(v)^B = \operatorname{tr.deg} \Bbbk(X)^B$. If the restriction of v to $\Bbbk(X)^B$ is not trivial, then by Lemma 4.3, $\operatorname{tr.deg} \Bbbk(v)^B < \operatorname{tr.deg} \Bbbk(X)^B$, a contradiction. Thus v is central, and Y is a source of X. A one-dimensional subtorus $S \subseteq Aut_G(X) \subseteq Aut_G(G/H) = W(H)$ provided by Lemma 4.5 yields the extension of H of the same complexity. This completes the proof.

Remark 4.6. If H is an observable subgroup and $W(H)$ contains non-trivial subtorus, then the formula $a(G/H) = \max_{H_1} c(G/H_1)$ can be obtained by the same arguments. If $W(H)$ is either finite or unipotent, then our proof gives the inequality $a(G/H) \leq c(G/H) - 1$.

In this context we would like to present a reformulation of a problem firstly posed in [2].

Problem. *Let V be a finite-dimensional G-module and v be a vector in V. Suppose that the group $Aut_G(Gv)$ is finite. Is it true that Gv is closed in V?*

If Gv is affine then the answer is positive by Luna's theorem. Hence the problem is to prove that if $W(H)$ is finite for an observable subgroup H then H is reductive.

5. Proof of Theorem 1.1.

The inclusion $A \subset \mathbb{C}[G]$ induces a dominant morphism of affine varieties $G \to \operatorname{Spec} A$. Hence $\operatorname{Spec} A$ can be considered as an affine embedding of a homogeneous space G/H. The quotient A/I is the algebra of polynomial functions on a G-invariant closed irreducible subvariety $Y \subseteq \operatorname{Spec} A$. Thus $\operatorname{tr.deg} (Q(A/I))^G \leq \operatorname{mod}(X)$ and $\operatorname{mod}(X) \leq a(G/H)$. We shall prove that $a(G/H) \leq \frac{1}{2}(\dim G - \operatorname{rk} G) - 1$ and this estimate is exact.

Note that $c(G/\{e\}) = \frac{1}{2}(\dim G - \operatorname{rk} G)$. Consider three possible cases.

1) H is finite and $W(H)$ is finite. Here $a(G/H) = 0$.

2) H is finite and $W(H)$ is infinite. For any one-dimensional subtorus $T_1 \subset N_G(H)$ there exists a Borel subgroup B of G which does not contain T_1 and there is a B-orbit of dimension $\dim B$ on $G/(HT_1)$. This implies $c(G/(HT_1)) = c(G/\{e\}) - 1$. By Theorem 3.1, we have $a(G/H) = c(G/\{e\}) - 1$.

3) $\dim H$ is positive. In this case $a(G/H) \leq c(G/H) \leq c(G/\{e\}) - 1$.

The proof is completed.

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