DOI 10.1007/s10958-019-04619-w
Journal of Mathematical Sciences, Vol. 244, No. 2, January, 2020

ON DYNAMIC AGGREGATION SYSTEMS

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Abstract. We consider consecutive aggregation procedures for individual preferences $\mathfrak{c} \in \mathfrak{C}_r(A)$ on a set of alternatives A, $|A| \geq 3$: on each step, the participants are subject to intermediate collective decisions on some subsets B of the set A and transform their a priori preferences according to an adaptation function $\mathcal{A}$. The sequence of intermediate decisions is determined by a lot J, i.e., an increasing (with respect to inclusion) sequence of subsets B of the set of alternatives. An explicit classification is given for the clones of local aggregation functions, each clone consisting of all aggregation functions that dynamically preserve a symmetric set $\mathfrak{D} \subseteq \mathfrak{C}_2(A)$ with respect to a symmetric set of lots $\mathcal{J}$. On the basis of this classification, it is shown that a clone $\mathcal{F}$ of local aggregation functions that preserves the set $\mathfrak{R}_2(A)$ of rational preferences with respect to a symmetric set $\mathcal{J}$ contains nondictatorial aggregation functions if and only if $\mathcal{J}$ is a set of maximal lots, in which case the clone $\mathcal{F}$ is generated by the majority function. On the basis of each local aggregation function f, lot J, and an adaptation function $\mathcal{A}$, one constructs a nonlocal (in general) aggregation function $f_{J,\mathcal{A}}$ that imitates a consecutive aggregation procesure. If f dynamically preserves a set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ with respect to a set of lots $\mathcal{J}$, then the aggregation function $f_{J,\mathcal{A}}$ preserves the set $\mathfrak{D}$ for each lot $J \in \mathcal{J}$. If $\mathfrak{D} = \mathfrak{R}_2(A)$, then the adaptation function can be chosen in such a way that in any profile $\mathbf{c} \in (\mathfrak{R}_2(A))^n$, the Condorcet winner (if it exists) would coincide with the maximal element with respect to the preferences $f_{J,\mathcal{A}}(\mathbf{c})$ for each maximal lot J and f that dynamically preserves the set of rational preferences with respect to the set of maximal lots.

1. Basic Definitions

Consider a finite nonempty set (of alternatives) A and let r be a fixed positive integer. In order to avoid trivial cases, we assume that $2 \leq r < |A|$, unless there are some other explicit constraints. The set of all r-element subsets of A is denoted by $[A]^r$:

$$[B]^r \rightleftharpoons \bigl\{C \subseteq B\colon |C| = r\bigr\}.$$

According to [1], [2], for each set $B \subseteq A$ (individual) *r-preferences* on B are modeled by r-choice functions, i.e., functions $\mathfrak{c}\colon [B]^r \to B$ satisfying the condition

$$\bigl(\forall p \in [B]^r\bigr)\, \mathfrak{c}(p) \in p.$$

The set of all r-choice functions on B is denoted by $\mathfrak{C}_r(B)$. Note that the formal set-theoretic definition of a function implies that for any set X there is a unique (empty) function $\varnothing\colon \varnothing \to X$, and therefore, if $|B| < r$, then $\mathfrak{C}_r(B) = \{\varnothing\}$ (this fact will be used in what follows).

An r-choice function $\mathfrak{c} \in \mathfrak{C}_r(B)$ is called *rational*, if there is a linear order $\prec$ on B such that for any set $p \in [B]^r$, the value $\mathfrak{c}(p)$ is a maximal element of p with respect to the order $\prec$. The set of all rational r-choice functions on B is denoted by $\mathfrak{R}_r(B)$.

Remark 1. $\mathfrak{R}_2(B) = \mathfrak{C}_2(B)$ for any two-element set $B \subseteq A$.

In this paper, we mainly address the case $r = 2$ (the results of [2,3] show that the general case hardly adds any new combinatorial effects), in which any preference $\mathfrak{c}$ can be identified with the binary connected asymmetric relation

$$P_{\mathfrak{c}} \rightleftharpoons \bigl\{(a,b) \in B^2\colon a \neq b \text{ and } \mathfrak{c}(\{a,b\}) = b\bigr\}$$

or the *complete directed graph* (*tournament*) $\Gamma_{\mathfrak{c}} \rightleftharpoons (B, P_{\mathfrak{c}})$.

Translated from Trudy Seminara imeni I. G. Petrovskogo, No. 32, pp. 257–282, 2019.

 1072–3374/20/2442–0278 © 2020 Springer Science+Business Media, LLC

Proposition 1. *For any 2-choice function on B, $|B| \geq 2$, the following conditions are equivalent*:

(1) *the function $\mathfrak{c}$ is rational*;
(2) *the relation $P_{\mathfrak{c}}$ is that of a strict linear order*;
(3) $\mathfrak{c}(\{\mathfrak{c}(\{x,y\}),z\}) = \mathfrak{c}(\{x,\mathfrak{c}(\{y,z\})\})$ *for all $x,y,z \in B$ ($x \neq y$ and $y \neq z$).*

For each permutation σ of the set A, a set $B \subseteq A$, and a function $\mathfrak{c} \in \mathfrak{C}_r(B)$, the symbol σB stands for the set

$$\{\sigma(x)\colon x \in B\},$$

and the symbol $\mathfrak{c}_\sigma$ denotes the r-choice function on σB defined as

$$\mathfrak{c}_\sigma(p) = \sigma^{-1}\big(\mathfrak{c}(\sigma p)\big)$$

for all $p \in [B]^r$. For each set $\mathfrak{D} \subseteq \mathfrak{C}_r(B)$ and a permutation σ of the set A, let $\mathfrak{D}_\sigma \rightleftharpoons \{\mathfrak{c}_\sigma\colon \mathfrak{c} \in \mathfrak{D}\}$. A set

$$\mathbb{D} \subseteq \bigcup_{B\subseteq A} \mathscr{P}\big(\mathfrak{C}_r(B)\big)$$

is called *symmetric* if

$$\mathfrak{D} \in \mathbb{D} \Rightarrow \mathfrak{D}_\sigma \in \mathbb{D}$$

for any set $\mathfrak{D} \subseteq \mathfrak{C}_r(B)$ and a permutation σ of the set A. A set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ is said to be *symmetric* if the singleton $\{\mathfrak{D}\}$ is symmetric.

For any sets $C \subseteq B \subseteq A$ and a function $\mathfrak{c} \in \mathfrak{C}_r(B)$, the symbol $\mathfrak{c}_{[C]}$ denotes the restriction $\mathfrak{c} \restriction [C]^r$ of the function $\mathfrak{c}$ to the set $[C]^r$. For any set $\mathfrak{D} \subseteq \mathfrak{C}_r(B)$, let $\mathfrak{D}_{[C]} \rightleftharpoons \{\mathfrak{c}_{[C]}\colon \mathfrak{c} \in \mathfrak{D}\}$. Note that if $|C| < r$ and $\mathfrak{c} \in C_r(B)$, then $\mathfrak{c}_{[C]} = \varnothing$, and therefore, $\mathfrak{D}_{[C]} = \{\varnothing\}$ for any nonempty set $\mathfrak{D} \subseteq \mathfrak{C}_r(B)$.

Definition 1. An *adaptation function* (of r-preferences on the set A) is any function

$$\mathcal{A}\colon \mathfrak{C}_r(A) \times \left(\bigcup_{B\subseteq A} \mathfrak{C}_r(B)\right) \to \mathfrak{C}_r(A)$$

that satisfies the following conditions for all sets $B \subseteq A$ and functions $\mathfrak{c} \in \mathfrak{C}_r(A)$ and $\mathfrak{d} \in \mathfrak{C}_r(B)$:

(1) $\mathcal{A}(\mathfrak{c},\mathfrak{d})_{[B]} = \mathfrak{d}$;
(2) if $\mathfrak{c}_{[B]} = \mathfrak{d}$, then $\mathcal{A}(\mathfrak{c},\mathfrak{d}) = \mathfrak{c}$.

For each set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$, a set $B \subseteq A$ and a function $\mathfrak{d} \in \mathfrak{C}_r(B)$, let $\mathcal{A}(\mathfrak{D},\mathfrak{d}) \rightleftharpoons \{\mathcal{A}(\mathfrak{c},\mathfrak{d})\colon \mathfrak{c} \in \mathfrak{D}\}$. An adaptation function $\mathcal{A}$ *preserves* a set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ if

$$\mathcal{A}(\mathfrak{D},\mathfrak{d}) \subseteq \mathfrak{D}$$

for all sets $B \subseteq A$ and functions $\mathfrak{d} \in \mathfrak{D}_{[B]}$. Note that the definition of the adaptation function implies that the condition $\mathcal{A}(\mathfrak{D},\mathfrak{d}) \subseteq \mathfrak{D}$ is equivalent to: $\mathcal{A}(\mathfrak{D},\mathfrak{d}) = \mathfrak{D}$ for any function $\mathfrak{d} \in \mathfrak{D}_{[B]}$, $B \subseteq A$.

Example 1. Let $r = 2$ and $|A| = 3$. Define an adaptation function $\mathcal{A}$ as follows. Let $B = \{x,y\} \in [A]^2$, $\mathfrak{c} \in \mathfrak{R}_2(A)$ and $\mathfrak{d} \in \mathfrak{R}_2(B) = \mathfrak{C}_2(B)$. For simplicity, instead of rational functions of functions $\mathfrak{c}$, we will be speaking of the corresponding linear orders $P_{\mathfrak{c}}$. If x,y are "neighboring" alternatives with respect to $P_{\mathfrak{c}}$, then the relation $P_{\mathcal{A}(\mathfrak{c},\mathfrak{d})}$ is a linear order that extends the order $P_{\mathfrak{d}}$ and "keeps the element $z \in A \setminus \{x,y\}$ on its place". If x,y are extreme alternatives with respect to $P_{\mathfrak{c}}$ and have "exchanged positions" with respect to the order $P_{\mathfrak{d}}$, then the relation $P_{\mathcal{A}(\mathfrak{c},\mathfrak{d})}$ is a linear order that extends $P_{\mathfrak{d}}$ and "moves the middle element $z \in A \setminus \{x,y\}$ to the maximal position". In the remaining cases, we define the function $\mathcal{A}$ in an arbitrary manner (in agreement with the conditions of Definition 1).

A function $\mathcal{A}$ on the set $\mathfrak{R}_2(A) \times \Big(\bigcup\limits_{B\in[A]^2} \mathfrak{R}_2(B)\Big)$ can be visually represented by the following table. Let $P_{\mathfrak{c}} = x < y < z$. Then

$P_{\mathfrak{d}}$	$P_{\mathcal{A}(\mathfrak{c},\mathfrak{d})}$	$P_{\mathfrak{d}}$	$P_{\mathcal{A}(\mathfrak{c},\mathfrak{d})}$
$x<y$	$x<y<z$	$y<x$	$y<x<z$
$y<z$	$x<y<z$	$z<y$	$x<z<y$
$x<z$	$x<y<z$	$z<x$	$z<x<y$

It is easy to see that $\mathcal{A}$ is a symmetric adaptation function that preserves the set $\mathfrak{R}_2(A)$.

Consider a fixed nonempty finite set $N=\{1,2,\dots,n\}$ (of agents). To avoid trivial cases, assume that $n\geq 2$. By definition, a dynamic *profile* (of agents) is any pair $(\mathbf{c},\mathcal{A})$, where $\mathbf{c}\in\big(\mathfrak{C}_r(A)\big)^n$ and $\mathcal{A}$ as an adaptation function of r-preferences on the set A.

An *aggregation function* (of r-preferences on the set $B\subseteq A$) is, by definition, any function

$$f\colon \big(\mathfrak{C}_r(B)\big)^n\to\mathfrak{C}_r(B).$$

An aggregation function satisfies

(1) *conservativity condition* (**K**) if

$$f(\mathfrak{c}_1,\mathfrak{c}_2,\dots,\mathfrak{c}_n)(p)\in\big\{\mathfrak{c}_1(p),\mathfrak{c}_2(p),\dots,\mathfrak{c}_n(p)\big\}$$

for all $\mathfrak{c}_1,\mathfrak{c}_2,\dots,\mathfrak{c}_n\in\mathfrak{C}_r(B)$ and $p\in[B]^r$;

(2) the *first condition of independence of irrelevant alternatives* ($\mathbf{IIA}_1$) if

$$\big(\mathfrak{c}_1(p),\mathfrak{c}_2(p),\dots,\mathfrak{c}_n(p)\big)=\big(\mathfrak{c}'_1(p),\mathfrak{c}'_2(p),\dots,\mathfrak{c}'_n(p)\big)\implies f(\mathfrak{c}_1,\mathfrak{c}_2,\dots,\mathfrak{c}_n)(p)=f(\mathfrak{c}'_1,\mathfrak{c}'_2,\dots,\mathfrak{c}'_n)(p)$$

for all $\mathfrak{c}_1,\mathfrak{c}_2,\dots,\mathfrak{c}_n,\mathfrak{c}'_1,\mathfrak{c}'_2,\dots,\mathfrak{c}'_n\in\mathfrak{C}_r(B)$ and $p\in[B]^r$.

(3) the *second condition of independence of irrelevant alternatives* ($\mathbf{IIA}_2$) if

$$\big(\mathfrak{c}_1(p),\mathfrak{c}_2(p),\dots,\mathfrak{c}_n(p)\big)=\big(\mathfrak{c}_1(q),\mathfrak{c}_2(q),\dots,\mathfrak{c}_n(q)\big)\implies f(\mathfrak{c}_1,\mathfrak{c}_2,\dots,\mathfrak{c}_n)(p)=f(\mathfrak{c}_1,\mathfrak{c}_2,\dots,\mathfrak{c}_n)(q)$$

for all $\mathfrak{c}_1,\mathfrak{c}_2,\dots,\mathfrak{c}_n\in\mathfrak{C}_r(B)$ and $p,q\in[B]^r$.

Remark 2. For $r\leq 2$, the second condition of independence of irrelevant alternatives follows from conservativity, and the conservativity condition is equivalent to the unanimity condition **U**:

$$\mathfrak{c}_1(p)=\mathfrak{c}_2(p)=\dots=\mathfrak{c}_n(p)=a\implies f(\mathfrak{c}_1,\mathfrak{c}_2,\dots,\mathfrak{c}_n)(p)=a$$

for all $\mathfrak{c}_1,\mathfrak{c}_2,\dots,\mathfrak{c}_n\in\mathfrak{C}_r(B)$, $p\in[B]^2$ and $a\in p$.

Aggregation functions satisfying conditions **K**, $\mathbf{IIA}_1$, and $\mathbf{IIA}_2$ are called *local*[1]. The set of all local aggregation functions $f\colon\big(\mathfrak{C}_r(B)\big)^n\to\mathfrak{C}_r(B)$ is denoted by $\mathcal{L}_{n,r}(B)$.

Local aggregation functions admit the following convenient description. Each sequence from the set B^n is (or is identified with) a function $\mathbf{a}\colon\{1,2,\dots,n\}\to B$; therefore, for any n-tuple $\mathbf{a}\in B^{<\omega}$, its domain and range are respectively denoted by the standard symbols $\operatorname{dom}\mathbf{a}$ and $\operatorname{ran}\mathbf{a}$. By B^n_r we denote the set $\{\mathbf{a}\in B^n\colon|\operatorname{ran}\mathbf{a}|=r\}$. The notations

$$B^n_{\leq r}\rightleftharpoons\bigcup_{k\leq r}B^n_k,\quad B^{<\omega}_{\leq r}\rightleftharpoons\bigcup_{n<\omega}B^n_{\leq r},\quad B^{<\omega}_{\geq r}\rightleftharpoons\bigcup_{r\leq n<\omega}B^n_{\geq r},$$

etc. should be understood in the natural sense. Any function $g\colon B^n_{\leq r}\to B$ satisfying the condition

$$(\forall\,(x_1,x_2,\dots,x_n)\in B^n_{\leq r})\bigvee_{1\leq i\leq n}\big(g(x_1,x_2,\dots,x_n)=x_i\big)$$

is called an (n-place) *conservative r-function* on the set B.

Proposition 2. *Let $|B|\geq r$. Then for any aggregation function $f\colon\big(\mathfrak{C}_r(B)\big)^n\to\mathfrak{C}_r(B)$ satisfying conditions* **K**, $\mathbf{IIA}_1$, *and* $\mathbf{IIA}_2$, *there is a unique n-place conservative r-function $\hat f$ on B such that*

$$f(\mathfrak{c}_1,\mathfrak{c}_2,\dots,\mathfrak{c}_n)(p)=\hat f\big(\mathfrak{c}_1(p),\mathfrak{c}_2(p),\dots,\mathfrak{c}_n(p)\big)$$

for all $\mathfrak{c}_1,\mathfrak{c}_2,\dots,\mathfrak{c}_n\in\mathfrak{C}_r(B)$ and $p\in[B]^r$.

[1]For $r=2$, the locality conditions considered in the present paper are in complete agreement with those of [4–6].

For each local aggregation function f of r-preferences on the set B and any set $C \subseteq B$, $|C| \geq r$, one can correctly define a local aggregation function $f_{[C]}$ of r-preferences on C as follows: for any $\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n \in \mathfrak{C}_r(C)$,

$$f_{[C]}(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n) = f(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_n),$$

where $\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_n$ are arbitrary r-choice functions on B whose restrictions to $[C]^r$ coincide with the functions $\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n$, respectively. It is easy to see that

$$\hat{f}_{[C]} = \hat{f} \upharpoonright C^n_{\leq r}.$$

From now on, we are going to drop the "hat" $\hat{}$ in $\hat{f}$, identifying each local aggregation function f with the corresponding conservative r-function $\hat{f}$.

Given an aggregation function $f\colon \big(\mathfrak{C}_r(B)\big)^n \to \mathfrak{C}_r(B)$ and a set $\mathfrak{D} \subseteq \mathfrak{C}_r(B)$, let

$$f(\mathfrak{D}) \rightleftharpoons \big\{f(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n)\colon \mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n \in \mathfrak{D}\big\}.$$

An aggregation function f *preserves* the set $\mathfrak{D}$ if

$$f(\mathfrak{D}) \subseteq \mathfrak{D}.$$

Remark 3. Each aggregation function $f\colon \big(\mathfrak{C}_r(B)\big)^n \to \mathfrak{C}_r(B)$ preserves the empty set and the set $\mathfrak{C}_r(B)$. Any aggregation function f that satisfies the unanimity condition **U** preserves each one-element set. Moreover, in this case, f preserves $\mathfrak{D}$ if and only if $f(\mathfrak{D}) = \mathfrak{D}$.

For local aggregation functions f, the operation $f(\mathfrak{D})$ and the relation of f preserving a set $\mathfrak{D}$ can be naturally extended to sets $\mathfrak{D} \subseteq \mathfrak{C}_r(C)$ by letting

$$f(\mathfrak{D}) \rightleftharpoons \big\{f_{[C]}(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n)\colon \mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n \in \mathfrak{D}\big\}.$$

For each local aggregation function $f\colon \big(\mathfrak{C}_r(B)\big) \to \mathfrak{C}_r(B)$, we denote by $\mathrm{Inv}(f)$ the set of all sets

$$\mathfrak{D} \in \bigcup_{C \subseteq B} \mathscr{P}\big(\mathfrak{C}_r(C)\big)$$

such that $\mathfrak{D}$ is preserved by f[1].

Remark 4. For each local aggregation function f of r-preferences on a set B, the set $\mathrm{Inv}(f)$ contains the empty set, all sets $\mathfrak{C}_r(C)$, $C \subseteq B$, and all one-element sets $\mathfrak{D} \in \bigcup_{C \subseteq B} \mathscr{P}\left(\mathfrak{C}_r(C)\right)$. Moreover, the set $\mathrm{Inv}(f)$ is closed with respect to restrictions, i.e., if $\mathfrak{D} \in \mathrm{Inv}(f) \cap \mathscr{P}\left(\mathfrak{C}_r(C)\right)$ and $D \subseteq C$, then $\mathfrak{D}_{[D]} \in \mathrm{Inv}(f)$.

From now on, we are going to drop the subscript $[C]$ in the expression $f_{[C]}$, assuming that each local aggregation function of r-preferences on a set B automatically defines local aggregation functions of r-preferences on any set $C \subseteq B$.

For each set $\mathcal{F}$ of local aggregation functions, let

$$\mathrm{Inv}(\mathcal{F}) = \bigcap_{f \in \mathcal{F}} \mathrm{Inv}(f).$$

If $\mathfrak{D} \in \mathrm{Inv}(\mathcal{F})$, we say that the set $\mathcal{F}$ *preserves* the set $\mathfrak{D}$.

Definition 2. Any pair $(f, \mathcal{A})$, where f is a local aggregation function and $\mathcal{A}$ is an adaptation function of r-preferences on the set A, is called a *dynamic aggregation system* (over the set of r-preferences on the set A). The set of all dynamic aggregation systems over the set of r-preferences on the set A is denoted by $V_r(A)$.

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[1] In the theory of closed classes of discrete functions, the symbol $\mathrm{Inv}(f)$ denotes the set of predicates preserved by f. As shown in [3], the relation of a function f preserving a predicate P and the relation of a function f preserving a set of functions F are closely related notions (essentially, one is obtained from the other by translation into another language). Therefore, using the symbol $\mathrm{Inv}(f)$ in the new sense is unlikely to cause confusion.

A *lot* on a set A is defined as any subset of $\mathscr{P}(A)$ that is linearly ordered by inclusion and contains the sets $\varnothing$ and A. Obviously, any lot J allows for a (unique) natural enumeration $J = \{A_0, A_1, \ldots, A_m\}$, $m < \omega$, satisfying the condition

$$1 \le i < j \le m \Rightarrow A_i \subsetneq A_j$$

(therefore, $A_0 = \varnothing$ and $A_m = A$). In what follows, the expression

$$J = \{A_0, A_1, \ldots, A_m\}$$

presumes the above condition.

The set of all lots $J = \{A_0, A_1, \ldots, A_m\}$ on A with the condition $|A_1| \ge r$ is denoted by $\mathcal{J}_r(A)$. For each lot $J = \{A_0, A_1, \ldots, A_m\}$ and a permutation σ of the set A, the symbol J_σ stands for the lot $\{\sigma A_0, \sigma A_1, \ldots, \sigma A_m\}$. Obviously, if $J \in \mathcal{J}_r(A)$, then $\sigma J \in \mathcal{J}_r(A)$. A set $\mathcal{J} \subseteq \mathcal{J}(A)$ of lots on A is said to be *symmetric* if

$$J \in \mathcal{J} \Rightarrow J_\sigma \in \mathcal{J}$$

for any lot J on the set A.

The main notion of the present paper is that of a flow on a set of preferences $\mathfrak{D}$ with respect to a lot J (given a scheme of consecutive aggregation).

Definition 3. Given a dynamic aggregation system $S = (f, \mathcal{A}) \in V_r(A)$, a set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$, and a lot

$$J = \{A_0, A_1, \ldots, A_m\} \in \mathcal{J}_r(A),$$

the *flow* $\Pi_S(\mathfrak{D}, J)$ (from the set $\mathfrak{D}$ with respect to the lot J) is a pair

$$\big(\mathfrak{C}_S(\mathfrak{D}, J), \mathbb{F}_S(\mathfrak{D}, J)\big)$$

such that:

(1) $\mathfrak{C}_S(\mathfrak{D}, J) \subseteq \bigcup\limits_{B \in J} \mathfrak{C}_r(B)$, and the set $\mathfrak{C}_S(\mathfrak{D}, J) \cap \mathfrak{C}_r(A_0)$ is nonempty (i.e., $\mathfrak{C}_S(\mathfrak{D}, J) \cap \mathfrak{C}_r(A_0) \ne \{\varnothing\}$);

(2) $\mathbb{F}_S(\mathfrak{D}, J)$ coincides with the family $\{\mathfrak{D}_\mathfrak{c}\} \subseteq \bigcup\limits_{B \in J} \mathscr{P}\big(\mathfrak{C}_r(B)\big)$ indexed by functions $\mathfrak{c} \in \mathfrak{C}_S(\mathfrak{D}, J) \setminus \mathfrak{C}_r(A)$;

(3) for all k, $0 \le k \le m-1$, and $\mathfrak{c} \in \mathfrak{C}_S(\mathfrak{D}, J)$, we have:
   (a) $\mathfrak{c} \in \mathfrak{C}_r(A_k) \implies \mathfrak{D}_\mathfrak{c} = \mathcal{A}(\mathfrak{D}, \mathfrak{c})_{[A_{k+1}]}$,
   (b) $\mathfrak{c} \in \mathfrak{C}_r(A_{k+1}) \iff \mathfrak{c} \in f(\mathfrak{D}_\mathfrak{d})$ for some function $\mathfrak{d} \in \mathfrak{C}_S(\mathfrak{D}, J) \cap \mathfrak{C}_r(A_k)$.

Proposition 3. *Conditions* (1)–(4) *of Definition* 3 *specify a single flow, namely,*

$$\Pi_S(\mathfrak{D}, J) = \big(\mathfrak{C}_S(\mathfrak{D}, J), \mathbb{F}_S(\mathfrak{D}, J)\big).$$

In this situation:

(1) *the set $\mathfrak{C}_S(\mathfrak{D}, J)$ is closed with respect to restrictions to sets $B \in J$, i.e., for all $i, j \in \{0, 1, \ldots, m\}$ and $\mathfrak{c} \in \mathfrak{C}_S(\mathfrak{D}, J) \cap \mathfrak{C}_r(A_j)$, we have*

$$i \le j \implies \mathfrak{c}_{[A_i]} \in \mathfrak{C}_S(\mathfrak{D}, J);$$

(2) *for all i, $0 \le i \le m-1$, and $\mathfrak{c} \in \mathfrak{C}_S(\mathfrak{D}, J) \cap \mathfrak{C}_r(A_i)$, we have*

$$\mathfrak{D}_\mathfrak{c} \subseteq \mathfrak{C}_r(A_{i+1}) \quad \textit{and} \quad (\mathfrak{D}_\mathfrak{c})_{[A_i]} = \big(f(\mathfrak{D}_\mathfrak{c})\big)_{[A_i]} = \{\mathfrak{c}\};$$

(3) $\mathfrak{D}_\mathfrak{c} \cap \mathfrak{D}_\mathfrak{d} \ne \varnothing \iff f(\mathfrak{D}_\mathfrak{c}) \cap f(\mathfrak{D}_\mathfrak{d}) \ne \varnothing \iff \mathfrak{c} = \mathfrak{d}$ *for all* $\mathfrak{c}, \mathfrak{d} \in \mathfrak{C}_S(\mathfrak{D}, J) \setminus \mathfrak{C}_r(A)$,

Definition 4. A dynamic aggregation system $S = (f, \mathcal{A}) \in V_r(A)$ *preserves* a set of preferences $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ with respect to a lot $J \in \mathcal{J}_r(A)$ if:

(1) the adaptation function $\mathcal{A}$ preserves the set $\mathfrak{D}$;

(2) the aggregation function f preserves each set $\mathfrak{E} \in \mathbb{F}_S(\mathfrak{D}, J)$, i.e.,

$$\mathbb{F}_S(\mathfrak{D}, J) \subseteq \mathrm{Inv}(f).$$

A dynamic aggregation system S *preserves* a set of preferences $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ with respect to a set of lots $\mathcal{J} \subseteq \mathcal{J}_r(A)$ if it preserves the set $\mathfrak{D}$ with respect to each lot $J \in \mathcal{J}$.

Proposition 4. *If a dynamic aggregation system S* preserves *a set of preferences $\mathfrak{D}$ with respect to a lot $J = \{A_0, A_1, \ldots, A_m\}$, then the flow $\Pi_S(\mathfrak{D}, J) = \big(\mathfrak{C}_S(\mathfrak{D}, J), \mathbb{F}_S(\mathfrak{D}, J)\big)$ "becomes trivial":*

(1) $\mathfrak{C}_S(\mathfrak{D}, J) = \bigcup\limits_{k \le m} \mathfrak{D}_{[A_k]}$,

(2) *if* $\mathfrak{c} \in \mathfrak{D}_{[A_k]}$ *and* $k < m$, *then* $\mathfrak{D}_{\mathfrak{c}} = \{\mathfrak{d} \in \mathfrak{D}_{[A_{k+1}]} \colon \mathfrak{d}_{[A_k]} = \mathfrak{c}\}$.

Proof. This result is easily established by induction in i $(0 \le i < m)$. □

Proposition 5. *Given a lot $J = \{A_0, A_1, \ldots, A_m\} \in \mathcal{J}_r(A)$, let $\mathcal{A}_1$ and $\mathcal{A}_2$ be adaptation functions of r-preferences on the set A that preserve a set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$. Then for any local aggregation function f of r-preferences on A, the dynamic aggregation system $S_1 = (f, \mathcal{A}_1)$ preserves the set $\mathfrak{D}$ with respect to the lot J if and only if the dynamic aggregation system $S_2 = (f, \mathcal{A}_2)$ preserves the set $\mathfrak{D}$ with respect to the lot $\mathcal{J}$.*

Proof. Let system $S_1 = (f, \mathcal{A}_1)$ preserve $\mathfrak{D}$ with respect to J. For each $i \in \{1, 2\}$, consider the flows

$$\Pi_{S_i}(\mathfrak{D}, J) = \big(\mathfrak{C}_{S_i}(\mathfrak{D}, J), \mathbb{F}_{S_i}(\mathfrak{D}, J)\big).$$

Set

$$\mathbb{F}_{S_i}(\mathfrak{D}, J) = \{\mathfrak{D}^i_{\mathfrak{c}}\}, \quad \mathfrak{c} \in \mathfrak{C}_{S_i}(\mathfrak{D}, J) \setminus \mathfrak{C}_r(A).$$

Now, using Proposition 4 and induction in i $(0 \le i < m)$, one can easily prove conjunction of the following three statements:

(a) $\mathfrak{C}_{S_1}(\mathfrak{D}, J) \cap \mathfrak{C}_r(A_i) = \mathfrak{C}_{S_2}(\mathfrak{D}, J) \cap \mathfrak{C}_r(A_i)$,

(b) $\mathfrak{D}^1_{\mathfrak{c}} = \mathfrak{D}^2_{\mathfrak{c}}$ for any function $\mathfrak{c} \in \mathfrak{C}_{S_1}(\mathfrak{D}, J) \cap \mathfrak{C}_r(A_i)$,

(c) f preserves the set $\mathfrak{D}^2_{\mathfrak{c}}$ for each function $\mathfrak{c} \in \mathfrak{C}_{S_2}(\mathfrak{D}, J) \cap \mathfrak{C}_r(A_i)$.

Obviously, this suffices in order to prove Proposition 5. □

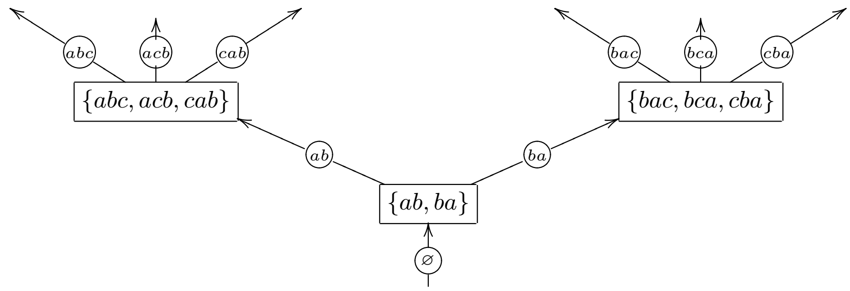
Definition 5. A local aggregation function f of r-preferences on a set A is said to *dynamically preserve* a set of preferences $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ with respect to a set of lots $\mathcal{J} \in \mathcal{J}_r(A)$ if the dynamic aggregation system $(f, \mathcal{A})$ preserves the set $\mathfrak{D}$ with respect to the set $\mathcal{J}$ for some (any) adaptation function $\mathcal{A}$ of r-preferences on the set A that preserves the set $\mathfrak{D}$. The set of all local aggregation functions of r-preferences on the set A that dynamically preserve the set $\mathfrak{D}$ with respect to the set $\mathcal{J}$ is denoted by $\mathrm{Pol}(\mathfrak{D}, \mathcal{J})$.

Example 2. It is convenient to represent the flow $\Pi_S(\mathfrak{D}, J) = \big(\mathfrak{C}_S(\mathfrak{D}, J), \mathbb{F}_S(\mathfrak{D}, J)\big)$ as a tree-shaped structure with a one-to-one correspondence between its vertices and sets $\mathfrak{E} \in \mathbb{F}_S(\mathfrak{D}, J)$ (with their names as labels of the vertices) and between its arcs and functions $\mathfrak{c} \in \mathfrak{C}_S(\mathfrak{D}, J)$ (whose names label the arcs). An arc $\mathfrak{c}$ issues from a vertex $\mathfrak{E}$ if $\mathfrak{c} \in f(\mathfrak{E})$ (according to statement (3) of Proposition 3, this set $\mathfrak{E}$ is uniquely defined) and enters the vertex $\mathfrak{D}_{\mathfrak{c}}$. The arc $\mathfrak{c} = \varnothing$ has no initial vertex and the arcs $\mathfrak{d} \in \mathfrak{C}_r(A)$ have no terminal vertices.

Let $A = \{a, b, c\}$ and let ∂ be the aggregation function according to the majority rule:

$$\hat{\partial}(x, y, z) = \begin{cases} x & \text{if } x = y, \\ z & \text{otherwise.} \end{cases}$$

The figure below represents the scheme for the flow of the function $\Pi_S(\mathfrak{D}, J)$ for the dynamic aggregation system $S = (\partial, \mathcal{A})$, where $\mathcal{A}$ is an arbitrary adaptation function from $\mathfrak{D} = \mathfrak{R}_2(\{a, b, c\})$ which preserves the set $\mathfrak{R}_2(A)$ with respect to the lot $J = \{\varnothing, \{a, b\}, \{a, b, c\}\}$. We make use of the trivial observation that a majority function dynamically preserves the set $\mathfrak{R}_2(\{a, b, c\})$ with respect to the lot J (and even with respect to the set of lots $\{J_\sigma \colon \sigma \text{ is a permutation of the set } A\}$). For brevity, any linear ordering $x_1 < x_2 < \cdots < x_k$ we write down as a sequence $x_1 x_2 \ldots x_k$.



Proposition 6. *Given a symmetric set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ and a symmetric set $\mathcal{J} \subseteq \mathcal{J}_r(A)$, let*

$$f\colon \big(\mathfrak{C}_r(A)\big)^n \to \mathfrak{C}_r(A)$$

be a local function preserving the set $\mathfrak{D}$ with respect to the set $\mathcal{J}$. Then the set $\bigcup\limits_{J\in\mathcal{J}} \mathbb{F}_S(\mathfrak{D}, J)$ is symmetric.

2. The Clone Method and the Main Structural Theorem

The clone method in the theory of collective choice was essentially introduced by S. Shalah in [1] and developed by the authors in [2, 7]. This method is based on the fact that the set of aggregation functions preserving a set (or a class of sets) of preferences is closed with respect to composition and contains all projections ("dictatorial" aggregation functions), i.e., is a clone. This observation allows the elaborate tools of the theory of closed classes of discrete functions to be used in the mathematical theory of collective choice.

Recall that a conservative n-place r-function on a set A is, by definition, any function

$$g\colon A^n_{\le r} \to A$$

satisfying the condition

$$(\forall\, (x_1, x_2, \dots, x_n) \in A^n_{\le r}) \bigvee_{1\le i\le n} (g(x_1, x_2, \dots, x_n) = x_i).$$

The set of all n-ary conservative r-functions on A is denoted by $\mathcal{K}^n_r(A)$. The union $\bigcup_{n<\omega} \mathcal{K}^n_r(A)$ is denoted by $\mathcal{K}_r(A)$.

For any integer i, $1 \le i \le n$, the function e^n_i defined by

$$e^n_i(x_1, x_2, \dots, x_n) = x_i$$

is called an (n-place i-th) *r-projection* on the set A. Obviously, $e^n_i \in \mathcal{K}^n_r(A)$. The aggregation function corresponding to the r-projection is called a *dictatorial aggregation function* (or *dictatorial aggregation rule*).

Let

$$f_1, f_2, \dots, f_n \in \mathcal{K}^m_r(A).$$

It is easy to check that for any positive integer n and any sequence $\mathbf{x}$ from the set $A^m_{\le r}$, the sequence

$$\big(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_n(\mathbf{x})\big)$$

belongs to the class $A^m_{\le r}$. Therefore, for any $f \in \mathcal{K}^n_r(A)$ and

$$f_1, f_2, \dots, f_n \in \mathcal{K}^m_r(A),$$

the relations

$$h(\mathbf{x}) = f\big(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_n(\mathbf{x})\big), \quad \mathbf{x} \in A^m_{\le r},$$

correctly define a function $h \in \mathcal{K}^m_r(A)$, which is called *composition* of the functions f and $f_1, f_2, \dots, f_n$ and is denoted by $f(f_1, f_2, \dots, f_n)$.

A conservative *r-clone* on the set A is, by definition, any set $\mathcal{F} \subseteq \mathcal{K}_r(A)$ that contains all projections and is closed with respect to composition.

For each r-function $f \in \mathcal{K}_r^n(A)$ and a permutation σ of the set A, by f_σ we denote a function on $A^n_{\leq r}$ defined by

$$f_\sigma(\mathbf{x}) = \sigma^{-1} f(\sigma \mathbf{x})$$

for all $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in A^n_{\leq r}$, where $\sigma x = \big(\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_n)\big)$. It is easy to check that f_σ is well-defined and belongs to the set $\mathcal{K}_r^n(A)$. A clone $\mathcal{F} \subseteq \mathcal{K}_r(A)$ is said to be *symmetric*, if

$$f \in \mathcal{F} \Rightarrow f_\sigma \in \mathcal{F}$$

for all functions $f \in K_r(A)$ and permutations σ of A.

Theorem 1. *Given a nonempty set A and an integer r, $1 \leq r \leq |A|$, let $\mathbb{D} \subseteq \bigcup\limits_{B \subseteq A} \mathcal{P}\big(\mathfrak{C}_r(B)\big)$. Then*:

(1) *the set $\mathcal{F}(\mathbb{D})$ of all conservative r-functions $f \in \mathcal{K}_r(A)$ preserving each set $\mathfrak{D} \in \mathbb{D}$ is a conservative r-clone*;
(2) *if $\mathbb{D}$ is symmetric, then the clone $\mathcal{F}(\mathbb{D})$ is symmetric.*

Corollary 1. *For each $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ and $\mathcal{J} \subseteq \mathcal{J}_r(A)$, the set $\mathcal{F} = \mathrm{Pol}(\mathfrak{D}, \mathcal{J})$ is a conservative r-clone. Moreover, if $\mathfrak{D}$ and $\mathcal{J}$ are symmetric sets, then the clone $\mathcal{F}$ is symmetric.*

Our next aim is to show that in the case of $r = 2$, for any symmetric sets $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ and $\mathcal{J} \subseteq \mathcal{J}_r(A)$, the clone $\mathcal{F} = \mathrm{Pol}(\mathfrak{D}, \mathcal{J})$ has a fairly simple structure. More precisely, we are going to show that from the standpoint of preservation of functions $\mathfrak{c} \in \mathfrak{C}_r(A)$, the set of all clones $\mathcal{F} = \mathrm{Pol}(\mathfrak{D}, \mathcal{J})$ (over all symmetric sets $\mathfrak{D}$ and $\mathcal{J}$) splits into finitely many classes, each admitting a fairly effective description. This description allows us to explicitly submit the clone $\mathrm{Pol}(\mathfrak{D}, \mathcal{J})$ in the case of $\mathfrak{D} = \mathfrak{R}_2(A)$. An exhaustive description of symmetric conservative clones and their invariant sets can be found in [7] and [3].

For any $B \subseteq A$, a set $\mathfrak{D} \subseteq \mathfrak{C}_r(B)$ is called *trivial* if there exist a set $Q \subseteq [B]^r$ and a function $\mathfrak{d}\colon Q \to B$ such that

$$\mathfrak{D} = \big\{\mathfrak{c} \in \mathfrak{C}_r(B) \colon \mathfrak{c} \restriction Q = \mathfrak{d}\big\}.$$

For instance, any one-element set $\mathfrak{D} \subseteq \mathfrak{C}_r(B)$ is trivial. Moreover, the empty set and $\mathfrak{C}_r(B)$ are trivial; apart from these two, no other set $\mathfrak{D} \subseteq \mathfrak{C}_r(B)$ can be both symmetric and trivial.

Proposition 7. *Any local aggregation function*

$$f \colon \big(\mathfrak{C}_r(A)\big)^n \to \mathfrak{C}_r(A)$$

preserves each trivial set

$$\mathfrak{D} \in \bigcup_{B \subseteq A} \mathscr{P}\big(\mathfrak{C}_r(B)\big).$$

A set $\mathbb{D} \subseteq \bigcup\limits_{B \subseteq A} \mathscr{P}\big(\mathfrak{C}_r(B)\big)$ is called *trivial*, if it consists of trivial sets $\mathfrak{D}$.

The majority function $\partial \in \mathcal{K}_2^3(A)$ on an arbitrary set A that has been defined in Example 2 can also be specified by the identity

$$\partial(x, x, y) = \partial(x, y, x) = \partial(y, x, x) = x.$$

Let us define a function $\ell \in \mathcal{K}_2^3(A)$ by the identity

$$\ell(x, x, y) = \ell(x, y, x) = \ell(y, x, x) = y.$$

In the case of $|A| = 3$, a 2-function $f \in \mathcal{K}_2(A)$ (of arbitrary arity) is called *even* if $f_\sigma = f$ for any even permutation σ of A.

In the case of $|A| = 4$, a 2-function $f \in \mathcal{K}_2(A)$ (of arbitrary arity) is called *Kleinian* if $f_\sigma = f$ for any permutation σ of A from the Klein four-group.

Example 3. The table below represents an even function $ev \in \mathcal{K}_2^2(\{a,b,c\})$ and a Klein function $kl \in \mathcal{K}_2^2(\{a,b,c,d\})$.

ev	a	b	c
a	a	b	a
b	b	b	c
c	a	c	c

kl	a	b	c	d
a	a	b	c	a
b	a	b	b	d
c	a	c	c	d
d	d	b	c	d

Theorem 2 (on the structure of conservative symmetric 2-clones with a finite support)**.** *Given a finite set A, $|A| \geq 2$, and an integer r, $2 \leq r \leq |A|$, let $\mathcal{F} \subseteq \mathcal{K}_2(A)$ be a symmetric conservative 2-clone. Then one of the following conditions holds.*

(1) *The clone $\mathcal{F}$ consists only of projections, and therefore, it preserves any set $\mathfrak{D} \subseteq \mathfrak{C}_r(B)$, $B \subseteq A$.*
(2) *The clone $\mathcal{F}$ is generated by the function ∂, and therefore, it preserves a set $\mathfrak{D} \subseteq \mathfrak{C}_r(B)$, $B \subseteq A$, if and only if this set is preserved by ∂.*
(3) *The clone $\mathcal{F}$ is generated by the function ℓ, and therefore, it preserves a set $\mathfrak{D} \subseteq \mathfrak{C}_r(B)$, $B \subseteq A$, if and only if this set is preserved by ℓ.*
(4) *The clone $\mathcal{F}$ is generated by the functions ∂ and ℓ, and therefore, it preserves a set $\mathfrak{D} \subseteq \mathfrak{C}_r(B)$, $B \subseteq A$, if and only if it is preserved by both ∂ and ℓ.*
(5) *The clone $\mathcal{F}$ preserves only trivial sets $\mathfrak{D} \subseteq \mathfrak{C}_r(B)$, $B \subseteq A$.*
(6) *$|A| = 3$ and the clone $\mathcal{F}$ contains the function ev.*
(7) *$|A| = 4$ and the clone $\mathcal{F}$ contains the function kl.*

Corollary 2 (reduction theorem for invariant sets of preferences)**.** *Given a set A, $|A| \geq 5$, and a nontrivial symmetric set*

$$\mathbb{D} \subseteq \bigcup_{B \subseteq A} \mathscr{P}\big(\mathfrak{C}_r(B)\big),$$

let $f\colon \big(\mathfrak{C}_2(A)\big)^n \to \mathfrak{C}_2(A)$ be a nondictatorial aggregation function and $\mathbb{D} \subseteq \operatorname{Inv}(f)$. Then

$$\mathbb{D} \subseteq \operatorname{Inv}(\partial) \quad \text{or} \quad \mathbb{D} \subseteq \operatorname{Inv}(\ell).$$

Remark 5. A more detailed analysis of the structure of conservative symmetric 2-clones[1] shows that each condition 5–7 of Theorem 2 holds for more than one conservative symmetric r-clone, in general. However, for the purposes of this paper it suffices to have the above slightly rougher classification.

Proof of Theorem 2. Let $\mathcal{F}$ be a conservative symmetric clone on A. First, we examine the set of binary 2-functions $f \in \mathcal{F}$.

For each pair of sequences $(\mathbf{a}, \mathbf{b}) \in A_2^2 \times A_2^2$, we define a sequence $\mathrm{t}(\mathbf{a}, \mathbf{b}) \in 2 \cup 2^2 \cup \{2\}$, called the *type* of pair $(\mathbf{a}, \mathbf{b})$. Let $\mathbf{a} = a_0 a_1$ and $\mathbf{b} = b_0 b_1$. Then, for all $i, j \in \{0,1\}$, set

$$\mathrm{t}(\mathbf{a}, \mathbf{b}) = \begin{cases} 0 & \text{if } a_0 = b_0 \text{ and } a_1 = b_1, \\ 1 & \text{if } a_0 = b_1 \text{ and } a_1 = b_0, \\ ij & \text{if } a_i = b_j \text{ and } a_{1-i} \neq b_{1-j}, \\ 2 & \text{if } \operatorname{ran} \mathbf{a} \cap \operatorname{ran} \mathbf{b} = \varnothing. \end{cases}$$

Obvioiyusly, the type is defined for each pair $(\mathbf{a}, \mathbf{b}) \in A_2^2 \times A_2^2$.

For each $i \in \{0,1\}$, define by $\rhd_i$ the binary relation on the set A_2^2 defined by

$$\mathbf{a} \rhd_i \mathbf{b} \iff \big(\big(\forall f \in \mathcal{F} \cap \mathcal{K}_2^2(A)\big)\, f(\mathbf{a}) = a_i \implies f(\mathbf{b}) = b_i\big)$$

for all $\mathbf{a} = a_0 a_1, \mathbf{b} = b_0 b_1 \in A_2^2$. For each sequence $\mathbf{a} = a_0 a_1 \in A_2^2$, denote by $\bar{\mathbf{a}}$ the sequence $a_1 a_0$.

[1]See [7].

Lemma 1. *The relation $\rhd_i$ is reflexive and transitive. Moreover, for each $i \in \{0,1\}$, sequences $\mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}' \in A^2_2$, and permutation σ of A, we have*

(1) $\mathbf{a} \rhd_i \mathbf{b} \implies \sigma\mathbf{a} \rhd_i \sigma\mathbf{b}$,
(2) $\mathbf{a} \rhd_i \mathbf{b} \implies \bar{\mathbf{a}} \rhd_{1-i} \bar{\mathbf{b}}$,
(3) $\mathbf{a} \rhd_i \mathbf{b} \implies \mathbf{b} \rhd_{1-i} \mathbf{a}$,
(4) $t(\mathbf{a}, \mathbf{b}) = t(\mathbf{a}', \mathbf{b}') \implies (\mathbf{a} \rhd_i \mathbf{b} \implies \mathbf{a}' \rhd_i \mathbf{b}')$.

Proof. It reduces to formal verification. Statement (4) follows directly from statement (1). □

Lemma 2. *The only possible cases are the following*:

(1) $(\forall\, i < 2)\, (\forall\, \mathbf{x}, \mathbf{y} \in A^2_2)\ \mathbf{x} \rhd_i \mathbf{y}$,
(2) $(\forall\, i < 2)\, (\forall\, \mathbf{x}, \mathbf{y} \in A^2_2)\ \mathbf{x} \rhd_i \mathbf{y} \leftrightarrow \mathrm{t}(\mathbf{x}, \mathbf{y}) = 0$,
(3) $(\forall i < 2)\, (\forall\, \mathbf{x}, \mathbf{y} \in A^2_2)\, \mathbf{x} \rhd_i \mathbf{y} \leftrightarrow \mathrm{t}(\mathbf{x}, \mathbf{y}) \in \{0, 1\}$,
(4) $|A| = 4 \wedge (\forall\, i < 2)\, (\forall\, \mathbf{x}, \mathbf{y} \in A^2_2)\, \mathbf{x} \rhd_i \mathbf{y} \leftrightarrow \mathrm{t}(\mathbf{x}, \mathbf{y}) \in \{0, 1, 2\}$,
(5) $|A| = 3 \wedge (\forall\, i < 2)\, (\forall\, \mathbf{x}, \mathbf{y} \in A^2_2)\, \mathbf{x} \rhd_i \mathbf{y} \leftrightarrow \mathrm{t}(\mathbf{x}, \mathbf{y}) \in \{0, 01, 10\}$.

Proof. Fix any i from $\{0, 1\}$. Suppose that the relation $\rhd_i$ contains a pair (by statement (4) of Lemma 1, any pair) $\mathbf{a}, \mathbf{b} \in A^2_2$ of type 00. Let $\mathbf{a} = a_0a_1$ and $\mathbf{b} = a_0b_1$. Then we have:

(a) $a_0a_1 \rhd_i a_0b_1$ (assumption),
(b) $a_0b_1 \rhd_{1-i} a_0a_1$ from (a) (by statement (2) of Lemma 1),
(c) $b_1a_0 \rhd_i a_1a_0$ from (b) (by statement (3) of Lemma 1),
(d) $a_0b_1 \rhd_i a_1b_1$ from (c) (by statement (4) of Lemma 1),
(e) $a_0a_1 \rhd_i a_1b_1$ from (a) and (c) (by transitivity),
(f) $a_1b_1 \rhd_i b_1a_0$ from (e) (by statement (4) of Lemma 1),
(g) $a_0a_1 \rhd_i b_1a_0$ from (a) and (f) (by transitivity),
(h) $a_0b_1 \rhd_i a_1a_0$ from (g) (by statement (4) of Lemma 1,
(i) $a_0a_1 \rhd_i a_1a_0$ from (a) and (h) (by transitivity).

Note that the sequences from statements (c), (e), (g), (i) are of the respective types 11, 10, 01, 1. Recalling that $\rhd_i$ is reflexive, and taking into account statement (4) of Lemma 1, we have

$$\mathbf{x} \rhd_i \mathbf{y} \text{ for all } (\mathbf{x}, \mathbf{y}) \text{ such that } t(\mathbf{x}, \mathbf{y}) \neq 2.$$

For $|A| = 3$, everything has been proved. For $|A| \geq 4$, we choose $c \in A \setminus \{a_0, a_1, b_1\}$ and go on as follows:

(j) $a_0b_1 \rhd_i b_1c$ from (e) (by statement (4) of Lemma 1),
(k) $a_0a_1 \rhd_i b_1c$ from (a) and (j) (by transitivity).

The sequences from statement (k) are of type 2. Thus, in the situation under consideration, case (1) is realized.

If relation $\rhd_i$ contains a pair of type 11, similar reasoning should be used. In what follows, we assume that none of the pairs of type 00 or 11 belong to the relation $\rhd_i$.

Suppose that $\rhd_i$ contains a pair (and therefore, any pair) $\mathbf{a}, \mathbf{b} \in A^2_2$ of type 01. Let $\mathbf{a} = a_0a_1$ and $\mathbf{b} = a_1b_1$. Then we have:

(a′) $a_0a_1 \rhd_i a_1b_1$ (assumption),
(b′) $a_1b_1 \rhd_i b_1a_0$ from (a') (by statement (4) of Lemma 1),
(c′) $a_0a_1 \rhd_i b_1a_0$ from (a') and (b') (by transitivity).

It follows that $\mathbf{x} \rhd_i \mathbf{y}$ for all pairs $(\mathbf{x}, \mathbf{y})$ of type 10.

If $\mathbf{x} \rhd_i \mathbf{y}$ for a pair $(\mathbf{x}, \mathbf{y})$ of type 1, then:

(d′) $a_1a_0 \rhd_i a_0a_1$ (assumption),
(e′) $a_1a_0 \rhd_i a_1b_1$ from (a') and (d') (by transitivity).

Therefore, there is a pair of type 00 that belongs to $\rhd_i$, which is a contradiction.

Thus, relation $\rhd_i$ contains all p[airs of types 0, 01, 10 and does not contain those of types 1, 00, 11. For $|A| = 3$, case (5) is realized. If $|A| \geq 4$, then we choose $c \in A \setminus \{a_0, a_1, b_1\}$ and go on as follows:

(f′) $a_1b_1 \rhd_i b_1c$ from (a') (by statement (4) of Lemma 1),
(g′) $a_0a_1 \rhd_i b_1c$ from (a') and (f') (by transitivity),
(h′) $b_1c \rhd_i a_1a_0$ from (g') (by statement (4) of Lemma 1),
(i′) $a_0a_1 \rhd_i a_1a_0$ from (a') and (h') (by transitivity).

The pair from statement (i′) is of type 1, which is a contradiction.

For a pair of type 10 belonging to the relation $\rhd_i$, the reasoning is similar. In what follows, assume that the relation $\rhd_i$ contains pairs of types 0, 1, 2 only.

Suppose that $\rhd_i$ contains a pair (and therefore, any pair) $\mathbf{a}, \mathbf{b} \in A_2^2$ of type 2. Let $\mathbf{a} = a_0a_1$ and $\mathbf{b} = b_0b_1$. Then:

(a″) $a_0a_1 \rhd_i b_0b_1$ (assumption),
(b″) $b_0b_1 \rhd_i a_1a_0$ from (a″) (by statement (4) of Lemma 1,
(c″) $a_0a_1 \rhd_i a_1a_0$ from (a″) and (b″) (by transitivity).

The pair from statement (c″) is of type 1. For $|A| = 4$, we have case (4). For $|A| \geq 5$, we take $c \in A \setminus \{a_0, a_1, b_0, b_1\}$ and go on as follows:

(d″) $b_0b_1 \rhd_i ca_0$ from (a″) (by statement (4) of Lemma 1),
(e″) $a_0a_1 \rhd_i ca_0$ from (a″) and (d″) (by transitivity).

The pair from statement (e″) is of type 01, which is a contradiction.

From now on, assume that relation $\rhd_i$ contains only pairs of types 0, 1. Hence, by statement (4) of Lemma 1, it follows that one of the cases (2) or (3) holds. □

Now, let us consider each of the cases (1)–(5) of Lemma 1. If case (2) or (3) takes place, then the 2-clone $\mathcal{F}$ satisfies the following condition:

Δ^2: For any pairs $\mathbf{a}, \mathbf{b} \in A_2^2$, $\operatorname{ran}\mathbf{a} \neq \operatorname{ran}\mathbf{b}$, and any elements $a \in \operatorname{ran}\mathbf{a}$ and $b \in \mathbf{b}$, there is a function $f \in \mathcal{F} \cap \mathcal{K}_2^2(A)$ such that

$$f(\mathbf{a}) = a,\ f(\mathbf{b}) = b \text{ and } f(x, x) = x \text{ for all } x \in A.$$

A function $f\colon A^n \to A$ satisfying the condition $f(x, x, \dots, x) = x$ is called *idempotent.*

Lemma 3. *Suppose that a conservative* 2*-clone* $\mathcal{F}$ *on a finite set* A *satisfies condition* Δ^2*. Then* $\mathcal{F}$ *preserves only trivial sets* $\mathfrak{D} \subseteq \mathfrak{C}_2(B)$*,* $B \subseteq A$ *(i.e., condition* (5) *of Theorem* 2 *holds).*

Proof. Let $B \subseteq A$ and suppose that a conservative 2-clone $\mathcal{F}$ satisfying condition Δ^2 preserves a nontrivial set $\mathfrak{D} \subseteq \mathfrak{C}_2(B)$. Let Q_0 be a maximal subset of $[B]^2$ with respect to inclusion and

$$\mathfrak{c} \restriction Q_0 = \mathfrak{d} \restriction Q_0$$

for all $\mathfrak{c}, \mathfrak{d} \in \mathfrak{D}$. Denote by $\mathfrak{c}_0$ the function $\mathfrak{c} \restriction Q_0$, where $\mathfrak{c}$ is an arbitrary function from $\mathfrak{D}$.

Note that the nontriviality of $\mathfrak{D}$ implies that the cardinality of $[B]^2 \setminus Q_0$ is not less than 2.

Arguing by induction with respect to the cardinality of Q, $Q_0 \subseteq Q \subseteq [B]^2$, let us show that for any Q, the set $\mathfrak{D}$ contains all functions $\mathfrak{c} \in \mathfrak{C}_2(B)$ such that

$$\mathfrak{c} \restriction Q_0 = \mathfrak{c}_0.$$

Thereby, we come to a contradition with the nontriviality of the set $\mathfrak{D}$ for $Q = [B]^2$.

The basis of induction ($Q = Q_0$) is obvious. Assuming that our statement holds for all sets Q, $Q_0 \subseteq Q \subseteq [B]^2$, of cardinality k, $|Q_0| \leq k < |[B]^2|$, let us prove it for an arbitrary Q', $Q_0 \subseteq Q' \subseteq [B]^2$, of cardinality $k+1$. Let $\mathfrak{d}$ be an arbitrary function from $\mathfrak{C}_2(A)$ such that $\mathfrak{d} \restriction Q_0 = \mathfrak{c}_0$. We have to show that the set $\mathfrak{D}$ contains a function $\mathfrak{d}'$ such that

$$\mathfrak{d}' \restriction Q' = \mathfrak{d} \restriction Q'.$$

If $|Q'| = 1$, then $Q_0 = \varnothing$, and our statement holds by the definition of Q_0. Assume that $|Q'| \geq 2$. Take arbitrary two-element sets $p, q \in Q'$ and let $\mathfrak{d}(p) = a$, $\mathfrak{d}(q) = b$. Let $\{c\} = p \setminus q$ and $\{d\} = q \setminus p$. Thus, $\{a, c\} \neq \{b, d\}$.

By the induction assumption, the set $\mathfrak{D}$ contains functions $\mathfrak{d}_{p,a}$, $\mathfrak{d}_{p,c}$, $\mathfrak{d}_{q,b}$ and $\mathfrak{d}_{q,d}$, which:

(1) coincide with $\mathfrak{d}$ on $Q \setminus \{p, q\}$;
(2) satisfy the relations $\mathfrak{d}_{p,a}(p) = a$, $\mathfrak{d}_{p,c}(p) = c$, $\mathfrak{d}_{q,b}(q) = b$, $\mathfrak{d}_{q,d}(q) = d$.

Condition Δ^2 ensures that the 2-clone $\mathcal{F}$ contains idempotent functions f_1, f_2, f_3 such that:

(1) $f_1\big(a, \mathfrak{d}_{q,d}(p)\big) = a$ and $f_1(\mathfrak{d}_{p,a}(q), d) = d$;
(2) $f_2\big(c, \mathfrak{d}_{q,b}(p)\big) = c$ and $f_2(\mathfrak{d}_{p,c}(q), b) = b$;
(3) $f_3(a, c) = a$ and $f_3(d, b) = b$.

Consider the function $\mathfrak{d}' = f_3\big(f_1(\mathfrak{d}_{p,a}, \mathfrak{d}_{q,d}), f_2(\mathfrak{d}_{p,c}, \mathfrak{d}_{q,b})\big)$. It belongs to $\mathfrak{D}$, since the 2-clone $\mathcal{F}$ preserves the set $\mathfrak{D}$. The function d' coincides with $\mathfrak{d}$ on $Q' \setminus \{p, q\}$, since f_1, f_2, f_3 are idempotent functions. Finally, we have

$$\mathfrak{d}'(p) = f_3\Big(f_1\big(a, \mathfrak{d}_{q,d}(p)\big),\ f_2\big(c, \mathfrak{d}_{q,b}(p)\big)\Big) = f_3(a, c) = a = \mathfrak{d}(p),$$
$$\mathfrak{d}'(q) = f_3\big(f_1(\mathfrak{d}_{p,a}(q), d), f_2(\mathfrak{d}_{p,c}(q), b)\big) = f_3(d, b) = b = \mathfrak{d}(q). \qquad \square$$

Suppose now that case (1) of Lemma 2 holds. Then all two-place functions $f \in \mathcal{F}$ are projections.

Lemma 4. *Let $\mathcal{F}$ be a conservative 2-clone on an arbitrary set A and let all two-place functions $f \in \mathcal{F}$ be projections. Then, for any sequence $\mathbf{a} = a_1 a_2 \ldots a_n \in A_2^n$ and a function $\sigma\colon A \to A$, we have*

$$f(\sigma\mathbf{a}) = \sigma\big(f(\mathbf{a})\big).$$

Proof. First, we note that the statement of this lemma holds for an arbitrary 2-projection f. Let $\mathbf{b} = (b_1, b_2)$ be a sequence of all distinct elements from $\operatorname{ran}\mathbf{a}$. Setting $\tau = \mathbf{b}^{-1}\mathbf{a}$, consider the function

$$f' = f\big(e^n_{\tau(1)}, e^n_{\tau(2)}, \ldots, e^n_{\tau(n)}\big).$$

Obviously, $f' \in \mathcal{F}$. Moreover, f' is a two-place function. By the assumption of this lemma, f' is a projection. Then we have

$$\sigma\big(f(\mathbf{a})\big) = \sigma\big(f'(\mathbf{b})\big) = f'(\sigma\mathbf{b}) = f\big(e^t_{\tau(1)}(\sigma\mathbf{b}), e^t_{\tau(2)}(\sigma\mathbf{b}), \ldots, e^t_{\tau(n)}(\sigma\mathbf{b})\big)$$
$$= f\Big(\sigma\big(b_{\tau(1)}\big),\ \sigma\big(b_{\tau(2)}\big), \ldots, \sigma\big(b_{\tau(n)}\big)\Big) = f(\sigma\mathbf{b}\tau) = f(\sigma\mathbf{a}). \quad \square$$

It is easy to check that for any $B \subseteq A$, the set

$$\mathcal{F}_B \rightleftharpoons \bigcup_{n<\omega} \{f \upharpoonright B^n \colon f \in \mathcal{F}\}$$

is a conservative 2-clone on B. Without loss of generality, we can assume that the set A contains $E_2 = \{0, 1\}$. Then the 2-clone $\mathcal{F}_{E_2}$ is a Post class contained in T_{01} (i.e., consisting of functions that preserve $\mathbf{0}$ and $\mathbf{1}$). Denote this Post class by P.

Consider a function $\sigma\colon A \to A$ such that $\sigma(0) = 1$, $\sigma(1) = 0$. From Lemma 4, it follows that each function $g \in \mathrm{P}$ is self-dual. Using the classification of Post classes (see [8–10]), we conclude that P coincides with one of the classes O_1, D_1, D_2, L_4, where O_1 is the class of all projections, D_2 is the class generated by the function ∂ on E_2, L_2 is the class generated by the function ℓ on E_2, and D_1 is the class generated by ∂ and ℓ on E_2.

Again making use of Lemma 4, we come to one of the cases (1)–(4) of Theorem 2.

It remains to consider cases (4) and (5) of Lemma (2).

Direct verification shows that in case (5), each two-place function $f \in \mathcal{F}$ is even, Moreover, each even two-place function $f \in \mathcal{F}$ is uniquely determined by its values on an arbitrary pair $(p, \bar{p}) \in A_2^2 \times A_2^2$. Therefore, there exist altogether four even two-place functions: two projections, ev, and ev_σ, where σ is an arbitrary transposition on the set A. Thus, if the 2-clone $\mathcal{F}$ consists not merely of projections, then condition (6) of Theorem 2 holds.

In a similar way, in case (4), it is easy to show that any two-place 2-function $f \in \mathcal{F}$ is a Klein function. Moreover, it is easy to check that for any two-place Klein 2-function f the following can be claimed: f is a projection or there exist three distinct elements $u, v, w \in A$ such that

$$f(u,v) = v,\ f(v,w) = w \text{ and } f(u,w) = u.$$

Then, for a suitable permutation σ of A, we have $f_\sigma = kl$, which implies condition (7) of Theorem 2. □

A set $\mathfrak{D} \subseteq \mathfrak{C}_2(A)$ is called *wide*, if for each $B \subsetneq A$ and $\mathfrak{d} \in \mathfrak{D}_{[B]}$, the set $\mathfrak{D}$ contains at least two distinct functions $\mathfrak{c}'$ and $\mathfrak{c}''$ such that $\mathfrak{c}'_{[B]} = \mathfrak{c}''_{[B]} = \mathfrak{d}$.

For instance, the set $\mathfrak{R}_2(A)$ is wide. As an example of a non-wide symmetric set $\mathfrak{D} \subseteq \mathfrak{C}_2(A)$ one can take the set of all functions $\mathfrak{c} \in \mathfrak{C}_2(A)$ "with a winner":

$$\mathfrak{c} \in \mathfrak{D} \iff (\exists\, a \in A)\big(\forall\, x \in A \setminus \{a\}\big)\, \mathfrak{c}\big(\{x,a\}\big) = a.$$

Corollary 3. *Given a nonempty nontrivial wide symmetric set of 2-preferences $\mathfrak{D} \subseteq \mathfrak{C}_2(A)$ and a nonempty symmetric set $\mathcal{J} \subseteq \mathcal{J}_2(A)$, the clone $\mathcal{F} = \operatorname{Pol}(\mathfrak{D}, \mathcal{J})$ satisfies at least one of the conditions* (1)–(4) *or* (6)–(7) *of Theorem* 2.

Proof. Let us choose a lot $J = \{A_0, A_1, \dots, A_m\} \in \mathcal{J}$, an adaptation function $\mathcal{A}$ preserving the set $\mathfrak{D}$, and an aggregation function $f \in \mathcal{F}$. Consider the flow $\Pi_S(\mathfrak{D}, J) = \big(\mathfrak{C}_S(\mathfrak{D}, J), \mathbb{F}_S(\mathfrak{D}, J)\big)$, where $S = (f, \mathcal{A})$. It suffices to show that under the conditions of Corollary 3, the set $\mathbb{F}_S(\mathfrak{D}, J)$ is nontrivial.

Let $\mathbb{F}_S(\mathfrak{D}, J) = \{\mathfrak{D}_\mathfrak{c}\}$, $\mathfrak{c} \in \mathfrak{C}_S(\mathfrak{D}, J) \setminus \mathfrak{C}_2(A)$. Suppose that the set $\mathbb{F}_S(\mathfrak{D}, J)$ is trivial. Using induction in i, $0 \le i \le m$, we are going to show that for any i:

(a) $\mathfrak{C}_S(\mathfrak{D}, J) \cap \mathfrak{C}_2(A_i) = \mathfrak{C}_2(A_i)$,

(b) if $i < m$ and $\mathfrak{c} \in \mathfrak{C}_S(\mathfrak{D}, J) \cap \mathfrak{C}_2(A_i)$, then

$$\mathfrak{D}_\mathfrak{c} = \{\mathfrak{d} \in \mathfrak{C}_2(A_{i+1})\colon \mathfrak{C}_{[A_i]} = \mathfrak{c}\}.$$

Indeed, a) $\mathfrak{C}_S(\mathfrak{D}, J) \cap \mathfrak{C}_2(A_0) = \{\varnothing\} = \mathfrak{C}_2(A_0)$ and b) $\mathfrak{D}_\varnothing = \mathfrak{D}_{[A_1]} = \mathfrak{C}_2(A_1)$, since the set $\mathfrak{D}$ is nonempty and symmetric. The basic induction assertion is proved. Suppose that the statement holds for $i = l < m$. Let $\mathfrak{c}$ be an arbitrary function from $\mathfrak{C}_2(A_{l+1})$. Set $\mathfrak{d} = \mathfrak{c}_{[A_l]}$. By the induction assumption, we have $\mathfrak{d} \in \mathfrak{C}_S(\mathfrak{D}, J)$ and $\mathfrak{D}_\mathfrak{d} = \{\mathfrak{e} \in \mathfrak{C}_2(A_{l+1})\colon \mathfrak{C}_{[A_l]} = \mathfrak{d}\}$. Therefore, $\mathfrak{c} \in \mathfrak{D}_\mathfrak{d} = f(\mathfrak{D}_\mathfrak{d}) \subseteq \mathfrak{C}_S(\mathfrak{D}, J) \cap \mathfrak{C}_2(A_{l+1})$.

Further, let $l + 1 < m$. Suppose that the set $\mathfrak{D}_\mathfrak{c}$ is trivial, i.e., there exist a set $Q \subseteq [A_{l+2}]^2$ and a function $\mathfrak{e}\colon Q \to A_{l+1}$ such that $\mathfrak{D}_\mathfrak{c} = \{\mathfrak{f} \in \mathfrak{C}_r(A_{l+2})\colon \mathfrak{f} \restriction Q = \mathfrak{e}\}$. On the other hand, by Proposition 4, we have $\mathfrak{D}_\mathfrak{c} = \{\mathfrak{f} \in \mathfrak{D}_{[A_{l+2}]}\colon \mathfrak{f}_{[A_{l+1}]} = \mathfrak{c}\}$. Since the set $\mathfrak{D}$ is wide, we have $Q = [A_{l+1}]^2$ and $\mathfrak{e} = \mathfrak{c}$. This proves the induction step assertion.

Now, since $\mathfrak{C}_S(\mathfrak{D}, J) \cap \mathfrak{C}_2(A) = \mathfrak{D}$ (see Proposition 4), we have $\mathfrak{D} = \mathfrak{C}_2(A)$, which is in contradition with the nontriviality of $\mathfrak{D}$. □

Now we are in a position to describe all aggregation functions of 2-preferences f that dynamically preserve the set $\mathfrak{R}_2(A)$ of rational 2-preferences with respect to an arbitrary symmetrical set of lots $\mathcal{J}$. The result obtained is in some sense opposite to the Arrow impossibility theorem and even the Condorcet theorem, since in essential cases, such functions f are precisely the functions generated by the majority function.

A lot $J = (A_0, A_1, \dots, A_m)$ is called *maximal* if $|A_1| = 2$ and $|A_{k+1} \setminus A_k| = 1$ for all k, $1 \le k \le m - 1$. By $\mathcal{J}_{\max}(A)$ we denote the set of all maximal lots on A. Obviously, the set $\mathcal{J}_{\max}$ is symmetric.

Theorem 3 (on the impossibility/possibility of rational preferences for dynamic aggregation systems)**.** *Given a finite set A, $|A| \ge 3$, and a nonempty symmetric set of lots $\mathcal{J} \subseteq \mathcal{J}_r(A)$, the following can be claimed*:

1. *if $\mathcal{J} \ne \mathcal{J}_{\max}(A)$, then there are no nondictatorial aggregation functions that dynamically preserve the set $\mathfrak{R}_2(A)$ with respect to $\mathcal{J}$*;
2. *if $\mathcal{J} = \mathcal{J}_{\max}(A)$, then for any local aggregation function of 2-preferences f the following statements are equivalent*:

(a) f *dynamically preserves the set* $\mathfrak{R}_2(A)$ *with respect to* $\mathcal{J}$,
(b) f *belongs to a* 2*-clone* $\mathcal{F} \subseteq \mathcal{K}_2(A)$ *generated by the majority function* ∂.

Proof. The first claim follows immediately from the Arrow theorem [11], since for each lot $J = \{A_0, A_1, \dots, A_n\}$ with $|A_{i+1} \setminus A_i| \geq 2$ for some i, $1 \leq i \leq m-1$, the condition $f \in \mathrm{Pol}(\mathfrak{R}_2(A), J)$ implies that f preserves the set of all rational 2-preferences on $(A_{i+1} \setminus A_i) \cup \{a\}$, where $a \in A_i$.

The second claim can be proved with the help of Corollary 3 by simple verification of the fact that any set $\{c \in \mathfrak{R}_2(B) \colon \mathfrak{c}_{[B]} = \mathfrak{d}\}$, where $C \subseteq B \subseteq A$, $|B \setminus C| = 1$ and $\mathfrak{d} \in \mathfrak{R}_2(C)$, is preserved by ∂ and not preserved by ℓ, ev, or kl. □

3. Nonlocal Aggregation Based on a Dynamical Affregation System

In the preceding section, we have focused on the possibility for a local aggregation function f to preserve a set of r-preferences $\mathfrak{D}$ under dynamic aggregation, assuming that on each step the profile of "voters" is formed anew. Now we are going to consider dynamic aggregation for which the dynamic profile of participants remains fixed. In contrast to the preceding section, here the role of the adaptation function $\mathcal{A}$ becomes essential.

Definition 6. Consider an adaptation function $\mathcal{A}$ of r-preferences on A, an aggregation function

$$f \colon \big(\mathfrak{C}_r(A)\big)^n \to \mathfrak{C}_r(A),$$

and a lot

$$J = \{A_0, A_1, \dots, A_m\} \in \mathcal{J}_r(A).$$

Let

$$\mathbf{c} = (\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_n) \in \big(\mathfrak{C}_r(A)\big)^n.$$

Define the sequences

$$(\mathfrak{d}^k)_{0 \leq k \leq m} \in \mathfrak{C}_r(A)_{[A_0]} \times \mathfrak{C}_r(A)_{[A_1]} \times \dots \times \mathfrak{C}_r(A)_{[A_m]},$$
$$(\mathbf{c}^k)_{1 \leq k \leq m} \in \big(\mathfrak{C}_r(A)_{[A_1]}\big)^n \times \big(\mathfrak{C}_r(A)_{[A_2]}\big)^n \times \dots \times \big(\mathfrak{C}_r(A)_{[A_m]}\big)^n$$

as follows:

(1) $\mathfrak{d}_k = f(\mathbf{c}_k)$ for all k, $1 \leq k \leq m$;
(2) $\mathbf{c}_{k+1} = \big(\mathcal{A}(\mathfrak{c}_1, \mathfrak{d}_k), \mathcal{A}(\mathfrak{c}_2, \mathfrak{d}_k), \dots, \mathcal{A}(\mathfrak{c}_n, \mathfrak{d}_k)\big)_{[A_{k+1}]}$ for all k, $1 \leq k \leq m-1$

(the sequences $(\mathfrak{d}^k)_{0 \leq k \leq m}$ and $(\mathbf{c}^k)_{1 \leq k \leq m}$ are defined uniquely, since $\mathfrak{d}_0 = \varnothing$).

The function $\mathfrak{d}_m$ is called the *result of consecutive aggregation according to the rule* f *on the* n*-tuple* $\mathbf{c}$ *with respect to the lot* J *and the adaptation function* $\mathcal{A}$. Fixing a lot J and an adaptation function $\mathcal{A}$, for each aggregation function f we define an aggregation function $f_{J,\mathcal{A}} \colon (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$ that maps each n-tuple $\mathbf{c} \in (\mathfrak{C}_r(A))^n$ to the result of consecutive aggregation according to the rule f on the n-tuple $\mathbf{c}$ with respect to the lot J and the adaptation function $\mathcal{A}$.

Proposition 8. *Suppose that a local aggregation function* $f\big(\mathfrak{C}_r(A)\big)^n \to \mathfrak{C}_r(A)$ *dynamically preserves a set* $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ *with respect to the set* $\mathcal{J} \subseteq \mathcal{J}_r(A)$. *Then, for each lot* $J \in \mathcal{J}$ *and an adaptation function* $\mathcal{A}$ *preserving the set* $\mathfrak{D}$, *the function* $f_{J,\mathcal{A}}$ *preserves the set* $\mathfrak{D}$. *In particular, for each* 2*-function* f *generated by the majority function* ∂, *a lot* $J \in \mathcal{J}_{\max}(A)$, *and an adaptation function* $\mathcal{A}$ *preserving the set* $\mathfrak{R}_2(A)$, *the function* $f_{J,\mathcal{A}}$ *preserves the set* $\mathfrak{R}_2(A)$.

Remark 6. The function $f_{J,\mathcal{A}}$ is nonlocal, in general.

A natural question arises: given a function $f \in \mathrm{Pol}(\mathfrak{D}, \mathcal{J})$, is it possible to choose an adaptation function $\mathcal{A}$ in such a way the function $f_{J,\mathcal{A}}$ would provide a "just result" for each n-tuple $\mathbf{c} \in \mathfrak{D}^n$? To some extent, the following theorem answers this question in the case of $\mathfrak{D} = \mathfrak{R}_2(A)$.

Each local aggregation 2-function $f\colon A^n_{\le 2} \to A$ is uniquely determined by the set of *local decisive coalitions* $C^f_{a,b} \subseteq \mathscr{P}(\{1,2,\dots,n\})$, $a, b \in A$, $a \neq b$:

$$f(\mathbf{a}) = a \iff \{i\colon a_i = a\} \in C^f_{\operatorname{ran} a}$$

for each sequence $\mathbf{a} = (a_1, a_2, \dots, a_n) \in A^n_2$ and an alternative $a \in \operatorname{ran} \mathbf{a}$. If coalitions $C^f_{a,b}$ are independent of the pair of alternatives (a, b), the function f is called *neutral*. For all neutral functions f, we write C^f instead of $C^f_{a,b}$. It is easy to check that all 2-functions f generated by the majority rule ∂ are neutral.

For each profile

$$\mathbf{c} = (\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_n) \in \big(\mathfrak{C}_2(A)\big)^n,$$

we define its *top element* with respect to a local neutral aggregation function f as an element $a \in A$ such that for any $x \in A \setminus \{a\}$, we have

$$\{i\colon \mathfrak{c}_i(\{x, a\}) = a\} \in C^f.$$

Theorem 4. *There is an adaptation function of* 2*-preferences* $\mathcal{A}$ *such that*:

(1) $\mathcal{A}$ *preserves the set* $\mathfrak{R}_2(A)$,

(2) *for any* 2*-function* f *generated by the majority rule* ∂*, a lot* $J \in \mathscr{J}_{\max}(A)$*, a profile*

$$\mathbf{c} = (\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_n) \in \big(\mathfrak{R}_2(A)\big)^n,$$

and an element $a \in A$*, it can be claimed that if* a *is a top element of the profile* $\mathbf{c}$ *with respect to* f*, then* a *is a maximal element with respect to the linear order corresponding to preferences* $f_{\mathcal{A},J}(\mathbf{c})$*.*

Proof. Instead of functions $\mathfrak{c} \in \mathfrak{R}_2(A)$, we will be dealing with the corresponding linear orders $P_\mathfrak{c}$. A linear order

$$x_1 < x_2 < \cdots < x_k$$

will be written as a sequence $x_1 x_2 \dots x_k$.

For each set $B \subseteq A$, element $a \in A \setminus B$, linear order $\prec_B$ on B and linear order $\prec_A$ on $B \cup \{a\}$, set:

(1) $T(\prec_B, \prec_A, a) = \big\{b \in B\colon (\exists\, c \in B)\ b \preceq_B c \prec_A a\big\}$,

(2) $\prec_B \oplus_a \prec_A = \prec_B \cup \{(x, a)\colon x \in T(\prec_B, \prec_A, a)\} \cup \{(a, x)\colon x \in B \setminus T(\prec_B, \prec_A, a)\}$

(here, $b \preceq_B c$ stands for $b \prec_B c \vee b = c$). Obviously, $\prec_B \oplus_a \prec_A$ is a linear order on the set $B \cup \{a\}$.

Further, let $\prec_{A \setminus B} = a_1 a_2 \dots a_k$ be the restriction of the linear order $\prec_A$ to the set $A \setminus B$. Set

(1) $\prec_0 = \prec_B$,

(2) $\prec_{i+1} = \prec_i \oplus_{a_{i+1}} \prec_A$ for all $i,\ 0 \le i \le k-1$,

(3) $\mathcal{A}(\prec_B, \prec_A) = \prec_k$

(here we identify rational preferences with the corresponding linear orders).

Let us extend the adaptation function $\mathcal{A}$ in an arbitrary way to the set of irrational preferences and show that it posesses the desired properties.

We use the notation from Definition 6. With the help of induction in k, $1 \le k \le m$, it is easy to show that if a is a top element of the profile $\mathbf{c}$ with respect to the function f and $a \in A_k$, then a is a maximal element with respect to the order corresponding to preferences $\mathfrak{d}_k$. The case $k = m$ proves Theorem 4. □

Remark 7. Theorem 4 implies that if the profile $\mathbf{c}$ has a Condorcet winner $a \in A^1$, then a is a maximal element of the linear order $P_\mathfrak{c}$, where $\mathfrak{c} = \partial^n_{\mathcal{A},J}(\mathbf{c})$, ∂^n is an n-place majority function (n is odd), J is an arbitrary maximal lot, and $\mathcal{A}$ is the function described in the proof of Theorem 4. Here, however, the relation

$$\partial^n(\mathbf{c}) = \partial^n_{\mathcal{A},J}(\mathbf{c})$$

may not hold.

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[1] See [12].

Example 4. For $A = \{a, b, c, d\}$, let the profile $\mathbf{c}$ consist of rational preferences $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3$ corresponding to the orders $P_{\mathfrak{c}_1} = cadb$, $P_{\mathfrak{c}_2} = bdac$, $P_{\mathfrak{c}_3} = dabc$. It is easy to check that the preferences $\mathfrak{d} = \partial(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3)$ are rational and correspond to the order $P_{\mathfrak{d}} = dabc$.

Consider the lot $J = \{A_0, A_1, A_2, A_3\}$ with $A_0 = \varnothing$, $A_1 = \{b, c\}$, $A_2 = \{a, b, c\}$, $A_3 = \{a, b, c, d\}$. The calculations from Definition 6 are represented in the table below (with rational aggregation functions being replaced by the corresponding linear orders).

k	A_k	$\mathcal{A}(\mathfrak{d}_k, \mathfrak{c}_1)$	$\mathcal{A}(\mathfrak{d}_k, \mathfrak{c}_2)$	$\mathcal{A}(\mathfrak{d}_k, \mathfrak{c}_3)$	$\mathfrak{c}_1^k$	$\mathfrak{c}_2^k$	$\mathfrak{c}_3^k$	$\mathfrak{d}^{k+1}$
0	$\varnothing$	$cadb$	$bdac$	$dabc$	cb	bc	bc	bc
1	$\{b, c\}$	$bcad$	$bdac$	$dabc$	bca	bac	abc	bac
2	$\{a, b, c\}$	$bacd$	$bdac$	$dbac$	$bacd$	$bdac$	$dbac$	$bdac$

Thus, the preferences $\partial_{\mathcal{A},J}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3)$ correspond to the order $bdac$, i.e., $\partial_{\mathcal{A},J}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3) \neq \partial(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3)$.

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