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Generalization of Cauchy invariants for equatorial β-plane flows

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ABSTRACT

Three Lagrangian invariants are shown to exist for flows in the equatorial region in the β - plane approximation. They extend the Cauchy invariants to a non-rotating fluid. The relationship between these generalized invariants and the results following from Kelvin's and Ertel's theorems is ascertained. Explicit expressions of the invariants for equatorially trapped waves and equatorial Gerstner waves are presented.

1. Introduction

Lagrangian variables are rarely used in geophysical hydrodynamics. There is even no formulation of equations in Lagrangian form in the classical book (Pedlosky, 1987). At the same time, quite recently Constantin (2012b) used the Lagrangian approach in the β-plane approximation and made an exact analytical description of equatorially trapped waves generalizing the classical Gerstner solution (Gerstner, 1809; Lamb, 1932; Constantin, 2011). That result stimulated an increasing interest in Lagrangian description of wave motions in an equatorial region (see Constantin, 2013, 2014; Constantin and Germain, 2013; Henry, 2013, 2016, 2017; Hsu, 2014; Ionescu-Kruse, 2016; Constantin and Monismith, 2017; Kluczek, 2017; Rodrguez-Sanjurjo, 2017 and references therein). In the present paper, I address Lagrangian invariants of such flows.

In a Lagrangian description, the coordinates X, Y, Z of a fluid particle are considered to be functions of their labels q, s, r. The continuity equation for an incompressible fluid is written in the form

$$\frac{D(X, Y, Z)}{D(q, s, r)} = \frac{D(X_0, Y_0, Z_0)}{D(q, s, r)} = S_0(q, s, r). \tag{1}$$

Here, S_0 is a time independent function. If we take as labels the original particle coordinates X_0, Y_0, Z_0, then $S_0 = 1$. In a general case, S_0 depends on coordinate labeling. According to the requirement of one-to-one mapping of the Euler and Lagrangian variables, it will not vanish to zero in the flow region. The function S_0 is an integral of motion or a Lagrangian invariant. It is present in the basic equations but its explicit form is found from the solution, which is a specific feature of Lagrangian description.

In 1815, Cauchy pointed to the existence of three more Lagrangian invariants related to the equations of fluid motion. They are currently referred to as the Cauchy invariants (Abrashkin et al., 1996; Zakharov and Kuznetsov, 1997; Bennett, 2006; Kuznetsov, 2006; Frisch and Villone, 2014; Besse and Frisch, 2017). The constancy of the Cauchy invariants is less known than the famous Kelvin's theorem about conservation of velocity circulation. However, both conservation laws have the same meaning. The difference between them is that Kelvin's theorem concerns conservation of the integral quantity, i.e. velocity circulation, whereas the Cauchy invariant is local but expressing the same constancy.

Cauchy invariants were rediscovered more than once. The background of the problem can be found in detail in (Frisch and Villone, 2014; see also Salmon, 1988). The conservation of these invariants is a consequence of special symmetry – the so-called relabeling symmetry (Salmon, 1988; Zakharov and Kuznetsov, 1997; Bennett, 2006) implying a change of labels q, s, r for each fluid particle. It is apparent, however, that no change of labels can affect the dynamics of a fluid. The role of such a symmetry was first realized in the works of Eckart (1938, 1960) and Newcomb (1967) and was presented in detail by Salmon (1988). Its consequences are the Thomson (Kelvin) theorem about the conservation of circulation, the frozen vortex theorem, and the Ertel theorem.

The goal of the present paper is to find generalizations of the Cauchy invariants for a nonuniformly rotating fluid in an equatorial plane. The paper is organized as follows. In Section 2 equations in Lagrangian form are derived and explicit expressions for generalized Cauchy invariants are found. Their relationship with Kelvin's and Ertel's theorems is considered in Sections 3 and 4, respectively. The form of the invariants for equatorially trapped and Gerstner waves is found in the concluding section.

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2. Governing Lagrangian equations and their invariants

Choose a rotating framework with the origin at a point on the Earth's surface. Let X, Y, Z be Cartesian coordinates, with the spatial variable X corresponding to longitude, the variable Y to latitude, and the variable Z to the local vertical, respectively. The momentum equations for the β-plane approximation in the equatorial region have the following form (Pedlosky, 1987; Constantin, 2012a, 2012b):

$$\frac{du}{dt}+2\Omega w-\beta Yv=-\frac{1}{\rho}p_X, \tag{2}$$

$$\frac{dv}{dt}+\beta Yu=-\frac{1}{\rho}p_Y, \tag{3}$$

$$\frac{dw}{dt}-2\Omega u=-\frac{1}{\rho}p_Z-g, \tag{4}$$

where t is time, Ω is the speed of eastward rotation of the Earth (taken to be a sphere of radius R) round the polar axis, $\beta=2\Omega/R$, g is gravitational acceleration, ρ is water density, p is pressure, and u, v, w are fluid velocity components.

We use the Lagrangian framework for the description of the flow. The equations of motion (2)–(4) in Lagrangian variables are written as

$$X_{tt}X_q+Y_{tt}Y_q+Z_{tt}Z_q+(2\Omega Z_t-\beta YY_t)X_q+\beta YX_tY_q-2\Omega X_tZ_q=-\frac{p_q}{\rho}-gZ_q,$$
$$X_{tt}X_s+Y_{tt}Y_s+Z_{tt}Z_s+(2\Omega Z_t-\beta YY_t)X_s+\beta YX_tY_s-2\Omega X_tZ_s=-\frac{p_s}{\rho}-gZ_s,$$
$$X_{tt}X_r+Y_{tt}Y_r+Z_{tt}Z_r+(2\Omega Z_t-\beta YY_t)X_r+\beta YX_tY_r-2\Omega X_tZ_r=-\frac{p_r}{\rho}-gZ_r.$$

Here, we use $u=X_t, v=Y_t, w=Z_t$. By cross differentiation of this system it is possible to exclude the pressure:

$$\frac{\partial}{\partial t}\left[\frac{D(X_t,X)}{D(s,r)}+\frac{D(Y_t,Y)}{D(s,r)}+\frac{D(Z_t,Z)}{D(s,r)}+2\Omega\frac{D(Z,X)}{D(s,r)}+\beta Y\frac{D(X,Y)}{D(s,r)}\right]=0,$$
$$\frac{\partial}{\partial t}\left[\frac{D(X_t,X)}{D(r,q)}+\frac{D(Y_t,Y)}{D(r,q)}+\frac{D(Z_t,Z)}{D(r,q)}+2\Omega\frac{D(Z,X)}{D(r,q)}+\beta Y\frac{D(X,Y)}{D(r,q)}\right]=0,$$
$$\frac{\partial}{\partial t}\left[\frac{D(X_t,X)}{D(q,s)}+\frac{D(Y_t,Y)}{D(q,s)}+\frac{D(Z_t,Z)}{D(q,s)}+2\Omega\frac{D(Z,X)}{D(q,s)}+\beta Y\frac{D(X,Y)}{D(q,s)}\right]=0.$$

Thus, we have three integrals of motion

$$\frac{D(X_t,X)}{D(s,r)}+\frac{D(Y_t,Y)}{D(s,r)}+\frac{D(Z_t,Z)}{D(s,r)}+2\Omega\frac{D(Z,X)}{D(s,r)}+\beta Y\frac{D(X,Y)}{D(s,r)}=S_1(q,s,r), \tag{5}$$

$$\frac{D(X_t,X)}{D(r,q)}+\frac{D(Y_t,Y)}{D(r,q)}+\frac{D(Z_t,Z)}{D(r,q)}+2\Omega\frac{D(Z,X)}{D(r,q)}+\beta Y\frac{D(X,Y)}{D(r,q)}=S_2(q,s,r), \tag{6}$$

$$\frac{D(X_t,X)}{D(q,s)}+\frac{D(Y_t,Y)}{D(q,s)}+\frac{D(Z_t,Z)}{D(q,s)}+2\Omega\frac{D(Z,X)}{D(q,s)}+\beta Y\frac{D(X,Y)}{D(q,s)}=S_3(q,s,r). \tag{7}$$

Direct differentiation of Eqs. (5)–(7) demonstrates that

$$\frac{\partial S_1}{\partial q}+\frac{\partial S_2}{\partial s}+\frac{\partial S_3}{\partial r}=0,$$

i.e. divergence of the vector $\vec{S}\{S_1,S_2,S_3\}$ in Lagrangian coordinates is zero. This means that an expression for invariants cannot be specified arbitrarily.

If there is no rotation ($\Omega=0;\beta=0$), then the system reduces to the following set of equations

$$\frac{D(X_t,X)}{D(s,r)}+\frac{D(Y_t,Y)}{D(s,r)}+\frac{D(Z_t,Z)}{D(s,r)}=S_{10}(q,s,r),$$
$$\frac{D(X_t,X)}{D(r,q)}+\frac{D(Y_t,Y)}{D(r,q)}+\frac{D(Z_t,Z)}{D(r,q)}=S_{20}(q,s,r),$$
$$\frac{D(X_t,X)}{D(q,s)}+\frac{D(Y_t,Y)}{D(q,s)}+\frac{D(Z_t,Z)}{D(q,s)}=S_{30}(q,s,r).$$

The subscript “0” indicates that the invariants are calculated in a non-rotating fluid. Those expressions were discovered by Cauchy, and the functions S_{10}, S_{20}, S_{30} are referred to as the Cauchy invariants (Abrashkin et al., 1996; Zakharov and Kuznetsov, 1997; Bennett, 2006; Kuznetsov, 2006; Frisch and Villone, 2014; Besse and Frisch, 2017).

3. The physical meaning of the invariants

Consider the generalized Cauchy invariants corresponding to $\Omega\neq 0$, $\beta\neq 0$. Let us replace in Eqs. (5)–(7) the derivatives of X_t, Y_t, Z_t with respect to q, s, r by the derivatives with respect to X, Y, Z:

$$\xi\frac{D(Y,Z)}{D(s,r)}+(\eta+2\Omega)\frac{D(Z,X)}{D(s,r)}+(\zeta+\beta Y)\frac{D(X,Y)}{D(s,r)}=S_1, \tag{8}$$

$$\xi\frac{D(Y,Z)}{D(r,q)}+(\eta+2\Omega)\frac{D(Z,X)}{D(r,q)}+(\zeta+\beta Y)\frac{D(X,Y)}{D(r,q)}=S_2, \tag{9}$$

$$\xi\frac{D(Y,Z)}{D(q,s)}+(\eta+2\Omega)\frac{D(Z,X)}{D(q,s)}+(\zeta+\beta Y)\frac{D(X,Y)}{D(q,s)}=S_3, \tag{10}$$

where $\vec{\omega}=\vec{\omega}(\xi,\eta,\zeta)$ is the vorticity vector with the components

$$\xi=w_Y-v_Z,\quad \eta=u_Z-w_X,\quad \zeta=v_X-u_Y.$$

Let us take as an example a closed curve bounding in the sr plane a rectangle having sides δs and δr at the moment of time $t=0$ and designate by A, B, C the areas of its projections onto the coordinate planes at the time moment t:

$$A=|d\vec{Y}\times d\vec{Z}|=\frac{D(Y,Z)}{D(s,r)}\delta s\delta r,$$
$$B=|d\vec{Z}\times d\vec{X}|=\frac{D(Z,X)}{D(s,r)}\delta s\delta r,$$
$$C=|d\vec{X}\times d\vec{Y}|=\frac{D(X,Y)}{D(s,r)}\delta s\delta r.$$

Then Eq. (8) may be written in the form

$$\xi A+(\eta+2\Omega)B+(\zeta+\beta Y)C=S_1\delta s\delta r.$$

This expression describes constancy of the vector flow $\vec{\omega}+\vec{\omega}_*$ (where $\vec{\omega}_*=\vec{\omega}_*(0{,}2\Omega,\beta Y)$ through an infinitely small moving contour formed by $\delta s\delta r$ particles. Analogously, we can show that Eq. (9) defines the condition of a constant flow through contour $\delta r\delta q$, and Eq. (10) through contour $\delta q\delta s$. The combination of these results leads us to the conclusion that the vector flow $\vec{\omega}+\vec{\omega}_*$ around an arbitrary, moving closed contour persists to be constant. Or, according to the Stokes formula, circulation around a closed contour is retained, which is the same.

Our considerations may be reversed by deriving from the condition of time-independent functions S_1, S_2, S_3 Kelvin's theorem about circulation around a closed contour. Indeed, let us choose Lagrangian variables so that the particles forming the fluid contour should lie, for example, in the plane of s, r variables. Then, by virtue of the invariance of S_1, the circulation speed around it will persist to be constant.

The same result may be obtained using Euler variables. Eqs. (2)–(4) in vector form are written as

$$\frac{\partial\vec{V}}{\partial t}+(\vec{V}\nabla)\vec{V}+\vec{\omega}_*\times\vec{V}=-\frac{\nabla p}{\rho},$$

where $\vec{V}=\vec{V}(u,v,w)$ is velocity. By applying the curl operator and making simple transformations we obtain

$$\frac{d(\vec{\omega} + \vec{\omega}_*)}{dt} = ((\vec{\omega} + \vec{\omega}_*)\nabla)\vec{V}. \tag{11}$$

Eq. (11) is a generalization of the Helmholtz equation to the case of fluid motion in an equatorial region. Instead of the flow vorticity vector, (11) contains an absolute vortex, that is a sum of vorticity and vector $\vec{\omega}_*$ related to the action of the Coriolis force. Expression (11) represents the condition of absolute vortex freezing into a moving fluid, from which follows that the frozen-in vortex flow remains constant (Lamb, 1932; Kochin et al., 1964).

Thus, an absolute vortex in a nonuniformly rotating fluid is analogous to vorticity, and the physical meaning of the generalized invariants S_1, S_2, S_3 is identical to that for the Cauchy invariants.

4. The relation to Ertel's theorem

Consider labeling variables as functions of Euler coordinates:

$$q = q(X, Y, Z), \quad s = s(X, Y, Z), \quad r = r(X, Y, Z).$$

By solving the system of equations

$$q_X X_q + q_Y Y_q + q_Z Z_q = 1,$$
$$q_X X_s + q_Y Y_s + q_Z Z_s = 0,$$
$$q_X X_r + q_Y Y_r + q_Z Z_r = 0,$$

we find

$$q_X = S_0^{-1}\frac{D(Y, Z)}{D(s, r)}, \quad q_Y = S_0^{-1}\frac{D(Z, X)}{D(s, r)}, \quad q_Z = S_0^{-1}\frac{D(X, Y)}{D(s, r)} \quad .$$

here S_0 is the Jacobian (1) value. With these relations taken into consideration, expression (8) is written as

$$S_1 = S_0((\vec{\omega} + \vec{\omega}_*)\nabla q). \tag{12}$$

Two other invariants are represented analogously:

$$S_2 = S_0((\vec{\omega} + \vec{\omega}_*)\nabla s), \quad S_3 = S_0((\vec{\omega} + \vec{\omega}_*)\nabla r). \tag{13}$$

The labels q, s, r of a single fluid particle do not change in the course of its motion; therefore, the structure of Eqs. (12), (13) is analogous to Ertel's theorem for potential vorticity (Pedlosky, 1987; Truesdell, 1951; Viudez, 1999). However, in our formulas we use an absolute vortex instead of conventional vorticity.

5. Application to exact solutions

The results of the general theory will be tested on an example of the known exact solutions for equatorial waves. Suppose that a fluid particle is moving in the $Y = s$ plane. Then, the equations of hydrodynamics (1), (5)–(7) may be written as

$$\frac{\partial}{\partial t}\frac{D(X, Z)}{D(q, r)} = 0, \tag{14}$$

$$\frac{\partial}{\partial t}\left[\frac{D(X_t, X)}{D(r, q)} + \frac{D(Z_t, Z)}{D(r, q)} + 2\Omega\frac{D(Z, X)}{D(r, q)}\right] = 0, \tag{15}$$

$$\frac{\partial}{\partial t}\left[\frac{D(X_t, X)}{D(s, r)} + \frac{D(Z_t, Z)}{D(s, r)} + 2\Omega\frac{D(Z, X)}{D(s, r)} - \beta s X_r\right] = 0, \tag{16}$$

$$\frac{\partial}{\partial t}\left[\frac{D(X_t, X)}{D(q, s)} + \frac{D(Z_t, Z)}{D(q, s)} + 2\Omega\frac{D(Z, X)}{D(q, s)} + \beta s X_q\right] = 0. \tag{17}$$

The first of them is the continuity equation. For the sake of simplicity, the order of following of the equations of motion are changed. Eqs. (14), (15) do not contain differentiation with respect to s. The third term in Eq. (15) coincides, to an accuracy of the constant multiplier, with Eq. (14) and may be omitted. Then the system of two Eqs. (14), (15) will coincide with the equations of plane motion of a perfect incompressible fluid, but the functions X, Z will still depend on the third coordinate s.

Constantin (2012b) found an exact solution to the system of Eqs. (14)–(17):

$$X = q - \frac{1}{k}e^{\chi}\sin\theta, \quad Y = s, \quad Z = r + \frac{1}{k}e^{\chi}\cos\theta, \tag{18}$$

$$\chi = [r - f(s)], \quad \theta = k(q - ct), \quad f(s) = \frac{\beta}{2(kc+2\Omega)}s^2, \quad c = \frac{\sqrt{\Omega^2 + kg} - \Omega}{k}.$$

This solution describes equatorial surface waves traveling eastward with speed c; in (18) k is wave number. They are periodic spatial waves whose amplitude decreases exponentially in meridional direction. That is why they are called trapped waves.

The expressions for generalized invariants of the Cauchy waves (18) have the following form

$$S_1 = 0,$$
$$S_2 = 2\Omega - 2(kc + \Omega)e^{2\chi}, \tag{19}$$
$$S_3 = \beta s\left[1 - \frac{2(kc + \Omega)}{kc+2\Omega}e^{2\chi}\right].$$

The zonal component of vector $\vec{S}$ is equal to zero. The vorticity for the waves (18) in this case is defined by (Constantin, 2012b):

$$\vec{\omega} = S_0^{-1}\{-skc^2g^{-1}\beta e^{\chi}\sin\theta,\ -2kce^{2\chi},\ skc^2g^{-1}\beta(e^{\chi}\cos\theta - e^{2\chi})\},$$
$$S_0 = 1 - e^{2\chi}. \tag{20}$$

All of its three components are non-zero, and the zonal and vertical components depend on time. Comparison of expressions (19) and (20) demonstrates the difference between the vorticity vector (vector of absolute vorticity $\vec{\omega}_*$) and the vector of Lagrangian invariants.

Expression (18) is a unique example of an exact solution of equations of fluid motion in an equatorial region. This solution permits generalization to the case of a uniform zonal near-surface flow (Henry, 2013, 2016) and may be used for describing not only surface gravity waves but also internal waves affecting thermocline dynamics (Constantin, 2013; Constantin and Germain, 2013; Constantin, 2014).

The case $\beta = 0$ corresponds to the f-plane approximation and Eqs. (14), (15) are equivalent to the equations of two-dimensional hydrodynamics. In this case, solution (18) describes equatorial Gerstner waves that differ from waves in a fluid at rest only by the form of dispersion relation, hence by the propagation velocity (see the last equation in (18)). The generalized Cauchy invariants for Gerstner waves have the form

$$S_1 = S_3 = 0, \quad S_2 = 2\Omega - 2(kc + \Omega)e^{2\chi},$$

and the vorticity components are defined by

$$\omega_X = \omega_Z = 0, \quad \omega_Y = -\frac{2kce^{2\chi}}{1 - e^{2\chi}}.$$

Comparisons of these relations shows that for plane waves the vector of the generalized invariants and the vorticity vector are qualitatively similar: their zonal and vertical components are equal to zero, and the meridional components are time independent.

The solution for equatorial Gerstner waves may be extended to the case of uniform zonal and arbitrary meridional current (Kluczek, 2017), as well as to the case of a two-layer fluid (Hsu, 2014; Rodrguez-Sanjurjo, 2017). It is worthy of note that different generalizations of Gerstner's solution are also used in the studies of zonal waves in middle and higher geographical latitudes (Constantin and Monismith, 2017; Vitek, 1969, 1993).

Lagrangian invariants in the approximate model of the atmospheric front were used by Vitek (1993) for assessing its characteristics. This example is a good reason to believe that the obtained theoretical result will find practical application.

6. Conclusions

A system of equations in Lagrangian form has been derived for equatorial flows in the β-plane approximation. It has been shown that the system has three integrals of motion referred to as the generalized Cauchy invariants. The properties of the invariants have been studied on an example of known exact solutions for equatorial waves.

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