

On the Canonical Embeddings of Certain Homogeneous Spaces

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To A. L. Onishchik on his 70th birthday.

ABSTRACT. We study equivariant affine embeddings of homogeneous spaces and their equivariant automorphisms. We prove the solvability of any connected group of equivariant automorphisms for an affine embedding with a fixed point and finitely many orbits. This is applied to studying the orbital decomposition for algebraic monoids and canonical embeddings of quasiaffine homogeneous spaces, i.e., those affine embeddings associated with the coordinate algebras of homogeneous spaces, provided the latter algebras are finitely generated. We pay special attention to the canonical embeddings of quotient spaces of reductive groups modulo the unipotent radicals of parabolic subgroups. For these varieties, we describe the orbital decomposition, compute the modality of the group action, and find out which of them are smooth. We also describe minimal ambient modules for these canonical embeddings provided that the acting group is simply connected.

1. Introduction

Let G be a connected reductive algebraic group over an algebraically closed field \mathbb{k} , and H a closed subgroup of G . It was proved by Matsushima [Ma60] and Onishchik [On60] that the homogeneous space G/H is affine if and only if H is reductive. (For a simple proof, see [Lu73, §2]; a characteristic-free proof is given in [Ri77].) The subgroup H is said to be *observable* in G if the homogeneous space G/H is a quasiaffine variety. For a description of observable subgroups; see [Gr97], [Su88]. In particular, any reductive subgroup is observable.

Let us recall that H is a *Grosshans subgroup* in G if H is observable and the algebra of regular functions $\mathbb{k}[G/H]$ is finitely generated. This class of subgroups was considered by Grosshans [Gr73], [Gr83], [Gr97] in connection with the Hilbert 14th problem. In particular, it is proved in [Gr83] that the unipotent radical P_u of a parabolic subgroup P of G is a Grosshans subgroup.

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Let H be a Grosshans subgroup in G . The *canonical embedding* of the homogeneous space G/H is the affine G -variety $\text{CE}(G/H) = \text{Spec } \mathbb{k}[G/H]$ corresponding to the affine algebra $\mathbb{k}[G/H]$. It is easy to see that $\text{CE}(G/H)$ is a normal affine variety with an open G -orbit isomorphic to G/H , and the complement of the open orbit has codimension ≥ 2 . Moreover, these properties characterize $\text{CE}(G/H)$ up to G -equivariant isomorphism (for more details see [Gr97]).

The goal of this paper is to study the canonical embeddings of the homogeneous spaces G/P_u . Such embeddings form a remarkable class of affine quasihomogeneous G -varieties. They provide a geometric point of view at the properties of the algebra $\mathbb{k}[G]^{P_u}$.

We begin with the following general result on equivariant automorphisms of an affine embedding $G/H \hookrightarrow X$: if X contains a G -fixed point and only finitely many G -orbits, then the connected part of the group $\text{Aut}_G(X)$ is solvable (Theorem 1). It is easy to prove that the group of equivariant automorphisms of $\text{CE}(G/H)$ is naturally isomorphic to $N_G(H)/H$. We deduce that the number of G -orbits in $\text{CE}(G/P_u)$ is infinite, except the trivial cases (Proposition 2).

A detailed description of $\text{CE}(G/P_u)$ is obtained in Section 3 under the assumption $\text{char } \mathbb{k} = 0$.

In fact, our approach works for any affine embedding $G/P_u \hookrightarrow X$ with the maximal possible group of G -equivariant automorphisms (equal to the Levi subgroup L of P). Such affine embeddings X are classified by finitely generated semigroups S of G -dominant weights having the property that all highest weights of tensor products of simple L -modules with highest weights in S belong to S , too. Furthermore, every choice of the generators $\lambda_1, \dots, \lambda_m \in S$ gives rise to a natural G -equivariant embedding $X \hookrightarrow \text{Hom}(V^{P_u}, V)$, where V is the sum of simple G -modules of highest weights $\lambda_1, \dots, \lambda_m$; see Theorem 2. The convex cone Σ^+ spanned by S is nothing else but the dominant part of the cone Σ spanned by the weight polytope of V^{P_u} ; see 3.2.

We prove that the $(G \times L)$ -orbits in X are in bijection with those faces of Σ whose interiors contain dominant weights, the orbit representatives being given by the projectors onto the subspaces of V^{P_u} spanned by eigenvectors with eigenweights in a given face (Theorem 3). Also we compute the stabilizers of these points in $G \times L$ and in G , and the modality of the action $G : X$. Smooth embeddings are classified by Theorem 4.

These results are applied to canonical embeddings $X = \text{CE}(G/P_u)$ as follows. The semigroup S here consists of all dominant weights, and Σ is the span of the dominant Weyl chamber by the Weyl group of L . From Theorem 3 we deduce that $(G \times L)$ -orbits in X are in bijection with the subdiagrams in the Dynkin diagram of G such that no connected component of such a subdiagram is contained in the Dynkin diagram of L . In terms of these diagrams, we compute the stabilizers and the modality of $G : X$; see Corollary 2. From Theorem 4, a classification of smooth canonical embeddings stems (Corollary 3).

The techniques used in the description of affine $(G \times L)$ -embeddings of G/P_u are parallel to those developed in [Ti03] for the study of equivariant compactifications of reductive groups. This analogy becomes more transparent in view of the bijection between these affine embeddings $G/P_u \hookrightarrow X$ and algebraic monoids M with the group of invertibles L , given by $X = \text{Spec } \mathbb{k}[G \times^P M]$ (Proposition 7).

Finally, we describe the G -module structure on the tangent space of $\text{CE}(G/P_u)$ at the G -fixed point, assuming that G is simply connected and simple. This space is obtained from $\bigoplus_i \text{Hom}(V_i^{P_u}, V_i)$, where V_i are the fundamental simple G -modules, by removing certain summands according to an explicit algorithm; see Theorem 5. The tangent space at the fixed point is at the same time the minimal ambient G -module for $\text{CE}(G/P_u)$. Its dual space is canonically isomorphic to the linear span of a minimal homogeneous generating set for the algebra $\mathbb{k}[G]^{P_u}$, which is positively graded unless $P = G$.

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2. Equivariant automorphisms

Let G/H be a homogeneous space. Denote by $N_G(H)$ the normalizer of H in G . The group $\text{Aut}_G(G/H)$ of G -equivariant automorphisms of G/H is isomorphic to $N_G(H)/H$, where nH acts on G/H by $nH * gH = gn^{-1}H$, $\forall n \in N, g \in G$.

Recall that an *affine embedding* of a homogeneous space G/H is an affine G -variety X containing a point $x \in X$ such that the orbit Gx is dense in X and the orbit morphism $G \rightarrow Gx$, $g \mapsto gx$ induces an isomorphism between G/H and Gx . In this situation we use the notation $G/H \hookrightarrow X$. The embedding is said to be *trivial* if $Gx = X$.

2.1. Automorphisms. For an embedding $G/H \hookrightarrow X$, the group $\text{Aut}_G(X)$ preserves the open orbit and may be regarded as a (closed) subgroup of $N_G(H)/H$.

It is natural to ask which subgroups of $N_G(H)/H$ can be realized as $\text{Aut}_G(X)$, where X is as above. Let us list some results in this direction, assuming $\text{char } \mathbb{k} = 0$:

- (1) if G/H is a spherical homogeneous space, then $\text{Aut}_G(X) = N_G(H)/H$ for any affine embedding $G/H \hookrightarrow X$; see e.g., [AT01, §5];
- (2) if $G = \text{SL}(2)$, $H = \{e\}$, then for any nontrivial normal affine embedding the group $\text{Aut}_G(X)$ is a Borel subgroup in $N_G(H)/H \cong \text{SL}(2)$ [Po73];
- (3) if H is reductive, then the following conditions are equivalent (cf. [AT01, Proposition 2]):
 - for any nontrivial affine embedding $G/H \hookrightarrow X$ one has $\dim \text{Aut}_G(X) < \dim N_G(H)/H$;
 - $N_G(H)/H$ is a semisimple group.

Indeed, let L^0 denote the connected component of unity in an algebraic group L . An affine embedding $G/H \hookrightarrow X$ such that $\dim \text{Aut}_G(X) = \dim N_G(H)/H$ may be regarded as a \widehat{G} -equivariant embedding of \widehat{G}/\widehat{H} , where $\widehat{G} = G \times (N_G(H)/H)^0$ and $\widehat{H} = \{(n, nH) \mid n \in N_G(H), nH \in (N_G(H)/H)^0\}$. If $N_G(H)/H$ is not semisimple, then $(N_G(H)/H)^0$ contains a central one-dimensional torus S , whence $N_{\widehat{G}}(\widehat{H})/\widehat{H} \supseteq \widehat{S} = \{e\} \times S$. Let $\widehat{N} \subseteq N_{\widehat{G}}(\widehat{H})$ be the extension of \widehat{S} by \widehat{H} . Then there exists a nontrivial embedding $\widehat{G}/\widehat{H} \hookrightarrow X = \widehat{G} \times^{\widehat{N}} \mathbb{A}^1$, where the quotient torus $\widehat{S} = \widehat{N}/\widehat{H}$ acts on \mathbb{A}^1 by homotheties. This proves the direct implication. The converse implication stems from Luna's theorem [Lu75], since $N_{\widehat{G}}(\widehat{H})/\widehat{H}$ is finite if $N_G(H)/H$ is semisimple.

The main result of this section may be regarded as a partial generalization of part (2).

THEOREM 1. *Let $G/H \hookrightarrow X$ be an affine embedding with a finite number of G -orbits and with a G -fixed point. Then the group $\text{Aut}_G(X)^0$ is solvable.*

We begin the proof with the following

LEMMA 1. *Let X be an affine variety with an action of a connected semisimple group S . Suppose that there is a point $x \in X$ and a one-parameter subgroup $\gamma : \mathbb{k}^\times \rightarrow S$ such that $\lim_{t \rightarrow 0} \delta(t)x$ exists in X for any subgroup δ conjugate to γ . Then x is a $\gamma(\mathbb{k}^\times)$ -fixed point.*

PROOF. Let T be a maximal torus in S containing $\gamma(\mathbb{k}^\times)$. It is known (for example, see [PV89]) that X can be realized as a closed S -stable subvariety in V for a suitable S -module V . Let $x = x_{\lambda_1} + \cdots + x_{\lambda_n}$ be the weight decomposition of x (with respect to T) with weights $\lambda_1, \dots, \lambda_n$. The one-parameter subgroups of T form the lattice $\mathfrak{X}_*(T)$ dual to the character lattice $\mathfrak{X}(T)$. The existence of $\lim_{t \rightarrow 0} \gamma(t)x$ in X means that all pairings $\langle \gamma, \lambda_i \rangle$ are nonnegative. Let $\gamma_1, \dots, \gamma_m$ be all the translates of γ under the action of the Weyl group $W = N_S(T)/T$. By assumption, $\langle \gamma_j, \lambda_i \rangle \geq 0$ for any $i = 1, \dots, n$, $j = 1, \dots, m$, hence $\langle \gamma_1 + \cdots + \gamma_m, \lambda_i \rangle \geq 0$. Since $\gamma_1 + \cdots + \gamma_m = 0$, one has $\langle \gamma_j, \lambda_i \rangle = 0$ for any i, j . This shows that the points x_{λ_i} (and x) are $\gamma(\mathbb{k}^\times)$ -fixed. \square

The following proposition is a generalization of [Gr83, Theorem 4.3].

PROPOSITION 1. *Suppose that $G/H \hookrightarrow X$ is an affine embedding with a non-trivial G -equivariant action of a connected semisimple group S . Then the orbit $S*x$ is closed in X , $\forall x \in G/H$.*

PROOF. We may assume $x = eH$. If $S*x$ is not closed, then, by [Ke78, Theorem 1.4], there is a one-parameter subgroup $\gamma : \mathbb{k}^\times \rightarrow S$ such that the limit

$$\lim_{t \rightarrow 0} \gamma(t) * x$$

exists in X and does not belong to $S*x$. Replacing S by a finite cover, we may assume that S embeds in $N_G(H)$ (and thus in G) with a finite intersection with H . By the definition of $*$ -action, one has $\gamma(t) * x = \gamma(t^{-1})x$. For any $s \in S$ the limit

$$\lim_{t \rightarrow 0} (s\gamma(t)) * x = \lim_{t \rightarrow 0} \gamma(t^{-1})s^{-1}x$$

exists. Hence $\lim_{t \rightarrow 0} s\gamma(t^{-1})s^{-1}x$ exists, too. This shows that for any one-parameter subgroup δ of S , conjugate to $-\gamma$, $\lim_{t \rightarrow 0} \delta(t)x$ exists in X . Lemma 1 implies that $x = \lim_{t \rightarrow 0} \gamma(t) * x$, and this contradiction proves Proposition 1. \square

PROOF OF THE THEOREM. Suppose $\text{Aut}_G(X)^0$ is not solvable. Then there is a connected semisimple group S acting on X G -equivariantly. By Proposition 1, any $(S, *)$ -orbit in the open G -orbit of X is closed in X . In particular, the $(S, *)$ -action on X is stable.

Let X_1 be the closure of a G -orbit in X . Since G has a finite number of orbits in X , the variety X_1 is $(S, *)$ -stable. Applying the above arguments to X_1 , we show that any $(S, *)$ -orbit in X is closed. But in this case all $(S, *)$ -orbits have the same dimension $\dim S$. On the other hand, a G -fixed point is an $(S, *)$ -orbit, a contradiction. \square

COROLLARY 1 (of the proof). *Let X be an affine G -variety with an open G -orbit. Suppose that*

- (1) *a semisimple group S acts on X effectively and G -equivariantly;*
- (2) *the dimension of a closed G -orbit in X is less than $\dim S$.*

Then the number of G -orbits in X is infinite.

REMARK 1. Condition (2) is essential. Indeed, let H be a one-dimensional unipotent root subgroup of $G = \mathrm{SL}(n)$. Then $X = \mathrm{CE}(G/H) \cong \mathrm{SL}(n) \times^{\mathrm{SL}(2)} \mathbb{A}^2$, where $\mathrm{SL}(2)$ embeds in $\mathrm{SL}(n)$ as the standard 3-dimensional simple subgroup containing H , has two orbits, and $S = \mathrm{SL}(n-2) \subset N_G(H)/H$.

In many cases, Theorem 1 may be used to show that the group $\mathrm{Aut}_G(X)$ cannot be very big.

2.2. Canonical embeddings. Now we are going to apply the results obtained to the study of the canonical embedding $\mathrm{CE}(G/P_u)$. Fix a pair $T \subset B$, where B is a Borel subgroup in G and T is a maximal torus. We shall consider a parabolic subgroup $P \supseteq B$.

REMARK 2. The commutator subgroup $G' \subseteq G$ is the maximal semisimple subgroup in G . It is easy to see that $\mathrm{CE}(G/P_u) = G \times^{G'} \mathrm{CE}(G'/P_u)$ is the homogeneous fibration over G/G' with fiber $\mathrm{CE}(G'/P_u)$. Thus without loss of generality we may assume G to be semisimple.

Furthermore, $\mathrm{CE}(G/P_u) = \mathrm{CE}(\tilde{G}/\tilde{P}_u)/\tilde{Z}$, where $\tilde{G} \rightarrow G = \tilde{G}/\tilde{Z}$ is the simply connected covering, and $\tilde{P} \subseteq \tilde{G}$ is the preimage of P . Passing to the quotient modulo a finite central subgroup preserves many features of $\mathrm{CE}(G/P_u)$ (for instance, the orbit dimensions, the modality of the G -action, normality, etc). Therefore we may assume in many questions that G is simply connected.

Then $G = \prod_i G_i$, $P = \prod_i P_i$, where G_i are simple factors. It follows that $\mathrm{CE}(G/P_u) = \prod_i \mathrm{CE}(G_i/(P_i)_u)$, and we may assume G to be a simply connected simple algebraic group.

The following proposition gives a partial answer to a question posed in [Ar03].

PROPOSITION 2. *The number of G -orbits in $\mathrm{CE}(G/P_u)$ is finite if and only if either $P \cap G_i = G_i$ or $P \cap G_i = B \cap G_i$ for each simple factor $G_i \subseteq G$.*

PROOF. By Remark 2 we may assume that G is simple. If $P = G$, then $P_u = \{e\}$ and $\mathrm{CE}(G/P_u) = G$. If $P = B$, then $P_u = U$ is a maximal unipotent subgroup of G , the variety $\mathrm{CE}(G/P_u)$ is spherical [Br97, 2.1], and any spherical variety contains a finite number of G -orbits [Br97].

To prove the converse implication, let us fix some notation. Let L be the Levi subgroup of P containing T , Z the center of L , and S the maximal semisimple subgroup of L .

LEMMA 2. *If $P \neq G$, then $\mathrm{CE}(G/P_u)$ contains a G -fixed point.*

PROOF OF THE LEMMA. It is possible to find a one-parameter subgroup $\gamma : \mathbb{k}^\times \rightarrow Z$, such that the pairing of γ with any nonzero dominant weight is positive (see, e.g., Remark 3 below). The 1-torus $\gamma(\mathbb{k}^\times)$, regarded as a subgroup of $\mathrm{Aut}_G(G/P_u)$, defines a G -invariant grading on $\mathbb{k}[G/P_u]$.

The homogeneous subalgebra $\mathbb{k}[G/U] \subseteq \mathbb{k}[G/P_u]$ has a multigrading $\mathbb{k}[G/U] = \bigoplus_\lambda E(\lambda)$ by eigenspaces of the T -action from the right, which are called *dual Weyl*

modules, so that $\deg E(\lambda) = \langle \gamma, \lambda \rangle$. It is known that $E(\lambda) \neq 0$ iff λ is dominant, and $\dim E(\lambda) = 1 \iff \lambda = 0$ (see, e.g., [Gr97, §12]). Hence the grading is nonnegative on $\mathbb{k}[G/U]$, and only constant functions have degree 0.

The same is true for the algebra $A = \mathbb{k}[S * M]$, where M is any homogeneous generating set of $\mathbb{k}[G/U]$, since the $*$ -actions of S and γ commute. However $\mathbb{k}[G/P_u]$ is the integral closure of A [Gr83, 5.4] ($A = \mathbb{k}[G/P_u]$ if $\text{char } \mathbb{k} = 0$; cf. 3.1). It easily follows that the grading is nonnegative on $\mathbb{k}[G/P_u]$, and the positive part of this grading is a G -stable maximal ideal in $\mathbb{k}[G/P_u]$. \square

Note that the group $N_G(P_u)/P_u$ is isomorphic to L . Hence if $P \neq B$, then S acts on $\text{CE}(G/P_u)$ effectively and G -equivariantly. By Theorem 1, the number of G -orbits in $\text{CE}(G/P_u)$ is infinite. \square

2.3. Reductive monoids. Finally, consider an application of Theorem 1 to another remarkable class of affine embeddings.

PROPOSITION 3. *Let M be an algebraic monoid with zero such that its group of invertible elements $G(M)$ is reductive. Then the number of left $G(M)$ -cosets in M is finite if and only if M is commutative.*

PROOF. It is known that M is an affine variety [Rit98] and $G(M)$ is open in M [Vi95]. If M is commutative, then $G(M)$ is commutative, hence M is a toric variety and the number of $G(M)$ -orbits in M is finite.

Otherwise, $G(M)$ contains a semisimple subgroup S , and the action of S on M by right multiplication is $G(M)$ -equivariant. The zero element is a $G(M)$ -fixed point, and we conclude by Theorem 1. \square

3. The canonical embedding of G/P_u

In this section we obtain a detailed description of $\text{CE}(G/P_u)$, assuming that $\text{char } \mathbb{k} = 0$. Our basic idea is to treat G/P_u as a homogeneous space under $G \times L$, in the notation of the previous section. The action is defined by $(g, h)xP_u = gxh^{-1}P_u$, $\forall g, x \in G, h \in L$.

It is clear that $X = \text{CE}(G/P_u)$ is a $(G \times L)$ -equivariant affine embedding of G/P_u . More generally, we shall describe the structure of an arbitrary affine $(G \times L)$ -embedding of G/P_u and deduce results concerning the canonical embedding as a particular case.

3.1. The coordinate algebra. One easily sees from the Bruhat decomposition that a Borel subgroup of $G \times L$ has an open orbit in G/P_u , i.e., G/P_u is a spherical homogeneous $(G \times L)$ -space. Alternatively, one can deduce that G/P_u is spherical from the multiplicity-free property for the isotypic decomposition of $\mathbb{k}[G/P_u]$, which we are going to describe.

Let $V(\lambda) = V_G(\lambda)$ denote the simple G -module of highest weight λ with respect to B , where λ is any vector from the semigroup $\mathfrak{X}^+ = \mathfrak{X}_G^+$ of B -dominant weights. The set of positive/negative roots with respect to B is denoted by $\Delta^\pm = \Delta_G^\pm$, and $\Pi = \Pi_G \subseteq \Delta_G^+$ is the set of simple roots. The respective sets of coroots are denoted by $\check{\Delta}^\pm, \check{\Pi}$. Let $C = C_G$, and let \check{C} be the dominant Weyl chambers in $\mathfrak{X}(T) \otimes \mathbb{Q}$ and $\mathfrak{X}_*(T) \otimes \mathbb{Q}$, respectively. Recall that C, \check{C} are fundamental chambers of the Weyl group $W = W_G = N_G(T)/T$.

The group G itself can be regarded as a homogeneous space $(G \times G)/\text{diag } G$, where the left and right copies of G act by left and right translations, respectively. The $(G \times G)$ -isotypic decomposition of $\mathbb{k}[G]$ is well known:

PROPOSITION 4 ([Kr85, II.3.1, Satz 3]). $\mathbb{k}[G] = \bigoplus_{\lambda \in \mathfrak{X}^+} \mathbb{k}[G]_{(\lambda)}$, where the subspace $\mathbb{k}[G]_{(\lambda)} \cong V(\lambda)^* \otimes V(\lambda)$ is the linear span of the matrix elements of the representation $G : V(\lambda)$.

Now the isotypic decomposition of $\mathbb{k}[G/P_u]$ is provided by passing to P_u -invariants from the right on the right-hand side of the above decomposition. Note that $V(\lambda)^{P_u} \cong V_L(\lambda)$ is isomorphic to the simple L -module of highest weight λ .

PROPOSITION 5. There is a $(G \times L)$ -module decomposition

$$\mathbb{k}[G/P_u] = \bigoplus_{\lambda \in \mathfrak{X}^+} \mathbb{k}[G/P_u]_{(\lambda)},$$

where $\mathbb{k}[G/P_u]_{(\lambda)} \cong V(\lambda)^* \otimes V_L(\lambda)$ is the linear span of the matrix elements of the linear maps $V(\lambda)^{P_u} \rightarrow V(\lambda)$ induced by $g \in G$, regarded as regular functions on G/P_u .

PROOF. In view of Proposition 4 and the above remark, it suffices to note that the space of matrix elements of linear maps $V(\lambda)^{P_u} \rightarrow V(\lambda)$ equals $(\mathbb{k}[G]_{(\lambda)})^{P_u}$ (invariants under right translations). \square

Next we describe the multiplicative structure of $\mathbb{k}[G/P_u]$.

PROPOSITION 6. There is a decomposition

$$\mathbb{k}[G/P_u]_{(\lambda)} \cdot \mathbb{k}[G/P_u]_{(\mu)} = \mathbb{k}[G/P_u]_{(\lambda+\mu)} \oplus \bigoplus_i \mathbb{k}[G/P_u]_{(\lambda+\mu-\beta_i)}$$

for all $\lambda, \mu \in \mathfrak{X}^+$, where $\lambda + \mu - \beta_i$ runs over the highest weights of all "lower" irreducible components in the L -module decomposition

$$V_L(\lambda) \otimes V_L(\mu) = V_L(\lambda + \mu) \oplus \cdots,$$

so that $\beta_i \in \mathbb{Z}_+ \Pi_L$.

PROOF. $\mathbb{k}[G/P_u]_{(\lambda)} \cdot \mathbb{k}[G/P_u]_{(\mu)}$ is spanned by the products of matrix elements of linear maps $V(\lambda)^{P_u} \rightarrow V(\lambda)$ and $V(\mu)^{P_u} \rightarrow V(\mu)$ induced by $g \in G$, i.e., by matrix elements of $V(\lambda)^{P_u} \otimes V(\mu)^{P_u} \rightarrow V(\lambda) \otimes V(\mu)$. But $V(\lambda)^{P_u} \otimes V(\mu)^{P_u} \cong V_L(\lambda) \otimes V_L(\mu)$, and each L -highest weight vector occurring on the left-hand side is a G -highest weight vector at the same time, because it is fixed by P_u . It generates a simple G -submodule $V(\lambda + \mu - \beta) \subseteq V(\lambda) \otimes V(\mu)$ and a simple L -submodule $V_L(\lambda + \mu - \beta) = V(\lambda + \mu - \beta)^{P_u}$, where $\beta \in \mathbb{Z}_+ \Pi_L$. The latter L -submodule is mapped to $V(\lambda + \mu - \beta)$ by $g \in G$. Therefore the above space of matrix elements for tensor products is spanned by all the $\mathbb{k}[G/P_u]_{(\lambda+\mu-\beta)}$. \square

REMARK 3. Let γ be any vector in the interior of the cone $\check{C} \cap \Pi_L^\perp \cap \langle \check{\Pi} \rangle$. For instance, one may take $\gamma = \check{\rho}^L$, the sum of the fundamental coweights corresponding to simple roots from $\Pi \setminus \Pi_L$, or $\gamma = \check{\rho}_G - \check{\rho}_L = \frac{1}{2} \sum_{\alpha \in \Delta_G^+ \setminus \Delta_L^+} \check{\alpha}$. (Here $\check{\rho}_G = \frac{1}{2} \sum_{\alpha \in \Delta_G^+} \check{\alpha}$ is half the sum of positive coroots, or equivalently, the sum of fundamental coweights of a reductive group G .) Then $\langle \gamma, C \rangle \geq 0$, and the inequality is strict on $C \setminus \{0\}$ if G is simple. (This is because $C \setminus \{0\}$ is contained in the interior of the cone $\mathbb{Q}_+ \Pi$ dual to \check{C} , for indecomposable root systems.)

Replacing γ by a multiple, we may assume that $\gamma \in \mathfrak{X}_*(T)$ defines a one-parameter subgroup $\mathbb{k}^\times \rightarrow Z$. This 1-torus defines an invariant nonnegative algebra grading of $\mathbb{k}[G/P_u]$ via the action by right translations of an argument, so that $\deg \mathbb{k}[G/P_u]_{(\lambda)} = \langle \gamma, \lambda \rangle$, $\forall \lambda \in \mathfrak{X}^+$. If G is simple, then $\mathbb{k}[G/P_u]_0 = \mathbb{k}$; cf. Lemma 2.

Moreover, the γ -action defines a vector space grading of $V(\lambda)$ and a G -module grading of $\mathbb{k}[G]$ such that $V(\lambda)^{P_u}$ and $\mathbb{k}[G/P_u]_{(\lambda)}$ are the homogeneous components of maximal degree. In fact, the weight polytope of $V(\lambda)^{P_u}$ is the face of the weight polytope of $V(\lambda)$, where the linear function $\langle \gamma, \cdot \rangle$ reaches its maximal value.

It follows that $\mathbb{k}[G/P_u]_{(\lambda)} \cdot \mathbb{k}[G/P_u]_{(\mu)}$ is the homogeneous component of maximal degree in $\mathbb{k}[G]_{(\lambda)} \cdot \mathbb{k}[G]_{(\mu)}$. The isotypic decomposition of the latter product space is a well-known particular case of Proposition 6. Taking the maximal degree means that we must choose only those direct summands with $\langle \gamma, \lambda + \mu - \beta_i \rangle = \langle \gamma, \lambda \rangle + \langle \gamma, \mu \rangle \iff \beta_i \perp \gamma \iff \beta_i \in \mathbb{Z}_+ \Pi_L$. The respective simple L -modules are exactly those occurring in the decomposition of $V(\lambda)^{P_u} \otimes V(\mu)^{P_u}$. Thus the particular case of Proposition 6 implies the general one.

REMARK 4. Since G/P_u is a spherical homogeneous space under $G \times L$, the powerful theory of spherical varieties [Br97], [Kn91] can be applied to the study of its equivariant embeddings. For instance, it is easy to deduce from Proposition 6 that the *valuation cone* of G/P_u equals $-\check{C}_L$, and the *colors* are identified with simple coroots of G . Now it follows from the general theory that *normal* affine $(G \times L)$ -embeddings $G/P_u \hookrightarrow X$ are in bijection with convex polyhedral cones generated by $\check{\Pi}$ and finitely many vectors from $-\check{C}_L$, $(G \times L)$ -orbits in X correspond to faces of such a cone with interiors intersecting $-\check{C}_L$, etc.

However, in this paper we prefer to give a more elementary treatment of affine embeddings of G/P_u based on properties of their coordinate algebras and on explicit embeddings into ambient vector spaces. Our approach is similar to that of [Ti03], [Ti03a, 3.3–3.4] for projective group completions and reductive monoids.

3.2. Affine embeddings. Affine $(G \times L)$ -embeddings $X \hookrightarrow G/P_u$ are determined by their coordinate algebras $\mathbb{k}[X]$, which are $(G \times L)$ -stable finitely generated subalgebras of $\mathbb{k}[G/P_u]$ with the quotient field $\mathbb{k}(G/P_u)$. Since $\mathbb{k}[G/P_u]$ is multiplicity free, we have $\mathbb{k}[X] = \bigoplus_{\lambda \in S} \mathbb{k}[G/P_u]_{(\lambda)}$, where $S \subseteq \mathfrak{X}^+$ is a finitely generated semigroup such that $\mathbb{Z}S = \mathfrak{X}(T)$. Proposition 6 implies that all highest weights $\lambda + \mu - \beta$ of $V_L(\lambda) \otimes V_L(\mu)$ belong to S whenever $\lambda, \mu \in S$.

The variety X is normal iff $\mathbb{k}[X]^{U \times (U \cap L)}$ is integrally closed in its quotient field [Kr85, Theorem III.3.3-2]. But the latter algebra is just the semigroup algebra of S , which is integrally closed iff $S = \Sigma^+ \cap \mathfrak{X}(T)$ is the semigroup of all lattice vectors in the polyhedral cone $\Sigma^+ = \mathbb{Q}_+ S$. For example, for $X = \text{CE}(G/P_u)$ we have $S = \mathfrak{X}^+ = C \cap \mathfrak{X}(T)$, $\Sigma^+ = C$.

By Proposition 6, a $(G \times L)$ -stable subspace $\mathbb{k}[G/P_u]_{(\lambda_1)} \oplus \cdots \oplus \mathbb{k}[G/P_u]_{(\lambda_m)}$ generates $\mathbb{k}[X]$ iff S is L -generated by $\lambda_1, \dots, \lambda_m$ in the sense of the following

DEFINITION 1. We say that $\lambda_1, \dots, \lambda_m$ L -generate S if S consists of all highest weights $k_1 \lambda_1 + \cdots + k_m \lambda_m - \beta$ of L -modules $V_L(\lambda_1)^{\otimes k_1} \otimes \cdots \otimes V_L(\lambda_m)^{\otimes k_m}$, $k_1, \dots, k_m \in \mathbb{Z}_+$. (In particular, any generating set L -generates S .)

Since $\mathbb{k}[G/P_u]_{(\lambda)}^* \cong \text{Hom}(V(\lambda)^{P_u}, V(\lambda))$, we have a $(G \times L)$ -equivariant closed immersion $X \hookrightarrow \bigoplus_{i=1}^m \text{Hom}(V(\lambda_i)^{P_u}, V(\lambda_i))$ in this case. Conversely, any equivariant immersion

$$G/P_u \hookrightarrow \bigoplus_{i=1}^m \text{Hom}(V(\lambda_i)^{P_u}, V(\lambda_i))$$

gives rise to an affine embedding

$$X = \overline{G/P_u} \subseteq \bigoplus_{i=1}^m \text{Hom}(V(\lambda_i)^{P_u}, V(\lambda_i))$$

with weight semigroup S L -generated by $\lambda_1, \dots, \lambda_m$.

Put $V = V(\lambda_1) \oplus \dots \oplus V(\lambda_m)$; then $V^{P_u} \cong V_L(\lambda_1) \oplus \dots \oplus V_L(\lambda_m)$. Since a multiple of each $\mu \in C \cap \text{conv } W_L\{\lambda_1, \dots, \lambda_m\}$ eventually occurs as an L -highest weight in $(V^{P_u})^{\otimes k}$ by [Ti03, Lemma 1], we have $\Sigma^+ = C \cap \mathbb{Q}_+(W_L\{\lambda_1, \dots, \lambda_m\})$.

We sum up the above discussion in the following theorem:

THEOREM 2. *There is a bijection between affine $(G \times L)$ -equivariant embeddings $X \hookrightarrow G/P_u$ and subsemigroups $S \subseteq \mathfrak{X}^+$ L -generated by finitely many weights $\lambda_1, \dots, \lambda_m \in \mathfrak{X}^+$ and such that $\mathbb{Z}S = \mathfrak{X}(T)$. There is a natural equivariant embedding*

$$X \hookrightarrow \bigoplus_{i=1}^m \text{Hom}(V(\lambda_i)^{P_u}, V(\lambda_i)),$$

$$eP_u \mapsto (\mathbf{1}_{V(\lambda_1)^{P_u}}, \dots, \mathbf{1}_{V(\lambda_m)^{P_u}})$$

for any L -generating set of S . The convex cone spanned by S is $\Sigma^+ = C \cap \mathbb{Q}_+(W_L\{\lambda_1, \dots, \lambda_m\})$. The variety X is normal iff $S = \Sigma^+ \cap \mathfrak{X}(T)$.

EXAMPLE 1. If G is semisimple and simply connected, then there is a natural inclusion

$$\text{CE}(G/P_u) \subseteq \bigoplus_{i=1}^l \text{Hom}(V(\omega_i)^{P_u}, V(\omega_i)),$$

where $\omega_1, \dots, \omega_l$ are the fundamental weights of G .

3.3. Relation to reductive monoids. One observes that the classification of affine embeddings of G/P_u is given in the same terms as the classification of algebraic monoids with the group of invertibles L [Vi95], [Ti03a, 3.3]. Here is a geometric explanation of this coincidence.

The group L embeds in G/P_u as the orbit of eP_u . Let M be the closure of L in X . Under the embedding $X \hookrightarrow \text{Hom}(V^{P_u}, V)$, M embeds in $\text{End } V^{P_u}$ as an algebraic submonoid with the group of invertibles L . As $V^{P_u} \cong \bigoplus_{i=1}^m V_L(\lambda_i)$, we have $\mathbb{k}[M] = \bigoplus_{\lambda \in S} \mathbb{k}[L]_{(\lambda)}$, where S is the semigroup L -generated by $\lambda_1, \dots, \lambda_m$.

There is a natural birational proper map $G \times^P M \rightarrow X$, $(g, z) \mapsto gz$, where P acts on M through its quotient group $L \cong P/P_u$ by left translations. Moreover, X is recovered from M as $\text{Spec } \mathbb{k}[G \times^P M]$. Indeed, $\mathbb{k}[G \times^P M] = (\mathbb{k}[G] \otimes \mathbb{k}[M])^P = \bigoplus_{\lambda, \mu} (\mathbb{k}[G]_{(\mu)} \otimes \mathbb{k}[M]_{(\lambda)})^P$, where P acts on G by right translations and on M as above. But $(\mathbb{k}[G]_{(\mu)} \otimes \mathbb{k}[M]_{(\lambda)})^P \cong V(\mu)^* \otimes (V(\mu) \otimes V_L(\lambda)^*)^P \otimes V_L(\lambda) \cong V(\lambda)^* \otimes V_L(\lambda)$ whenever $\lambda = \mu$, and 0, otherwise. Therefore $\mathbb{k}[G \times^P M] = \mathbb{k}[X]$.

Conversely, let $M \leftarrow L$ be any algebraic monoid, and $S \subseteq \mathfrak{X}_L^+$ its weight semigroup. The same reasoning as above shows that $\mathbb{k}[G \times^P M] \cong \bigoplus_{\lambda \in S \cap C} \mathbb{k}[G/P_u]_{(\lambda)}$.

In other words, affine embeddings of G/P_u correspond to algebraic monoids, whose group of invertible elements is L and the weight semigroup consists of G -dominant weights.

There is a bijective correspondence between the following sets: $\{(G \times L)$ -orbits in $X\}$, $\{(G \times L)$ -stable prime ideals in $\mathbb{k}[X]\}$, $\{\text{faces } \Gamma \subseteq \Sigma^+ \text{ such that } \nu \notin \Gamma \text{ whenever } V_L(\nu) \hookrightarrow V_L(\lambda) \otimes V_L(\mu), \lambda \in S, \mu \in S \setminus \Gamma\}$, $\{(L \times L)$ -stable prime ideals in $\mathbb{k}[M]\}$, $\{(L \times L)$ -orbits in $M\}$. Thus we have finally proved

PROPOSITION 7. *There is a bijection between affine $(G \times L)$ -embeddings $X \hookrightarrow G/P_u$ and algebraic monoids M with the group of invertibles L and the weight semigroup $S \subseteq \mathfrak{X}_G^+$, given by $X = \text{Spec } \mathbb{k}[G \times^P M]$. The natural proper birational map $G \times^P M \rightarrow X$ yields a bijection between $(G \times L)$ -orbits in X and $(L \times L)$ -orbits in M preserving inclusions of orbit closures.*

EXAMPLE 2. Let $G = \text{GL}(n)$, and let P be the stabilizer of a d -subspace in \mathbb{k}^n . Then G/P_u embeds in the variety of complexes

$$X = \left\{ \mathbb{k}^d \xrightarrow{A_1} \mathbb{k}^n \xrightarrow{A_2} \mathbb{k}^{n-d} \mid A_2 A_1 = 0 \right\}$$

so that $eP_u \mapsto (A_1^0, A_2^0)$, where A_1^0 is the inclusion and A_2^0 the projector with respect to a fixed decomposition $\mathbb{k}^n = \mathbb{k}^d \oplus \mathbb{k}^{n-d}$. Here $L = \text{GL}(d) \times \text{GL}(n-d)$ is the stabilizer of this decomposition.

In the above notation, we may take $V = \mathbb{k}^n \oplus (\mathbb{k}^n)^*$, so that $V^{P_u} = \mathbb{k}^d \oplus (\mathbb{k}^{n-d})^*$, and $X \hookrightarrow \text{Hom}(V^{P_u}, V)$, $(A_1, A_2) \mapsto A_1 \oplus A_2^*$. It follows that $M = \text{Mat}(d) \times \text{Mat}(n-d) = \{(A_1, A_2) \mid \text{Im } A_1 \subseteq \mathbb{k}^d \subseteq \text{Ker } A_2\}$. The map $G \times^P M \rightarrow X$ is given by $(g, A_1, A_2) \mapsto (gA_1, A_2 g^{-1})$. The weight semigroup S is freely generated by $\pi_1, \dots, \pi_d, \pi_1^*, \dots, \pi_{n-d}^*$, where π_i is the highest weight of $\bigwedge^i \mathbb{k}^n$ and π_i^* is the dual highest weight. The $(G \times L)$ -orbits in X , as well as $(L \times L)$ -orbits in M , are determined by the numerical invariant $(\text{rk } A_1, \text{rk } A_2)$, and the inclusion of orbit closures corresponds to the product order on these pairs.

3.4. Orbits. Our aim is to describe the orbital decomposition of $X \hookrightarrow G/P_u$. Let us recall some basic notions and introduce some notation.

DEFINITION 2. The *generic modality* of the action $G : X$ of an algebraic group on an irreducible variety is the number

$$\begin{aligned} d_G(X) &= \text{codim}_X Gx \quad (x \in X \text{ a general point}) \\ &= \min_{x \in X} \text{codim}_X Gx = \text{tr.deg } \mathbb{k}(X)^G. \end{aligned}$$

The *modality* of $G : X$ is the maximum number of parameters which a continuous family of G -orbits in X depends on, i.e.,

$$\text{mod}_G X = \max_{Y \subseteq X} d_G(Y),$$

where Y runs over all G -stable irreducible subvarieties of X . (Note that X has finitely many orbits iff $\text{mod}_G X = 0$.)

Let $\Sigma = \mathbb{Q}_+(W_L\{\lambda_1, \dots, \lambda_m\})$ be the convex cone generated by the weight polytope of V^{P_u} . Note that $\Sigma = W_L \Sigma^+$, $\Sigma^+ = \Sigma \cap C$.

For any face $\Gamma \subseteq \Sigma$, let $V_\Gamma \subseteq V^{P_u}$ be the sum of T -eigenspaces with eigenweights in Γ , and $e_\Gamma : V^{P_u} \rightarrow V_\Gamma$ the T -equivariant projector.

For any subset $\Phi \subseteq \Pi$, let L_Φ denote the standard Levi subgroup with the system of simple roots Φ , L'_Φ its commutator subgroup, P_Φ the standard parabolic subgroup generated by L_Φ and B , and P_Φ^- the opposite parabolic subgroup. If $N \subseteq \mathfrak{X}(T) \otimes \mathbb{Q}$ is a subspace such that there exists $\gamma \in \check{C}$, $\gamma \perp N$, $\gamma \not\perp \alpha$, $\forall \alpha \in \Delta \setminus N$, then $\Phi = \Pi \cap N$ is the base of the root subsystem $\Delta \cap N$, and we put $L_N = L_\Phi$, etc. For any sublattice $\Lambda \subseteq \mathfrak{X}(T)$, denote by $T^\Lambda \subseteq T$ the diagonalizable group which is the common kernel of all characters $\lambda \in \Lambda$.

Suppose that Γ is a face of Σ whose interior intersects C . Put $|\Gamma| = \langle \Delta_L \rangle \cap \langle \Gamma \rangle$, $\|\Gamma\| = |\Gamma| \oplus \langle \Gamma \rangle^\perp$, $\langle \Gamma \rangle_{\mathbb{Z}} = \sum_{\lambda_i \in \Gamma} \mathbb{Z}\lambda_i + (\mathbb{Z}\Delta_L \cap \langle \Gamma \rangle)$ (a sublattice generating $\langle \Gamma \rangle$).

The following theorem is a counterpart of the results of [Ti03, §9].

THEOREM 3. *The $(G \times L)$ -orbits $Y \subseteq X$ are in bijection with the faces $\Gamma \subseteq \Sigma$ whose interiors intersect C . The inclusion of faces corresponds to the inclusion of orbit closures. The orbit $Y = Y_\Gamma$ is represented by e_Γ . The stabilizers are:*

$$(G \times L)_{e_\Gamma} = \left[(P_{\|\Gamma\|})_{\mathfrak{u}} \times (L \cap P_{\|\Gamma\|}^-)_{\mathfrak{u}} \right] \\ \times \left[\left(L'_{\langle \Gamma \rangle^\perp} T^{\langle \Gamma \rangle_{\mathbb{Z}}} \times (L \cap L_{\langle \Gamma \rangle^\perp})' T^{\langle \Gamma \rangle_{\mathbb{Z}}} \right) \cdot \text{diag } L_{|\Gamma|} \right], \\ G_{e_\Gamma} = (P_{\|\Gamma\|})_{\mathfrak{u}} \rtimes L'_{\langle \Gamma \rangle^\perp} T^{\langle \Gamma \rangle_{\mathbb{Z}}}.$$

All G -orbits in Y are isomorphic and permuted transitively by L . The (generic) modality of Y is:

$$d_G(Y) = \dim L \cap (P_{\|\Gamma\|})_{\mathfrak{u}}.$$

PROOF. By Proposition 7, $(G \times L)$ -orbits in X are in bijection with $(L \times L)$ -orbits in $M = \bar{L} \subseteq \text{End } V^{P_{\mathfrak{u}}}$. Therefore it suffices to describe the orbits for the action $L \times L : M$. This description goes back to Putcha and Renner. In particular, one finds that the projectors e_Γ form a complete set of orbit representatives (cf. [Ti03, Theorem 8]). Let us give an outline of an elementary proof.

First observe that \bar{T} intersects all $(L \times L)$ -orbits in M . (For $\mathbb{k} = \mathbb{C}$, the easiest way to see it is to close in M the Cartan decomposition $L = KTK$, where $K \subset L$ is a maximal compact subgroup. For an arbitrary \mathbb{k} , one may consider the Iwahori decomposition of $G(\mathbb{k}((t)))$ instead; see [Br97, 2.4, Exemple 2].) Next, it is easy to deduce from affine toric geometry that T -orbits in \bar{T} are represented by e_Γ over all faces $\Gamma \subseteq \Sigma$.

But one sees from the structure of $(L \times L)_{e_\Gamma}$ that $((L \times L)_{e_\Gamma})^{\text{diag } T}$ is a union of T -orbits permuted by W_L transitively. Indeed, if $y = (g_1, g_2)e_\Gamma$ is fixed by $\text{diag } T$, then one may assume that $(g_1, g_2)^{-1}(\text{diag } T)(g_1, g_2)$ is contained in the maximal torus $(T^{\langle \Gamma \rangle_{\mathbb{Z}}} \times T^{\langle \Gamma \rangle_{\mathbb{Z}}}) \cdot \text{diag } T$ of $(L \times L)_{e_\Gamma}$. Hence $g_1, g_2 \in N_L(T)$ represent two elements $w_1, w_2 \in W_L$ acting on Γ equally, and $y = w_2 e_\Gamma w_2^{-1}$. Therefore, $(L \times L)$ -orbits in M are represented by those e_Γ corresponding to faces with interiors intersecting C .

The above reasoning also proves the assertion on inclusions of faces and orbit closures, since it is true for T -orbits in \bar{T} .

Now we compute the stabilizers. Let V'_Γ be the T -stable complement to V_Γ in $V^{P_{\mathfrak{u}}}$. For $(g, h) \in G \times L$ we have: $ge_\Gamma h^{-1} = e_\Gamma$ iff

- (1) $gV_\Gamma = V_\Gamma$;
- (2) $hV'_\Gamma = V'_\Gamma$;
- (3) the actions of g, h on $V_\Gamma \cong V^{P_{\mathfrak{u}}}/V'_\Gamma$ coincide.

The condition (1) means that $g \in P_{\|\Gamma\|}$. Indeed, for any $\alpha \in \Delta^+ \setminus \|\Gamma\|$ we have $\langle \check{\alpha}, \Gamma \rangle \geq 0$, and the strict inequality is achieved. Hence $(P_{\|\Gamma\|})_{\mathfrak{u}}$ fixes V_{Γ} pointwise, whereas no element of $(P_{\|\Gamma\|}^-)_{\mathfrak{u}}$ preserves V_{Γ} . By definition, V_{Γ} is $L_{|\Gamma|}$ -stable. On the other hand, it is easy to see that adding roots $\alpha \in \|\Gamma\| \setminus |\Gamma|$ moves the weights of $V(\lambda_i) \cap V_{\Gamma}$ outside the weight polytope of $V(\lambda_i)$, $\forall i = 1, \dots, m$. Hence the respective root vectors act on V_{Γ} trivially, i.e., $\alpha \perp \Gamma$. This means that $\Delta \cap \|\Gamma\| = (\Delta \cap |\Gamma|) \sqcup (\Delta \cap \langle \Gamma \rangle^{\perp})$ is a disjoint orthogonal union, $L_{\|\Gamma\|} = L_{|\Gamma|} \cdot L_{\langle \Gamma \rangle^{\perp}}$, and $L'_{\langle \Gamma \rangle^{\perp}}$ fixes V_{Γ} pointwise.

Similar arguments show that (2) $\iff h \in L \cap P_{\|\Gamma\|}^-$, and the subgroup $(L \cap P_{\|\Gamma\|}^-)_{\mathfrak{u}} \rtimes (L \cap L_{\langle \Gamma \rangle^{\perp}})'$ acts on $V^{P_{\mathfrak{u}}}/V'_{\Gamma}$ trivially. Thus after factoring out the kernels of the actions, we may assume $g, h \in L_{|\Gamma|} \subseteq L$. But $L_{|\Gamma|}$ acts on $V_{\Gamma} \cong V^{P_{\mathfrak{u}}}/V'_{\Gamma}$ with kernel $T^{(\Gamma)z}$, $\langle \Gamma \rangle_z$ being the weight lattice of V_{Γ} . Hence (3) $\iff g \equiv h \pmod{T^{(\Gamma)z}}$, and we are done.

The formula for $G_{e_{\Gamma}}$ stems from that for $(G \times L)_{e_{\Gamma}}$ immediately. Since the L -action on Y commutes with the G -action, it permutes the G -orbits transitively, and all of them are isomorphic and, in particular, have the same dimension. Now $d_G(Y) = \dim Y - \dim G e_{\Gamma} = \dim(G \times L) - \dim(G \times L)_{e_{\Gamma}} - \dim G + \dim G_{e_{\Gamma}} = \dim L - \dim(L \cap P_{\|\Gamma\|}^-)_{\mathfrak{u}} - \dim(L \cap L_{\langle \Gamma \rangle^{\perp}})' T^{(\Gamma)z} - \dim L_{|\Gamma|}/T^{(\Gamma)z} = \dim L \cap (P_{\|\Gamma\|})_{\mathfrak{u}}$. \square

COROLLARY 2. *The $(G \times L)$ -orbits $Y \subseteq \text{CE}(G/P_{\mathfrak{u}})$ are in bijection with the subsystems of simple roots $\Pi_Y \subseteq \Pi$ such that no component of Π_Y is contained in Π_L . The stabilizers of Y in $G \times L$ and in G are:*

$$\left[(P_{\Pi_Y \cup (\Pi_L \setminus \partial \Pi_Y)})_{\mathfrak{u}} \times (L \cap P_{\Pi_L \setminus \partial \Pi_Y}^-)_{\mathfrak{u}} \right] \rtimes \left[(L'_{\Pi_Y} \times (L \cap L'_{\Pi_Y})) \cdot \text{diag } L_{\Pi_L \cap \Pi_Y} \right]$$

and $(P_{\Pi_Y \cup (\Pi_L \setminus \partial \Pi_Y)})_{\mathfrak{u}} \rtimes L'_{\Pi_Y}$.

Here $\partial \Pi_Y$ is the set of simple roots from $\Pi \setminus \Pi_Y$ neighboring with Π_Y on the Dynkin diagram of G . We have $d_G(Y) = \dim L \cap (P_{\Pi_L \setminus \partial \Pi_Y})_{\mathfrak{u}}$.

The modality $\text{mod}_G \text{CE}(G/P_{\mathfrak{u}}) = \max_Y d_G(Y)$ is reached on Y such that $\Pi_Y \supseteq \Pi \setminus \Pi_L$, and each component of Π_Y is obtained from a component of $\Pi \setminus \Pi_L$ by adding roots from Π_L in such a way that $\dim L_{\Pi_L^{(k)} \setminus \partial \Pi_Y} = \min$ for each component $\Pi_L^{(k)} \subseteq \Pi_L$. (In particular, $\text{mod}_G \text{CE}(G/P_{\mathfrak{u}}) = 0$ iff Π_L is a union of components of Π , which implies Proposition 2.)

PROOF. The orbits $Y \subseteq \text{CE}(G/P_{\mathfrak{u}})$ are in bijection with the faces $\Gamma \subseteq W_L C$ whose interiors intersect C . Then $\Gamma \cap C$ is the face of C of the same dimension $\dim \Gamma$, and the dual face of $\mathbb{Q}_+ \check{\Pi}$ is spanned by a certain subset $\check{\Pi}_Y \subseteq \check{\Pi}$, so that $\langle \beta_Y, W_L C \rangle \geq 0$ for a certain positive linear combination β_Y of $\check{\Pi}_Y$.

Suppose that a component $\Pi_Y^{(i)}$ of Π_Y is contained in Π_L . Let $\rho_{(i)} \in C$ be the sum of fundamental weights corresponding to simple roots in $\Pi_Y^{(i)}$, and $w_{(i)} \in W_L$ the longest element of the Weyl group of $\Pi_Y^{(i)}$. Then $w_{(i)} \rho_{(i)} \in W_L C$, but $\langle \beta_Y, w_{(i)} \rho_{(i)} \rangle = -\langle \beta_Y, \rho_{(i)} \rangle < 0$, a contradiction.

On the other hand, if no component of Π_Y is contained in Π_L , then it is easy to find a positive linear combination β_Y of $\check{\Pi}_Y$ such that $\langle \beta_Y, \Pi_L \rangle \leq 0$ (choosing sufficiently large coefficients in β_Y successively at $\check{\alpha} \in \check{\Pi}_Y \cap \check{\Pi}_L$ and finally at $\check{\alpha} \in \check{\Pi}_Y \setminus \check{\Pi}_L$), whence $\langle \beta_Y, W_L C \rangle \geq 0$.

We have $|\Gamma| = \langle \Pi_L \rangle \cap \Pi_Y^\perp$, $\langle \Gamma \rangle^\perp = \langle \Pi_Y \rangle$, whence $\|\Gamma\|$ is spanned by $\Pi_Y \sqcup (\Pi_L \cap \Pi_Y^\perp) = \Pi_Y \cup (\Pi_L \setminus \partial \Pi_Y)$. Choosing a sufficiently large generating set $\{\lambda_1, \dots, \lambda_m\}$ for \mathfrak{X}^+ , we see that $\langle \Gamma \rangle_{\mathbb{Z}} = \langle \Gamma \rangle \cap \mathfrak{X}(T)$, whence $T^{\langle \Gamma \rangle_{\mathbb{Z}}}$ is connected and is in fact a maximal torus in L'_{Π_Y} . This proves the formulas for the stabilizers and for $d_G(Y)$.

Finally, if we look for an Y with $d_G(Y) = \max$, i.e., $\dim L_{\Pi_L \setminus \partial \Pi_Y} = \min$, we may always include $\Pi \setminus \Pi_L$ in Π_Y in order to enlarge $\Pi_L \cap \partial \Pi_Y$ as much as possible. It remains to note that $\dim L_{\Pi_L \setminus \partial \Pi_Y} = \min \iff \dim L_{\Pi_L^{(k)} \setminus \partial \Pi_Y} = \min, \forall k$. \square

EXAMPLE 3. Let $G = \mathrm{SL}(n)$, and let P be the stabilizer of a hyperplane in \mathbb{k}^n . Then $\mathrm{CE}(G/P_u) = \mathrm{Mat}(n, n-1)$ with the G -action by left multiplication. Let $\alpha_1, \dots, \alpha_{n-1}$ be the simple roots of G . The group $L \cong \mathrm{GL}(n-1)$ acts on $\mathrm{Mat}(n, n-1)$ by right multiplication, $\Pi_L = \{\alpha_1, \dots, \alpha_{n-2}\}$.

The possible choices for Π_Y are $\Pi_Y = \{\alpha_k, \dots, \alpha_{n-1}\}$, $1 \leq k \leq n$. The respective orbit Y consists of all matrices of rank $k-1$. We have $L_{\Pi_L \setminus \partial \Pi_Y} \cong \mathrm{GL}(k-1) \times \mathrm{GL}(n-k)$, and $d_G(Y) = (k-1)(n-k)$. The latter formula can be derived directly from the observation that a G -orbit in Y is formed by all matrices of rank $k-1$ with given linear dependencies between the columns. The space of linear dependencies depends on $(k-1)(n-k)$ parameters, which are the coefficients of linear expressions of all columns through the basic ones. (In fact, the respective moduli space is nothing else but the Grassmannian of $(n-k)$ -subspaces in \mathbb{k}^{n-1} .) The maximum value of $d_G(Y)$ is reached for $k = \lfloor (n+1)/2 \rfloor$, and $\mathrm{mod}_{\mathrm{SL}(n)} \mathrm{Mat}(n, n-1) = \lfloor (n-1)^2/4 \rfloor$.

We illustrate all the above results by another example of a canonical embedding.

EXAMPLE 4. Let $G = \mathrm{Sp}(2l)$, and let P be the stabilizer of a Lagrangian subspace $\mathbb{k}^l \subset \mathbb{k}^{2l}$. A complementary Lagrangian subspace is canonically isomorphic to $(\mathbb{k}^l)^*$, the pairing with \mathbb{k}^l being given by the symplectic form. Then $L = \mathrm{GL}(l)$ is the stabilizer of the decomposition $\mathbb{k}^{2l} = \mathbb{k}^l \oplus (\mathbb{k}^l)^*$.

Let $X \subseteq \mathrm{Mat}(2l, l)$ be the set of all linear maps with isotropic image. Then X is an affine embedding of G/P_u , so that eP_u is mapped to the identity map $\mathbb{k}^l \rightarrow \mathbb{k}^l \subset \mathbb{k}^{2l}$. In the notation of 3.3, we have $M = \mathrm{Mat}(l)$, and $G \times^P M \rightarrow X$, $(g, A) \mapsto g \begin{pmatrix} A \\ 0 \end{pmatrix}$, is the multiplication map.

Let $\alpha_1, \dots, \alpha_l$ be the simple roots, and $\omega_1, \dots, \omega_l$ the fundamental weights of G , in the standard order. Then $\Pi_L = \{\alpha_1, \dots, \alpha_{l-1}\}$, and ω_i , regarded as a dominant weight of L , is the highest weight of $\bigwedge^i \mathbb{k}^l$, $\forall i$. It follows that S is generated by $\omega_1, \dots, \omega_l$, since $S \ni \omega_1$. Therefore $X = \mathrm{CE}(G/P_u)$.

The $(G \times L)$ -orbits $Y \subset X$ (and $(L \times L)$ -orbits in M) consist of all matrices of a given rank k , $0 \leq k \leq l$. We have $\Pi_Y = \{\alpha_{k+1}, \dots, \alpha_l\}$, $L_{\Pi_L \setminus \partial \Pi_Y} \cong \mathrm{GL}(k) \times \mathrm{GL}(l-k)$, $d_G(Y) = k(l-k)$, and $\mathrm{mod}_G X = \lfloor l^2/4 \rfloor$. The reasoning is similar to that of Example 3.

3.5. Smoothness. Now we classify those affine embeddings of G/P_u which are smooth.

EXAMPLE 5. Here are three basic examples of smooth embeddings $X \hookrightarrow G/P_u$:

- (1) The embedding $X = \mathrm{Mat}(n, n-1)$ of Example 3 is smooth.
- (2) The embedding $X = \mathrm{Mat}(n)$ of $G = \mathrm{GL}(n)$ is smooth. (Here $P = L = G$.)
- (3) The group G itself is a smooth embedding of G . (Again $P = L = G$.)

Our next result shows that these are the only nontrivial examples of smooth affine embeddings.

THEOREM 4. *Any smooth affine $(G \times L)$ -embedding of G/P_u is of the form $X = G_0 \times^{Z_0} X_\perp$. Here $G = (G_0 \times G_\perp)/Z_0$ is the quotient of a product of two reductive groups by a finite central diagonally embedded subgroup, $P \supseteq G_0$, and the embedding $G_\perp/P_u \hookrightarrow X_\perp$ is the direct product of several embeddings (1), (2) of Example 5, where the actions on the factors of type (1) are possibly shifted by some characters of G_\perp .*

PROOF. The idea of the proof is similar to that of [Ti03, Theorem 9].

Let Γ_0 be the minimal face (i.e., maximal linear subspace) in Σ . Then $\langle \gamma, \Gamma_0 \cap C \rangle \geq 0$ for γ as in Remark 3. Since γ is fixed by W_L , we obtain $\langle \gamma, \Gamma_0 \rangle \geq 0 \implies \gamma \perp \Gamma_0$. It follows that Γ_0 is orthogonal to each component of Π not contained in Π_L .

Therefore each component of Π is either contained in $|\Gamma_0|$ or orthogonal to Γ_0 . Since X is smooth, hence normal, $\langle \Gamma_0 \rangle_Z = \Gamma_0 \cap \mathfrak{X}(T)$, whence $T_\perp = T^{\langle \Gamma_0 \rangle_Z}$ is a torus. Put $G_\perp = L'_{\langle \Gamma_0 \rangle_\perp} T_\perp$, and $G_0 = L'_{|\Gamma_0|} T_0$, where the subtorus $T_0 \subseteq T$ extends the maximal torus of $L'_{|\Gamma_0|}$ in such a way that $T = (T_0 \times T_\perp)/\text{diag } Z_0$, $Z_0 = T_0 \cap T_\perp$ being a finite central subgroup of G . Then $G = (G_0 \times G_\perp)/\text{diag } Z_0$.

It may happen that $\Sigma \cap \gamma^\perp \neq \Gamma_0$. However, the interior of the cone dual to $\Sigma \cap \gamma^\perp$ contains a nonzero vector $\gamma_0 \perp \Pi$: otherwise this interior is separated from Π^\perp by a linear function $\langle \cdot, \lambda \rangle$ for some $\lambda \in (\Sigma \cap \gamma^\perp \setminus \Gamma_0) \cap \langle \Pi \rangle \subseteq \Sigma \cap (\Pi_L) \setminus \Gamma_0 = \emptyset$, a contradiction. Replacing γ by a multiple of $\gamma + \gamma_0$, with γ_0 sufficiently small, we may assume that $\gamma \in \mathfrak{X}_*(Z)$ and $\langle \gamma, \Sigma \setminus \Gamma_0 \rangle > 0$.

By Theorem 3, the face Γ_0 corresponds to the closed $(G \times L)$ -orbit $Y_0 \ni e_{\Gamma_0}$. The γ -action by right translations of an argument yields an equivariant retraction $X \rightarrow Y_0$, $x \mapsto \lim_{t \rightarrow \infty} \gamma(t) * x$, $e_\Sigma \mapsto e_{\Gamma_0}$. Thus we have $X = (G \times L) \times^{(G \times L)_{e_{\Gamma_0}}} X_\perp$, $X_\perp = (G \times L)_{e_{\Gamma_0}} e_\Sigma \ni e_{\Gamma_0}$. But $(G \times L)_{e_{\Gamma_0}} = (G_\perp \times (L \cap G_\perp)) \cdot \text{diag } G_0$, and $\text{diag } G_0$ acts on X_\perp trivially as a normal subgroup in the stabilizer of the open orbit. Thus $X = G_0 \times^{Z_0} X_\perp$, where $X_\perp \hookrightarrow G_\perp/P_u$ is an embedding with a fixed point.

In the sequel, we assume that $X \subseteq \bigoplus_{i=1}^m \text{Hom}(V(\lambda_i)^{P_u}, V(\lambda_i))$ itself contains the fixed point 0. The γ -action contracts the ambient vector space on the right hand side to 0. After renumbering the λ_i 's, we assume that

$$T_0 X = \bigoplus_{i=1}^p \text{Hom}(V(\lambda_i)^{P_u}, V(\lambda_i))$$

for some $p \leq m$. Since X is smooth and contracted to 0 by γ , it projects onto $T_0 X$ isomorphically.

Let $e_{P_u} \mapsto (e_1, \dots, e_p)$ under this isomorphism. Then e_i has the dense $(G \times L)$ -orbit in $\text{Hom}(V(\lambda_i)^{P_u}, V(\lambda_i))$ and commutes with L , whence by Schur's lemma e_i is a nonzero scalar operator on $V(\lambda_i)^{P_u}$. After rescaling the above isomorphism, we assume that $e_i = \mathbf{1}_{V(\lambda_i)^{P_u}}$.

Let $G_i \subseteq \text{GL}(V(\lambda_i))$ be the image of G , and P_i, L_i the images of P, L . Then $G_i e_i$ is dense in $\text{Hom}(V(\lambda_i)^{P_u}, V(\lambda_i))$. It follows that the orbit of the highest weight vector is dense in $V(\lambda_i)$, whence G_i acts on $\mathbb{P}(V(\lambda_i))$ transitively. By [On62], [St82], $G_i = \text{GL}(V(\lambda_i))$, $\text{SL}(V(\lambda_i))$, $\text{Sp}(V(\lambda_i))$, or $\text{Sp}(V(\lambda_i)) \cdot \mathbb{k}^\times$, $P_i \neq G_i$ in the second case, and P_i fixes the highest weight line in the last two cases (so that $V(\lambda_i)^{P_u} \neq V(\lambda_i)$ and $\dim V(\lambda_i)^{P_u} = 1$, respectively).

Two simple components of G never project to the same G_i nontrivially (because their images must commute). However, there might exist a simple component

of G projecting to several G_i 's nontrivially. Let $i = i_1, \dots, i_q$ be the respective indices, and G_{i_1, \dots, i_q} , P_{i_1, \dots, i_q} , L_{i_1, \dots, i_q} the images of G, P, L in $G_{i_1} \times \dots \times G_{i_q}$. Then G'_{i_1, \dots, i_q} is simple, $\dim Z(G_{i_1, \dots, i_q}) \leq q$, the orbit $G_{i_1, \dots, i_q}(e_{i_1}, \dots, e_{i_q})$ is dense in $\bigoplus_{k=1}^q \text{Hom}(V(\lambda_{i_k})^{P_u}, V(\lambda_{i_k}))$, and the stabilizer of $(e_{i_1}, \dots, e_{i_q})$ in $G'_{i_1, \dots, i_q} \cap L_{i_1, \dots, i_q}$ is trivial. In particular, we have the inequality

$$\dim G_{i_1, \dots, i_q} - \dim (P_{i_1, \dots, i_q})_u \geq \sum_{k=1}^q \dim V(\lambda_{i_k}) \cdot \dim V(\lambda_{i_k})^{P_u},$$

which is strict whenever $\dim(L_{i_1, \dots, i_q})_{(e_{i_1}, \dots, e_{i_q})} > 0$. This leaves the following possibilities:

- (1) $G_{i_k} = \text{GL}(n) = P_{i_k}$, $V(\lambda_{i_k}) = \mathbb{k}^n$, $q + (n^2 - 1) \geq qn^2$;
- (2) $G_{i_k} = \text{GL}(n)$ or $\text{SL}(n)$, $V(\lambda_{i_k}) = \mathbb{k}^n$, P_{i_k} is the stabilizer of the hyperplane in \mathbb{k}^n given by vanishing of the last coordinate, $q + (n^2 - 1) - (n - 1) \geq qn(n - 1)$;
- (3) $G_{i_k} = \text{GL}(n)$ or $\text{SL}(n)$, $V(\lambda_{i_k}) = \mathbb{k}^n$ or $(\mathbb{k}^n)^*$ (both cases occur), P_{i_k} is the stabilizer of the subspace in \mathbb{k}^n generated by the first d basic vectors, $q + (n^2 - 1) - d(n - d) \geq nd + n(n - d) + (q - 2)n$.

In all cases we have either $q = 1$ or, in the last two cases, $q = n = 2$, and the inequalities become equalities. But in the latter situation $\dim Z(G_{i_1, \dots, i_q}) = q$, and it is easy to see that $\dim(L_{i_1, \dots, i_q})_{(e_{i_1}, \dots, e_{i_q})} > 0$, a contradiction.

Thus $G \hookrightarrow G_1 \times \dots \times G_p$, $G' = G'_1 \times \dots \times G'_p$, and $G(e_1, \dots, e_p)$ is dense in $\bigoplus_{i=1}^p \text{Hom}(V(\lambda_i)^{P_u}, V(\lambda_i))$, with stabilizer P_u . Now an easy dimension count shows that each triple $(G_i, P_i, V(\lambda_i))$ belongs to case (1) or (2), and $\dim Z(G)$ is the number of occurrences of (1). Thus $X = X_1 \times \dots \times X_p$, and each $X_i = \text{Hom}(V(\lambda_i)^{P_u}, V(\lambda_i))$ is an embedding of $G_i/(P_i)_u$, in case (1), or $G'_i/(P_i)_u$, in case (2). \square

COROLLARY 3. *The canonical embedding $\text{CE}(G/P_u)$ is smooth iff $G = (G_0 \times G_\perp)/Z_0$ is the quotient of a product of two reductive groups by a finite central diagonally embedded subgroup, $G_\perp = G_1 \times \dots \times G_p$, $G_i = \text{SL}(n_i)$ ($i > 0$), $P \supseteq G_0$, and $P \cap G_i$ are the stabilizers of hyperplanes (or lines) in \mathbb{k}^{n_i} .*

3.6. Tangent spaces. Finally, we describe the tangent space $T_0 \text{CE}(G/P_u)$ of $\text{CE}(G/P_u)$ at the unique G -fixed point 0 , assuming that G is simple and $P \neq G$ (see Lemma 2). The G -module structure of this tangent space provides information on ambient G -modules for $\text{CE}(G/P_u)$; namely $T_0 \text{CE}(G/P_u)$ is the smallest one.

As $\mathbb{k}[\text{CE}(G/P_u)]$ is nonnegatively graded by a one-parameter subgroup $\gamma \in \mathfrak{X}_*(Z)$ so that $\mathbb{k}[\text{CE}(G/P_u)]_0 = \mathbb{k}$ (see Remark 3), the space $T_0 \text{CE}(G/P_u)$ is dual to the linear span of a minimal system of homogeneous generators for $\mathbb{k}[\text{CE}(G/P_u)]$. Thus to describe $T_0 \text{CE}(G/P_u)$ is the same thing as to find the minimal homogeneous generating subspace for $\mathbb{k}[\text{CE}(G/P_u)]$, or to find the minimal L -generating set for \mathfrak{X}^+ .

For simplicity, we assume that G is simply connected. Then \mathfrak{X}^+ is freely generated by the fundamental weights $\omega_1, \dots, \omega_l$, and it suffices to find out which ω_i are L -generated by the other fundamental weights.

Let $\alpha_1, \dots, \alpha_l$ be the simple roots of G , and $\check{\alpha}_i, \check{\omega}_i$ denote the simple coroots and the fundamental coweights, respectively.

DEFINITION 3. The *singularity* of a Dynkin diagram is either the node of branching or the node representing the long root neighboring with a short one.

The Z -action by right translations of an argument defines an invariant algebra multigrading of $\mathbb{k}[G/P_u]$ so that $\mathbb{k}[G/P_u]_{(\lambda)}$ has the weight $\lambda|_Z$. A choice of $\gamma \in \mathfrak{X}_*(Z) \otimes \mathbb{Q} = \Pi_L^\perp = \langle \tilde{\omega}_i \mid \alpha_i \notin \Pi_L \rangle$ yields a specialization of this multigrading, so that $\deg \mathbb{k}[G/P_u]_{(\lambda)} = \langle \gamma, \lambda \rangle$; cf. Remark 3. (The degrees might be rational numbers; however, multiplying γ by a sufficiently large number yields an integer grading.) For brevity, we shall speak about the *degree* of λ with respect to γ .

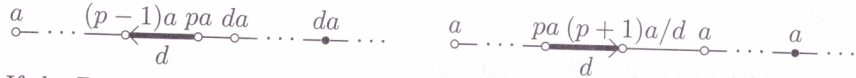
Put $\bar{\lambda} = \lambda|_{T \cap L'}$, $\forall \lambda \in \mathfrak{X}^+$. Then $\bar{\omega}_i$ is a fundamental weight of the commutator group L' whenever $\alpha_i \in \Pi_L$, or zero, otherwise. Note that $V_L(\lambda) \hookrightarrow V_L(\lambda_1) \otimes \dots \otimes V_L(\lambda_n)$ iff $V(\bar{\lambda}) \hookrightarrow V(\bar{\lambda}_1) \otimes \dots \otimes V(\bar{\lambda}_n)$ and $\deg \lambda = \deg \lambda_1 + \dots + \deg \lambda_n$ with respect to $\forall \gamma \in \mathfrak{X}_*(Z) \otimes \mathbb{Q}$.

The degrees with respect to the generators $\tilde{\omega}_i$ are determined in terms of the matrix $(\langle \tilde{\omega}_i, \omega_j \rangle)_{i,j=1}^l$, which is the inverse transpose of the Cartan matrix of G . These matrices are computed in [OV88, Table 2]. The i th row of this matrix represents the degrees d_1, \dots, d_l of $\omega_1, \dots, \omega_l$ with respect to $\tilde{\omega}_i$. Let us label the nodes of the Dynkin diagram by these degrees. An inspection of the inverse transposed Cartan matrices yields the following observation:

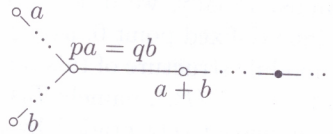
- The labels of nodes in a segment from an extreme node up to either α_i or the singularity form a sequence $a, 2a, \dots, pa$,



- If the Dynkin diagram has no branching, then the nodes after the singularity up to α_i are labeled by da, \dots, da or $(p+1)a/d, a, \dots, a$, where d is the multiplicity of the “thick” edge, depending on whether α_i is a long root or not,



- If the Dynkin diagram has the branching, then the nodes at the branches not containing α_i are labeled by $a, 2a, \dots, pa$ and $b, 2b, \dots, qb$ as above, and the nodes at the third branch from the singularity up to α_i are labeled by a decreasing arithmetic progression $pa = qb, a + b, \dots$,



THEOREM 5. Suppose that G is simple simply connected and $P \neq G$. Then $T_0 \text{CE}(G/P_u)$ is the $(G \times L)$ -stable subspace of

$$\bigoplus_{i=1}^l \text{Hom}(V(\omega_i)^{P_u}, V(\omega_i))$$

obtained by removing certain summands using the following procedure.

- (1) Take any $\alpha_k \in \Pi_L$ represented by an extreme node of the Dynkin diagram of G .

- (2) Remove subsequently all the i th summands corresponding to α_i which follow after α_k at the Dynkin diagram until you pass the first instance of $\alpha_i \notin \Pi_L$ or the singularity,



- (3) If G is simply laced, and at least two branches of the Dynkin diagram are contained in Π_L , then continue removing the summands along the third branch after the singularity as in (2) until, in the case $G = \mathbf{E}_l$, the removed segment becomes longer than both other branches,



- (4) If G is not simply laced, and you have passed the singularity along the direction to long roots, then continue removing summands as in (2),



EXAMPLES. Let $G = \mathbf{E}_8$, and let P be the projective stabilizer of a highest weight vector in $V(\omega_1)$, in the enumeration of [OV88, Table 1]. Then $L' = \mathbf{E}_7$, with the simple roots corresponding to the black nodes of the diagram:



We have $\dim \text{CE}(G/P_u) = 191$, but the minimal ambient G -module is

$$\begin{aligned} & (V(\omega_1) \otimes V_L(\omega_1)^*) \oplus (V(\omega_2) \otimes V_L(\omega_2)^*) \\ & \oplus (V(\omega_7) \otimes V_L(\omega_7)^*) \oplus (V(\omega_8) \otimes V_L(\omega_8)^*) \end{aligned}$$

of dimension $248 \cdot 1 + 30380 \cdot 56 + 3875 \cdot 133 + 147250 \cdot 912 = 136508903$.

Now take $G = \mathbf{F}_4$ and P the projective stabilizer of a highest weight vector in $V(\omega_1)$ again. Then $L' = \text{Spin}(7)$, with the simple roots corresponding to the black nodes of the diagram:



We have $\dim \text{CE}(G/P_u) = 37$, and the minimal ambient G -module

$$(V(\omega_1) \otimes V_L(\omega_1)^*) \oplus (V(\omega_2) \otimes V_L(\omega_2)^*) \oplus (V(\omega_4) \otimes V_L(\omega_4)^*)$$

has dimension $26 \cdot 1 + 273 \cdot 8 + 52 \cdot 7 = 2574$.

PROOF. The space $\bigoplus_i \text{Hom}(V(\omega_i)^{P_u}, V(\omega_i))$ is dual to $\bigoplus_i \mathbb{k}[G/P_u]_{(\omega_i)}$, a generating subspace of $\mathbb{k}[G/P_u]$. To obtain the tangent space, it suffices to remove summands corresponding to ω_i which are L -generated by the others.

First observe that if $\alpha_i \notin \Pi_L \sqcup \partial\Pi_L$, then ω_i is not L -generated by the other fundamental weights. Indeed, specialize the multigrading of $\mathbb{k}[G/P_u]$ using $\check{\alpha}_i$. Then $\deg \omega_i = 1$, but $\deg \omega_j = 0, \forall j \neq i$.

Secondly, ω_i is L -generated by the other ω_j 's iff it is L_k -generated by the other ω_j 's such that $\alpha_j \in \Pi_{L_k}$, where L_k is one of the simple factors of L . Indeed, each dominant weight L -generated by ω_j 's is the sum of dominant weights L_k -generated by ω_j such that $\alpha_j \in \Pi_{L_k}$, over all simple factors $L_k \subseteq L$, and of

a dominant weight generated by ω_j such that $\alpha_j \notin \Pi_L$. However, specializing the multigrading of $\mathbb{k}[G/P_u]$ to a nonnegative grading such that $\mathbb{k}[G/P_u]_0 = \mathbb{k}$ (Remark 3) shows that ω_j 's do not L_k -generate 0. The assertion follows, because ω_i cannot be decomposed as a nontrivial sum of dominant weights. Thus we may assume that Π_L is indecomposable.

In order to verify that certain ω_j are L -generated by the others (as asserted in Theorem 5), we use the following formulas [OV88, Table 5]:

- (1) $V(\bar{\omega}_1)^{\otimes i} \leftrightarrow V(\bar{\omega}_i)$ for:
 $L' = \mathrm{SL}(m), \mathrm{Sp}(2m), 1 \leq i \leq m; L' = \mathrm{Spin}(2m+1), 1 \leq i < m;$
 $L' = \mathrm{Spin}(2m), 1 \leq i \leq m-2.$
- (2) $V(\bar{\omega}_1) \otimes V(\bar{\omega}_{m-1}) \leftrightarrow V(0)$ for $L' = \mathrm{SL}(m).$
- (3) $V(\bar{\omega}_1)^{\otimes 2} \leftrightarrow V(0)$ for $L' = \mathrm{Sp}(2m), \mathrm{Spin}(2m).$
- (4) $V(\bar{\omega}_1) \otimes V(\bar{\omega}_{m-1}) \leftrightarrow V(\bar{\omega}_m)$ for $L' = \mathrm{Spin}(2m).$
- (5) $V(\bar{\omega}_m)^{\otimes 2} = V(2\bar{\omega}_m) \oplus V(\bar{\omega}_{m-1}) \oplus \cdots \oplus V(\bar{\omega}_1) \oplus V(0)$
for $L' = \mathrm{Spin}(2m+1).$
- (6) $V(\bar{\omega}_m)^{\otimes 2} = V(2\bar{\omega}_m) \oplus V(\bar{\omega}_{m-2}) \oplus V(\bar{\omega}_{m-4}) \oplus \cdots$
for $L' = \mathrm{Spin}(2m).$
- (7) $V(\bar{\omega}_{m-1}) \otimes V(\bar{\omega}_m) = V(\bar{\omega}_{m-1} + \bar{\omega}_m) \oplus V(\bar{\omega}_{m-3}) \oplus V(\bar{\omega}_{m-5}) \oplus \cdots$
for $L' = \mathrm{Spin}(2m).$
- (8) $V(\bar{\omega}_{m-1})^{\otimes 2} \leftrightarrow V(\bar{\omega}_{m-2}) \oplus V(\bar{\omega}_{m-5})$ for $L' = \mathbf{E}_m, m = 6, 7.$
- (9) $V(\bar{\omega}_m)^{\otimes 2} \leftrightarrow V(\bar{\omega}_{m-3})$ for $L' = \mathbf{E}_m, m = 6, 7.$
- (10) $V(\bar{\omega}_{m-1}) \otimes V(\bar{\omega}_m) \leftrightarrow V(\bar{\omega}_{m-4})$ for $L' = \mathbf{E}_m, m = 6, 7.$

Here the fundamental weights of L are numbered according to [OV88, Table 1]. The respective relations between degrees are easily verified using the above description of degrees with respect to fundamental coweights. Note that it suffices to consider degrees with respect to $\bar{\omega}_i$ such that $\alpha_i \in \partial\Pi_L$, because $\mathfrak{X}_*(Z) \otimes \mathbb{Q} = \langle \bar{\omega}_i, \check{\alpha}_j \mid \alpha_i \in \partial\Pi_L, \alpha_j \notin \Pi_L \sqcup \partial\Pi_L \rangle$ and the degrees of fundamental weights corresponding to roots in $\Pi_L \sqcup \partial\Pi_L$ with respect to $\check{\alpha}_j \notin \check{\Pi}_L \sqcup \partial\check{\Pi}_L$ are zero.

For instance, suppose that $\Pi = \mathbf{E}_l, \Pi_L = \mathbf{D}_{l-1}$. Let us enumerate the simple roots of G as in the picture:

$$\begin{array}{ccccccc} a & 2a & \dots & (l-3)a & = 2b & a+b & 2a \\ \bullet & \times & & \times & & \times & \times \\ \alpha_1 & \alpha_2 & & \alpha_{l-3} & & \alpha_{l-1} & \alpha_l \\ & & & \downarrow & & & \\ & & & b \bullet & \alpha_{l-2} & & \end{array}$$

We consider the degrees with respect to $\bar{\omega}_l$. Using (1) and $d_i = id_1$, we verify that ω_i are L -generated by $\omega_1, 1 \leq i \leq l-3$. By (4) and $d_1 + d_{l-2} = d_{l-1}$, we see that ω_{l-1} is L -generated by ω_1, ω_{l-2} . Finally, (3) and $d_l = 2d_1$ imply that ω_l is L -generated by ω_1 .

It remains to prove that the remaining fundamental weights are not L -generated by the others. We shall use the following observation from the representation theory of $\mathrm{SL}(m)$:

$$(\dagger) \quad V(\bar{\omega}_{j_1}) \otimes \cdots \otimes V(\bar{\omega}_{j_n}) \leftrightarrow V(\bar{\omega}_i) \implies j_1 + \cdots + j_n \geq i, \\ 1 \leq i, j_1, \dots, j_n \leq m.$$

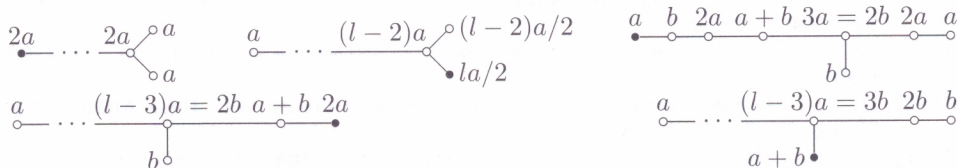
(Here $\bar{\omega}_m = 0$ and the other $\bar{\omega}_j$ are the fundamental weights of $\mathrm{SL}(m)$ in the standard order.) In the sequel, we shall frequently apply (\dagger) to $L' = \mathrm{SL}(m)$ in the following way: it often happens that the conclusion of (\dagger) implies $d_{j_1} + \cdots + d_{j_n} > d_i$, whence ω_i is not L -generated by the other ω_j 's.

First suppose that the Dynkin diagram of G has no branching.

Fix any $\alpha_m \notin \Pi_L$ and consider the degrees of fundamental weights with respect to $\tilde{\omega}_m$ on one of the segments of $\Pi \setminus \{\alpha_m\}$. From the above description of degrees, we easily see that $d_i < d_j + d_k$ unless α_j, α_k are farther from α_m than α_i is. Hence on this segment each ω_i could be L -generated only by fundamental weights corresponding to the roots on the other side of α_i than α_m . We immediately deduce that if Π_L does not contain an extreme node of (the Dynkin diagram of) Π , then no fundamental weight is L -generated by the others.

Now assume that α_m is a short root and look at the degrees on the segment from an extreme node to α_m containing the singularity. For $\Pi = \mathbf{C}_l, \mathbf{F}_4, \mathbf{G}_2$ we have $d_i < d_j + d_k$ whenever α_i is short; hence the fundamental weights corresponding to the short roots after the singularity up to $\partial\Pi_L$ are not L -generated by the others. The same assertion for the unique short root α_l of $\Pi = \mathbf{B}_l$ stems from (†).

Next, suppose that the Dynkin diagram of G has the branching. We consider the degrees with respect to $\tilde{\omega}_m$ such that α_m corresponds to the extreme node of a ray of the Dynkin diagram. For reader's convenience we indicate these degrees in the diagrams, where the black node corresponds to α_m (the picture for α_m on the long ray of \mathbf{E}_l is obtained from that for \mathbf{E}_8 by cutting off $8 - l$ subsequent nodes on the long ray, starting with the extreme node):



We enumerate the simple roots and fundamental weights of G according to [OV88, Table 1].

One sees from this picture that the fundamental weights ω_i corresponding to extreme nodes of the two other rays, as well as ω_i for α_m on the long ray of \mathbf{E}_l and $i < l - 5$, are not L -generated by the others. Indeed, for α_m on a short ray of \mathbf{D}_l or on the middle ray of \mathbf{E}_l we can use (†) or the fact that a semispinor weight of \mathbf{D}_{l-1} is not \mathbf{D}_{l-1} -generated by $\tilde{\omega}_1$. In all other cases we have $d_i < d_j + d_k$, $1 \leq j, k \leq l$.

Now assume that the extreme nodes of at least two rays are not in Π_L .

If one of these rays is long, then looking at the degrees with respect to the $\tilde{\omega}_m$ we see that except for the weights on the third ray, which could be L -generated by the one at the extreme node, the only possibilities for L -generation are: $\Pi = \mathbf{D}_l$, $d_i = 2d_l$, $V_L(\omega_i) \hookrightarrow V_L(\omega_{l-1})^{\otimes 2}$, $1 \leq i < l - 2$; $\Pi = \mathbf{E}_8$, $d_5 = 2d_2$, $V_L(\omega_5) \hookrightarrow V_L(\omega_2)^{\otimes 2}$; $\Pi = \mathbf{E}_8$, $d_4 = d_2 + d_7$, $V_L(\omega_4) \hookrightarrow V_L(\omega_2) \otimes V_L(\omega_7)$; $\Pi = \mathbf{E}_l$, $d_{l-5} = 2d_{l-1}$, $V_L(\omega_{l-5}) \hookrightarrow V_L(\omega_{l-1})^{\otimes 2}$. However, these possibilities are excluded by considering the degrees with respect to the extreme node of the second ray.

Otherwise, consider the degrees with respect to $\tilde{\omega}_m$ such that α_m is at the extreme node of the short ray.

For $\Pi = \mathbf{D}_l$, $V_L(\omega_i) \hookrightarrow V_L(\omega_{j_1}) \otimes \cdots \otimes V_L(\omega_{j_n})$, $j_1, \dots, j_n \leq l - 2 < i$, implies $d_i = d_{j_1} + \cdots + d_{j_n}$. However, considering the degrees with respect to the extreme node of the second ray violates this equality, a contradiction.

For $\Pi = \mathbf{E}_l$, the only possibilities for L -generation are: $l = 6$, $d_4 = d_6 = 2d_1$, and $V_L(\omega_4)$ or $V_L(\omega_6)$ is contained in $V_L(\omega_1)^{\otimes 2}$. However these possibilities are

excluded by considering the degrees with respect to the extreme node of the second ray.

We conclude that no fundamental weights on a segment between two nodes of the Dynkin diagram not contained in Π_L are L -generated by the others, except possibly the one at the singularity provided that one of the rays is contained in Π_L . This completes the proof. \square

REMARK 5. Our results immediately extend to the case where G is semisimple and simply connected; see Remark 2. The general case looks more complicated because the structure of \mathfrak{X}^+ is more involved. It would be interesting to solve the problem in full generality.

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