

On ultrafilter extensions of first-order models and ultrafilter interpretations

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Abstract

There exist two known concepts of ultrafilter extensions of first-order models, both in a certain sense canonical. One of them [1] comes from modal logic and universal algebra, and in fact goes back to [2]. Another one [3, 4] comes from model theory and algebra of ultrafilters, with ultrafilter extensions of semigroups [5] as its main precursor. By a classical fact of general topology, the space of ultrafilters over a discrete space is its largest compactification. The main result of [3, 4], which confirms a canonicity of this extension, generalizes this fact to discrete spaces endowed with an arbitrary first-order structure. An analogous result for the former type of ultrafilter extensions was obtained in [6]. Results of such type are referred to as extension theorems.

After a brief introduction to this area, we offer a uniform approach to both types of extensions. It is based on the idea to extend the extension procedure itself. We propose a generalization of the standard concept of first-order models in which functional and relational symbols are interpreted rather by ultrafilters over sets of functions and relations than by functions and relations themselves, and an appropriate semantic for generalized models of this form. We provide two specific operations which turn generalized models into ordinary ones, establish necessary and sufficient conditions under which the latter are the two canonical ultrafilter extensions of some ordinary models, and provide a topological characterization of generalized models. Defining a natural concept of homomorphisms between generalized models, we generalize a restricted version of the extension theorem to generalized models. To formulate the full version, we provide even a wider concept of ultrafilter interpretations together with their semantic based on limits of ultrafilters, and show that the former concept can be identified, in a certain way, with a partial case of the latter; moreover, the new concept absorbs the ordinary concept of models. We provide two more specific operations which turn generalized models in the narrow sense into ones in the wide sense, and establish necessary and sufficient conditions under which generalized models in the wide sense are the images of ones in the narrow sense under these operations, and also are two canonical ultrafilter extensions of some ordinary models. Finally, we define homomorphisms between generalized models in the wide sense, and establish for them three full versions of the extension theorem.

The results of first three sections of this paper were partially announced in [7].

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1. Introduction

In this section, we recall main definitions and facts concerning ultrafilter extensions of arbitrary maps, relations, and first-order models. All results mentioned here are established in various previous papers, so we omit their proofs. The section provides also some (of necessity incomplete) historical information.

Fix a first-order language and consider an arbitrary model \mathfrak{A} in the language:

$$\mathfrak{A} = (X, F, \dots, R, \dots)$$

with the universe X , operations F, \dots , and relations R, \dots . Let us define an abstract *ultrafilter extension* of \mathfrak{A} as a model \mathfrak{A}' in the same language of form

$$\mathfrak{A}' = (\beta X, F', \dots, R', \dots)$$

where βX is the set of ultrafilters over X (one lets $X \subseteq \beta X$ by identifying each $x \in X$ with the principal ultrafilter given by x), and operations F', \dots and relations R', \dots on βX extend F, \dots and R, \dots , respectively. There are essentially *two* known ways to extend relations by ultrafilters, and *one* to extend maps. Partial cases of these extensions were discovered by various authors in different time and different areas, typically, without a knowledge of parallel studies in adjacent areas.

Recall that βX carries a natural topology generated by basic open sets

$$\tilde{A} = \{\mathfrak{u} \in \beta X : A \in \mathfrak{u}\}$$

for all $A \subseteq X$. Easily, the sets are also closed, so the space βX is zero-dimensional. Moreover, βX is compact, Hausdorff, extremally disconnected (the closure of any open set is open), and the largest compactification of the discrete space X . This means that X is dense in βX and every (trivially continuous) map h of X into any compact Hausdorff space Y uniquely extends to a continuous map \tilde{h} of βX into Y :

$$\begin{array}{ccc} \beta X & & \\ \uparrow & \searrow \tilde{h} & \\ X & \xrightarrow{h} & Y \end{array}$$

by letting for all $\mathfrak{u} \in \beta X$,

$$\tilde{h}(\mathfrak{u}) = y \text{ where } \{y\} = \bigcap_{A \in \mathfrak{u}} \text{cl}_Y h'' A.$$

(As usual, $\text{cl}_S A$ is the closure of A in S , and $f'' A$ is the image of A under f .) The largest compactification of Tychonoff spaces was discovered independently by Čech [8] and M. Stone [9]; then Wallman [10] did the same for T_1 spaces (by using ultrafilters on lattices of closed sets); see [5, 11, 12] for more information.

The ultrafilter extension of a unary relation R on a set X is exactly the basic (cl)open set \tilde{R} , and the ultrafilter extension of a unary map $F : X \rightarrow Y$, where Y is a compact Hausdorff space (for operations F on X we let $Y = \beta X$ as $X \subseteq \beta X$), is exactly its continuous extension \tilde{F} . Thus in the unary case, the procedure gives classical objects known since 1930s. As for maps and relations of greater arities, several instances of their ultrafilter extensions were discovered only in 1960s.

Ultrafilter extensions of maps. Studying ultraproducts, Kochen [13] and Frayne, Morel, and Scott [14] considered a “multiplication” of ultrafilters, which actually is the ultrafilter extension of the n -ary operation of taking n -tuples. They shown that the successive iteration of ultrapowers by ultrafilters $\mathbf{u}_1, \dots, \mathbf{u}_n$ is isomorphic to a single ultrapower by their “product”. This has led to the general construction of iterated ultrapowers, invented by Gaifman and elaborated by Kunen, which has become common in model theory and set theory (see [15, 16]).

Ultrafilter extensions of semigroups appeared in 1960s as subspaces of function spaces; the first explicit construction of the ultrafilter extension of a group is due to Ellis [17]. In 1970s Galvin and Glazer applied them to give an easy proof of what now known as Hindman’s Finite Sums Theorem; the key idea was to use ultrafilters that are idempotent w.r.t. the extended operation. Then the method was developed by Bergelson, Blass, van Douwen, Hindman, Protasov, Strauss, and many others, and provided numerous Ramsey-theoretic applications in number theory, algebra, topological dynamics, and ergodic theory. The book [5] is a comprehensive treatise of this area, with an historical information. This technique was recently applied for obtaining analogous results for certain non-associative algebras (see [18, 19]).

Ultrafilter extensions of arbitrary n -ary maps have been introduced independently in recent works by Goranko [1] and Saveliev [3, 4]. For a map $F : X_1 \times \dots \times X_n \rightarrow Y$, the extended map $\tilde{F} : \beta X_1 \times \dots \times \beta X_n \rightarrow \beta Y$ is defined by letting

$$\tilde{F}(\mathbf{u}_1, \dots, \mathbf{u}_n) = \{A \subseteq Y : \{x_1 \in X_1 : \dots \{x_n \in X_n : F(x_1, \dots, x_n) \in A\} \in \mathbf{u}_n \dots\} \in \mathbf{u}_1\}.$$

One can simplify this cumbersome notation by introducing *ultrafilter quantifiers*. For every ultrafilter \mathbf{u} over a set X and formula $\varphi(x, \dots)$ with parameters x, \dots valuated over X , let

$$(\forall^{\mathbf{u}}x) \varphi(x, \dots) \text{ mean } \{x : \varphi(x, \dots)\} \in \mathbf{u}.$$

In fact, such quantifiers are a special kind of second-order quantifiers: $(\forall^{\mathbf{u}}x)$ is equivalent to $(\forall A \in \mathbf{u})(\exists x \in A)$, and also (since \mathbf{u} is ultra) to $(\exists A \in \mathbf{u})(\forall x \in A)$. Note also that ultrafilter quantifiers are self-dual, i.e. $\forall^{\mathbf{u}}$ and $\exists^{\mathbf{u}}$ coincide; they generally do not commute with each other, i.e. $(\forall^{\mathbf{u}}x)(\forall^{\mathbf{v}}y)$ and $(\forall^{\mathbf{v}}y)(\forall^{\mathbf{u}}x)$ are generally not equivalent; and if \mathbf{u} is the principal ultrafilter given by $a \in X$ then $(\forall^{\mathbf{u}}x)\varphi(x, \dots)$ is reduced to $\varphi(a, \dots)$.

Now the definition above can be rewritten as follows:

$$\tilde{F}(\mathbf{u}_1, \dots, \mathbf{u}_n) = \{A \subseteq Y : (\forall^{\mathbf{u}_1}x_1) \dots (\forall^{\mathbf{u}_n}x_n) F(x_1, \dots, x_n) \in A\}.$$

The map \tilde{F} can be also described as the composition of the ultrafilter extension of taking n -tuples, which maps $\beta X_1 \times \dots \times \beta X_n$ into $\beta(X_1 \times \dots \times X_n)$, and the continuous extension of F considered as a unary map, which maps $\beta(X_1 \times \dots \times X_n)$ into βY .

Ultrafilter extensions of relations. One type of ultrafilter extensions of relations goes back to a seminal paper by Jónsson and Tarski [2] where they have been appeared implicitly, in terms of representations of Boolean algebras with operators. For binary relations, their representation theory was rediscovered in modal logic by Lemmon [20] who credited much of this work to Scott (see footnote 6 on p. 204); see also [21]. Goldblatt and Thomason [22] (where Section 2 was entirely due to Goldblatt) used this to characterize modal definability; the term “ultrafilter extension” has been coined probably in the subsequent work by van Benthem [23] (for modal definability see also [24, 25]). Later Goldblatt [26] considered the extension of n -ary relations in the context of universal algebra and model theory.

Let us give a definition that is equivalent to one appeared in [1] (or [2] and [26]): for a relation $R \subseteq X_1 \times \dots \times X_n$, the extended relation $R^* \subseteq \beta X_1 \times \dots \times \beta X_n$ is defined by letting

$$R^*(\mathbf{u}_1, \dots, \mathbf{u}_n) \text{ iff} \\ (\forall A_1 \in \mathbf{u}_1) \dots (\forall A_n \in \mathbf{u}_n) (\exists x_1 \in A_1) \dots (\exists x_n \in A_n) R(x_1, \dots, x_n).$$

Another type of ultrafilter extensions of n -ary relations has been recently discovered in [3, 4]:

$$\tilde{R}(\mathbf{u}_1, \dots, \mathbf{u}_n) \text{ iff} \\ \{x_1 \in X_1 : \dots \{x_n \in X_n : R(x_1, \dots, x_n) \in A\} \in \mathbf{u}_n \dots\} \in \mathbf{u}_1,$$

or rewriting this via ultrafilter quantifiers,

$$\tilde{R}(\mathbf{u}_1, \dots, \mathbf{u}_n) \text{ iff } (\forall^{u_1} x_1) \dots (\forall^{u_n} x_n) R(x_1, \dots, x_n).$$

By decoding ultrafilter quantifiers, this also can be rewritten by

$$\tilde{R}(\mathbf{u}_1, \dots, \mathbf{u}_n) \text{ iff} \\ (\forall A_1 \in \mathbf{u}_1) (\exists x_1 \in A_1) \dots (\forall A_n \in \mathbf{u}_n) (\exists x_n \in A_n) R(x_1, \dots, x_n),$$

whence it is clear that

$$\tilde{R} \subseteq R^*.$$

If R is a unary relation, both extensions, \tilde{R} and R^* , coincide with the basic open set given by R (and with $\text{cl}_{\beta X} R$, the closure of R in the space βX). If a binary relation R is functional, then R^* (but not \tilde{R}) coincides with the above-defined extension of R as a unary map; this fact does not for relations of bigger arities. An easy instance of the \sim -extensions, where R are linear orders, was studied in [27].

A systematic comparative study of both extensions (for binary R) is undertaken in [6]. In particular, there is shown that the $*$ - and the \sim -extensions have a dual character w.r.t. relation-algebraic operations: the $*$ -extension commutes with composition and inversion but not Boolean operations except for union, while the \sim -extension commutes with all Boolean operations but neither composition nor inversion. Also [6] provides topological characterizations of \tilde{R} and R^* in terms of appropriate closure operations and in terms of Vietoris-type topologies (regarding R as multi-valued maps).

Ultrafilter extensions of models. Ultrafilter extensions of arbitrary first-order models were defined and studied for the first time independently in [1] and in [3] with two distinct versions of extended relations: Goranko considered models with the $*$ -extensions of relations and Saveliev with their \sim -extensions. Here we shall consider both types of extensions; for a given model \mathfrak{A} denote them by \mathfrak{A}^* and $\tilde{\mathfrak{A}}$, respectively.¹ Thus for an arbitrary model $\mathfrak{A} = (X, F, \dots, R, \dots)$ we let

$$\mathfrak{A}^* = (\beta X, \tilde{F}, \dots, R^*, \dots) \quad \text{and} \quad \tilde{\mathfrak{A}} = (\beta X, \tilde{F}, \dots, \tilde{R}, \dots).$$

Since for any relation R we have $\tilde{R} \subseteq R^*$, the following observation is obvious:

¹ Another notation was used in [1] where \mathfrak{A}^* was denoted by $\mathbf{U}(\mathfrak{A})$ and in [3, 4, 28] where $\tilde{\mathfrak{A}}$ was denoted by $\beta\mathfrak{A}$.

Theorem 1. For any model \mathfrak{A} with the universe X the identity map on βX is a homomorphism of $\widetilde{\mathfrak{A}}$ onto \mathfrak{A}^* :

$$\begin{array}{ccc} \widetilde{\mathfrak{A}} & \xrightarrow{\text{id}} & \mathfrak{A}^* \\ & \searrow & \nearrow \\ & \mathfrak{A} & \end{array}$$

The following theorem has been appeared in [3] and called the First Extension Theorem in [4]:

Theorem 2. Let \mathfrak{A} and \mathfrak{B} be two models in the same signature. If h is a homomorphism between \mathfrak{A} and \mathfrak{B} , then the continuous extension \widetilde{h} is a homomorphism between $\widetilde{\mathfrak{A}}$ and $\widetilde{\mathfrak{B}}$:

$$\begin{array}{ccc} \widetilde{\mathfrak{A}} & \xrightarrow{\widetilde{h}} & \widetilde{\mathfrak{B}} \\ \uparrow & & \uparrow \\ \mathfrak{A} & \xrightarrow{h} & \mathfrak{B} \end{array}$$

Theorem 2 on the \sim -extensions is a precise counterpart of Theorem 3 on the $*$ -extensions, a principal result of [1]:

Theorem 3. Let \mathfrak{A} and \mathfrak{B} be two models in the same signature. If h is a homomorphism between \mathfrak{A} and \mathfrak{B} , then the continuous extension \widetilde{h} is a homomorphism between \mathfrak{A}^* and \mathfrak{B}^* :

$$\begin{array}{ccc} \mathfrak{A}^* & \xrightarrow{\widetilde{h}} & \mathfrak{B}^* \\ \uparrow & & \uparrow \\ \mathfrak{A} & \xrightarrow{h} & \mathfrak{B} \end{array}$$

Moreover, both theorems remain true for isomorphic embeddings and some other model-theoretic interrelations (see [1, 3, 4]). It was shown in [28] that Theorem 2 does not hold for elementary embeddings, moreover, the ultrafilter extensions of a model and an its elementary submodel can be even non-elementarily equivalent.

Theorem 2 is actually a partial case of a much stronger result of [3], called the Second Extension Theorem in [4]. To formulate this, we need the following concepts introduced in [3].

Let X_1, \dots, X_n, Y be topological spaces, and let $A_1 \subseteq X_1, \dots, A_{n-1} \subseteq X_{n-1}$. An n -ary function $F : X_1 \times \dots \times X_n \rightarrow Y$ is *right continuous w.r.t. A_1, \dots, A_{n-1}* iff for each i , $1 \leq i \leq n$, and every $a_1 \in A_1, \dots, a_{i-1} \in A_{i-1}$ and $x_{i+1} \in X_{i+1}, \dots, x_n \in X_n$, the unary map

$$x \mapsto F(a_1, \dots, a_{i-1}, x, x_{i+1}, \dots, x_n)$$

of X_i into Y is continuous. An n -ary relation $R \subseteq X_1 \times \dots \times X_n$ is *right open (right closed, right clopen, etc.) w.r.t. A_1, \dots, A_{n-1}* iff for each i , $1 \leq i \leq n$, and every $a_1 \in A_1, \dots, a_{i-1} \in A_{i-1}$ and $x_{i+1} \in X_{i+1}, \dots, x_n \in X_n$, the set

$$\{x \in X_i : R(a_1, \dots, a_{i-1}, x, x_{i+1}, \dots, x_n)\}$$

is open (closed, clopen, etc.) in X_i .

Theorem 4 ([3, 4]) describes topological properties of the \sim -extensions and serves as a base of Theorem 5, the Second Extension Theorem of [4]. (A very partial case of the latter theorem, in which the models under consideration are semigroups, has been appeared in [29], Theorem 4.5.3.)

Theorem 4. *Let \mathfrak{A} be a model. In the extension $\widetilde{\mathfrak{A}}$, all operations are right continuous and all relations right clopen w.r.t. the universe of \mathfrak{A} .*

Theorem 5. *Let \mathfrak{A} and \mathfrak{C} be two models in the same signature, h a homomorphism of \mathfrak{A} into \mathfrak{C} , and let \mathfrak{C} be endowed with a compact Hausdorff topology in which all operations are right continuous, and all relations are right closed, w.r.t. the image of the universe of \mathfrak{A} under h . Then \widetilde{h} is a homomorphism of $\widetilde{\mathfrak{A}}$ into \mathfrak{C} :*

$$\begin{array}{ccc} \widetilde{\mathfrak{A}} & & \\ \uparrow & \text{---} \widetilde{h} & \\ \mathfrak{A} & \xrightarrow{h} & \mathfrak{C} \end{array}$$

Theorem 2 (for homomorphisms) easily follows: take $\widetilde{\mathfrak{B}}$ as such a \mathfrak{C} . The main meaning of Theorem 5 is that it generalizes the mentioned classical Čech–Stone result to the case when the underlying discrete space X carries an arbitrary first-order structure.

A natural question is whether the $*$ -extensions are also canonical in a similar sense. The answer is positive; two following theorems are counterparts of Theorems 4 and 5, respectively (essentially both have been proved in [6]). Recall that a set is *regular closed* iff it is the closure of an open set.

Theorem 6. *Let \mathfrak{A} be a model. In the extension \mathfrak{A}^* , all relations are regular closed, namely, the closures of the relations in \mathfrak{A} (while all operations are right continuous w.r.t. the universe of \mathfrak{A} as before).*

Theorem 7. *Let \mathfrak{A} and \mathfrak{C} be two models in the same signature, h a homomorphism of \mathfrak{A} into \mathfrak{C} , and let \mathfrak{C} be endowed with a compact Hausdorff topology in which all operations are right continuous w.r.t. the image of the universe of \mathfrak{A} under h , and all relations are closed. Then \widetilde{h} is a homomorphism of \mathfrak{A}^* into \mathfrak{C} .*

$$\begin{array}{ccc} \mathfrak{A}^* & & \\ \uparrow & \text{---} \widetilde{h} & \\ \mathfrak{A} & \xrightarrow{h} & \mathfrak{C} \end{array}$$

Similarly, Theorem 3 (for homomorphisms) follows from Theorem 7. The latter also generalizes the Čech–Stone result for discrete spaces to discrete models but with a narrow class of target models \mathfrak{C} : having relations rather closed than right closed in Theorem 5. In the sequel, we shall refer to Theorems 2 and 3 as the *First Extension Theorems*, and to stronger Theorems 5 and 7 as the *Second Extension Theorems*, for the $*$ - and \sim -types of ultrafilter extensions, respectively. Let us point out that in all these extension theorems the converse implication “if \widetilde{h} is a homomorphism of an ultrafilter extension \mathfrak{A}' then h is a homomorphism of \mathfrak{A} ” is also true but trivial since \mathfrak{A} is a submodel of \mathfrak{A} . We note also that the Second Extension Theorems are based on an “abstract extension theorem” describing certain conditions on models, their submodels, homomorphisms, and topological properties, under which such a homomorphism lifts from such a submodel to the whole model. The theorem will be used in our paper, too; we shall formulate it later on (Theorem 45).

We end this introductory section with topological characterizations of both types of ultrafilter extensions of relations and ultrafilter extensions of maps into discrete spaces and into compact Hausdorff spaces.

Theorem 8. *Let X_1, \dots, X_n, Y be discrete spaces, Z a compact Hausdorff space, and let the sets $\beta X_1, \dots, \beta X_n, Y$ be endowed with the standard topology on ultrafilters, and $\beta X_1 \times \dots \times \beta X_n$ with the usual product topology. Then*

- (i) $Q \subseteq \beta X_1 \times \dots \times \beta X_n$ is \tilde{R} for some $R \subseteq X_1 \times \dots \times X_n$ iff Q is right clopen w.r.t. X_1, \dots, X_{n-1} ;
- (ii) $Q \subseteq \beta X_1 \times \dots \times \beta X_n$ is R^* for some $R \subseteq X_1 \times \dots \times X_n$ iff Q is regular closed;
- (iii) $G : \beta X_1 \times \dots \times \beta X_n \rightarrow \beta Y$ is \tilde{F} for some $F : X_1 \times \dots \times X_n \rightarrow Y$ iff G is right continuous and right open w.r.t. X_1, \dots, X_{n-1} ;
- (iv) $G : \beta X_1 \times \dots \times \beta X_n \rightarrow Z$ is \tilde{H} for some $H : X_1 \times \dots \times X_n \rightarrow Z$ iff G is right continuous w.r.t. X_1, \dots, X_{n-1} .

Moreover, all the four extension operations: $R \mapsto \tilde{R}$, $R \mapsto R^*$, $F \mapsto \tilde{F}$, $H \mapsto \tilde{H}$, are bijections.

This theorem shows that Theorems 4 and 6 in fact *characterize* the $*$ - and \sim -extensions via their topological properties (and the same will follow from Theorems 28 and 29 later).

The subsequent text is organized as follows.

In Section 2, we develop a topological technique that allows us to define an ultrafilter extension of the procedure of ultrafilter extension itself. This concept is crucial for our article. Based on it, we provide a uniform approach to both types of ultrafilter extensions of relations (Theorem 15), and furthermore, in Section 3, we define an ultrafilter interpretation of first-order syntax, under which functional and relational symbols are interpreted rather by ultrafilters over sets of functions and relations than by their elements. We define generalized models using ultrafilter evaluations of variables and ultrafilter interpretations and an appropriate semantic for them. We provide two specific operations, e and E , which turn generalized models into ordinary ones, establish necessary and sufficient conditions under which the latter are two canonical ultrafilter extensions of some ordinary models (Theorem 28), and give a topological characterization of generalized models (Theorem 29). Defining a natural concept of homomorphisms between generalized models, we establish the First Extension Theorem for generalized models (Theorem 32) and a stronger variant of it (Theorem 33).

In Section 4, we define an even wider concept of generalized models together with their semantic based on limits of ultrafilters, and show that this new concept absorbs the ordinary concept of models with the usual semantic (Theorem 34) as well as our previous concept of generalized models with their semantic (Theorem 35). We provide two more specific operations, i and I , which turn generalized models in the narrow sense into ones in the wide sense, show how they relate to the operations e and E via their limits in appropriate topologies (Theorems 37 and 43), and establish necessary and sufficient conditions under which generalized models in the wide sense are the images of ones in the narrow sense under i and I , and also are two canonical ultrafilter extensions of some ordinary models (Theorems 39 and 44). Finally, we define homomorphisms between generalized models in the wide sense, and establish for them an “abstract extension theorem” (Theorem 46) and two Second Extension Theorems (Theorems 48 and 50). In Section 5, we conclude the article by posing some problems and tasks.

A part of the results mentioned in Sections 1–3 was announced in [7]; here we provide complete proofs of all our results.²

²In [7], it was erroneously stated that the set of right continuous maps forms a compact Hausdorff space w.r.t. the pointwise convergence topology; actually, the intended topology was a *restricted* pointwise convergence topology, as explained in details below.

2. Extending the ultrafilter extension procedure

A purpose of this section is to provide a uniform approach to both types of ultrafilter extensions: the smaller \sim -extensions and larger $*$ -extensions. For this, we shall develop some ideas and machinery which will lead us in the next section to certain structures, called there generalized models, generalizing ultrafilter extensions of each of the two types.

We shall give an alternative description of the $*$ -extension of relations in terms of the basic (cl)open sets and the continuous extension of maps. The crucial idea is to consider continuous extension of the procedure of ultrafilter extension itself, i.e. a *self-application* of the procedure. Let us clarify what is the idea precisely. For simplicity, consider firstly unary maps, for which the ultrafilter extensions are just the continuous extensions. To make the notation easier, let us denote the operation of continuous extension of maps by ext ; i.e. $\text{ext}(f)$ is another notation for \tilde{f} :

$$\text{ext}(f) = \tilde{f}.$$

So if we consider (unary) maps of X into Y , then ext is a map of Y^X into $C(\beta X, \beta Y)$, the set of all continuous functions of βX into βY . If $C(\beta X, \beta Y)$ would be endowed with some compact Hausdorff topology, then we could extend the map ext to a (unique) continuous map $\tilde{\text{ext}}$ of $\beta(Y^X)$ into $C(\beta X, \beta Y)$:

$$\begin{array}{ccc} \beta(Y^X) & & \\ \uparrow & \searrow \tilde{\text{ext}} & \\ Y^X & \xrightarrow{\text{ext}} & C(\beta X, \beta Y) \end{array}$$

We are going to show that such a topology on $C(\beta X, \beta Y)$ exists, and in fact, is a weaker version of the pointwise convergence topology (while the standard full version of the topology is not compact, as explained in the first remark after Lemma 12). Furthermore, as we shall see, the same approach will work in the case of n -ary maps (and relations, which can be reduced to maps).

Restricted pointwise convergence topology. Let X and Y be topological spaces and $A \subseteq X$. Define a topology on the set Y^X of all maps of X into Y by letting the family of sets $O_{a,B} = \{f \in Y^X : f(a) \in B\}$ for all $a \in A$ and all $B \subseteq Y$ which are open in Y , as an open subbase. We shall call it the *A-pointwise convergence* topology. Clearly, if $A = X$ then it is the usual pointwise convergence topology, which, as well-known (see e.g. [12]), coincides with the standard (Tychonoff) product topology.

If we consider $Y^{X_1 \times \dots \times X_n}$ as the set of n -ary maps, and choose subsets $A_1 \subseteq X_1, \dots, A_n \subseteq X_n$, the topology with an open subbase consisting of sets

$$O_{a_1, \dots, a_n, B} = \{f \in Y^{X_1 \times \dots \times X_n} : f(a_1, \dots, a_n) \in B\}$$

for all $a_1 \in A_1, \dots, a_n \in A_n$ and all $B \subseteq Y$ which are open in Y , will be called the (A_1, \dots, A_n) -pointwise convergence topology. Although it is the same that the set of unary maps of $X_1 \times \dots \times X_n$ into Y endowed with the $A_1 \times \dots \times A_n$ -pointwise convergence topology, we shall use this terminology to emphasize when we shall say about n -ary maps.

Let $1 \leq i \leq j \leq n$. For any $f : X_1 \times \dots \times X_n \rightarrow Y$, $\mathbf{a} \in X_1 \times \dots \times X_{i-1}$, and $\mathbf{u} \in X_{j+1} \times \dots \times X_n$, the map

$$f_{\mathbf{a}}^{\mathbf{u}} : X_i \times \dots \times X_j \rightarrow Y$$

is defined by letting

$$f_{\mathbf{a}}^{\mathbf{u}}(x_1, \dots, x_{j-i+1}) = f(\mathbf{a}, x_1, \dots, x_{j-i+1}, \mathbf{u})$$

for all $x_1 \in X_1, \dots, x_{j-i+1} \in X_{j-i+1}$. We omit the sub- and superscripts whenever the sequences \mathbf{a} and \mathbf{u} respectively are empty.

Let cur be the *currying* (or *evaluation*) map taking any $f : X_1 \times \dots \times X_n \rightarrow Y$ with $n \geq 2$ to the map $\text{cur}(f) : X_n \rightarrow Y^{X_1 \times \dots \times X_{n-1}}$ such that

$$\text{cur}(f)(x) = f^x.$$

(A more precise term would be the *right currying* but we prefer the shorter one.) Clearly, the map cur is bijective.

Let for any positive $n < \omega$, topological spaces X_1, \dots, X_n, Y , and sets $A_1 \subseteq X_1, \dots, A_{n-1} \subseteq X_{n-1}$,

$$RC_{A_1, \dots, A_{n-1}}(X_1, \dots, X_n, Y)$$

denote the set of n -ary maps $f : X_1 \times \dots \times X_n \rightarrow Y$ that are right continuous w.r.t. A_1, \dots, A_{n-1} , which we consider with the topology of (A_1, \dots, A_n) -pointwise convergence.

Lemma 9. *If $f \in RC_{A_1, \dots, A_{n-1}}(X_1, \dots, X_n, Y)$ then*

$$\text{cur}(f) \in C(X_n, RC_{A_1, \dots, A_{n-2}}(X_1, \dots, X_{n-1}, Y)).$$

Proof. By definition of currying, $\text{cur}(f)$ maps X_n into $RC_{A_1, \dots, A_{n-2}}(X_1, \dots, X_{n-1}, Y)$. Let us verify that it is continuous. Pick any $\mathbf{a} \in X_1 \times \dots \times X_{n-1}$, open set B in Y , and consider the subbasic open set $O_{\mathbf{a}, B} = \{h \in RC_{A_1, \dots, A_{n-2}}(X_1, \dots, X_{n-1}, Y) : h(\mathbf{a}) \in B\}$ in the space $RC_{A_1, \dots, A_{n-1}}(X_1, \dots, X_n, Y)$. We have:

$$\text{cur}(f)(u) \in O_{\mathbf{a}, B} \text{ iff } \text{cur}(f)(u)(\mathbf{a}) \in B \text{ iff } f_{\mathbf{a}}(u) \in B.$$

The set $\text{cur}(f)^{-1}(O_{\mathbf{a}, B}) = (f_{\mathbf{a}})^{-1}(B)$ is open since the map $f_{\mathbf{a}}$ is continuous. \square

Lemma 10. *For any positive $n < \omega$, topological spaces X_1, \dots, X_n , their dense subsets $D_1 \subseteq X_1, \dots, D_n \subseteq X_n$, and Hausdorff space Y ,*

- (i) *if maps $f, g \in RC_{D_1, \dots, D_{n-1}}(X_1, \dots, X_n, Y)$ coincide on $D_1 \times \dots \times D_n$, then they coincide everywhere,*
- (ii) *the space $RC_{D_1, \dots, D_{n-1}}(X_1, \dots, X_n, Y)$ endowed with the (D_1, \dots, D_{n-1}) -pointwise convergence topology, is Hausdorff.*

Proof. For brevity, let us denote the space $RC_{D_1, \dots, D_{k-1}}(X_1, \dots, X_k, Y)$ by RC_k . We argue by induction on n . For induction basis, see [12], Theorem 2.1.9. Assume we have already proven the claim for $n = k$. Let us prove this for $n = k + 1$.

Let maps $f, g \in RC_{k+1}$ coincide on $D_1 \times \dots \times D_{k+1}$. Then for each $\mathbf{a} \in D_{k+1}$ the maps $f^{\mathbf{a}}$ and $g^{\mathbf{a}}$ coincide on $D_1 \times \dots \times D_k$ and are right continuous w.r.t. the D_i . By induction hypothesis, $f^{\mathbf{a}} = g^{\mathbf{a}}$. Hence, the maps $\text{cur}(f), \text{cur}(g) : X_{k+1} \rightarrow RC_k$ coincide on D_{k+1} . By Lemma 9, the maps are continuous, while by induction hypothesis the space RC_k is Hausdorff. Hence, $\text{cur}(f) = \text{cur}(g)$ again by [12], Theorem 2.1.9. Therefore, $f = g$ since cur is bijective.

Furthermore, this shows that the space RC_{k+1} is Hausdorff. Indeed, let $f, g \in RC_{k+1}$ and $f \neq g$. Then, by the just proven fact, $f(\mathbf{a}) \neq g(\mathbf{a})$ for some $\mathbf{a} \in D_1 \times \dots \times D_{k+1}$. Since Y is Hausdorff, pick any disjoint open neighborhoods $A, B \subseteq Y$ of $f(\mathbf{a})$ and $g(\mathbf{a})$. Then the sets $F = \{h \in RC_{k+1} : h(\mathbf{a}) \in A\}$ and $G = \{h \in RC_{k+1} : h(\mathbf{a}) \in B\}$ are disjoint open neighborhoods of $f \in F$ and $g \in G$. \square

Lemma 11. *Let X_1, \dots, X_n be discrete spaces and Y a compact Hausdorff space. The set*

$$RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$$

of n -ary maps of $\beta X_1 \times \dots \times \beta X_n$ into Y which are right continuous w.r.t. X_1, \dots, X_{n-1} , endowed with the (X_1, \dots, X_n) -pointwise convergence topology, is homeomorphic to the space $Y^{X_1 \times \dots \times X_n}$ endowed with the usual pointwise convergence topology. Therefore, the space is compact Hausdorff; moreover, it is zero-dimensional iff so is Y .

Proof. Let us verify that the map ext , which takes each n -ary f in $Y^{X_1 \times \dots \times X_n}$ to its extension $\text{ext}(f) = \tilde{f}$ in $RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$, is a homeomorphism. The fact that ext is injective is trivial, and that ext is surjective follows from Lemma 10 (since each X_i is dense in βX_i and Y is Hausdorff): whenever $g \in RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$ and $f = g \upharpoonright (X_1 \times \dots \times X_n)$, then $\tilde{f} = g$. Finally, the fact that it preserves in both directions open sets belonging to the subbases of the spaces, is immediate by the definition of the (X_1, \dots, X_n) -pointwise convergence topology. Therefore, the space is homeomorphic to the usual product space of Y , hence, by the Tychonoff theorem, is compact Hausdorff, and moreover, the zero-dimensionality iff so is Y (see e.g. [12]). \square

Lemma 12. *Let X_1, \dots, X_n be discrete spaces, Y a compact Hausdorff space, and $S \subseteq Y$ dense in Y . Then the set $\{\tilde{f} : f \in S^{X_1 \times \dots \times X_n}\}$ is dense in the space $RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$ endowed with the (X_1, \dots, X_n) -pointwise convergence topology.*

Proof. Let $k < \omega$, pick for all $i < k$ arbitrary $\mathbf{a}_i \in X_1 \times \dots \times X_n$ and $B_i \subseteq Y$ open in Y , and show that, whenever the basic open set $\bigcap_{i < k} O_{\mathbf{a}_i, B_i}$ is nonempty, then it contains a point \tilde{f} for some $f \in S^{X_1 \times \dots \times X_n}$. Note that if some of the \mathbf{a}_i coincide, say, $\mathbf{a}_i = \mathbf{a}_j$ for all $i, j \in A$ and some $A \subseteq k$, then $\bigcap_{i \in A} B_i$ is nonempty whenever so is $\bigcap_{i < k} O_{\mathbf{a}_i, B_i}$. So we can assume w.l.g. that all the \mathbf{a}_i are distinct. Then any $f \in S^{X_1 \times \dots \times X_n}$ satisfying $f(\mathbf{a}_i) \in B_i \cap S$ for all $i < k$, is as required. \square

Question. Does this remain true, moreover, for the full pointwise convergence topology? The answer is affirmative for unary maps, i.e. the set $\{\tilde{f} : f \in S^X\}$ is dense in $C(\beta X, Y)$. What happens for binary maps? (Problem 1 in Section 5.)

Remarks. One may ask whether the usual (unrestricted) pointwise convergence topology on the set $RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$ is compact, or equivalently, whether the set forms a closed subspace of the compact Hausdorff space $Y^{\beta X_1 \times \dots \times \beta X_n}$ with the Tychonoff product topology. If this would be the case, we could use this more standard topology for our purpose. Let us show that the answer is in the negative, even for unary maps.

1. The set $C(\beta X, \beta Y)$ endowed with the pointwise convergence topology is not compact.

It suffices to verify that for an arbitrary map $h : \beta X \rightarrow \beta Y$ there exists an ultrafilter \mathfrak{f} over $C(\beta X, \beta Y)$ converging to h (to recall related facts the reader can look at the beginning of Section 4). Since w can be discontinuous, this will show that $C(\beta X, \beta Y)$ is not closed in $\beta Y^{\beta X}$. Consider the family

$$\begin{aligned} \mathcal{F} &= \{O_{\mathfrak{u}, \tilde{S}} : \mathfrak{u} \in \beta X \text{ and } h(\mathfrak{u}) \in \tilde{S}\} \\ &= \{\{f \in C(\beta X, \beta Y) : S \in f(\mathfrak{u})\} : \mathfrak{u} \in \beta X \text{ and } S \in h(\mathfrak{u})\}. \end{aligned}$$

Let us check that \mathcal{F} is centered. It suffices to show that for any positive $n \in \omega$, ultrafilters $\mathfrak{u}_0, \dots, \mathfrak{u}_{n-1}$ over X , and non-empty subsets S_0, \dots, S_{n-1} of X , there exists a map $f \in C(\beta X, \beta Y)$ satisfying

$$S_0 \in f(\mathfrak{u}_0), \dots, S_{n-1} \in f(\mathfrak{u}_{n-1}).$$

To see, pick arbitrary pairwise disjoint sets A_0, \dots, A_{n-1} such that $A_0 \in \mathbf{u}_0, \dots, A_{n-1} \in \mathbf{u}_{n-1}$, elements $s_0 \in S_0, \dots, s_{n-1} \in S_{n-1}$, and consider a map $g : X \rightarrow Y$ such that $g(x) = s_i$ whenever $x \in A_i, i < n$, and $g(x) = y$, where y is a fixed element of Y , otherwise. (Actually, g on the set $X \setminus (A_0 \cup \dots \cup A_{n-1})$ could be defined arbitrarily.) Let $f = \tilde{g}$, so $f \in C(\beta X, \beta Y)$. For each $i < n$ we have $A_i \subseteq g^{-1}(S_i)$ and $A_i \in \mathbf{u}_i$, therefore, $g^{-1}(S_i) \in \mathbf{u}_i$, and so, $S_i \in \tilde{g}(\mathbf{u}_i) = f(\mathbf{u}_i)$. Thus the map f witnesses that the family \mathcal{F} is centered.

Now extend \mathcal{F} to an ultrafilter $\mathfrak{f} \in \beta C(\beta X, \beta Y)$. It is clear that \mathfrak{f} converges to the map h , as required.

2. Since we know that $C(\beta X, \beta Y)$ with the X -pointwise convergence topology is compact while with the (full) pointwise convergence topology is not, we may ask what is the map $f \in C(\beta X, \beta Y)$ such that the ultrafilter \mathfrak{f} defined above converges to f in the weaker (restricted) topology. It is not difficult to show that $f = \widetilde{h \upharpoonright X}$. Note that $(\beta Y)^{\beta X}$ with the X -pointwise convergence topology is compact (since it is compact even with the stronger pointwise convergence topology). The map r of this compact space onto its compact subspace $C(\beta X, \beta Y)$, defined by letting for all $h \in (\beta Y)^{\beta X}$

$$r(h) = \widetilde{h \upharpoonright X}$$

is a natural retraction. However, $(\beta Y)^{\beta X}$ with the X -pointwise convergence topology is not Hausdorff nor even a T_0 -space since, whenever $h \in (\beta Y)^{\beta X}$ is discontinuous, then the points h and $r(h)$ are distinct but have the same neighborhoods (it suffices to consider subbasic neighborhoods, and for any $a \in X$ and open $B \subseteq \beta Y$ we have $h \in O_{a,B}$ iff $r(h) \in O_{a,B}$).

3. These observations hold in a general setting, for n -ary maps into any compact Hausdorff space Y : the full pointwise convergence topology on $RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$ is not compact, while the $(\beta X_1, \dots, \beta X_n)$ -pointwise convergence topology on $Y^{\beta X_1 \times \dots \times \beta X_n}$ is compact but not T_0 , and the map r defined by letting for all $h \in Y^{\beta X_1 \times \dots \times \beta X_n}$

$$r(h) = \text{ext}(h \upharpoonright (X_1 \times \dots \times X_n))$$

is a natural retraction of $Y^{\beta X_1 \times \dots \times \beta X_n}$ onto $RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$.

Self-application of the extension operation. Now we are ready to define the continuous extension $\widetilde{\text{ext}}$ of the map ext in a general form. Let X_1, \dots, X_n be discrete spaces and Y a compact Hausdorff space. Recall that for any n -ary map f of $X_1 \times \dots \times X_n$ into Y , $\text{ext}(f)$ is \tilde{f} , the extension of f to ultrafilters which is right continuous w.r.t. principal ultrafilters:

$$\text{ext} : \beta(Y^{X_1 \times \dots \times X_n}) \rightarrow RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y).$$

By Lemma 11, the set $RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$ endowed with the (X_1, \dots, X_n) -pointwise convergence topology is a compact Hausdorff space. Therefore, ext extends to a unique continuous map $\widetilde{\text{ext}}$ on ultrafilters over the set $Y^{X_1 \times \dots \times X_n}$:

$$\begin{array}{ccc} \beta(Y^{X_1 \times \dots \times X_n}) & & \\ \uparrow & \dashrightarrow \widetilde{\text{ext}} & \\ Y^{X_1 \times \dots \times X_n} & \xrightarrow{\text{ext}} & RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y) \end{array}$$

Remark. Alternatively, we can first define $\widetilde{\text{ext}}$ on ultrafilters over the set of unary maps and then extend it to $\widetilde{\text{ext}}$ on ultrafilters over the set of n -ary maps by induction on n by using currying.

For this, we first note that the one-to-one correspondence between the sets $Y^{X_1 \times \dots \times X_n \times X_{n+1}}$ and $(Y^{X_1 \times \dots \times X_n})^{X_{n+1}}$ given by cur induces the one-to-one correspondence between the sets of

ultrafilters over them, which takes each ultrafilter $\mathfrak{f} \in \beta(Y^{X_1 \times \dots \times X_n \times X_{n+1}})$ to the ultrafilter $\mathfrak{f}' = \{\text{cur} \text{ " } A : A \in \mathfrak{f}\} \in \beta((Y^{X_1 \times \dots \times X_n})^{X_{n+1}})$. Or else, \mathfrak{f}' can be defined via the continuous extension of currying:

$$\begin{array}{ccc} \beta(Y^{X_1 \times \dots \times X_{n+1}}) & \xrightarrow{\widetilde{\text{cur}}} & \beta((Y^{X_1 \times \dots \times X_n})^{X_{n+1}}) \\ \uparrow & & \uparrow \\ Y^{X_1 \times \dots \times X_{n+1}} & \xrightarrow{\text{cur}} & (Y^{X_1 \times \dots \times X_n})^{X_{n+1}} \end{array}$$

Since cur is a bijection, it is easily follows that so is $\widetilde{\text{cur}}$, and for all $\mathfrak{f} \in \beta(Y^{X_1 \times \dots \times X_n \times X_{n+1}})$ we have $\widetilde{\text{cur}}(\mathfrak{f}) = \mathfrak{f}'$.

Now, for $n = 1$, we extend $\text{ext} : Y^X \rightarrow C(\beta X, Y)$ to $\widetilde{\text{ext}} : \beta(Y^X) \rightarrow C(\beta X, Y)$. And assuming that $\widetilde{\text{ext}}$ has been already defined for n , we can define $\widetilde{\text{ext}}(\mathfrak{f})$ by letting

$$\widetilde{\text{ext}}(\mathfrak{f})(\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{u}_{n+1}) = \widetilde{\text{ext}}(\widetilde{\text{ext}}(\mathfrak{f}')(\mathbf{u}_{n+1}))(\mathbf{u}_1, \dots, \mathbf{u}_n)$$

since $\widetilde{\text{ext}}$ has been already defined on \mathfrak{f}' and $\widetilde{\text{ext}}(\mathfrak{f}')(\mathbf{u}_{n+1})$ by induction hypothesis.

Question. One can offer another, alternative way to extend the ultrafilter extension procedure by considering it as the map not into the space of right continuous maps but into set of all maps with the usual product topology. Thus for any discrete X_1, \dots, X_n and compact Hausdorff Y , let ext be a map of the discrete space $Y^{X_1 \times \dots \times X_n}$ into $Y^{\beta X_1 \times \dots \times \beta X_n}$ endowed with the usual product topology (or equivalently, the usual, unrestricted pointwise convergence topology). As the range is a compact Hausdorff space, the map ext continuously extends to $\widetilde{\text{ext}}$ (in the new sense):

$$\begin{array}{ccc} \beta(Y^{X_1 \times \dots \times X_n}) & & \\ \uparrow & \dashrightarrow \widetilde{\text{ext}} & \\ Y^{X_1 \times \dots \times X_n} & \xrightarrow{\text{ext}} & Y^{\beta X_1 \times \dots \times \beta X_n} \end{array}$$

Unlike the previous construction, now some ultrafilters $\mathfrak{f} \in \beta(Y^{X_1 \times \dots \times X_n})$ are mapped into maps $\widetilde{\text{ext}}(\mathfrak{f}) \in Y^{\beta X_1 \times \dots \times \beta X_n}$ that no longer are right continuous w.r.t. principal ultrafilters (as explained in the remarks above). However, these maps $\widetilde{\text{ext}}(\mathfrak{f})$ are still close to those: any neighborhood of $\widetilde{\text{ext}}(\mathfrak{f})$ contains some right continuous map; this is because

$$\widetilde{\text{ext}} \text{ " } \beta(Y^{X_1 \times \dots \times X_n}) = \text{cl}_{Y^{\beta X_1 \times \dots \times \beta X_n}}(\text{ext} \text{ " } Y^{X_1 \times \dots \times X_n}).$$

Is this version of $\widetilde{\text{ext}}$ surjective? This is the case iff the image of ext is dense in the space; see the question after Lemma 12.

Can this version of $\widetilde{\text{ext}}$ lead to some interesting possibilities? (Problem 2 in Section 5.)

Lemma 13. *For any positive $n < \omega$, discrete spaces X_1, \dots, X_n , and compact Hausdorff space Y , the continuous map $\widetilde{\text{ext}}$ of the space $\beta(Y^{X_1 \times \dots \times X_n})$ into the space $RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$ is surjective and, whenever at least one of the X_i is infinite, non-injective.*

Proof. To simplify the notation, let RC denote the space $RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$ endowed with the (X_1, \dots, X_n) -pointwise convergence topology. Pick any $f \in \text{RC}$, let

$$Z = \{(\mathbf{a}, B) : \mathbf{a} \in X_1 \times \dots \times X_n, B \text{ is open in } Y, \text{ and } f(\mathbf{a}) \in B\},$$

and consider the following family \mathcal{F} of subsets of $Y^{X_1 \times \dots \times X_n}$:

$$\mathcal{F} = \{\{g \in Y^{X_1 \times \dots \times X_n} : g(\mathbf{a}) \in B\} : (\mathbf{a}, B) \in Z\}.$$

The family \mathcal{F} is centered; this can be stated by arguments similar to those in the first remark after Lemma 12. We are going to prove the following key property of the family \mathcal{F} :

$$\widetilde{\text{ext}}(\mathfrak{f}) = f \text{ for all } \mathfrak{f} \in \beta(Y^{X_1 \times \dots \times X_n}) \text{ such that } \mathcal{F} \subseteq \mathfrak{f}.$$

The lemma will be deduced from this property: since the argument works for all $f \in \text{RC}$, the property shows that $\widetilde{\text{ext}}$ is surjective; and that $\widetilde{\text{ext}}$ is non-injective will be shown once two distinct such ultrafilters $\mathfrak{f}', \mathfrak{f}''$ will be constructed.

Let us verify the following equality:

$$\bigcap_{A \in \mathcal{F}} \text{cl}_{\text{RC}}\{\tilde{g} : g \in A\} = \{f\}.$$

First note that by Lemma 12, for every $A = \{g \in Y^{X_1 \times \dots \times X_n} : g(\mathbf{a}) \in B\}$ in \mathcal{F} we have

$$\text{cl}_{\text{RC}}\{\tilde{g} : g \in A\} = \text{cl}_{\text{RC}}\{h \in \text{RC} : h(\mathbf{a}) \in B\}.$$

Therefore,

$$f \in \bigcap_{A \in \mathcal{F}} \text{cl}_{\text{RC}}\{\tilde{g} : g \in A\}.$$

Next, toward a contradiction, assume that there exists $f' \in \text{RC}$ such that $f' \neq f$ and $f' \in \bigcap_{A \in \mathcal{F}} \text{cl}_{\text{RC}}\{\tilde{g} : g \in A\}$. By Lemma 10, there exists $\mathbf{b} \in X_1 \times \dots \times X_n$ such that $f(\mathbf{b}) \neq f'(\mathbf{b})$. As Y is Hausdorff, pick disjoint open neighborhoods U and U' of the points $f(\mathbf{b})$ and $f'(\mathbf{b})$, respectively. We have:

$$\begin{aligned} \bigcap_{A \in \mathcal{F}} \text{cl}_{\text{RC}}\{\tilde{g} : g \in A\} &= \bigcap_{(\mathbf{a}, B) \in \mathcal{Z}} \text{cl}_{\text{RC}}\{h \in \text{RC} : h(\mathbf{a}) \in B\} \\ &\subseteq \text{cl}_{\text{RC}}\{h \in \text{RC} : h(\mathbf{b}) \in U\} \\ &\subseteq \text{cl}_{\text{RC}}\{h \in \text{RC} : h(\mathbf{b}) \in Y \setminus U'\} = \{h \in \text{RC} : h(\mathbf{b}) \in Y \setminus U'\}, \end{aligned}$$

(where the last equality holds since the set $\{h \in \text{RC} : h(\mathbf{b}) \in Y \setminus U'\}$ is the complement in RC of the subbasic open set $\{h \in \text{RC} : h(\mathbf{b}) \in U'\} = O_{\mathbf{b}, U'}$). Therefore,

$$f' \notin \bigcap_{A \in \mathcal{F}} \text{cl}_{\text{RC}}\{\tilde{g} : g \in A\},$$

a contradiction. Thus we have verified that the equality is true.

Now the required key property of the family \mathcal{F} , i.e. that we have $\widetilde{\text{ext}}(\mathfrak{f}) = f$ whenever $\mathfrak{f} \supseteq \mathcal{F}$, is clearly follows from this equality. As observed above, this property immediately implies that $\widetilde{\text{ext}}$ is surjective; and to show that $\widetilde{\text{ext}}$ is also non-injective, it remains to construct two distinct ultrafilters $\mathfrak{f}', \mathfrak{f}'' \supseteq \mathcal{F}$.

Pick a family $\{B_{\mathbf{a}} : \mathbf{a} \in X_1 \times \dots \times X_n\}$ of subsets of Y such that $B_{\mathbf{a}} \neq Y$ and $f(\mathbf{a}) \in B_{\mathbf{a}}$ for all $\mathbf{a} \in X_1 \times \dots \times X_n$. The families

$$\begin{aligned} \mathcal{F}' &= \mathcal{F} \cup \left\{ \left\{ g \in Y^{X_1 \times \dots \times X_n} : (\forall \mathbf{a} \in X_1 \times \dots \times X_n) g(\mathbf{a}) \in B_{\mathbf{a}} \right\} \right\}, \\ \mathcal{F}'' &= \mathcal{F} \cup \left\{ \left\{ g \in Y^{X_1 \times \dots \times X_n} : (\exists \mathbf{a} \in X_1 \times \dots \times X_n) g(\mathbf{a}) \notin B_{\mathbf{a}} \right\} \right\} \end{aligned}$$

are both centered (the fact that \mathcal{F}'' is centered uses that one of the X_i is infinite). We extend them to two (automatically distinct) ultrafilters \mathfrak{f}' and \mathfrak{f}'' , respectively. By the key property of \mathcal{F} , we obtain

$$\widetilde{\text{ext}}(\mathfrak{f}') = \widetilde{\text{ext}}(\mathfrak{f}'') = f,$$

thus showing that $\widetilde{\text{ext}}$ is not injective. Note also that, since $f \in \text{RC}$ was chosen arbitrary, we have established a bit more: the preimage of *each* point in RC under the map $\widetilde{\text{ext}}$ consists of more than one point.

The lemma is proved. \square

Lemma 14. *Let X_1, \dots, X_n be discrete spaces, Y a compact Hausdorff space, $S \subseteq Y$, and $R \subseteq S^{X_1 \times \dots \times X_n}$. Then $\widetilde{\text{ext}}$ maps the closure of R in the space $\beta(S^{X_1 \times \dots \times X_n})$ onto the closure of R in the space $\text{RC}_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$ endowed with the (X_1, \dots, X_n) -pointwise convergence topology:*

$$\widetilde{\text{ext}} \text{“ cl}_{\beta(S^{X_1 \times \dots \times X_n})} R = \text{cl}_{\text{RC}_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)} \text{ext“ } R.$$

Proof. Again, to simplify notation, we temporarily let:

$$\begin{aligned} Z &= \beta(S^{X_1 \times \dots \times X_n}), \\ \text{RC} &= \text{RC}_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y). \end{aligned}$$

We consider Z with the standard topology on the space of ultrafilters, so it actually does not depend on the topology on S as a subspace of Y . The fact that Y is compact Hausdorff is used only to get the same properties of the topology on RC , which are essential to extend ext to $\widetilde{\text{ext}}$.

To prove the inclusion

$$\widetilde{\text{ext}} \text{“ cl}_Z R \subseteq \text{cl}_{\text{RC}} \text{ext“ } R,$$

recall that by the general definition of continuous extensions of unary maps, for any $\mathfrak{f} \in Z$ we have $\{\widetilde{\text{ext}}(\mathfrak{f})\} = \bigcap_{A \in \mathfrak{f}} \text{cl}_{\text{RC}} \text{ext“ } A$. Therefore,

$$\widetilde{\text{ext}} \text{“ cl}_Z R = \{\widetilde{\text{ext}}(\mathfrak{f}) : \mathfrak{f} \in \text{cl}_Z R\} = \bigcup_{\mathfrak{f} \in \text{cl}_Z R} \bigcap_{A \in \mathfrak{f}} \text{cl}_{\text{RC}} \text{ext“ } A.$$

But for any $\mathfrak{f} \in \text{cl}_Z R$ we have $R \in \mathfrak{f}$ and hence $\bigcap_{A \in \mathfrak{f}} \text{cl}_{\text{RC}} \text{ext“ } A \subseteq \text{cl}_{\text{RC}} \text{ext“ } R$, whence it follows

$$\bigcup_{\mathfrak{f} \in \text{cl}_Z R} \bigcap_{A \in \mathfrak{f}} \text{cl}_{\text{RC}} \text{ext“ } A \subseteq \text{cl}_{\text{RC}} \text{ext“ } R,$$

which gives the required inclusion.

To prove the converse inclusion

$$\text{cl}_{\text{RC}} \text{ext“ } R \subseteq \widetilde{\text{ext}} \text{“ cl}_Z R,$$

note that $\text{cl}_{\text{RC}} \widetilde{\text{ext}} \text{“ } R \subseteq \widetilde{\text{ext}} \text{“ cl}_Z R$ since the map $\widetilde{\text{ext}} : Z \rightarrow \text{RC}$ is closed as a continuous map of a compact space into a Hausdorff space (see e.g. [12], Corollary 3.1.11), and that $\widetilde{\text{ext}} \text{“ } R = \text{ext“ } R$ since R consists of principal ultrafilters over the set $S^{X_1 \times \dots \times X_n}$ (under our customary identification of elements with principal ultrafilters given by them).

The lemma is proved. \square

Now we are ready to give the promised alternative description of the $*$ -extension of relations. For simplicity, we formulate it only for the case when $X_1 = \dots = X_n = X$; nevertheless, this formulation does not lose generality since for given X_i we can take their union as such an X .

Theorem 15. *Let $R \subseteq X \times \dots \times X$ be any n -ary relation on a set X . Then its extension $R^* \subseteq \beta X \times \dots \times \beta X$ is (identified with) the image under $\widetilde{\text{ext}}$ of the basic set \widetilde{R} in the space $\beta(X^n)$ where R is considered as a unary relation on X^n :*

$$R^* = \{\widetilde{\text{ext}}(\mathfrak{r}) : R \in \mathfrak{r}\} = \widetilde{\text{ext}} \text{“ } \widetilde{R}.$$

Proof. By Theorem 6, $R^* = \text{cl}_{\beta X \times \dots \times \beta X} R$. As usual, the product space $\beta X \times \dots \times \beta X$ (n times) is identified with $(\beta X)^n$, so up to this identification we can let

$$R^* = \text{cl}_{(\beta X)^n} R.$$

We are going to use Lemma 14 by choosing appropriate discrete X_1, \dots, X_m , a compact Hausdorff Y , and $S \subseteq Y$. Let $m = 1$, let the space X_1 be n with the discrete topology, so $\beta X_1 = X_1 = n$, let the space Y be βX with the standard topology on the space of ultrafilters, and let S be X , so we have:

$$\beta(S^{X_1}) = \beta(X^n) \quad \text{and} \quad C(X_1, Y) = (\beta X)^n$$

(clearly, the X_1 -pointwise convergence topology on the latter set is the same that the full pointwise convergence topology). Now Lemma 14 gives us

$$\widetilde{\text{ext}} \text{“} \text{cl}_{\beta(X^n)} R = \text{cl}_{(\beta X)^n} \text{ext“} R.$$

But $\text{cl}_{\beta(X^n)} R = \widetilde{R}$ where R is considered as a unary relation on X^n (recall that if Z is discrete and $A \subseteq Z$, then the basic open set \widetilde{A} equals the closure $\text{cl}_Z A$), and furthermore, $\text{ext“} R = R$ (since $\widetilde{f} = f$ for all $f \in Y^n$ as $\beta n = n$). Putting all this together, we obtain:

$$R^* = \text{cl}_{(\beta X)^n} R = \text{cl}_{(\beta X)^n} \text{ext“} R = \widetilde{\text{ext“} \text{cl}_{\beta(X^n)} R} = \widetilde{\text{ext“} \widetilde{R}},$$

as required. \square

Although this characterization of the $*$ -extensions of relations is not simpler than one given by Theorem 6, its interest lies in a connection of these larger extensions with the smaller \sim -extensions of relations (by using also continuous extensions of maps). Other interrelations between the \sim - and $*$ -extensions of relations are established via Vietoris-type topologies in [6].

3. Ultrafilter interpretations

In this section, we define our main concepts: ultrafilter interpretations (of functional and relational symbols) and generalized models (involving ultrafilter evaluations and ultrafilter interpretations) together with their semantic. Then we provide two specific operations turning generalized models into ordinary ones, establish necessary and sufficient conditions under which the latter are two canonical ultrafilter extensions of some ordinary models, and give a topological characterization of generalized models. Finally, we define homomorphisms of generalized models and prove for them a version of the First Extension Theorem and an its refinement.

Generalized models. Using ultrafilters over maps in our previous considerations leads us to the following concept. Given a signature, we define an *ultrafilter* (or *generalized*) *interpretation* as a map ι that takes each n -ary functional symbol F to an ultrafilter over the set of n -ary operations on X , and each n -ary predicate symbol R to an ultrafilter over the set of n -ary relations on X ; let also v be an *ultrafilter valuation* of variables, i.e. a valuation which takes each variable x to an ultrafilter over a given set X :

$$v(x) \in \beta X, \quad \iota(F) \in \beta(X^{X \times \dots \times X}), \quad \iota(R) \in \beta \mathcal{P}(X \times \dots \times X).$$

We refer to the set $(\beta X, \iota(F), \dots, \iota(R), \dots)$ as a *generalized model*. Now we are going to define an appropriate satisfiability relation between generalized models and first-order formulas, which we will denote by the symbol \models .

First, given an interpretation ι of non-logical symbols, we expand any valuation v of variables to the map v_ι defined on all terms as follows. Let $\text{app} : X_1 \times \dots \times X_n \times Y^{X_1 \times \dots \times X_n} \rightarrow Y$ be the *application* operation:

$$\text{app}(a_1, \dots, a_n, f) = f(a_1, \dots, a_n).$$

Extend it to the map $\widetilde{\text{app}} : \beta X_1 \times \dots \times \beta X_n \times \beta(Y^{X_1 \times \dots \times X_n}) \rightarrow \beta Y$ right continuous w.r.t. the principal ultrafilters, in the usual way:

$$\begin{array}{ccc} \beta X_1 \times \dots \times \beta X_n \times \beta(Y^{X_1 \times \dots \times X_n}) & \xrightarrow{\widetilde{\text{app}}} & \beta Y \\ \uparrow & & \uparrow \\ X_1 \times \dots \times X_n \times Y^{X_1 \times \dots \times X_n} & \xrightarrow{\text{app}} & Y \end{array}$$

Let v_ι coincide with v on variables, and if v_ι has been already defined on terms t_1, \dots, t_n , we let

$$v_\iota(F(t_1, \dots, t_n)) = \widetilde{\text{app}}(v_\iota(t_1), \dots, v_\iota(t_n), \iota(F)).$$

Remark. We can consider, more generally, for any compact Hausdorff space Y the extension $\widetilde{\text{app}} : \beta X_1 \times \dots \times \beta X_n \times \beta(Y^{X_1 \times \dots \times X_n}) \rightarrow Y$ right continuous w.r.t. the principal ultrafilters:

$$\begin{array}{ccc} \beta X_1 \times \dots \times \beta X_n \times \beta(Y^{X_1 \times \dots \times X_n}) & & \\ \uparrow & \dashrightarrow^{\widetilde{\text{app}}} & \\ X_1 \times \dots \times X_n \times Y^{X_1 \times \dots \times X_n} & \xrightarrow{\text{app}} & Y \end{array}$$

though this is redundant for our immediate purposes.

Further, given a generalized model $\mathfrak{A} = (\beta X, \iota(F), \dots, \iota(R), \dots)$, define the satisfiability in \mathfrak{A} as follows. Let $\text{in} \subseteq X_1 \times \dots \times X_n \times \mathcal{P}(X_1 \times \dots \times X_n)$ be the *membership* predicate:

$$\text{in}(a_1, \dots, a_n, R) \text{ iff } (a_1, \dots, a_n) \in R.$$

Extend it to the relation $\widetilde{\text{in}} \subseteq \beta X_1 \times \dots \times \beta X_n \times \beta \mathcal{P}(X_1 \times \dots \times X_n)$ right clopen w.r.t. principal ultrafilters. Let

$$\mathfrak{A} \models t_1 = t_2 [v] \text{ iff } v_\iota(t_1) = v_\iota(t_2).$$

If $R(t_1, \dots, t_n)$ is an atomic formula in which R is not the equality predicate, we let

$$\mathfrak{A} \models R(t_1, \dots, t_n) [v] \text{ iff } \widetilde{\text{in}}(v_\iota(t_1), \dots, v_\iota(t_n), \iota(R)).$$

(Equivalently, we could define the satisfiability of atomic formulas by identifying predicates with their characteristic functions and using the satisfiability of equalities of the resulting terms.) Finally, if $\varphi(t_1, \dots, t_n)$ is obtained by negation, conjunction, or quantification from formulas for which \models has been already defined, we define $\mathfrak{A} \models \varphi [v]$ in the standard way.

When needed, we shall use variants of notation commonly used for ordinary models and satisfiability, for the generalized ones. E.g. for a generalized model \mathfrak{A} with the universe βX , a formula $\varphi(x_1, \dots, x_n)$, and elements $\mathbf{u}_1, \dots, \mathbf{u}_n$ of βX , the notation $\mathfrak{A} \models \varphi [\mathbf{u}_1, \dots, \mathbf{u}_n]$ means that φ is satisfied in \mathfrak{A} under a valuation taking the variables x_1, \dots, x_n to the ultrafilters $\mathbf{u}_1, \dots, \mathbf{u}_n$, respectively.

Generalized models actually generalize not all ordinary models but those that are ultrafilter extensions of some models. It is worth also pointing out that whenever a generalized interpretation is *principal*, i.e. all non-logical symbols are interpreted by principal ultrafilters, we naturally

identify it with the obvious ordinary interpretation with the same universe βX ; however, not every ordinary interpretation with the universe βX is of this form. Precise relationships between generalized models, ordinary models, and ultrafilter extensions will be described in Theorems 28 and 29. Let us also note in advance that in the last section, we shall define a wider concept of generalized models, which will cover (up to some natural identification) all ordinary models.

An ultrafilter valuation v is *principal* iff it takes any variable to a principal ultrafilter.

Lemma 16. *Let $\mathfrak{A} = (\beta X, \iota(F), \dots, \iota(R), \dots)$ and $\mathfrak{B} = (\beta X, \jmath(F), \dots, \jmath(R), \dots)$ be two generalized models in the same signature and having the same universe βX . If for all functional symbols F , predicate symbols R , variables x_1, \dots, x_n , and principal valuations v ,*

$$\begin{aligned} \widetilde{\text{app}}(v(x_1), \dots, v(x_n), \iota(F)) &= \widetilde{\text{app}}(v(x_1), \dots, v(x_n), \jmath(F)), \\ \widetilde{\text{in}}(v(x_1), \dots, v(x_n), \iota(R)) &\text{ iff } \widetilde{\text{in}}(v(x_1), \dots, v(x_n), \jmath(R)), \end{aligned}$$

then for all formulas φ , terms t_1, \dots, t_n , and valuations v ,

$$\mathfrak{A} \models \varphi(t_1, \dots, t_n) [v] \text{ iff } \mathfrak{B} \models \varphi(t_1, \dots, t_n) [v].$$

Proof. By induction on construction of formulas using the right continuity of $\widetilde{\text{app}}$ and the right clopenness of $\widetilde{\text{in}}$ w.r.t. X . \square

Corollary 17. *Given a generalized model $\mathfrak{A} = (\beta X, \iota(F), \dots, \iota(R), \dots)$, define a generalized model $\mathfrak{B} = (\beta X, \jmath(F), \dots, \jmath(R), \dots)$ in the same signature as follows: let \mathfrak{B} have the same universe βX , let \jmath coincide with ι on functional symbols, and for each predicate symbol R let $\jmath(R)$ be the principal ultrafilter given by the relation*

$$\{(a_1, \dots, a_n) \in X^n : \widetilde{\text{in}}(a_1, \dots, a_n, \iota(R))\}.$$

Then for all valuations v , formulas φ , and terms t_1, \dots, t_n ,

$$\mathfrak{A} \models \varphi(t_1, \dots, t_n) [v] \text{ iff } \mathfrak{B} \models \varphi(t_1, \dots, t_n) [v].$$

Proof. Lemma 16. \square

If X_1, \dots, X_n, Y are discrete spaces, let us say that an ultrafilter \mathfrak{f} over the set $Y^{X_1 \times \dots \times X_n}$ of n -ary maps is *pseudo-principal* iff $\widetilde{\text{app}}$ takes any n -tuple consisting of principal ultrafilters together with \mathfrak{f} to a principal ultrafilter:

$$a_1 \in X_1, \dots, a_n \in X_n \text{ implies } \widetilde{\text{app}}(a_1, \dots, a_n, \mathfrak{f}) \in Y.$$

Clearly, if the space Y is finite, then all ultrafilters in $\beta(Y^{X_1 \times \dots \times X_n})$ are pseudo-principal. (More generally, if we would defined $\widetilde{\text{app}}$ with the range in any compact Hausdorff Y , as proposed in the remark above, then all ultrafilters in $\beta(Y^{X_1 \times \dots \times X_n})$ were pseudo-principal.)

Lemma 18. *Let X_1, \dots, X_n, Y be discrete spaces. In $\beta(Y^{X_1 \times \dots \times X_n})$, every principal ultrafilter is pseudo-principal, and if Y and at least one of the X_i are infinite, then there exist pseudo-principal ultrafilters that are not principal as well as ultrafilters that are not pseudo-principal.*

Proof. Pick any $f \in Y^{X_1 \times \dots \times X_n}$. Let \mathcal{F} be the following family of subsets of the space $Y^{X_1 \times \dots \times X_n}$:

$$\mathcal{F} = \{\{g \in Y^{X_1 \times \dots \times X_n} : g(\mathbf{a}) = f(\mathbf{a})\} : \mathbf{a} \in X_1 \times \dots \times X_n\} \cup \{\{g \in Y^{X_1 \times \dots \times X_n} : g \neq f\}\}.$$

The family \mathcal{F} is centered (as at least one of the X_i is infinite), so pick any ultrafilter \mathfrak{f} over the set $Y^{X_1 \times \dots \times X_n}$ such that $\mathcal{F} \subseteq \mathfrak{f}$. Since $\bigcap \mathcal{F}$ empty, the ultrafilter \mathfrak{f} is non-principal. On the other hand, for every $\mathbf{a} = (a_1, \dots, a_n) \in X_1 \times \dots \times X_n$ we have:

$$\begin{aligned} S \in \widetilde{\text{app}}(a_1, \dots, a_n, \mathfrak{f}) &\text{ iff } (\forall^{a_1} x_1) \dots (\forall^{a_n} x_n) (\forall^{\mathfrak{f}} f) \text{ app}(x_1, \dots, x_n, f) \in S \\ &\text{ iff } (\forall^{a_1} x_1) \dots (\forall^{a_n} x_n) (\forall^{\mathfrak{f}} f) f(x_1, \dots, x_n) \in S \\ &\text{ iff } (\forall^{\mathfrak{f}} f) f(a_1, \dots, a_n) \in S \\ &\text{ iff } (\exists F \in \mathfrak{f}) (\forall f \in F) f(a_1, \dots, a_n) \in S. \end{aligned}$$

(The first equivalence follows from the definition of extensions of maps via ultrafilter quantifiers, the second holds by the definition of app , the third since a_1, \dots, a_n are principal, and the fourth decodes the definition of the $\forall^{\mathfrak{f}}$ quantifier.) Letting $S = \{f(a_1, \dots, a_n)\}$, we have $\widetilde{\text{app}}(a_1, \dots, a_n, \mathfrak{f}) = f(a_1, \dots, a_n) \in Y$, thus witnessing that \mathfrak{f} is pseudo-principal.

To construct a non-pseudo-principal ultrafilter, pick any $\mathbf{a} = (a_1, \dots, a_n) \in X_1 \times \dots \times X_n$ and $\mathfrak{u} \in \beta Y \setminus Y$ (as Y is infinite), and expand the centered family

$$\begin{aligned} \mathcal{G} &= \{ \{f \in Y^{X_1 \times \dots \times X_n} : f(\mathbf{a}) \in S\} : S \in \mathfrak{u} \} \\ &= \{O_{\mathbf{a}, \tilde{S}} : S \in \mathfrak{u}\} \end{aligned}$$

to an ultrafilter $\mathfrak{g} \supseteq \mathcal{G}$ over $Y^{X_1 \times \dots \times X_n}$. Calculations similar to those in the above give us

$$\widetilde{\text{app}}(a_1, \dots, a_n, \mathfrak{g}) = \mathfrak{u},$$

thus witnessing that \mathfrak{g} is not pseudo-principal. \square

A generalized interpretation ι is *pseudo-principal on functional symbols* iff $\iota(F)$ is a pseudo-principal ultrafilter for each functional symbol F (and then, for each term t).

Corollary 19. *Given a generalized model $\mathfrak{A} = (\beta X, \iota(F), \dots, \iota(R), \dots)$ with ι pseudo-principal on functional symbols, define a generalized model $\mathfrak{B} = (\beta X, \jmath(F), \dots, \jmath(R), \dots)$ in the same signature as follows: let \mathfrak{B} have the same universe βX , let \jmath coincide with ι on predicate symbols, and for each functional symbol F let $\jmath(F)$ be the principal ultrafilter given by the operation $f : X^n \rightarrow X$ defined by letting*

$$f(a_1, \dots, a_n) = \widetilde{\text{app}}(a_1, \dots, a_n, \iota(F)).$$

Then for all valuations v , formulas φ , and terms t_1, \dots, t_n ,

$$\mathfrak{A} \models \varphi(t_1, \dots, t_n)[v] \text{ iff } \mathfrak{B} \models \varphi(t_1, \dots, t_n)[v].$$

Proof. Lemma 16. \square

It follows that for every generalized model \mathfrak{A} whose interpretation is pseudo-principal on functional symbols, by replacing its relations as in Corollary 17 and its operations as in Corollary 19, one obtains an ordinary model \mathfrak{B} with the same universe such that for all formulas φ and elements $\mathfrak{u}_1, \dots, \mathfrak{u}_n$ of the universe, $\mathfrak{A} \models \varphi[\mathfrak{u}_1, \dots, \mathfrak{u}_n]$ iff $\mathfrak{B} \models \varphi[\mathfrak{u}_1, \dots, \mathfrak{u}_n]$.

We do not formulate this fact as a separate theorem since we shall be able to establish stronger facts soon. In Theorem 23, we shall establish that for any generalized model \mathfrak{A} , not necessarily with a pseudo-principal interpretation, one can construct a certain ordinary model $e(\mathfrak{A})$ satisfying the same formulas; and then, in Theorem 28, that whenever \mathfrak{A} has a pseudo-principal interpretation, $e(\mathfrak{A})$ is nothing but the \sim -extension of some model. In fact, in the latter case, $e(\mathfrak{A})$ coincides with \mathfrak{B} from the previous paragraph.

Operations e and E . Let us now define two operations, e and E , which turn generalized models into certain ordinary models that (as we shall see soon) generalize the $*$ - and \sim -extensions. Both operations take ultrafilters over n -ary maps to n -ary maps over ultrafilters, and ultrafilters over n -ary relations to n -ary relations over ultrafilters. Both operations are surjective and non-injective (Lemma 27).

The map e on ultrafilters over maps will be the map $\widetilde{\text{ext}}$ defined and discussed in Section 2. Now we extend $\widetilde{\text{ext}}$ to ultrafilters over relations by identifying n -ary relations with their n -ary characteristic functions into the discrete space $2 = \{0, 1\}$:

$$\begin{array}{ccc} \beta\mathcal{P}(X_1 \times \dots \times X_n) & & \\ \uparrow & \dashrightarrow^{\widetilde{\text{ext}}} & \\ \mathcal{P}(X_1 \times \dots \times X_n) & \xrightarrow{\text{ext}} & \{Q \subseteq \beta X_1 \times \dots \times \beta X_n : Q \text{ is right clopen w.r.t. } X_1, \dots, X_{n-1}\} \end{array}$$

(Recall that by Theorem 8(i), a subset of $\beta X_1 \times \dots \times \beta X_n$ is right clopen w.r.t. X_1, \dots, X_{n-1} iff it is of form \widetilde{R} for some n -ary subset R of $X_1 \times \dots \times X_n$.) Let the map e on ultrafilters over relations also coincide with the map $\widetilde{\text{ext}}$ on them. So in result we have:

$$\begin{aligned} e^{\ulcorner} \beta(Y^{X_1 \times \dots \times X_n}) &\subseteq \beta Y^{\beta X_1 \times \dots \times \beta X_n}, \\ e^{\ulcorner} \beta\mathcal{P}(X_1 \times \dots \times X_n) &\subseteq \mathcal{P}(\beta X_1 \times \dots \times \beta X_n). \end{aligned}$$

We observe that e and $\widetilde{\text{app}}$ (or $\widetilde{\text{in}}$) are expressed via each other:

Lemma 20. *Let X_1, \dots, X_n, Y be discrete spaces. For all $f \in \beta(Y^{X_1 \times \dots \times X_n})$, $\tau \in \beta\mathcal{P}(X_1 \times \dots \times X_n)$, and $\mathbf{u}_1 \in \beta X_1, \dots, \mathbf{u}_n \in \beta X_n$,*

$$\begin{aligned} e(f)(\mathbf{u}_1, \dots, \mathbf{u}_n) &= \widetilde{\text{app}}(\mathbf{u}_1, \dots, \mathbf{u}_n, f), \\ e(\tau)(\mathbf{u}_1, \dots, \mathbf{u}_n) &\text{ iff } \widetilde{\text{in}}(\mathbf{u}_1, \dots, \mathbf{u}_n, \tau). \end{aligned}$$

In other words,

$$\begin{aligned} e(f) &= \{(\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{v}) \in \beta X_1 \times \dots \times \beta X_n \times \beta Y : \widetilde{\text{app}}(\mathbf{u}_1, \dots, \mathbf{u}_n, f) = \mathbf{v}\}, \\ e(\tau) &= \{(\mathbf{u}_1, \dots, \mathbf{u}_n) \in \beta X_1 \times \dots \times \beta X_n : \widetilde{\text{in}}(\mathbf{u}_1, \dots, \mathbf{u}_n, \tau)\}. \end{aligned}$$

Proof. To simplify the notation, let RC be the space $RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, \beta Y)$ of n -ary maps on $\beta X_1 \times \dots \times \beta X_n$ into βY that are right continuous w.r.t. X_1, \dots, X_{n-1} , endowed with the (X_1, \dots, X_n) -pointwise convergence topology. By Lemma 11, RC is compact Hausdorff. Recall that for any $f \in \beta(Y^{X_1 \times \dots \times X_n})$ we have

$$e(f) = \widetilde{\text{ext}}(f) = g \in RC \text{ such that } \{g\} = \bigcap_{A \in f} \text{cl}_{RC} \{\tilde{f} : f \in A\},$$

and $\widetilde{\text{app}}^f$ is the n -ary map on $\beta X_1 \times \dots \times \beta X_n$ into βY defining by letting $\widetilde{\text{app}}^f(\mathbf{u}_1, \dots, \mathbf{u}_n) = \widetilde{\text{app}}(\mathbf{u}_1, \dots, \mathbf{u}_n, f)$ for all $\mathbf{u}_1 \in \beta X_1, \dots, \mathbf{u}_n \in \beta X_n$.

Note that both maps $\widetilde{\text{app}}^f$ and $e(f)$ are in RC (the first follows from the fact that $\widetilde{\text{app}}$ is right continuous w.r.t. X_1, \dots, X_{n-1} , the second holds since $\widetilde{\text{ext}}$ is a map into RC). Therefore, by Lemma 10, in order to show that they coincide, it suffices to verify that they coincide on principal ultrafilters.

For this, pick any $a_1 \in X_1, \dots, a_n \in X_n$ and $S \subseteq Y$. We have:

$$\begin{aligned} S \in \widetilde{\text{app}}^{\mathfrak{f}}(a_1, \dots, a_n) &\text{ iff } (\forall^{a_1} x_1) \dots (\forall^{a_n} x_n) (\forall^{\mathfrak{f}} f) \text{ app}(x_1, \dots, x_n, f) \in S \\ &\text{ iff } (\forall^{\mathfrak{f}} f) \text{ app}(a_1, \dots, a_n, f) \in S \\ &\text{ iff } (\forall^{\mathfrak{f}} f) f(a_1, \dots, a_n) \in S \\ &\text{ iff } \{f \in Y^{X_1 \times \dots \times X_n} : f(a_1, \dots, a_n) \in S\} \in \mathfrak{f}. \end{aligned}$$

(The first equivalence follows from the definition of extensions of maps via ultrafilter quantifiers, the second holds since a_1, \dots, a_n are principal, the third by the definition of app , and the fourth by the definition of the $\forall^{\mathfrak{f}}$ quantifier.) Therefore,

$$e(\mathfrak{f}) \in \text{cl}_{RC} \{ \tilde{f} \in RC : f \in Y^{X_1 \times \dots \times X_n} \text{ and } f(a_1, \dots, a_n) \in S \}.$$

As stated in Lemma 11, the space RC is zero-dimensional; in particular, the open set $O_{a_1, \dots, a_n, \tilde{S}} = \{g \in RC : g(a_1, \dots, a_n) \in \tilde{S}\}$ is closed (since its complement $RC \setminus O_{a_1, \dots, a_n, \tilde{S}} = O_{a_1, \dots, a_n, \beta Y \setminus \tilde{S}}$ is open too). It follows that

$$\text{cl}_{RC} \{ \tilde{f} \in RC : f \in Y^{X_1 \times \dots \times X_n} \text{ and } f(a_1, \dots, a_n) \in S \} \subseteq O_{a_1, \dots, a_n, \tilde{S}}.$$

Therefore, we obtain

$$e(\mathfrak{f}) \in O_{a_1, \dots, a_n, \tilde{S}},$$

or, in other words, $e(\mathfrak{f})(a_1, \dots, a_n) \in \tilde{S}$. The latter is clearly equivalent to $S \in e(\mathfrak{f})(a_1, \dots, a_n)$. Thus we get the inclusion $\widetilde{\text{app}}(a_1, \dots, a_n, \mathfrak{f}) \subseteq e(\mathfrak{f})(a_1, \dots, a_n)$. But since both $\widetilde{\text{app}}(a_1, \dots, a_n, \mathfrak{f})$ and $e(\mathfrak{f})(a_1, \dots, a_n)$ are *ultrafilters*, the inclusion actually gives the equality $\widetilde{\text{app}}(a_1, \dots, a_n, \mathfrak{f}) = e(\mathfrak{f})(a_1, \dots, a_n)$.

This proves the lemma for ultrafilters over sets of maps. The remaining claim about ultrafilters over sets of relations follows by replacing the relations with their characteristic functions. \square

Question. For which compact Hausdorff space Y , instead of βY with a discrete Y , does Lemma 20 remain true (providing that $\widetilde{\text{app}}$ is defined as in the remark above)? Does this hold at least for all zero-dimensional, or all extremally disconnected compact Hausdorff Y ? (Problem 3 in Section 5.)

Corollary 21. *Let X_1, \dots, X_n, Y be discrete spaces. The set of pseudo-principal ultrafilters is the preimage of the set $\{\tilde{f} : f \in Y^{X_1 \times \dots \times X_n}\}$ under the map e :*

$$\{\mathfrak{f} \in \beta(Y^{X_1 \times \dots \times X_n}) : \mathfrak{f} \text{ is pseudo-principal}\} = e^{-1} \{\tilde{f} : f \in Y^{X_1 \times \dots \times X_n}\}.$$

Recalling that $e = \widetilde{\text{ext}}$, that on the set $Y^{X_1 \times \dots \times X_n}$ (identified with principal ultrafilters) $\widetilde{\text{ext}}$ is just ext , and that $e \text{“} Y^{X_1 \times \dots \times X_n} = \{\tilde{f} : f \in Y^{X_1 \times \dots \times X_n}\}$, we can rewrite the set of pseudo-principal ultrafilters also by

$$e^{-1} e \text{“} Y^{X_1 \times \dots \times X_n} = \widetilde{\text{ext}}^{-1} \text{ext} \text{“} Y^{X_1 \times \dots \times X_n}.$$

Proof. Show first that if \mathfrak{f} is pseudo-principal, then $e(\mathfrak{f}) = \tilde{f}$ for some $f \in Y^{X_1 \times \dots \times X_n}$. By the definition of e ($= \widetilde{\text{ext}}$), always $e(\mathfrak{f})$ is a map belonging to the set $RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$. Since by Lemma 20 we have $e(\mathfrak{f})(\mathbf{u}_1, \dots, \mathbf{u}_n) = \widetilde{\text{app}}(\mathbf{u}_1, \dots, \mathbf{u}_n, \mathfrak{f})$, we see that the map e takes principal ultrafilters to principal ultrafilters whenever \mathfrak{f} is pseudo-principal. But then it follows from Lemma 10 that $e(\mathfrak{f})$ coincides with \tilde{f} if the map f is the restriction of $e(\mathfrak{f})$ to principal ultrafilters:

$$e(\mathfrak{f}) = \tilde{f} \text{ for } f = e(\mathfrak{f}) \upharpoonright (X_1 \times \dots \times X_n).$$

It remains to show the converse implication, i.e. that for every \tilde{f} there exists a pseudo-principal ultrafilter \mathfrak{f} with $e(\mathfrak{f}) = \tilde{f}$. For this, it clearly suffices to let \mathfrak{f} equal to the principal ultrafilter given by f . \square

Question. What are topological properties of the set of pseudo-principal ultrafilters in the space $\beta(Y^{X_1 \times \dots \times X_n})$? topological properties of its preimage under e , the set $\{\tilde{f} : f \in Y^{X_1 \times \dots \times X_n}\}$, in the space $RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, \beta Y)$ with the (X_1, \dots, X_n) -pointwise convergence topology (except for the fact that it is dense there, as stated in Lemma 12), or with the (usual) pointwise convergence topology? in the space $(\beta Y)^{\beta X_1 \times \dots \times \beta X_n}$ with the pointwise convergence topology?

Let us point out that objects naturally defined in terms of ultrafilter extensions often have rather hardly definable topological properties, see [30, 31]. (Problem 4 in Section 5.)

Corollary 22. *For all generalized models $\mathfrak{A} = (\beta X, \iota(F), \dots, \iota(R), \dots)$ and valuations v ,*

$$\begin{aligned} v_i(F(t_1, \dots, t_n)) &= e(\iota(F))(v_i(t_1), \dots, v_i(t_n)), \\ \mathfrak{A} \models R(t_1, \dots, t_n) [v] &\text{ iff } e(\iota(R))(v_i(t_1), \dots, v_i(t_n)). \end{aligned}$$

Proof. Lemma 20 with $X_1 = \dots = X_n = Y = X$. \square

For a generalized model $\mathfrak{B} = (\beta X, \mathfrak{f}, \dots, \mathfrak{r}, \dots)$, let

$$e(\mathfrak{B}) = (\beta X, e(\mathfrak{f}), \dots, e(\mathfrak{r}), \dots).$$

Note that $e(\mathfrak{B})$ is an ordinary model.

The following theorem is the first of the three main results of this section, it states that in point of view of the satisfaction of formulas, any generalized model \mathfrak{A} is not distinguished from the ordinary model $e(\mathfrak{A})$.

Theorem 23. *If \mathfrak{A} is a generalized model, then for all formulas φ and elements u_1, \dots, u_n of the universe of \mathfrak{A} ,*

$$\mathfrak{A} \models \varphi [u_1, \dots, u_n] \text{ iff } e(\mathfrak{A}) \models \varphi [u_1, \dots, u_n].$$

Proof. Induction on φ starting from Corollary 22. \square

Now we define the map E , which has the same domain that the map e does and also satisfying

$$\begin{aligned} E''\beta(Y^{X_1 \times \dots \times X_n}) &\subseteq \beta Y^{\beta X_1 \times \dots \times \beta X_n}, \\ E''\beta\mathcal{P}(X_1 \times \dots \times X_n) &\subseteq \mathcal{P}(\beta X_1 \times \dots \times \beta X_n), \end{aligned}$$

as follows: E and e coincide on $\beta(Y^{X_1 \times \dots \times X_n})$, and whenever $\mathfrak{r} \in \beta\mathcal{P}(X_1 \times \dots \times X_n)$ then we define

$$E(\mathfrak{r}) = \widetilde{\text{ext}}''\widetilde{\text{ext}}(\mathfrak{r}) = \{\widetilde{\text{ext}}(\mathfrak{q}) : \mathfrak{q} \in \widetilde{\text{ext}}(\mathfrak{r})\}$$

where \mathfrak{r} is considered as an ultrafilter over unary relations on $X_1 \times \dots \times X_n$ while \mathfrak{q} is considered as an ultrafilter over unary maps on n (and $\widetilde{\text{ext}}$ is consequently considered in two distinct meanings). Let us now explain the construction in details.

First, we consider $\mathcal{P}(X_1 \times \dots \times X_n)$ as the set of *unary* relations on $X_1 \times \dots \times X_n$. Then the map ext takes any subset R of $X_1 \times \dots \times X_n$ to the clopen subset \tilde{R} of $\beta(X_1 \times \dots \times X_n)$.

Therefore, the extended map $\widetilde{\text{ext}}$ takes any ultrafilter \mathfrak{r} over $\mathcal{P}(X_1 \times \dots \times X_n)$ to some clopen subset $Q = \widetilde{\text{ext}}(\mathfrak{r})$ of $\beta(X_1 \times \dots \times X_n)$:

$$\begin{array}{ccc} \beta\mathcal{P}(X_1 \times \dots \times X_n) & & \\ \uparrow & \searrow \widetilde{\text{ext}} & \\ \mathcal{P}(X_1 \times \dots \times X_n) & \xrightarrow{\text{ext}} & \{Q \subseteq \beta(X_1 \times \dots \times X_n) : Q \text{ is clopen}\} \end{array}$$

Next, we identify the product $X_1 \times \dots \times X_n$ with the set of unary maps f from the set n into $\bigcup_i X_i$ satisfying $f(i) \in X_{i+1}$ (for all $i < n$). Then the map ext takes any such f to the unary continuous map \tilde{f} from n into $\bigcup_i \beta X_i$ satisfying $f(i) \in \beta X_{i+1}$, and we identify the set of such maps \tilde{f} backwards with the product $\beta X_1 \times \dots \times \beta X_n$. Therefore, the extended map $\widetilde{\text{ext}}$ takes any ultrafilter \mathfrak{q} over $X_1 \times \dots \times X_n$ to some n -tuple $(\mathfrak{u}_1, \dots, \mathfrak{u}_n) = \widetilde{\text{ext}}(\mathfrak{q})$ in $\beta X_1 \times \dots \times \beta X_n$:

$$\begin{array}{ccc} \beta(X_1 \times \dots \times X_n) & & \\ \uparrow & \searrow \widetilde{\text{ext}} & \\ X_1 \times \dots \times X_n & \xrightarrow{\text{ext}} & \beta X_1 \times \dots \times \beta X_n \end{array}$$

(An analogous construction was previously used in Theorem 15.) In result, the set $Q = \widetilde{\text{ext}}(\mathfrak{r}) \subseteq \beta(X_1 \times \dots \times X_n)$ is mapped onto the set $\widetilde{\text{ext}}Q = E(\mathfrak{r}) \subseteq \beta X_1 \times \dots \times \beta X_n$. Since Q is clopen and the map $\widetilde{\text{ext}}$ is closed (as a continuous map between compact Hausdorff spaces), the resulting $E(\mathfrak{r})$ is a closed subset of the space $\beta X_1 \times \dots \times \beta X_n$.

Lemma 24. *Let $\mathfrak{r} \in \beta\mathcal{P}(X_1 \times \dots \times X_n)$. Then*

$$e(\mathfrak{r}) = \tilde{R} \text{ and } E(\mathfrak{r}) = R^*$$

for $R = e(\mathfrak{r}) \cap (X_1 \times \dots \times X_n) = E(\mathfrak{r}) \cap (X_1 \times \dots \times X_n) = \bigcap_{S \in \mathfrak{r}} \bigcup S$. Consequently,

$$e(\mathfrak{r}) \subseteq E(\mathfrak{r}).$$

We can write up this R more explicitly:

$$R = \{(a_1, \dots, a_n) \in X_1 \times \dots \times X_n : (\forall S \in \mathfrak{r}) (\exists Q \in S) Q(a_1, \dots, a_n)\}.$$

Proof. For $e(\mathfrak{r}) = \tilde{R}$, apply Lemma 20. For $E(\mathfrak{r}) = R^*$, note that $\mathfrak{q} \in \widetilde{\text{ext}}(\mathfrak{r})$ iff $\bigcap_{S \in \mathfrak{r}} \bigcup S \in \mathfrak{q}$. \square

For a generalized model $\mathfrak{B} = (\beta X, \mathfrak{f}, \dots, \mathfrak{r}, \dots)$, let

$$E(\mathfrak{B}) = (\beta X, E(\mathfrak{f}), \dots, E(\mathfrak{r}), \dots).$$

Then $E(\mathfrak{B})$, like $e(\mathfrak{B})$, is an ordinary model.

The following easy observation is similar to Theorem 1, and moreover, it turns out to be that theorem whenever the interpretation of \mathfrak{B} is pseudo-principal on functional symbols, as we shall see after Theorem 28.

Theorem 25. *For any generalized model \mathfrak{B} the identity map on its universe is a homomorphism of $e(\mathfrak{B})$ onto $E(\mathfrak{B})$:*

$$\begin{array}{ccc} & \mathfrak{B} & \\ e \swarrow & & \searrow E \\ e(\mathfrak{B}) & \xrightarrow{\text{id}} & E(\mathfrak{B}) \end{array}$$

Proof. Immediate from Lemma 24 since $\tilde{R} \subseteq R^*$ for all relations R . \square

Now we are going to establish two remaining main results of this section, Theorems 28 and 29. The first of them characterizes generalized models such that their e - and E -images are ultraextensions of ordinary models, while the second one characterizes ordinary models that are the e - and E -images of generalized models. Before this we prove two more auxiliary lemmas, which actually follow from the previously stated facts.

Lemma 26. *Let \mathfrak{A} be a generalized model with a pseudo-principal interpretation of functional symbols, and \mathfrak{B} the generalized model with a principal interpretation of functional symbols constructed from \mathfrak{A} as in Corollary 19. Then $e(\mathfrak{A}) = e(\mathfrak{B})$.*

Proof. Let ι and j be the interpretations in \mathfrak{A} and \mathfrak{B} , respectively. If F is a functional symbol, then the operations $e(\iota(F))$ and $e(j(F))$ are right continuous w.r.t. principal ultrafilters. Therefore, by Lemma 10, in order to show that they coincide, it suffices to verify that they coincide on principal ultrafilters.

If the symbol F is n -ary, let \mathbf{a} be any n -tuple of principal ultrafilters. We have:

$$\widetilde{\text{app}}(\mathbf{a}, \iota(F)) = j(F)(\mathbf{a}) = \text{app}(\mathbf{a}, j(F)) = \widetilde{\text{app}}(\mathbf{a}, j(F))$$

(the first equality holds by the definition of j from Corollary 19, the second as $j(F)$ is principal, and the third as $\widetilde{\text{app}}$ extends app). By Lemma 20,

$$e(\iota(F))(\mathbf{a}) = \widetilde{\text{app}}(\mathbf{a}, \iota(F)) \quad \text{and} \quad e(j(F))(\mathbf{a}) = \widetilde{\text{app}}(\mathbf{a}, j(F))$$

(that holds for n -tuples of non-principal ultrafilters as well). This completes the proof. \square

Lemma 27. *Both operations e and E are surjective and non-injective. More precisely,*

- (i) e (and E) on $\beta(Y^{X_1 \times \dots \times X_n})$ is a surjection onto $RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$,
- (ii) e on $\beta\mathcal{P}(X_1 \times \dots \times X_n)$ is a surjection onto $\{\tilde{R} \in \mathcal{P}(\beta X_1 \times \dots \times \beta X_n) : R \subseteq X_1 \times \dots \times X_n\}$
 $= \{Q \in \mathcal{P}(\beta X_1 \times \dots \times \beta X_n) : Q \text{ is right clopen w.r.t. } X_1, \dots, X_n\}$,
- (iii) E on $\beta\mathcal{P}(X_1 \times \dots \times X_n)$ is a surjection onto $\{R^* \in \mathcal{P}(\beta X_1 \times \dots \times \beta X_n) : R \subseteq X_1 \times \dots \times X_n\}$
 $= \{Q \in \mathcal{P}(\beta X_1 \times \dots \times \beta X_n) : Q \text{ is regular closed}\}$,

and each of the three maps is not an injection whenever at least one of the X_i is infinite.

Proof. Item (i) is Lemma 13; items (ii) and (iii) are immediate from Lemma 24; the non-injectivity is easy from the cardinality argument since both maps $R \mapsto \tilde{R}$ and $R \mapsto R^*$ are bijections. (Alternatively, (ii) can be obtained from (i) by replacing relations with their characteristic functions.) The equalities in (ii) and (iii) were stated in Theorem 8(i),(ii). \square

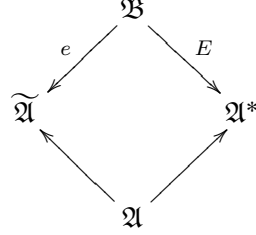
By Lemma 24, relations of the model $e(\mathfrak{B})$ are the \sim -extensions of some relations on X , while relations of the model $E(\mathfrak{B})$ are the $*$ -extensions of the same relations. Whether the whole models $e(\mathfrak{B})$ and $E(\mathfrak{B})$ are the ultrafilter extensions of some models depends only on the ultrafilter interpretation of functional symbols in \mathfrak{B} :

Theorem 28. *Let \mathfrak{B} be a generalized model with the universe βX . The following are equivalent:*

- (i) $e(\mathfrak{B}) = \tilde{\mathfrak{A}}$ for an ordinary model \mathfrak{A} with the universe X ,
- (ii) $E(\mathfrak{B}) = \mathfrak{A}^*$ for an ordinary model \mathfrak{A} with the universe X ,

(iii) the interpretation in \mathfrak{B} is pseudo-principal on functional symbols.

Moreover, the model \mathfrak{A} in (i) and (ii) is the same.



Proof. The implications from each of (i) and (ii) to (iii) are obvious: if the interpretation j in \mathfrak{B} is not pseudo-principal, then there are a functional symbol F and a sequence \mathbf{a} of principal ultrafilters over βX such that the operation $G = e(j(F))$ on βX takes \mathbf{a} to a non-principal ultrafilter $G(\mathbf{a})$ over X . Therefore, G is not of form \widetilde{f} for any operation f on X . Since G is the interpretation of F in both models $e(\mathfrak{B})$ and $E(\mathfrak{B})$, it follows that these models are not of form $\widetilde{\mathfrak{A}}$ and \mathfrak{A}^* for any ordinary model \mathfrak{A} .

Let us show now that, conversely, (iii) implies each of (i) and (ii). By Lemma 26, it suffices to handle the case when the pseudo-principal interpretation j in \mathfrak{B} is principal. So suppose this is the case and define an ordinary interpretation ι of the same language by letting, for all functional symbols F and predicate symbols R ,

$$\begin{aligned}
 \iota(F) &= G \text{ if the principal ultrafilter } j(F) \text{ over } X^{X \times \dots \times X} \text{ is given by } G, \\
 \iota(R) &= Q \text{ if } Q = e(j(R)) \cap (X \times \dots \times X).
 \end{aligned}$$

We have:

$$e(j(F)) = E(j(F)) = \widetilde{\iota(F)}$$

since $j(F)$ is principal and e (and E) on principal ultrafilters is ext, and

$$e(j(R)) = \widetilde{\iota(R)} \text{ and } E(j(R)) = (\iota(R))^*$$

by Lemma 24. Thus if \mathfrak{A} is the ordinary model given by ι , we obtain $e(\mathfrak{B}) = \widetilde{\mathfrak{A}}$ and $E(\mathfrak{B}) = \mathfrak{A}^*$, as required. \square

Finally, we point out that the fact whether an ordinary model with the universe βX is of form $e(\mathfrak{B})$, and whether it is of form $E(\mathfrak{B})$, for some generalized model \mathfrak{B} (clearly, with the same universe βX) depends only on its topological properties:

Theorem 29. *Let \mathfrak{A} be an ordinary model with the universe βX . Then:*

- (i) $\mathfrak{A} = e(\mathfrak{B})$ for a generalized model \mathfrak{B} iff in \mathfrak{A} all operations are right continuous w.r.t. X and all relations are right clopen w.r.t. X ,
- (ii) $\mathfrak{A} = E(\mathfrak{B})$ for a generalized model \mathfrak{B} iff in \mathfrak{A} all operations are right continuous w.r.t. X and all relations are regular closed.

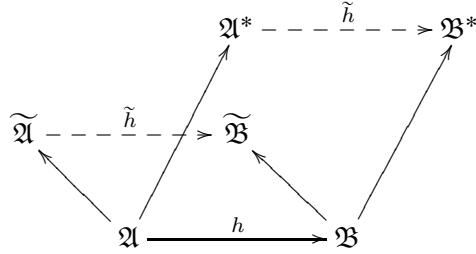
Proof. Lemma 27. \square

Since by Theorem 28, e and E applied to generalized models with pseudo-principal interpretations give the \sim - and $*$ -extensions of ordinary models, Theorem 29 can be considered as a generalization of Theorems 4 and 6.

First Extension Theorems. Here we discuss a possible generalization of the First Extension Theorems (Theorems 3 and 2) to generalized models. To start, let us restate both them in a single way as follows.

Theorem 30. *Let \mathfrak{A} and \mathfrak{B} be two models in the same signature, and let $h : X \rightarrow Y$ be a map between their universes. The following are equivalent:*

- (i) h is a homomorphism of \mathfrak{A} into \mathfrak{B} ,
- (ii) \tilde{h} is a homomorphism of $\tilde{\mathfrak{A}}$ into $\tilde{\mathfrak{B}}$,
- (iii) \tilde{h} is a homomorphism of \mathfrak{A}^* into \mathfrak{B}^* :



Proof. Theorems 2 and 3. □

This leads to a conclusion for our generalized models:

Lemma 31. *Let \mathfrak{U} and \mathfrak{V} be two generalized models in the same signature, and let $h : \beta X \rightarrow \beta Y$ be a map between their universes. The following are equivalent:*

- (i) h is a homomorphism of $e(\mathfrak{U})$ into $e(\mathfrak{V})$,
- (ii) h is a homomorphism of $E(\mathfrak{U})$ into $E(\mathfrak{V})$.

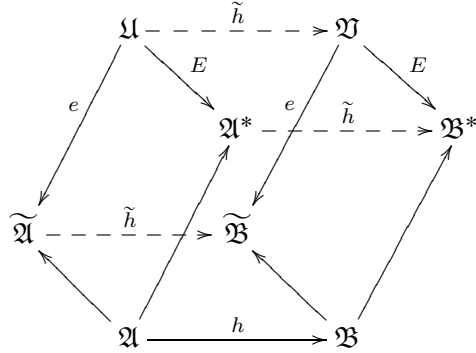
Proof. If \mathfrak{f} is an ultrafilter over operations, we have $e(\mathfrak{f}) = E(\mathfrak{f})$ by definition of e and E , hence the claim for homomorphisms w.r.t. operations holds trivially. If \mathfrak{r} is an ultrafilter over relations, we have $e(\mathfrak{r}) = \tilde{R}$ and $E(\mathfrak{r}) = R^*$ by Lemma 24, hence the claim for homomorphisms w.r.t. relations holds by Theorem 30. □

This observation leads to the following definition: If \mathfrak{U} and \mathfrak{V} are two generalized models in the same signature, we say that a map $h : \beta X \rightarrow \beta Y$ between their universes is a *homomorphism (of generalized models)* iff it is a homomorphism of $e(\mathfrak{U})$ into $e(\mathfrak{V})$ (or a homomorphism of $E(\mathfrak{U})$ into $E(\mathfrak{V})$, which is equivalent by Lemma 31). The concepts of *epimorphisms*, *isomorphic embeddings*, *submodels*, *elementary embeddings*, *elementary submodels*, *submodels*, *quotients*, etc., for generalized models are defined likewise.

The following can be considered as the First Extension Theorem for generalized models:

Theorem 32. *Let \mathfrak{U} and \mathfrak{V} be two generalized models in the same signature with the universes βX and βY , both having pseudo-principal interpretations on functional symbols, let \mathfrak{A} and \mathfrak{B} denote the models such that $\tilde{\mathfrak{A}} = e(\mathfrak{U})$ and $\tilde{\mathfrak{B}} = e(\mathfrak{V})$, and so $\mathfrak{A}^* = E(\mathfrak{U})$ and $\mathfrak{B}^* = E(\mathfrak{V})$, and let $h : X \rightarrow Y$. The following are equivalent:*

- (i) h is a homomorphism of \mathfrak{A} into \mathfrak{B} ,
- (ii) \tilde{h} is a homomorphism of \mathfrak{U} into \mathfrak{V} ,
- (iii) \tilde{h} is a homomorphism of $\tilde{\mathfrak{A}}$ into $\tilde{\mathfrak{B}}$,
- (iv) \tilde{h} is a homomorphism of \mathfrak{A}^* into \mathfrak{B}^* :



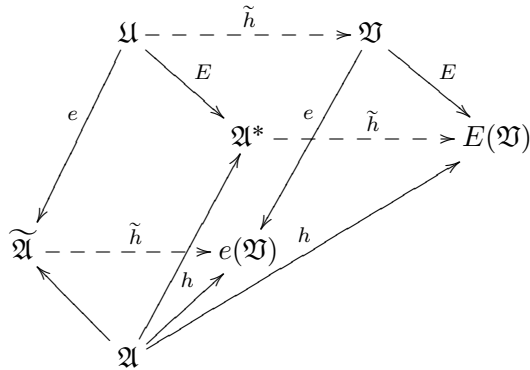
Proof. The equivalence of items (i) and (ii) follows from Theorem 28, Lemma 31, and the definition of homomorphisms of generalized models. The equivalence of items (i), (iii), and (iv) repeats Theorem 30. \square

For a generalized model \mathfrak{U} with the universe βX , the set X of principal ultrafilters forms a generalized submodel (and also ordinary submodels of $e(\mathfrak{U})$ and $E(\mathfrak{U})$) iff the interpretation in \mathfrak{U} is pseudo-principal on functional symbols; this can be added as item (iv) to Theorem 28. We shall call the submodel consisting of principal ultrafilters the *principal submodel*. Thus Theorem 32 can be reformulated by replacing “both having pseudo-principal interpretations” with “both having principal submodels”.

In fact, we can omit here the assumption about the pseudo-principality in the generalized model \mathfrak{V} by applying the Second Extension Theorems (Theorems 5 and 7):

Theorem 33. *Let \mathfrak{U} and \mathfrak{V} be two generalized models in the same signature with the universes βX and βY , let the interpretation of \mathfrak{U} be pseudo-principal on functional symbols with \mathfrak{A} the principal submodel (having the universe X), so $\tilde{\mathfrak{A}} = e(\mathfrak{U})$ and $\mathfrak{A}^* = E(\mathfrak{U})$, and let $h : X \rightarrow Y$. The following are equivalent:*

- (i) h is a homomorphism of \mathfrak{A} into $e(\mathfrak{V})$,
- (ii) h is a homomorphism of \mathfrak{A} into $E(\mathfrak{V})$,
- (iii) \tilde{h} is a homomorphism of \mathfrak{U} into \mathfrak{V} ,
- (iv) \tilde{h} is a homomorphism of $\tilde{\mathfrak{A}}$ into $e(\mathfrak{V})$,
- (v) \tilde{h} is a homomorphism of \mathfrak{A}^* into $E(\mathfrak{V})$:



Proof. The equivalence of items (i) and (iii) follows from Theorem 5 and Theorem 29(i), while the equivalence of items (ii) and (iii) follows from Theorem 7 and Theorem 29(ii). Finally, (ii) is equivalent to (iv) by Theorem 5 and to (v) by Theorem 7. \square

Observe that, by Theorem 29, whenever the interpretation of \mathfrak{B} is *not* pseudo-principal on functional symbols, then the models $e(\mathfrak{B})$ and $E(\mathfrak{B})$ are *not* of form $\tilde{\mathfrak{A}}$ and \mathfrak{A}^* for any ordinary model \mathfrak{A} ; nevertheless, these models still satisfy the conditions of Theorems 5 and 7 (playing the role of the model \mathfrak{C} there). Therefore, Theorem 33 is indeed more general than Theorem 32; it has a character intermediate between the First and the Second Extension Theorems. To have a reasonable generalization of Second Extension Theorems in the full form, we need to have a more general concept of generalized models; this is the subject of the next, last section of our article.

Remark. Theorems 30–32 remain true for epimorphisms and isomorphic embeddings, and Theorem 33 for epimorphisms. Also they can be stated for so-called homotopies and isotopies; these concepts (generalizing homomorphisms and isomorphisms) for ordinary models, together with both extension theorems, were introduced in [3] (and [4]). For generalized models they can be defined in the same way as this was done for homomorphisms and embeddings. Finally, versions for multi-sorted models (having rather many universes X_1, \dots, X_n than one universe X) can be also easily stated.

4. Wider ultrafilter interpretations

Here we discuss a possible generalization of the Second Extension Theorems (Theorems 5 and 7) to generalized models. For this, we should have a wider concept of generalized models which, on the one hand, would replace compact Hausdorff right topological models in these theorems, and on the other hand, would turn into our previous concept of generalized models whenever the universe is of form βY to include our versions of the First Extension Theorem for generalized models (Theorems 32 and 33). Also we should have a concept of satisfiability in these models which would turn into the satisfiability in our previous generalized model; recall that the latter can be redefined in terms of the map e (Theorem 23). Actually, our new concepts of generalized models and satisfiability will be wide enough to cover all ordinary models, not only ultrafilter extensions.

Generalized models in the wide sense. The new definition of generalized models requires only a minor modification of the former one. By an *ultrafilter* (or *generalized*) *interpretation* we still mean a map which takes functional and relational symbols to ultrafilters over operations and relations on a set X . But valuations of variables now will be in the set X itself, not in βX . Thus

$$\mathfrak{A} = (X, f, \dots, r, \dots)$$

is a *generalized model in the wide sense* iff X is the universe of \mathfrak{U} , i.e. individual variables are valued by elements of X , and if F is an n -ary functional symbol then it is interpreted by some $\mathfrak{f} \in \beta(X^{X^n})$, and if R is an n -ary predicate symbol then it is interpreted by some $\mathfrak{r} \in \beta\mathcal{P}(X^n)$. Generalized models described in Section 2 will be referred to as *generalized models in the narrow sense*. In the sequel, we shall sometimes omit the words “in the wide sense” but never “in the narrow sense”.

To revise the concept of satisfiability making it adequate for generalized models in this wider sense, we use the notion of convergence of ultrafilters. Recall that a filter \mathfrak{d} over a topological space X *converges* to a point $x \in X$ iff any neighborhood of x belongs to \mathfrak{d} . If a filter \mathfrak{d} converges to a unique point x then x is called the *limit* of \mathfrak{d} , in which case we shall write $\lim \mathfrak{d} = x$. As well-known, any filter over X converges to at most one point iff X is Hausdorff, and any *ultrafilter* over X converges to at least one point iff X is compact (see e.g. [12]; for the concept of \mathfrak{u} -limit with ultrafilters \mathfrak{u} see [5]). Note also that, even if X is compact Hausdorff, some filter over X that is not an ultrafilter may converge to *no* single point (consider e.g. any discrete X with $1 < |X| < \omega$ and the trivial filter consisting of X as its single element).

For a generalized model $\mathfrak{U} = (X, \mathfrak{f}, \dots, \mathfrak{r}, \dots)$, fix some topologies on the sets X^{X^n} and $\mathcal{P}(X^n)$ for all $n < \omega$ such that the signature has n -ary operations, respectively, relations. Let us say that \mathfrak{U} *converges* (w.r.t the family of topologies) to an ordinary model $\mathfrak{A} = (X, F, \dots, R, \dots)$ in the same signature iff for the interpretation of each symbol in \mathfrak{U} converges to one in \mathfrak{A} . Moreover, \mathfrak{A} is the *limit* of \mathfrak{U} iff the interpretation of \mathfrak{A} is the pointwise limit of one of \mathfrak{U} , in which case we write $\mathfrak{A} = \lim \mathfrak{U}$. Thus whenever the limits of all the ultrafilters $\mathfrak{f}, \dots, \mathfrak{r}, \dots$ exist then the generalized model \mathfrak{U} converges to its limit:

$$\lim \mathfrak{U} = (X, \lim \mathfrak{f}, \dots, \lim \mathfrak{r}, \dots),$$

which is an ordinary model in the same signature.

Let us say that a generalized model \mathfrak{U} endowed with topologies on its universe and the sets of operations and of relations on it, has a *well-defined satisfiability* iff there exists the limit of \mathfrak{U} , in which case we define it as the ordinary satisfiability in the limit:

$$\mathfrak{U} \models_{\lim} \varphi [v] \text{ iff } \lim \mathfrak{U} \models \varphi [v]$$

for all formulas φ and valuations v . We temporarily use the symbol \models_{\lim} for the renewed concept of satisfiability; after Theorem 35, which states that on generalized models in the narrow sense this concept coincides with the former one, we shall continue to use the former symbol \models .

Let us firstly show that all ordinary models and the satisfiability in them can be regarded as generalized models in the wide sense and the satisfiability defined via limits.

Theorem 34. *Any ordinary model \mathfrak{A} with the usual satisfaction relation \models is (up to a natural identification) a generalized model \mathfrak{U} with the satisfaction relation \models_{\lim} , so we have*

$$\mathfrak{U} \models_{\lim} \varphi [v] \text{ iff } \mathfrak{A} \models \varphi [v]$$

for all formulas φ and valuations v . Moreover, the same is true for ordinary models endowed with arbitrary topologies.

Proof. Define \mathfrak{U} as follows: let the universe of \mathfrak{U} coincide with one of \mathfrak{A} , which we denote by X , and let the interpretation in \mathfrak{U} be the principal interpretation giving with one in \mathfrak{A} , i.e. if an n -ary functional symbol is interpreted in \mathfrak{A} by $F \in X^{X^n}$ then it is interpreted in \mathfrak{U} by the principal ultrafilter over X^{X^n} given by F , and likewise for predicate symbols. We can suppose that all topologies on X^{X^n} and $\mathcal{P}(X^n)$ are discrete. Since any principal ultrafilter given by a point x has the limit x , we conclude that \models_{\lim} in \mathfrak{U} coincides with \models in \mathfrak{A} . Moreover, the same fact is true for every topologies on X^{X^n} and $\mathcal{P}(X^n)$, which proves the last claim of the theorem. \square

Now we are going to show that the wider concepts of generalized models and the satisfiability in them cover the former, narrow concepts. (Let us also note that the new concept is not exhausted by the two cases of ordinary models and generalized models in the narrow sense.)

Theorem 35. *Any generalized model \mathfrak{A} with the satisfaction relation in the narrow sense is, up to a natural identification, a generalized model \mathfrak{U} in the wide sense with the satisfaction relation defined via limits in certain appropriate topologies, so we have*

$$\mathfrak{U} \models_{\text{lim}} \varphi [v] \text{ iff } \mathfrak{A} \models \varphi [v]$$

for all formulas φ and valuations v .

Proof. We start by describing how to represent a generalized model \mathfrak{A} in the narrow sense by a certain generalized model \mathfrak{U} in the wide sense. Let βX be the universe of \mathfrak{A} , and suppose that the universe of \mathfrak{U} coincides with it. Now we must identify ultrafilters over the sets X^{X^n} and $\mathcal{P}(X^n)$ with certain ultrafilters over the sets $(\beta X)^{(\beta X)^n}$ and $\mathcal{P}((\beta X)^n)$, respectively. Let us provide a more general procedure, which will be referred to as the *identification map* and denoted by i .

Identification map i . For any positive $n < \omega$, discrete spaces X_1, \dots, X_n , compact Hausdorff space Y , and $S \subseteq Y$, we construct the map i taking ultrafilters over $S^{X_1 \times \dots \times X_n}$ to ultrafilters over $Y^{\beta X_1 \times \dots \times \beta X_n}$:

$$i: \beta(S^{X_1 \times \dots \times X_n}) \subseteq \beta(Y^{\beta X_1 \times \dots \times \beta X_n}).$$

The construction is going in two steps.

First, recall that the map ext provides a canonical one-to-one correspondence between the set $S^{X_1 \times \dots \times X_n}$ and its image

$$\text{ext}^{\text{“}} S^{X_1 \times \dots \times X_n} = \{\tilde{f} : f \in S^{X_1 \times \dots \times X_n}\} \subseteq RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y) \subseteq Y^{\beta X_1 \times \dots \times \beta X_n}.$$

This induces the bijection $+$ of $\beta(S^{X_1 \times \dots \times X_n})$ onto $\beta(\text{ext}^{\text{“}} S^{X_1 \times \dots \times X_n})$ taking each ultrafilter \mathfrak{f} over $S^{X_1 \times \dots \times X_n}$ to an ultrafilter \mathfrak{f}^+ over $\text{ext}^{\text{“}} S^{X_1 \times \dots \times X_n}$ by letting

$$\mathfrak{f}^+ = \{\text{ext}^{\text{“}} A : A \in \mathfrak{f}\}.$$

Second, for any $S \subseteq T$ we define the *lifting* map of βS into βT , by letting for all $\mathfrak{u} \in \beta S$,

$$\mathfrak{u}^T = \{B \subseteq T : A \in \mathfrak{u} \text{ and } B \supseteq A\}.$$

Define also the *projection* map of $\{\mathfrak{v} \in \beta T : S \in \mathfrak{v}\}$ into βS , by letting for all such \mathfrak{v} ,

$$\mathfrak{v}_S = \{A \cap S : A \in \mathfrak{v}\}.$$

Clearly, the domain of the projection is the range of the lifting, and moreover, $(\mathfrak{v}_S)^T = \mathfrak{v}$ and $(\mathfrak{u}^T)_S = \mathfrak{u}$, thus the lifting and the projection maps are two mutually inverse bijections. (Often one identifies these ultrafilters, thus considering βS as the closed subset of βT consisting of those ultrafilters over T that are concentrated on S , see e.g. [5], Section 3.3).

Now we define i as the composition of $+$ and lifting, thus for all $\mathfrak{f} \in S^{X_1 \times \dots \times X_n}$ we let

$$i(\mathfrak{f}) = (\mathfrak{f}^+)^{Y^{\beta X_1 \times \dots \times \beta X_n}}.$$

In result, for any ultrafilter \mathfrak{f} over $S^{X_1 \times \dots \times X_n}$, its image $i(\mathfrak{f})$ is an ultrafilter over $Y^{\beta X_1 \times \dots \times \beta X_n}$ which is concentrated on $\text{ext}^+ S^{X_1 \times \dots \times X_n} = \{\tilde{f} : f \in S^{X_1 \times \dots \times X_n}\}$.

Let us now expand the domain of the map i to ultrafilters over relations. We want to get i taking ultrafilters over $\mathcal{P}(X_1 \times \dots \times X_n)$ to ultrafilters over $\mathcal{P}(\beta X_1 \times \dots \times \beta X_n)$:

$$i^+ \beta \mathcal{P}(X_1 \times \dots \times X_n) \subseteq \beta \mathcal{P}(\beta X_1 \times \dots \times \beta X_n).$$

For this, we may identify n -ary relations with their characteristic functions, i.e. n -ary maps into $2 = \{0, 1\}$, where 2 is endowed with the discrete topology, and use the definition of i for ultrafilters over maps (with $Y = S = 2$). Equivalently, we might imitate the above construction: for each $\mathfrak{r} \in \beta \mathcal{P}(X_1 \times \dots \times X_n)$, we might turn it firstly to $\mathfrak{r}^+ \in \beta(\text{ext}^+ \mathcal{P}(X_1 \times \dots \times X_n))$ where $\text{ext}(R) = \tilde{R}$, so by Theorem 8,

$$\begin{aligned} \text{ext}^+ \mathcal{P}(X_1 \times \dots \times X_n) &= \{\tilde{R} : R \subseteq X_1 \times \dots \times X_n\} \\ &= \{Q \subseteq \beta X_1 \times \dots \times \beta X_n : Q \text{ is right clopen w.r.t. } X_1, \dots, X_n\}, \end{aligned}$$

by letting

$$\mathfrak{r}^+ = \{\text{ext}^+ A : A \in \mathfrak{r}\},$$

and secondly, by lifting the obtained ultrafilter to an ultrafilter over $\mathcal{P}(\beta X_1 \times \dots \times \beta X_n)$, thus letting

$$i(\mathfrak{r}) = (\mathfrak{r}^+)^{\mathcal{P}(\beta X_1 \times \dots \times \beta X_n)}.$$

In result, for any ultrafilter \mathfrak{r} over $\mathcal{P}(X_1 \times \dots \times X_n)$, its image $i(\mathfrak{r})$ is an ultrafilter over $\mathcal{P}(\beta X_1 \times \dots \times \beta X_n)$ which is concentrated on $\{Q \subseteq \beta X_1 \times \dots \times \beta X_n : Q \text{ is right clopen w.r.t. } X_1, \dots, X_n\}$.

Remark. In fact, the map $+$ is $\tilde{\text{ext}}$ for ext considered as a bijection between two discrete spaces, and thus a homeomorphism between the spaces of ultrafilters over them.

For maps these discrete spaces are $S^{X_1 \times \dots \times X_n}$ and $\text{ext}^+ S^{X_1 \times \dots \times X_n}$, so $\tilde{\text{ext}}$ is a homeomorphism between $\beta(S^{X_1 \times \dots \times X_n})$ and $\beta(\text{ext}^+ S^{X_1 \times \dots \times X_n})$, and $\mathfrak{f}^+ = \tilde{\text{ext}}(\mathfrak{f})$:

$$\begin{array}{ccc} \beta(S^{X_1 \times \dots \times X_n}) & \xrightarrow{+} & \beta(\text{ext}^+ S^{X_1 \times \dots \times X_n}) \\ \uparrow & & \uparrow \\ S^{X_1 \times \dots \times X_n} & \xrightarrow{\text{ext}} & \text{ext}^+ S^{X_1 \times \dots \times X_n} \end{array}$$

and analogously, for relations the discrete spaces are $\mathcal{P}(X_1 \times \dots \times X_n)$ and $\text{ext}^+ \mathcal{P}(X_1 \times \dots \times X_n)$, so $\tilde{\text{ext}}$ is a homeomorphism between $\beta \mathcal{P}(X_1 \times \dots \times X_n)$ and $\beta(\text{ext}^+ \mathcal{P}(X_1 \times \dots \times X_n))$, and $\mathfrak{r}^+ = \tilde{\text{ext}}(\mathfrak{r})$:

$$\begin{array}{ccc} \beta \mathcal{P}(X_1 \times \dots \times X_n) & \xrightarrow{+} & \beta(\text{ext}^+ \mathcal{P}(X_1 \times \dots \times X_n)) \\ \uparrow & & \uparrow \\ \mathcal{P}(X_1 \times \dots \times X_n) & \xrightarrow{\text{ext}} & \text{ext}^+ \mathcal{P}(X_1 \times \dots \times X_n) \end{array}$$

(cf. Remark before Lemma 13 explaining a similar situation with currying). Nevertheless, we use the symbol $+$ to avoid confusing with $\tilde{\text{ext}}$ for the map ext into a compact Hausdorff space Y , which will be also used in our arguments below.

Lemma 36. *The map i is a bijection between:*

- (i) *the set of all ultrafilters over $S^{X_1 \times \dots \times X_n}$ and the set of the ultrafilters over $Y^{\beta X_1 \times \dots \times \beta X_n}$ that are concentrated on $\text{ext}^+ S^{X_1 \times \dots \times X_n}$,*

(ii) the set of all ultrafilters over $\mathcal{P}(X_1 \times \dots \times X_n)$ and the set of the ultrafilters over $\mathcal{P}(\beta X_1 \times \dots \times \beta X_n)$ that are concentrated on $\text{ext} \mathcal{P}(X_1 \times \dots \times X_n)$.

Proof. For brevity, we let:

$$A = \beta(S^{X_1 \times \dots \times X_n}), \quad B = \beta(\text{ext} \mathcal{P}(X_1 \times \dots \times X_n)), \quad C = \{\mathfrak{g} \in \beta(Y^{\beta X_1 \times \dots \times \beta X_n}) : \text{ext} \mathcal{P}(X_1 \times \dots \times X_n) \in \mathfrak{g}\}.$$

As we have already pointed out, the map $+$ is a bijection of A onto B , and the lifting map is a bijection of B onto C . Therefore, i , as the composition of the two maps, is a bijection of A onto C , which proves item (i). Item (ii) is either proved similarly or obtained from (i) by replacing relations with their characteristic functions. (We may also note that these maps are homeomorphic embeddings.) \square

Let us now turn back to the discussed situation with generalized models in the former, narrow sense. In this case: all the discrete spaces X_1, \dots, X_n are equal to X , while the compact Hausdorff space Y is βX and its subset S is X or 2 in the cases of operations and relations of the model, respectively. (Recall that we identify elements of X with the principal ultrafilters given by them, so any n -ary operation on X is identified with a map of X^n into βX .) Now, for the generalized model $\mathfrak{A} = (\beta X, \mathfrak{f}, \dots, \mathfrak{r}, \dots)$ in the narrow sense, we let

$$i(\mathfrak{A}) = (\beta X, i(\mathfrak{f}), \dots, i(\mathfrak{r}), \dots),$$

and define \mathfrak{U} , the generalized model in the wide sense corresponding to \mathfrak{A} , as follows:

$$\mathfrak{U} = i(\mathfrak{A}).$$

It remains to verify that the new satisfiability, defined via limits, coincides with the old one. By Theorem 23, which states that for all formulas φ and valuations v , we have $\mathfrak{A} \models \varphi[v]$ iff $e(\mathfrak{A}) \models \varphi[v]$, the latter can be redefined via the map e . Therefore, it suffices to check that for all formulas φ and valuations v , we have

$$\lim \mathfrak{U} \models \varphi[v] \text{ iff } e(\mathfrak{A}) \models \varphi[v].$$

But actually, a stronger fact is true:

$$\lim \mathfrak{U} = e(\mathfrak{A}),$$

thus leading to the following result.

Theorem 37. *If \mathfrak{A} is a generalized model in the narrow sense, then $\lim i(\mathfrak{A}) = e(\mathfrak{A})$:*

$$\begin{array}{ccc} & i(\mathfrak{A}) & \\ & \nearrow i & \downarrow \text{lim} \\ \mathfrak{A} & & e(\mathfrak{A}) \\ & \searrow e & \end{array}$$

Proof. We must verify the equalities $\lim i(\mathfrak{f}) = e(\mathfrak{f})$ and $\lim i(\mathfrak{r}) = e(\mathfrak{r})$ for all $\mathfrak{f} \in \beta(X^{X \times \dots \times X})$ and $\mathfrak{r} \in \beta \mathcal{P}(X \times \dots \times X)$. This will be stated in the next, more general lemma.

Recall once more that the map e on ultrafilters over n -ary maps is $\widetilde{\text{ext}}$, the continuous extension of the map ext , where the latter, in turn, takes n -ary maps $f : X_1 \times \dots \times X_n \rightarrow Y$ of discrete

spaces X_1, \dots, X_n into a compact Hausdorff space Y , to their extensions $\tilde{f} = \text{ext}(f)$ that are right continuous w.r.t. X_1, \dots, X_{n-1} , and that these extensions form a compact Hausdorff space w.r.t. the (X_1, \dots, X_n) -pointwise convergence topology:

$$\begin{array}{ccc} \beta(Y^{X_1 \times \dots \times X_n}) & & \\ \uparrow & \searrow \widetilde{\text{ext}} & \\ Y^{X_1 \times \dots \times X_n} & \xrightarrow{\text{ext}} & RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y) \end{array}$$

The next lemma states that the map e is the composition of the identification map i and taking of limit.

Lemma 38. *Let X_1, \dots, X_n be discrete spaces, Y a compact Hausdorff space, and let the spaces $RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$ and $\{Q \subseteq \beta X_1 \times \dots \times \beta X_n : Q \text{ is right clopen w.r.t. } X_1, \dots, X_{n-1}\}$ be endowed with the (X_1, \dots, X_n) -pointwise convergence topologies. Then we have*

$$\lim i(f) = e(f) \text{ and } \lim i(\tau) = e(\tau)$$

for every ultrafilters $\mathfrak{f} \in \beta(Y^{X_1 \times \dots \times X_n})$ and $\tau \in \beta \mathcal{P}(X_1 \times \dots \times X_n)$.

Proof. For brevity, let RC denote the space $RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$. By Lemma 11, the space RC is compact Hausdorff. Hence, $i(f)$, which is an ultrafilter over the set $Y^{\beta X_1 \times \dots \times \beta X_n}$ concentrated on its subset RC and thus can be identified with its projection to RC , converges to a unique point of RC , i.e. has a limit in RC . We need to show that the limit is exactly the map $e(f)$.

Denote $e(f)$ by F . Then, since $e = \widetilde{\text{ext}}$ and $\text{ext} = \sim$, we get:

$$\{F\} = \bigcap_{A \in \mathfrak{f}} \text{cl}_{RC} \text{ext}^{\sim} A = \bigcap_{A \in \mathfrak{f}} \text{cl}_{RC} \{\tilde{f} : f \in A\}.$$

Therefore, for every $A \in \mathfrak{f}$ and any neighborhood O of the point F in the space, there exists $f \in A$ such that $\tilde{f} \in O$; here we use that the set $\text{ext}^{\sim} Y^{X_1 \times \dots \times X_n} = \{\tilde{f} : f \in Y^{X_1 \times \dots \times X_n}\}$ is dense in RC by Lemma 12.

Let us verify that, moreover, for any neighborhood O of $F \in RC$ the set $\{f \in Y^{X_1 \times \dots \times X_n} : \tilde{f} \in O\}$ is in \mathfrak{f} . Assume the converse: there exists a neighborhood O of F such that the set $\{f \in Y^{X_1 \times \dots \times X_n} : \tilde{f} \in O\}$ is not in \mathfrak{f} . Then, as \mathfrak{f} is an ultrafilter, the complement

$$A = \{f \in Y^{X_1 \times \dots \times X_n} : \tilde{f} \notin O\}$$

is in \mathfrak{f} . However, this contradicts to the above stated fact.

The case of ultrafilters over relations reduces to the case of ultrafilters over maps with $Y = 2$. The lemma is proved. \square

This proves Theorem 37. \square

Now the proof of Theorem 35 is complete. \square

Theorem 35 permits us to eliminate our temporary symbol \models_{lim} and use the former symbol \models also to denote the satisfaction in generalized models in the wide sense. Moreover, by Theorem 34 we might use the only ordinary symbol \models to denote the satisfaction in both ordinary and generalized models; we however prefer to retain the symbol \models for a convenience of reading.

Finally, we refine the first part of Theorem 35 (concerning rather models than the satisfaction relation) by characterizing the generalized models in the wide sense that correspond to those in the narrow sense:

Theorem 39. *Let \mathfrak{U} be a generalized model in the wide sense. Then:*

- (i) $\mathfrak{U} = i(\mathfrak{A})$ for some generalized model \mathfrak{A} in the narrow sense iff the universe of \mathfrak{U} is βX for some X and the interpretation takes all functional symbols to ultrafilters concentrated on $\text{ext}^{\ast} X^{X^n}$, and all relational symbols to ultrafilters concentrated on $\{Q \subseteq (\beta X)^n : Q \text{ is right clopen w.r.t. } X\}$;
- (ii) $\lim \mathfrak{U} = \widetilde{\mathfrak{A}}$ for some ordinary model \mathfrak{A} iff the universe of \mathfrak{U} is βX for some X and the interpretation takes all functional symbols to ultrafilters in $\{i(\mathfrak{f}) : \mathfrak{f} \in \beta(X^{X^n}) \text{ is pseudo-principal}\}$, and all relational symbols to ultrafilters concentrated on $\{Q \subseteq (\beta X)^n : Q \text{ is right clopen w.r.t. } X\}$.

Proof. Item (i) is immediate from Lemma 36; we recall only that the images of ultrafilters over $\mathcal{P}(X^n)$ under i are exactly ultrafilters over $\mathcal{P}((\beta X)^n)$ that are concentrated on $\text{ext}^{\ast} \mathcal{P}(X^n)$:

$$\begin{aligned} i^{\ast} \beta \mathcal{P}(X^n) &= \{\mathfrak{s} \in \beta \mathcal{P}((\beta X)^n) : \text{ext}^{\ast} \mathcal{P}(X^n) \in \mathfrak{s}\} \\ &= \{\mathfrak{s} \in \beta \mathcal{P}((\beta X)^n) : \{\tilde{R} : R \subseteq X^n\} \in \mathfrak{s}\} \\ &= \{\mathfrak{s} \in \beta \mathcal{P}((\beta X)^n) : \{Q \subseteq (\beta X)^n : Q \text{ is right clopen w.r.t. } X\} \in \mathfrak{s}\}. \end{aligned}$$

Item (ii) follows from item (i) and Theorem 28. □

Map I . Here we consider a variant of the map i , which we denote by I . This map relates to the operation E in the same way as the map i to the operation e does (which explains our choosing of the symbol I).

The map I has the same domain and range that the map i does:

$$\begin{aligned} I^{\ast} \beta(S^{X_1 \times \dots \times X_n}) &\subseteq \beta(Y^{\beta X_1 \times \dots \times \beta X_n}), \\ I^{\ast} \beta \mathcal{P}(X_1 \times \dots \times X_n) &\subseteq \beta \mathcal{P}(\beta X_1 \times \dots \times \beta X_n). \end{aligned}$$

and is defined as follows: on ultrafilters over $S^{X_1 \times \dots \times X_n}$ it coincides with i , and on ultrafilters over $\mathcal{P}(X_1 \times \dots \times X_n)$ it is defined likewise i except for taking \times instead of $+$, where \mathfrak{r}^{\times} uses rather cl than ext , i.e. turning $R \subseteq X_1 \times \dots \times X_n$ not to \tilde{R} but to R^* (recall that, by Theorem 8, cl is a bijection between all subsets of $X_1 \times \dots \times X_n$ and regular closed subsets of $\beta X_1 \times \dots \times \beta X_n$).

Thus for each $\mathfrak{r} \in \beta \mathcal{P}(X_1 \times \dots \times X_n)$, we might turn it firstly to $\mathfrak{r}^{\times} \in \beta(\text{cl}^{\ast} \mathcal{P}(X_1 \times \dots \times X_n))$ where $\text{cl}(R) = R^*$, so by Theorem 8,

$$\begin{aligned} \mathfrak{r}^{\times} &\in \beta \{R^* : R \subseteq X_1 \times \dots \times X_n\} \\ &= \beta \{Q \subseteq \beta X_1 \times \dots \times \beta X_n : Q \text{ is regular closed}\}, \end{aligned}$$

by letting

$$\mathfrak{r}^{\times} = \{\text{cl}^{\ast} A : A \in \mathfrak{r}\},$$

and secondly, by lifting the obtaining ultrafilter to an ultrafilter over $\mathcal{P}(\beta X_1 \times \dots \times \beta X_n)$, thus letting

$$I(\mathfrak{r}) = (\mathfrak{r}^{\times})^{\mathcal{P}(\beta X_1 \times \dots \times \beta X_n)}.$$

In result, for any ultrafilter \mathfrak{r} over $\mathcal{P}(X_1 \times \dots \times X_n)$, its image $I(\mathfrak{r})$ is an ultrafilter over $\mathcal{P}(\beta X_1 \times \dots \times \beta X_n)$ which is concentrated on $\{Q \subseteq \beta X_1 \times \dots \times \beta X_n : Q \text{ is regular closed}\}$. (To make an analogy between $+$ and \times more complete, we can also let that \times is defined on ultrafilters over $S^{X_1 \times \dots \times X_n}$ and coincides there with $+$; but in fact we do not need this.)

Remark. Again, the map \times is $\widetilde{\text{cl}}$ for cl considered as a bijection between two discrete spaces $\mathcal{P}(X_1 \times \dots \times X_n)$ and $\text{cl}^{\times} \mathcal{P}(X_1 \times \dots \times X_n)$, so $\widetilde{\text{cl}}$ is a homeomorphism between $\beta \mathcal{P}(X_1 \times \dots \times X_n)$ and $\beta(\text{cl}^{\times} \mathcal{P}(X_1 \times \dots \times X_n))$, and $\mathfrak{r}^{\times} = \widetilde{\text{cl}}(\mathfrak{r})$:

$$\begin{array}{ccc} \beta \mathcal{P}(X_1 \times \dots \times X_n) & \xrightarrow{\times} & \beta(\text{cl}^{\times} \mathcal{P}(X_1 \times \dots \times X_n)) \\ \uparrow & & \uparrow \\ \mathcal{P}(X_1 \times \dots \times X_n) & \xrightarrow{\text{cl}} & \text{cl}^{\times} \mathcal{P}(X_1 \times \dots \times X_n) \end{array}$$

Nevertheless, we use the symbol \times to keep the analogy with $+$.

Two next lemmas and the subsequent theorem are counterparts of Lemmas 36 and 38 and Theorem 37, respectively.

Lemma 40. *The map I is a bijection between:*

- (i) *the set of all ultrafilters over $S^{X_1 \times \dots \times X_n}$ and the set of the ultrafilters over $Y^{\beta X_1 \times \dots \times \beta X_n}$ that are concentrated on $\text{ext}^{\times} S^{X_1 \times \dots \times X_n}$,*
- (ii) *the set of all ultrafilters over $\mathcal{P}(X_1 \times \dots \times X_n)$ and the set of the ultrafilters over $\mathcal{P}(\beta X_1 \times \dots \times \beta X_n)$ that are concentrated on $\text{cl}_{\beta X_1 \times \dots \times \beta X_n}^{\times} \mathcal{P}(X_1 \times \dots \times X_n)$.*

Proof. Item (i) just repeats Lemma 36(i) since I coincides with i on ultrafilters over maps. For item (ii), let

$$\begin{aligned} A &= \beta \mathcal{P}(X_1 \times \dots \times X_n), \quad B = \beta(\text{cl}_{\beta X_1 \times \dots \times \beta X_n}^{\times} \mathcal{P}(X_1 \times \dots \times X_n)), \\ C &= \{ \mathfrak{s} \in \beta \mathcal{P}(\beta X_1 \times \dots \times \beta X_n) : \text{cl}_{\beta X_1 \times \dots \times \beta X_n}^{\times} \mathcal{P}(X_1 \times \dots \times X_n) \in \mathfrak{s} \}. \end{aligned}$$

(Here the closure $\text{cl}_{\beta X_1 \times \dots \times \beta X_n}^{\times}$ refers to the product topology where the spaces βX_i are endowed with their standard topologies, and the set $\text{cl}_{\beta X_1 \times \dots \times \beta X_n}^{\times} \mathcal{P}(X_1 \times \dots \times X_n)$ in the definition of B is considered as a discrete space.) As the map \times is a bijection of A onto B and the lifting map is a bijection of B onto C , the map I , which is the composition of the two maps, is a bijection of A onto C , thus proving (ii). \square

In what follows we consider the space $\beta X_1 \times \dots \times \beta X_n$ endowed with the usual product topology of the spaces βX_i and the set of regular closed sets in this space endowed with a compact Hausdorff topology. This topology is induced from the compact Hausdorff space $\mathcal{P}(X_1 \times \dots \times X_n)$, which we identify with the space $2^{X_1 \times \dots \times X_n}$ (where 2 is discrete and the space carries the usual product topology) by the natural bijection taking $R \subseteq X_1 \times \dots \times X_n$ to $R^* \subseteq \beta X_1 \times \dots \times \beta X_n$.

Recall also that by Lemma 11 (and its proof), the (X_1, \dots, X_n) -pointwise convergence topology on $RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$ can be induced from the product topology on $Y^{X_1 \times \dots \times X_n}$ by the bijection ext , which takes each $f \in Y^{X_1 \times \dots \times X_n}$ to $\tilde{f} \in Y^{\beta X_1 \times \dots \times \beta X_n}$. In particular, if $Y = 2$ then ext on relations (identified with their characteristic functions), which takes each $R \subseteq X_1 \times \dots \times X_n$ to $\tilde{R} \subseteq \beta X_1 \times \dots \times \beta X_n$, induces the above considered compact Hausdorff topology on $\mathcal{P}(\beta X_1 \times \dots \times \beta X_n)$. Therefore, we have three homeomorphic spaces: the space of subsets R of $X_1 \times \dots \times X_n$ and its images under the homeomorphisms ext and cl taking R, \tilde{R}, R^* into each others:

$$\begin{array}{ccc} \text{RClop} & \longleftrightarrow & \text{RegCl} \\ & \swarrow \text{ext} \quad \searrow \text{cl} & \\ & \mathcal{P}(X_1 \times \dots \times X_n) & \end{array}$$

where

$$\begin{aligned}
\text{RClop} &= \text{ext} \text{“} \mathcal{P}(X_1 \times \dots \times X_n) \\
&= \{ \tilde{R} : R \in \mathcal{P}(X_1 \times \dots \times X_n) \} \\
&= \{ Q \subseteq \beta X_1 \times \dots \times \beta X_n : Q \text{ is right clopen w.r.t. } X_1, \dots, X_{n-1} \}, \\
\text{RegCl} &= \text{cl} \text{“} \mathcal{P}(X_1 \times \dots \times X_n) \\
&= \{ R^* : R \in \mathcal{P}(X_1 \times \dots \times X_n) \} \\
&= \{ Q \subseteq \beta X_1 \times \dots \times \beta X_n : Q \text{ is regular closed} \}.
\end{aligned}$$

Question. Redefine the topology on the set RegCl as a restricted version of Vietoris topology (in an analogy with the restricted version of pointwise convergence topology turning out RC into a compact Hausdorff space homeomorphic to the product space $Y^{X_1 \times \dots \times X_n}$). Note that we cannot use the usual (unrestricted) Vietoris topology since in it, RegCl is not a closed subset of the space $\{Q \subseteq \beta X_1 \times \dots \times \beta X_n : Q \text{ is closed}\}$. (Problem 7 in Section 5.)

Lemma 41. *Let f be the homeomorphism between $\text{RClop} = \{\tilde{R} : R \in \mathcal{P}(X_1 \times \dots \times X_n)\}$ and $\text{RegCl} = \{R^* : R \in \mathcal{P}(X_1 \times \dots \times X_n)\}$ taking \tilde{R} to R^* . Then we have*

$$\begin{array}{ccc}
\beta \text{RClop} & \overset{\tilde{f}}{\dashrightarrow} & \beta \text{RegCl} \\
\uparrow i & & \uparrow I \\
\text{RClop} & \xrightarrow{f} & \text{RegCl} \\
\downarrow e & & \downarrow E \\
\beta \mathcal{P}(X_1 \times \dots \times X_n) & &
\end{array}$$

Proof. The equality $f \circ e = E$ follows from Theorem 24 and, in turn, implies the equality $\tilde{f} \circ i = I$. \square

Lemma 42. *Let X_1, \dots, X_n be discrete spaces, Y a compact Hausdorff space, and let the spaces $\text{RC} = \text{RC}_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$ and $\text{RegCl} = \{Q \subseteq \beta X_1 \times \dots \times \beta X_n : Q \text{ is regular closed}\}$ be endowed with the topology induced from $Y^{X_1 \times \dots \times X_n}$ and $\mathcal{P}(X_1 \times \dots \times X_n)$, respectively. Then we have*

$$\lim I(\mathfrak{f}) = E(\mathfrak{f}) \quad \text{and} \quad \lim I(\mathfrak{r}) = E(\mathfrak{r})$$

for every ultrafilters $\mathfrak{f} \in \beta(Y^{X_1 \times \dots \times X_n})$ and $\mathfrak{r} \in \beta \mathcal{P}(X_1 \times \dots \times X_n)$.

Proof. The first equality repeats the first equality in Lemma 38 as the topology on RC induced from $Y^{X_1 \times \dots \times X_n}$ coincides with the (X_1, \dots, X_n) -pointwise convergence topology by Theorem 11.

For the second equality, recall first a general fact: if $\mathfrak{u}, \mathfrak{v}$ are ultrafilters over sets A, B , respectively, both sets carries some topologies, and a map $g : A \rightarrow B$ is continuous, then $g(\lim \mathfrak{u}) = \lim \tilde{g}(\mathfrak{v})$ whenever both limits exist. It easily follows that if g is a homeomorphism, then $\lim \mathfrak{u} = g^{-1}(\lim \tilde{g}(\mathfrak{v}))$ and $\lim \mathfrak{v} = g(\lim \tilde{g}^{-1}(\mathfrak{u}))$. In our situation, we have:

$$\lim I(\mathfrak{r}) = f(\lim \tilde{f}^{-1}(I(\mathfrak{r}))) = f(\lim i(\mathfrak{r})) = f(e)(\mathfrak{r}) = E(\mathfrak{r})$$

where the first equality holds by this general fact, the second follows from Lemma 41, the third holds by Lemma 38, and the last again by Lemma 41. \square

For all generalized models $\mathfrak{A} = (\beta X, \mathfrak{f}, \dots, \mathfrak{r}, \dots)$ in the narrow sense, we let

$$I(\mathfrak{A}) = (\beta X, I(\mathfrak{f}), \dots, I(\mathfrak{r}), \dots).$$

Theorem 43. *If \mathfrak{A} is a generalized model in the narrow sense, then $\lim I(\mathfrak{A}) = E(\mathfrak{A})$:*

$$\begin{array}{ccc} & & I(\mathfrak{A}) \\ & \nearrow I & \downarrow \lim \\ \mathfrak{A} & & \\ & \searrow E & \\ & & E(\mathfrak{A}) \end{array}$$

Proof. Immediate from Lemma 42. □

We summarize the interplay between generalized models \mathfrak{A} in the narrow sense, their limits, and the operations i, I, e, E in the following diagram:

$$\begin{array}{ccc} i(\mathfrak{A}) & \overset{\tilde{f}}{\dashrightarrow} & I(\mathfrak{A}) \\ \lim \downarrow & \nearrow i & \downarrow \lim \\ e(\mathfrak{A}) & \xrightarrow{f} & E(\mathfrak{A}) \\ & \nwarrow e & \nearrow E \\ & & \mathfrak{A} \end{array}$$

Despite the fact that Theorem 43 is an E -analog of Theorem 37 used to get Theorem 35, the latter theorem has no such analog. This is due to an asymmetry between the operations e and E w.r.t. the satisfiability in generalized models in the narrow sense, as it has been defined: there is no E -analog of Theorem 23, which was also used in proving Theorem 35. Nonetheless, we are still able to get a counterpart of Theorem 39, which does not involve satisfiability:

Theorem 44. *Let \mathfrak{U} be a generalized model in the wide sense. Then:*

- (i) $\mathfrak{U} = I(\mathfrak{A})$ for some generalized model \mathfrak{A} in the narrow sense iff the universe of \mathfrak{U} is βX for some X and the interpretation takes all functional symbols to ultrafilters concentrated on $\text{ext}^{\llcorner} X^{X^n}$, and all relational symbols to ultrafilters concentrated on $\{Q \subseteq (\beta X)^n : Q \text{ is regular closed}\}$;
- (ii) $\lim \mathfrak{U} = \mathfrak{A}^*$ for some ordinary model \mathfrak{A} iff the universe of \mathfrak{U} is βX for some X and the interpretation takes all functional symbols to ultrafilters in $\{I(\mathfrak{f}) : \mathfrak{f} \in \beta(X^{X^n}) \text{ is pseudo-principal}\}$, and all relational symbols to ultrafilters concentrated on $\{Q \subseteq (\beta X)^n : Q \text{ is regular closed}\}$.

Proof. Item (i) is immediate from Lemma 40; recall that the images of ultrafilters over $\mathcal{P}(X^n)$ under I are exactly ultrafilters over $\mathcal{P}((\beta X)^n)$ that are concentrated on $\text{cl}_{\beta X_1 \times \dots \times \beta X_n} \llcorner \mathcal{P}(X^n)$:

$$\begin{aligned} i^{\llcorner} \beta \mathcal{P}(X^n) &= \{\mathfrak{s} \in \beta \mathcal{P}((\beta X)^n) : \text{cl}_{\beta X_1 \times \dots \times \beta X_n} \llcorner \mathcal{P}(X^n) \in \mathfrak{s}\} \\ &= \{\mathfrak{s} \in \beta \mathcal{P}((\beta X)^n) : \{R^* : R \subseteq X^n\} \in \mathfrak{s}\} \\ &= \{\mathfrak{s} \in \beta \mathcal{P}((\beta X)^n) : \{Q \subseteq (\beta X)^n : Q \text{ is regular closed}\} \in \mathfrak{s}\}. \end{aligned}$$

Item (ii) follows from item (i) and Theorem 28. □

Second Extension Theorems. Now we define homomorphisms of generalized models in the wide sense as homomorphisms of their limits: if \mathfrak{A} and \mathfrak{B} are two such models in the same signature with the universes X and Y , respectively, we say that a map $h : X \rightarrow Y$ is a *homomorphism (of generalized models in the wide sense)* iff it is a homomorphism of $\lim \mathfrak{A}$ into $\lim \mathfrak{B}$. Theorem 35 guaranties that for generalized models in the narrow sense, the new definition of homomorphism gives the same, up to the identification map i , that the former definition of homomorphisms from Section 3. Similar concepts (epimorphisms, isomorphic embeddings, elementary embeddings, submodels, elementary submodels, quotients, etc. of generalized models in the wide sense) are defined likewise and also coincide with the corresponding concepts for generalized models in the narrow sense.

The proofs of the Second Extension Theorems (Theorems 5 and 7) are based on Theorems 4 and 6, which describe the topological properties of the \sim - and $*$ -extensions, respectively, and a result called the “abstract extension theorem” in [4]. This result is rather about restrictions of continuous maps than about continuous extensions of maps, but it also states that such a map is a homomorphism of the whole models whenever it is a homomorphism of certain their submodels; we restate it in the next theorem:

Theorem 45. *Let \mathfrak{A} and \mathfrak{B} be two models in the same signature whose universes X and Y , respectively, both carry topologies, the topology on Y is Hausdorff, and let $D \subseteq X$ be a dense subset of X which forms a submodel \mathfrak{D} of \mathfrak{A} . Let, moreover, $h : X \rightarrow Y$ be a continuous map, and suppose that*

- (a) *all operations in \mathfrak{A} are right continuous w.r.t. D , and in \mathfrak{B} right continuous w.r.t. $h^{-1}D$,*
- (b) *one of two following items holds:*
 - (α) *all relations in \mathfrak{A} are right open w.r.t. D , and in \mathfrak{B} right closed w.r.t. $h^{-1}D$,*
 - (β) *all relations in \mathfrak{A} are regular closed, and in \mathfrak{B} closed, in the product topologies on X^n and Y^n (where n is the arity of a given relation), respectively, and h is a closed map.*

Then the following are equivalent:

- (i) *$h \upharpoonright D$ is a homomorphism of \mathfrak{D} into \mathfrak{B} ,*
- (ii) *h is a homomorphism of \mathfrak{A}*

$$\begin{array}{ccc}
 \mathfrak{A} & \xrightarrow{\quad h \quad} & \mathfrak{B} \\
 \uparrow & & \uparrow \\
 \mathfrak{D} & \xrightarrow{\quad h \upharpoonright D \quad} & \mathfrak{E}
 \end{array}$$

where \mathfrak{E} denotes the submodel of \mathfrak{B} with the universe $h^{-1}D$.

Proof. That (ii) implies (i) is trivial since \mathfrak{D} is a submodel of \mathfrak{A} . For the converse implication in the case (α), see [3] or [4], Theorem 4.1. The case (β) is obtained from the case (α) as follows.

If R is an n -ary relation on X belonging to the model \mathfrak{A} , consider R as a unary relation on X^n and note that under (i), the restriction $h \upharpoonright D$ is also a homomorphism between the model $(D^n, (\text{int}_{X^n} R) \cap D^n)$ and the model (Y^n, S) , where $\text{int}_{X^n} R$ is the interior of R in the product topology on X^n , so the set $(\text{int}_{X^n} R) \cap D^n = \text{int}_{D^n}(R \cap D^n)$ is an open unary relation on D^n , and S is the relation on Y interpreting the same predicate symbol that R doing and also considered as a unary relation on Y^n . By (α) we conclude that h is a homomorphism between $(X^n, \text{int}_{X^n} R)$ and (Y^n, S) . But as in the case (β) the map h is closed, we have: $h^{-1} \text{cl}_{Y^n} S \subseteq \text{cl}_{X^n} \text{int}_{X^n} R$, thus

h is a homomorphism between $(X^n, \text{cl}_{X^n} \text{int}_{X^n} R)$ and $(Y^n, \text{cl}_{Y^n} S)$. Finally, as under (β) , R is regular closed in X^n and S is closed in Y^n , we have $\text{cl}_{X^n} \text{int}_{X^n} R = R$ and $\text{cl}_{Y^n} S = S$, thus showing that h is a homomorphism between (X^n, R) and (Y^n, S) , and hence, between (X, R) and (Y, S) where the relations R and S are considered as n -ary. This gives (ii), completing the proof of the theorem. \square

Remark. The argument for proving (β) from (α) allows to obtain stronger statements. Instead of the assumption of (β) , it suffices to suppose that the interior of R is dense in the closure of R , i.e. the closure of R is regular closed: $\text{cl}_{X^n} \text{int}_{X^n} R = \text{cl}_{X^n} R$. This includes the cases of open as well as of regular closed R . Also instead of the assumption (α) , it suffices to suppose that R “has right dense interior w.r.t. D ”, i.e. that for each $i, 1 \leq i \leq n$, and every $a_1, \dots, a_{i-1} \in D$ and $x_{i+1}, \dots, x_n \in X$, the set $\{x \in X : (a_1, \dots, a_{i-1}, x, x_{i+1}, \dots, x_n) \in R\}$ has the interior dense in the closure of this set. As easy to see, in both cases the same proof works as well.

The same is applied to the following theorem.

The next result is an immediate analog of Theorem 45 for generalized models.

Theorem 46. *Let \mathfrak{U} and \mathfrak{V} be two generalized models in the same signature whose universes X and Y , respectively, both carry topologies, the topology on Y is Hausdorff, and let $D \subseteq X$ be a dense subset of X which forms a generalized submodel \mathfrak{D} of \mathfrak{U} . Let, moreover, $h : X \rightarrow Y$ be a continuous map, and suppose that for any $n < \omega$,*

- (a) *n -ary functional symbols are interpreted: in \mathfrak{U} by ultrafilters having limits in $RC_D(X^n, X)$, and in \mathfrak{V} by ultrafilters having limits in $RC_{h^{-1}D}(Y^n, Y)$,*
- (b) *one of two following items holds:*
 - (α) *n -ary predicate symbols are interpreted: in \mathfrak{U} by ultrafilters having limits in $\{R \subseteq X^n : R \text{ is right open w.r.t. } D\}$, and in \mathfrak{V} by ultrafilters having limits in $\{S \subseteq Y^n : S \text{ is right closed w.r.t. } h^{-1}D\}$,*
 - (β) *n -ary predicate symbols are interpreted: in \mathfrak{U} by ultrafilters having limits in $\{R \subseteq X^n : R \text{ is regular closed}\}$, and in \mathfrak{V} by ultrafilters having limits in $\{S \subseteq Y^n : S \text{ is closed}\}$, in the product topologies on X^n and Y^n , respectively, and h is a closed map.*

Then the following are equivalent:

- (i) $h \upharpoonright D$ is a homomorphism of \mathfrak{D} into \mathfrak{V} ,
- (ii) h is a homomorphism of \mathfrak{U} into \mathfrak{V} :

$$\begin{array}{ccc}
 \mathfrak{U} & \overset{h}{\dashrightarrow} & \mathfrak{V} \\
 \uparrow \text{lim} & & \uparrow \text{lim} \\
 \mathfrak{D} & \xrightarrow{h \upharpoonright D} & \mathfrak{E} \\
 \downarrow \text{lim} & & \downarrow \text{lim} \\
 & \text{lim } \mathfrak{U} \dashrightarrow \text{lim } \mathfrak{V} & \xrightarrow{h} \\
 & \uparrow & \uparrow \\
 & \text{lim } \mathfrak{D} & \xrightarrow{h \upharpoonright D} \text{lim } \mathfrak{E}
 \end{array}$$

where \mathfrak{E} denotes the generalized submodel of \mathfrak{V} with the universe $h^{-1}D$.

Proof. By definition, homomorphisms of generalized models \mathfrak{D} , \mathfrak{U} , and \mathfrak{V} are precisely homomorphisms of the ordinary models $\lim \mathfrak{D}$, $\lim \mathfrak{U}$, and $\lim \mathfrak{V}$. Now apply Theorem 45 to the latter three models. \square

Note that Theorem 46 includes Theorem 45 as a partial case by identifying operations and relations with principal ultrafilters given by them as in Theorem 34.

Before formulating an extension theorem for generalized models in the wide sense, let us state one more auxiliary result:

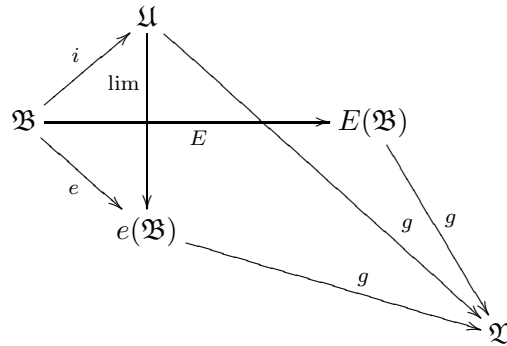
Lemma 47. *Let \mathfrak{U} and \mathfrak{V} be two generalized models in the same signature whose universes βX and Y , respectively, both carry topologies where the topology on βX is standard, let \mathfrak{U} coincide, up to the identification map i , with a generalized model \mathfrak{B} in the narrow sense (also having the universe βX), and let $g : \beta X \rightarrow Y$. Then the following are equivalent:*

- (i) g is a homomorphism of \mathfrak{U} into \mathfrak{V} ,
- (ii) g is a homomorphism of $e(\mathfrak{B})$ into \mathfrak{V} .

If moreover, the interpretation in \mathfrak{V} takes n -ary predicate symbols to ultrafilters having limits in $\{S \subseteq Y^n : S \text{ is closed}\}$ where Y^n carries the product topology, and the map g is closed, then the following item:

- (iii) g is a homomorphism of $E(\mathfrak{B})$ into \mathfrak{V} ,

also is equivalent to each of items (i) and (ii):



Proof. Before proving the equivalences, recall that by Theorem 34, the ordinary models $e(\mathfrak{B})$ and $E(\mathfrak{B})$ are identified with generalized models in the wide sense having the principal interpretations, and the limits of the principal ultrafilters over the sets of operations and relations are the operations and relations that generate them. The formulations of (i) and (ii) imply such an identification.

The equivalence of items (i) and (ii) requires no special assumptions about \mathfrak{B} , \mathfrak{V} , and g ; it is immediate from the following: our definition of homomorphisms of generalized models via their limits, the assumption $\mathfrak{U} = i(\mathfrak{B})$, and the equality $e(\mathfrak{B}) = \lim i(\mathfrak{B})$ stated in Theorem 37.

To prove that item (iii) under the additional assumption is also equivalent to (i) and (ii), we repeat the part of the proof of Theorem 45 that deduces the case (β) from the case (α) , taking into account Theorem 29 stating that all relations in $E(\mathfrak{B})$ are regular closed in the product topology on $(\beta X)^n$.

The proof is complete. \square

Now we are ready to formulate a version of the Second Extension Theorem for generalized models.

Theorem 48. Let $\mathfrak{U} = (\beta X, \mathfrak{f}, \dots, \mathfrak{r}, \dots)$ and $\mathfrak{V} = (Y, \mathfrak{g}, \dots, \mathfrak{s}, \dots)$ be two generalized models in the same signature, let βX carry its standard topology and Y a compact Hausdorff topology, let $h : X \rightarrow Y$, and suppose that

- (a) \mathfrak{U} coincides, up to the identification map i , with a generalized model \mathfrak{B} in the narrow sense, and the interpretation in \mathfrak{B} is pseudo-principal on functional symbols with \mathfrak{A} the principal submodel (having the universe X),
- (b) the interpretation in \mathfrak{V} takes all n -ary functional symbols to ultrafilters having limits in $RC_{h^{-1}X}(Y^n, Y)$, and all n -ary predicate symbols to ultrafilters having limits in $\{S \subseteq Y^n : S \text{ is right closed w.r.t. } h^{-1}X\}$, for any $n < \omega$.

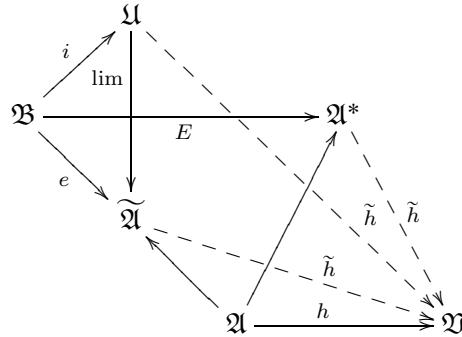
Then the following are equivalent:

- (i) h is a homomorphism of \mathfrak{A} into \mathfrak{V} ,
- (ii) \tilde{h} is a homomorphism of \mathfrak{U} into \mathfrak{V} ,
- (iii) \tilde{h} is a homomorphism of $\tilde{\mathfrak{A}}$ into \mathfrak{V} .

If moreover, the interpretation in \mathfrak{V} takes n -ary predicate symbols to ultrafilters having limits in $\{S \subseteq Y^n : S \text{ is closed}\}$ where Y^n carries the product topology, then the following item:

- (iv) \tilde{h} is a homomorphism of \mathfrak{A}^* into \mathfrak{V} ,

also is equivalent to each of items (i)–(iii):



Proof. The assumptions about \mathfrak{U} , \mathfrak{V} , and \tilde{h} of this theorem repeat the assumptions about \mathfrak{U} , \mathfrak{V} , and g of Lemma 47 with an extra requirement stating that the interpretation in \mathfrak{B} is pseudo-principal on functional symbols. So as the principal submodel of \mathfrak{B} is \mathfrak{A} , we have $e(\mathfrak{B}) = \tilde{\mathfrak{A}}$ and $E(\mathfrak{B}) = \mathfrak{A}^*$ by Theorem 28. Thus by Lemma 47, items (ii) and (iii) are equivalent, and under the additional assumption about \mathfrak{V} , item (iv) is also equivalent to each of them.

Let us now prove that (i) and (ii) are equivalent. It suffices to show that the models \mathfrak{U} and \mathfrak{V} satisfy the conditions of Theorem 46(α). For \mathfrak{V} , this is true by the assumption (b). As for \mathfrak{U} , by the assumption (a) we have $\mathfrak{U} = i(\mathfrak{B})$ with \mathfrak{B} a generalized model in the narrow sense. Furthermore, $i(\mathfrak{B})$ is a generalized model in the wide sense, and by Lemma 36, the interpretation of $i(\mathfrak{B})$ takes functional symbols to ultrafilters concentrated on $RC_X((\beta X)^n, \beta X)$, and relational symbols to ultrafilters concentrated on $\{Q \subseteq (\beta X)^n : Q \text{ is right clopen w.r.t. } X\}$. Therefore, the ultrafilters have limits in these sets endowed with the X -pointwise convergence topologies as the latter are compact Hausdorff by Lemma 11. Thus \mathfrak{U} also satisfies the conditions of Theorem 46(α), with the principal submodel \mathfrak{A} here as the submodel \mathfrak{D} from that theorem (again by identifying ordinary models with generalized models having the principal interpretations). This shows the equivalence of (i) and (ii), thus completing the proof. \square

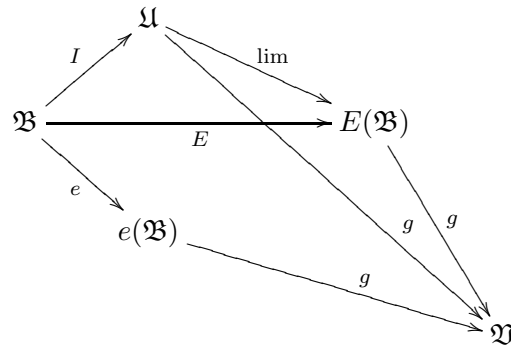
Finally, by changing i with I , we obtain the counterparts of Lemma 47 and Theorem 48:

Lemma 49. *Let \mathfrak{U} and \mathfrak{V} be two generalized models in the same signature whose universes βX and Y , respectively, both carry topologies where the topology on βX is standard, let $g : \beta X \rightarrow Y$, and suppose that*

- (a) $\mathfrak{U} = I(\mathfrak{B})$ for some generalized model \mathfrak{B} in the narrow sense (also having the universe βX),
- (b) the interpretation in \mathfrak{V} takes all n -ary functional symbols to ultrafilters having limits in $RC_{g^{\leftarrow X}}(Y^n, Y)$, and all n -ary predicate symbols to ultrafilters having limits in $\{S \subseteq S^n : R \text{ is closed}\}$ where Y^n carries the product topology.

Then the following are equivalent:

- (i) g is a homomorphism of \mathfrak{U} into \mathfrak{V} ,
- (ii) g is a homomorphism of $e(\mathfrak{B})$ into \mathfrak{V} ,
- (iii) g is a homomorphism of $E(\mathfrak{B})$ into \mathfrak{V} :



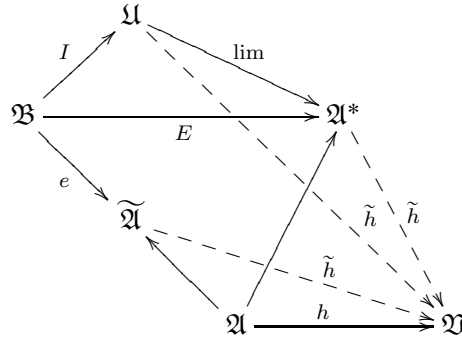
Proof. Again, by using Theorem 34, we identify the ordinary models $e(\mathfrak{B})$ and $E(\mathfrak{B})$ in (ii) and (iii) with the corresponding generalized models in the wide sense having the principal interpretations (and thus having $e(\mathfrak{B})$ and $E(\mathfrak{B})$ as their limits). The equivalence of items (i) and (iii) is immediate from the following: our definition of homomorphisms of generalized models via their limits, the assumption $\mathfrak{U} = I(\mathfrak{B})$, and the equality $e(\mathfrak{B}) = \lim I(\mathfrak{B})$ stated in Theorem 43. But then item (ii) is also equivalent to each of items (i) and (iii) since the identity map on βX is a homomorphism of $e(\mathfrak{B})$ onto $E(\mathfrak{B})$ by Theorem 25. The proof is complete. \square

Theorem 50. *Let $\mathfrak{U} = (\beta X, \mathfrak{f}, \dots, \mathfrak{r}, \dots)$ and $\mathfrak{V} = (Y, \mathfrak{g}, \dots, \mathfrak{s}, \dots)$ be two generalized models in the same signature, let βX carry its standard topology and Y a compact Hausdorff topology, let $h : X \rightarrow Y$, and suppose that*

- (a) $\mathfrak{U} = I(\mathfrak{B})$ for some generalized model \mathfrak{B} in the narrow sense, and the interpretation in \mathfrak{B} is pseudo-principal on functional symbols with \mathfrak{A} the principal submodel (having the universe X),
- (b) the interpretation in \mathfrak{V} takes all n -ary functional symbols to ultrafilters having limits in $RC_{h^{\leftarrow X}}(Y^n, Y)$, and all n -ary predicate symbols to ultrafilters having limits in $\{S \subseteq Y^n : S \text{ is closed}\}$ where Y^n carries the product topology.

Then the following are equivalent:

- (i) h is a homomorphism of \mathfrak{A} into \mathfrak{B} ,
- (ii) \tilde{h} is a homomorphism of \mathfrak{A} into \mathfrak{B} ,
- (iii) \tilde{h} is a homomorphism of $\tilde{\mathfrak{A}}$ into \mathfrak{B} ,
- (iv) \tilde{h} is a homomorphism of \mathfrak{A}^* into \mathfrak{B} :



Proof. The assumptions about \mathfrak{U} , \mathfrak{B} , and \tilde{h} of this theorem repeat the assumptions about \mathfrak{U} , \mathfrak{B} , and g of Lemma 49 with an extra requirement stating that the interpretation in \mathfrak{B} is pseudo-principal on functional symbols. So as the principal submodel of \mathfrak{B} is \mathfrak{A} , we have $e(\mathfrak{B}) = \tilde{\mathfrak{A}}$ and $E(\mathfrak{B}) = \mathfrak{A}^*$ by Theorem 28. Thus by Lemma 49, items (ii), (iii), and (iv) all are equivalent.

Let us now prove that (i) and (ii) are equivalent. It suffices to show that the models \mathfrak{U} and \mathfrak{B} satisfy the conditions of Theorem 46(β). For \mathfrak{B} , this is true by the assumption (b). As for \mathfrak{U} , by the assumption (a) we have $\mathfrak{U} = I(\mathfrak{B})$ with \mathfrak{B} a generalized model in the narrow sense. Furthermore, $I(\mathfrak{B})$ is a generalized model in the wide sense, and by Lemma 40, the interpretation of $I(\mathfrak{B})$ takes functional symbols to ultrafilters concentrated on $RC_X((\beta X)^n, \beta X)$, and relational symbols to ultrafilters concentrated on $\{Q \subseteq (\beta X)^n : Q \text{ is regular closed}\}$. Therefore, the ultrafilters have limits in these sets endowed with the corresponding compact Hausdorff topologies described above. Thus \mathfrak{U} also satisfies the conditions of Theorem 46(β), with the principal submodel \mathfrak{A} here as the submodel \mathfrak{D} from that theorem (again by identifying ordinary models with generalized models having the principal interpretations). This shows the equivalence of (i) and (ii), thus completing the proof. \square

Remark. Theorems 45–50 admits some variants and generalizations. E.g. they remain true for epimorphisms (since for any compact Hausdorff Y , if $h : X \rightarrow Y$ is such that $h''X$ is dense in Y , then $\tilde{h} : \beta X \rightarrow Y$ is surjective), as well as for homotopies and isotopies (in sense of [3], [4]), which can be defined for generalized models in the wide sense in the same way as this was done for homomorphisms and embeddings. Also versions for multi-sorted models (having rather many universes X_1, \dots, X_n than one universe X) can be easily stated.

5. Problems

This section contains the list of questions and tasks, including all ones previously posed here. A part of them is of rather technical character (1, 3, 4, and 7) while others are more program.

1. Does Lemma 12 remain true for the space $RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, Y)$ (or moreover, the space $Y^{\beta X_1 \times \dots \times \beta X_n}$) endowed with the full pointwise convergence topology? i.e. given discrete spaces X_1, \dots, X_n , a compact Hausdorff space Y , and a dense subset of Y , is the set

$$\text{ext } S^{X_1 \times \dots \times X_n} = \{\tilde{f} : f \in S^{X_1 \times \dots \times X_n}\}$$

dense in this space? It can be seen that the answer is affirmative for unary maps, i.e. the set $\{\tilde{f} : f \in S^X\}$ is dense in $C(\beta X, Y)$. What happens for binary maps?

2. Given discrete X_1, \dots, X_n and compact Hausdorff Y , let ext be a map of $Y^{X_1 \times \dots \times X_n}$ endowed with the discrete topology into $Y^{\beta X_1 \times \dots \times \beta X_n}$ endowed with the usual product topology (or equivalently, the usual pointwise convergence topology). As the range is a compact Hausdorff space, the map ext continuously extends to $\widetilde{\text{ext}}$:

$$\begin{array}{ccc} \beta(Y^{X_1 \times \dots \times X_n}) & & \\ \uparrow & \dashrightarrow^{\widetilde{\text{ext}}} & \\ Y^{X_1 \times \dots \times X_n} & \xrightarrow{\text{ext}} & Y^{\beta X_1 \times \dots \times \beta X_n} \end{array}$$

Can this alternative version of self-applying of the map ext lead to some interesting possibilities, including variants of the theory of generalized models?

Note that now $\widetilde{\text{ext}} \text{“} \beta(Y^{X_1 \times \dots \times X_n})$ does not coincide with $\text{ext} \text{“} Y^{X_1 \times \dots \times X_n}$ (unlike our previous situation); however, the latter set is still dense in the former:

$$\text{ext} \text{“} Y^{X_1 \times \dots \times X_n} \subset \widetilde{\text{ext}} \text{“} \beta(Y^{X_1 \times \dots \times X_n}) = \text{cl}_{Y^{\beta X_1 \times \dots \times \beta X_n}} (\text{ext} \text{“} Y^{X_1 \times \dots \times X_n}).$$

Also, is this version of $\widetilde{\text{ext}}$ surjective? This would be the case if the previous question in its stronger form, i.e. for the space $Y^{\beta X_1 \times \dots \times \beta X_n}$, had the affirmative answer.

3. For which compact Hausdorff spaces Y , instead of βY with a discrete Y , does Lemma 20 remain true, i.e. for any discrete X_1, \dots, X_n and the map $\widetilde{\text{app}}$ defined as in the remark in the beginning of Section 3:

$$\begin{array}{ccc} \beta X_1 \times \dots \times \beta X_n \times \beta(Y^{X_1 \times \dots \times X_n}) & & \\ \uparrow & \dashrightarrow^{\widetilde{\text{app}}} & \\ X_1 \times \dots \times X_n \times Y^{X_1 \times \dots \times X_n} & \xrightarrow{\text{app}} & Y \end{array}$$

the statements

$$\begin{aligned} e(\mathfrak{f})(\mathbf{u}_1, \dots, \mathbf{u}_n) &= \widetilde{\text{app}}(\mathbf{u}_1, \dots, \mathbf{u}_n, \mathfrak{f}), \\ e(\mathfrak{r})(\mathbf{u}_1, \dots, \mathbf{u}_n) &\text{ iff } \widetilde{\text{in}}(\mathbf{u}_1, \dots, \mathbf{u}_n, \mathfrak{r}) \end{aligned}$$

hold for all $\mathfrak{f} \in \beta(Y^{X_1 \times \dots \times X_n})$, $\mathfrak{r} \in \beta \mathcal{P}(X_1 \times \dots \times X_n)$, and $\mathbf{u}_1 \in \beta X_1, \dots, \mathbf{u}_n \in \beta X_n$? Does this hold at least for all compact Hausdorff spaces Y that are zero-dimensional, or extremally disconnected?

4. What are topological properties of the subset of the space $\beta(Y^{X_1 \times \dots \times X_n})$ consisting of pseudo-principal ultrafilters? Of the preimage of this set under e , i.e. the set $\{\tilde{f} : f \in Y^{X_1 \times \dots \times X_n}\}$, in the space $RC_{X_1, \dots, X_{n-1}}(\beta X_1, \dots, \beta X_n, \beta Y)$ with the (X_1, \dots, X_n) -pointwise convergence topology (except for the fact that it is dense there, as stated in Lemma 12), or with the (usual) pointwise convergence topology? in the space $(\beta Y)^{\beta X_1 \times \dots \times \beta X_n}$ with the pointwise convergence topology?

Often objects naturally defined in terms of ultrafilter extensions have rather hardly definable topological properties, as shown in [30, 31].

In two next questions we wonder about variants of the definition of the satisfiability in generalized models in the narrow sense.

5. Define an alternative satisfaction relation \models by using rather in^* than $\widetilde{\text{in}}$; i.e. if $R(t_1, \dots, t_n)$ is an atomic formula in which R is not the equality predicate, let

$$\mathfrak{A} \models R(t_1, \dots, t_n) [v] \text{ iff } \text{in}^*(v_i(t_1), \dots, v_i(t_n), \iota(P)).$$

Does this give a E -counterpart of the semantic of generalized models in the narrow sense? More precisely, is the following E -counterpart of Theorem 23 true: If \mathfrak{A} is a generalized model, then for all formulas φ and elements $\mathbf{u}_1, \dots, \mathbf{u}_n$ of the universe of \mathfrak{A} ,

$$\mathfrak{A} \models \varphi [\mathbf{u}_1, \dots, \mathbf{u}_n] \text{ iff } E(\mathfrak{A}) \models \varphi [\mathbf{u}_1, \dots, \mathbf{u}_n]$$

(with \models in this new sense)?

6. Another way to vary the definition \models is by letting

$$\mathfrak{A} \models R(t_1, \dots, t_n) [v] \text{ iff } (\forall^{\iota(R)} Q) \widetilde{Q}(v_i(t_1), \dots, v_i(t_n)).$$

This version looks less smooth. Does this, nevertheless, give something interesting?

7. To define the map I , we considered the set RegCl of regular closed subsets of the space $\beta X_1 \times \dots \times \beta X_n$ with a topology turning it into a space homeomorphic to the usual product space $2^{X_1 \times \dots \times X_n}$ with the discrete space 2. Redefine this topology on RegCl as a restricted version of Vietoris topology (in an analogy with the restricted version of pointwise convergence topology turning out RC into a compact Hausdorff space homeomorphic to the product space $Y^{X_1 \times \dots \times X_n}$). Note that in the usual Vietoris topology, the space $\{Q \subseteq \beta X_1 \times \dots \times \beta X_n : Q \text{ is closed}\}$ is compact Hausdorff but RegCl is not a closed subspace of it.

8. Investigate filter extensions of first-order models (as was started in [1] and [6]) and the corresponding concepts of filter interpretations and generalized models.

9. Investigate other possible types of ultrafilter extensions, besides the \sim - and $*$ -ones, isolate special features of the two canonical extensions among others (as was proposed at the end of [6]).

10. Investigate ultrafilter extensions of the syntax (ultrafilter extensions of languages, continuous extensions of evaluation and interpretation maps, ultrafilter extensions of the satisfaction relation).

11. Do the concepts of ultrafilter interpretations and generalized models lead to any interesting (e.g. Ramsey-theoretic) applications?

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