ON THE DESCRIPTION OF PARABOLIC NEWTON MAPS

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Abstract. A description of rational Newton maps in terms of the partial fraction decomposition of rational functions is obtained. Dynamics on parabolic immediate basins for rational Newton maps of entire functions have been studied. It is proved that every parabolic immediate basin contains invariant accesses to the parabolic fixed point at infinity. Moreover, among these accesses there exists a unique dynamically defined access where dynamics are attracted towards the parabolic fixed point, whereas for other accesses, if there is any, the dynamics are repelled.

1. Introduction

Let f be a complex polynomial or a transcendental entire function defined on the complex plane $\mathbb{C}$. A meromorphic function $N_f(z) := z - f(z)/f'(z)$ is called the Newton map of $f(z)$. In the literature, the Newton map of a polynomial $p(z)$ is also called the Newton method or the Newton-Raphson method applied to $p(z)$. The roots of a polynomial equation $p(z) = 0$ become the attracting fixed points of its Newton map $N_p(z)$. Hence searching for the roots of a complex polynomial is equivalent to searching for the attracting fixed points of the corresponding Newton map. At a starting point z_0 in the complex plane $\mathbb{C}$, we iteratively apply the Newton map $N_f(z)$ to produce the sequence

$$z_0, N_f(z_0), N_f^{\circ 2}(z_0), \dots, N_f^{\circ n}(z_0), \dots,$$

called the orbit of z_0. We write $F^{\circ n}$ for the nth iterate of a complex map F, for instance, $F^{\circ 2}(z) := F(F(z))$. If this orbit converges to a complex number ξ, then ξ is a fixed point of N_f. Moreover, the same orbit converges to a root of $f(z) = 0$. If the root is simple then the order of convergence is quadratic, thus the approximations rapidly approach the root. In this paper, we are interested in the case of rational Newton maps. We do not study here the numerical aspects of Newton's method (refer to the works [Sch, SS]) but rather consider the Newton maps as dynamical systems on the Riemann sphere $\hat{\mathbb{C}}$. More precisely, this paper consists of two results, a description of rational Newton maps $N_f(z)$ for $f(z) = p(z)e^{q(z)}$ with non-constant $q(z)$ in Theorem A and the study of accesses to ∞ on the open set of the complex plane where the above defined sequence converges in $\hat{\mathbb{C}}$ in Theorem B. This latter set is the union of basins of attraction of all attracting fixed points and the parabolic fixed point of N_f. It is open and belongs to the

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stable set called the Fatou set of N_f, the complement of which is called the Julia set, where the dynamics is chaotic (sensitive to initial conditions). The Fatou set of a holomorphic function F is defined as the set of $z \in \hat{\mathbb{C}}$ such that there exists U, an open neighborhood of z, where the iterates $\{F^{\circ n}|_U, n \geq 1\}$ form a normal family on U. The Fatou set of a rational function F coincides with the set of points $z \in \hat{\mathbb{C}}$ *stable in the sense of Lyapunov* for F.

Let us recall some basic definitions. For a fixed point $F(z) = z$ the quantity $\lambda = F'(z)$ is a local conformal invariant (under a conformal conjugacy) and is called the *multiplier* of z. The fixed point z is *attracting* if $|\lambda| < 1$, in particular, *superattracting* if $\lambda = 0$, *repelling* if $|\lambda| > 1$, *indifferent* if $|\lambda| = 1$, in this case let $\lambda = e^{2\pi i\theta}$, then *rationally indifferent* (also called parabolic) if $\theta \in \mathbb{Q}$, *irrationally indifferent* if $\theta \notin \mathbb{Q}$. The following is known for fixed points. Attracting fixed points and irrationally indifferent fixed points, if the function is locally linearisable, belong to the Fatou set. Repelling and parabolic fixed points belong to the Julia set. Irrationally indifferent fixed points, if the function is not locally linearisable, belong to the Julia set as well. The multiplier is also defined for periodic points and the same classification and the corresponding properties are true for them as well. The multiplicity of a fixed point z_0 is defined as the multiplicity of z_0 as a root of the fixed point equation $F(z) = z$. If $z = 0$ is a parabolic fixed point with the multiplier $+1$ then the Taylor series expansion of F near the origin is $F(z) = z + az^{m+1} + O(z^{m+2})$, where $a \neq 0$ is a complex number and $m + 1 \geq 2$ is the multiplicity of the parabolic fixed point at the origin. The number m in this case is called the parabolic multiplicity of a parabolic fixed point.

Every rational function of degree at least 2 has a fixed point that is either repelling or parabolic with the multiplier $+1$. We call this type of fixed points *weakly repelling*. If such a point is unique then the Julia set is connected by theorem of Shishikura [Sh]. For Newton maps the point at ∞ is the only weekly repelling fixed point, as a corollary, the Julia sets for all rational Newton maps is connected.

Denote $\deg(F, z)$ the local degree of a function F at a point z. Denote $C_F = \{z | \deg(F, z) > 1\}$ and $P_F = \overline{\bigcup_{n\geq 1} F^{\circ n}(C_F)}$ the set of critical points and the post-critical set of F respectively. For holomorphic functions, the set of critical points is exactly the set where the derivative vanishes. A function F is called *post-critically finite* if P_F is finite. A function F is called *geometrically finite* if the intersection of P_F with the Julia set is a finite set.

For geometrically finite rational Newton maps the Julia sets are locally connected, it is a corollary of a result in [T], where it asserts that for geometrically finite rational functions the Julia set is locally connected if it is connected.

The rational Newton maps can be described in terms of multiplies of fixed points as in the following theorem in [RS].

Theorem 1.1. *Let $N : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function. It is the Newton map of an entire function $f : \mathbb{C} \to \mathbb{C}$ if and only if for each fixed point ξ of N there is an integer $m = m_\xi \geq 1$ such that $N'(\xi) = (m-1)/m$. In this case there exists a constant $c \in \mathbb{C}\backslash\{0\}$ such that $f = c \cdot e^{\int \frac{1}{\zeta - N(\zeta)} d\zeta}$. Entire functions f and g have the same Newton map if and only if $f = c \cdot g$ for some constant $c \in \mathbb{C}\backslash\{0\}$.*

The point at ∞ is a removable singularity for some Newton maps making them rational functions on $\hat{\mathbb{C}}$ [RS].

Theorem 1.2. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Its Newton map N_f is a rational function if and only if there are polynomials p and q such that f has the form $f = pe^q$. Let m and n be the degrees of p and q, respectively. For $n = 0$ and $m \geq 2$, the point at ∞ is repelling with the multiplier $m/(m-1)$. For the pair $n = 0$ and $m = 1$, N_f is constant. For $n > 0$, the point at ∞ is parabolic with the multiplier $+1$ and the multiplicity $n + 1 \geq 2$.*

2. Results

The following criterion, based on partial fraction decomposition of rational functions, allows us easily check whether or not a given rational map is a Newton map.

Theorem A (Description of Newton maps)**.** *Let $N : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function of $\deg(N) \geq 2$. Let ∞ be its weakly repelling fixed point. Assume $\frac{1}{z-N(z)} = \sum_{i=1}^{k} r_i(\frac{1}{z-z_i}) + s(z)$ is given, the partial fraction decomposition for $\frac{1}{z-N(z)}$ over the field of complex numbers $\mathbb{C}$, where z_i runs over all distinct fixed points of N in $\mathbb{C}$, where $r_i = r_i(z)$, for $1 \leq i \leq k$, and $s = s(z)$ are polynomials. Then N is a Newton map of an entire function if and only if there exist integers $m_i \geq 1$ such that $r_i(z) \equiv m_i \cdot z + r_i(0)$. In this case, let $p = p(z) := \prod_{i=1}^{k}(z - z_i)^{m_i}$ (if $N(z)$ does not have any fixed point in $\mathbb{C}$, we let $p := 1$) and $q = q(z) := \int_0^z \left(s(w) + \sum_{i=1}^{k} r_i(0)\right) dw$ be polynomials, then $N = N_{pe^q}$ and $\deg(N) = k + \deg(q) = \deg(\frac{1}{z-N(z)}) + 1$.*

Proof. Let $N : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function of degree $d \geq 2$, without loss of generality assume that ∞ is a weekly repelling fixed point (we can always conjugate N by a suitable Möbius map sending its weekly repelling fixed point to ∞). As a rational function, $\frac{1}{z-N(z)}$ has $k \geq 0$ poles at points $z_1, \ldots, z_k$ in $\mathbb{C}$, which are exactly the fixed points of N in $\mathbb{C}$. Let the partial fraction decomposition of $\frac{1}{z-N(z)}$ be the following

$$\frac{1}{z - N(z)} = \sum_{i=1}^{k} r_i(\frac{1}{z - z_i}) + s(z), \tag{2.1}$$

where $r_i = r_i(z)$, for $1 \leq i \leq k$, and $s = s(z)$ are polynomials, (see for details Chapter 1.4 in [A]). Without loss of generality, we can normalize the polynomials $r_i(z)$ such that their constant terms vanish: $r_i(0) = 0$, for every $1 \leq i \leq k$, or we can add their constant terms to the polynomial $s(z)$.

Assume that $r_i(z) \equiv m_i \cdot z$ for integers $m_i \geq 1$. Then the partial fraction decomposition (2.1) can be rewritten as

$$\frac{1}{z - N(z)} = \sum_{i=1}^{k} \frac{m_i}{z - z_i} + s(z).$$

If we denote polynomials $p = p(z) := \prod_{i=1}^{k}(z - z_i)^{m_i}$ and $q = q(z) := \int_0^z s(w)dw$, from the latter we obtain $N(z) = z - p(z)/(p'(z) + p(z) \cdot q'(z))$. It follows that N is the Newton map of the entire function pe^q by the uniqueness of Newton maps (Theorem 1).

Conversely, let N be the Newton map of an entire function $f = pe^q$, where p and q are polynomials. Let $p(z) := \prod_{i=1}^{k}(z - z_i)^{m_i}$, where z_i runs over all distinct roots of p, then we obtain

$$\frac{1}{z-N_f(z)}=\frac{f'}{f}=\frac{p'e^q+pq'e^q}{pe^q}=\frac{p'+pq'}{p}=\frac{p'}{p}+q'=\sum_{i=1}^{k}\frac{m_i}{z-z_i}+q'(z).$$

The uniqueness of partial fraction decompositions for rational functions yields the required result.

It remains to show that the degree of the Newton map is equal the number of distinct roots of p plus the degree of q. We have

$$N_{pe^q}(z)=z-\frac{p(z)}{p'(z)+p(z)\cdot q'(z)}=\frac{z(p'(z)+p(z)\cdot q'(z))-p(z)}{p'(z)+p(z)\cdot q'(z)}.$$

The degree may drop if the numerator and denominator of the latter has some cancellation factor, i.e., the polynomial equations $z(p'(z)+p(z)q'(z))-p(z)=0$ and $p'(z)+p(z)q'(z)=0$ have a common solution for some $z=z_0$. Plugging the second into the first, we get $p(z_0)=0$. Combining it with the second, we derive $p'(z_0)=0$. These mean that $z=z_0$ is a multiple root of p and then $\deg(N)=k+\deg(q)$, where k is the number of distinct roots of p. Similarly, $\deg(\frac{1}{z-N_f(z)})=\deg(\frac{p'+pq'}{p})=\deg(N)-1=k+\deg(q)-1$. □

The following notion is the main object when one studies Newton maps. Since we are dealing with a general family of rational Newton maps, we allow ∞ to be a parabolic fixed point.

Definition 2.1 (Basin of Attraction)**.** Let ξ be an attracting or a parabolic fixed point of F. Denote $\mathcal{A}(\xi)=\operatorname{int}\{z\in\hat{\mathbb{C}}:\lim_{n\to\infty}F^{\circ n}(z)=\xi\}$ the *basin* of ξ defined as the interior of the set of starting points that converge to ξ under the iterates of F. Denoted $\mathcal{A}^\circ(\xi)$ the *immediate basin* of ξ that is the forward invariant connected component of the basin $\mathcal{A}(\xi)$.

Definition 2.2 (Invariant access to ∞)**.** Let $\mathcal{A}^\circ$ be the immediate basin of a fixed point $\xi\in\mathbb{C}$ or the parabolic fixed point at ∞ of the Newton map N_{pe^q}. Fix a point $z_0\in\mathcal{A}^\circ$, and consider a curve $\Gamma:[0,\infty)\to\mathcal{A}^\circ$ with $\Gamma(0)=z_0$ and $\lim_{t\to\infty}\Gamma(t)=\infty$. Its homotopy class (with endpoints fixed) within $\mathcal{A}^\circ$ defines an *access to* ∞ in $\mathcal{A}^\circ$. An access is called invariant if together with its every representative Γ the image $N_{pe^q}(\Gamma)$ also belongs to the same access.

In the case of an attracting immediate basin $\mathcal{A}^\circ(\xi)$, the end point z_0 of the homotopy is at the attracting fixed point ξ.

Remark 2.3. For parabolic immediate basins the invariant accesses are also well defined. Indeed, if we choose a point $z_0\neq\infty$ as one of the end points of the homotopy (the other end is at ∞), then within the immediate basin, by considering a composition of some curve joining z_0 and $N_{pe^q}(z_0)$ and then following the image curve $N_{pe^q}(\Gamma)$, we always stay in the same homotopy class of the curve Γ thanks to the simply connectivity of the immediate basins.

The invariant accesses are defined for immediate basins of attracting fixed points for Newton maps of polynomials in [HSS]. Our definition of an invariant access for these domains coincides with theirs and generalizes this notion to the parabolic immediate basins. In Figure 1, the immediate basins for the quartic parabolic Newton map with all accesses to ∞ are depicted.

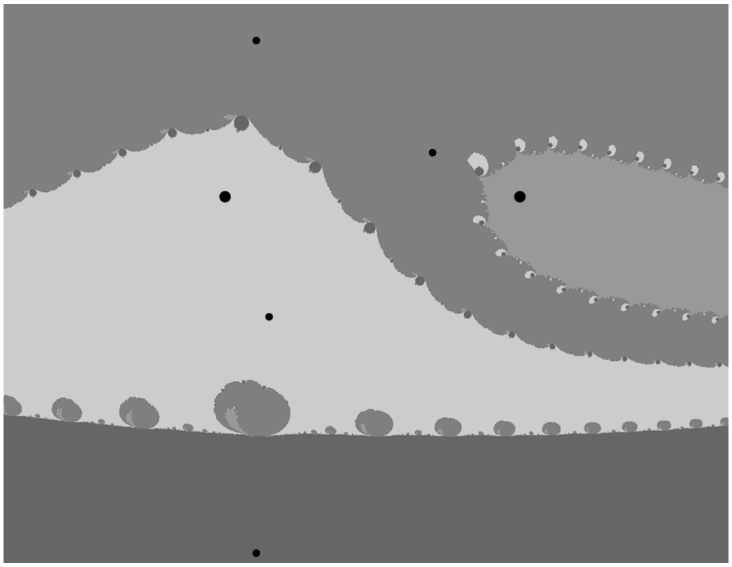

Figure 1. The Julia set of a quartic Newton map. Top and bottom gray regions are the two distinct parabolic immediate basins of ∞. Left and right central regions are attracting immediate basins. Critical points are drawn as black dots with the two heavy dots representing the two superattracting fixed points. There are 2 critical points in the left central immediate basin and in the gray basin on the top with the corresponding 2 invariant accesses to ∞ in each.

The following main theorem gives the structure of immediate basins of rational Newton maps. This result is obtained in the author's PhD thesis in [Ma1] and recently this result were generalized for meromorphic Newton maps in [BFJK] in full generality. In our proof, following [HSS], we show that all invariant accesses come in one-to-one correspondence for each fixed point of the corresponding proper map of the unit disk. Moreover, for parabolic immediate basins there always exists a distinguished access called the *dynamical invariant access to* ∞. To obtain this access in the parabolic immediate basin, we can take any curve η, starting at some z_0 and ending at $N_{pe^q}(z_0)$, and considering the homotopy class of the curve $\Gamma := \bigcup_{n\geq 0} N_{pe^q}^{\circ n}(\eta)$. This curve lands at ∞ and is forward invariant under N_{pe^q}. This notion of dynamical invariant access is used as a main ingredient to construct a natural bijection between distinct spaces of postcritically finite and postcritically minimal Newton maps using parabolic surgery in [Ma1, Ma2, Ma3]. Similar to postcritically finite Newton maps of polynomials, the postcritically minimal Newton maps are structurally important in the corresponding parameter plane.

For most cubic Newton maps of polynomials the Julia sets are locally connected as it was observed by Roesch in [R]. Very recently, the local connectivity issue for Newton maps of polynomials has been completely resolved by Drach and Schleicher in [DS]. They proved that for N_p, the Newton maps of polynomials, of all degrees, the Julia set is locally connected at every point z, except possibly when z belongs to, or is mapped to, the Julia set of some renormalizable polynomial-like restriction of N_p. For the cases of locally connected Julia sets the proof given below simplifies as we can apply parabolic surgery on Newton maps of polynomials and obtain parabolic Newton maps and the accesses of the former transform to the accesses of the latter (for the details refer to [Ma2]).

Theorem B (Invariant accesses to ∞). *Let N_{pe^q} be a rational Newton map of degree $d \geq 2$ and $\mathcal{A}^\circ$ be an immediate basin of an attracting or parabolic fixed point of N_{pe^q}. Assume $\mathcal{A}^\circ$ contains $k \geq 1$ critical points of N_{pe^q} counting with multiplicities, then $N_{pe^q}|_{\mathcal{A}^\circ}$ is a branched covering map of degree $k+1 \geq 2$, and $\mathcal{A}^\circ$ has exactly k invariant accesses to ∞. If $\mathcal{A}^\circ$ is a parabolic immediate basin then among its k invariant accesses to ∞ there always exists a dynamical invariant access that is unique.*

Proof. Let N_{pe^q} a rational Newton map and $\mathcal{A}^\circ$ its immediate basin be given. There are two cases; case of basins for attracting fixed points or case of basins for the parabolic fixed point at ∞. The first case for Newton maps of polynomials was treated in Proposition 6 in [HSS]. Since the arguments in the proof use only the local dynamics of the function within the basin, their results are also valid for attracting immediate basins of rational Newton maps N_{pe^q} with non-constant q.

It remains to prove the theorem for parabolic immediate basins of rational Newton maps N_{pe^q} with $\deg(q) > 0$. Following [HSS], denote $\mathbb{D} = \{z, |z| < 1\}$ the unit disk and its boundary $\mathbb{S}^1 = \{z, |z| = 1\}$ the unit circle, and let $\mathcal{A}^\circ$ be one of the parabolic immediate basins of ∞. Consider the Riemann map $\psi : \mathcal{A}^\circ \to \mathbb{D}$ uniquely determined by $\psi(c) = 0$ and $\psi'(c) > 0$, where c is any point in $\mathcal{A}^\circ$. Then the composition $f = \psi \circ N_{pe^q} \circ \psi^{-1}$ is a proper map of the unit disc $\mathbb{D}$ with the degree, which is equal to the degree of the restriction $N_{pe^q}|_{\mathcal{A}^\circ}$. The critical points of N_{pe^q} in $\mathcal{A}^\circ$ are mapped to the critical points of f in $\mathbb{D}$ preserving multiplicities. Assume that N_{pe^q} has $k \geq 1$ critical points in $\mathcal{A}^\circ$ counting with multiplicities, there is at least one critical point in every immediate basin. The connectivity of the Julia set implies that $\mathcal{A}^\circ$ is simply connected, then by the Riemann-Hurwitz formula the degree of the restriction $N_{pe^q}|_{\mathcal{A}^\circ}$ is $k+1$.

As a proper self map of $\mathbb{D}$, f is a Blaschke product (the product of a finite number of conformal automorphisms of $\mathbb{D}$, see [M]), hence it has an extension to $\hat{\mathbb{C}}$ by reflection, denote the extension again by f. The degree of the restriction $N_{pe^q}|_{\mathcal{A}^\circ}$ and of f coincide, which is $k+1$. Then f has $k+2$ fixed points, one of which is a double parabolic (of multiplicity 3 or of parabolic multiplicity 2). Since we have a parabolic dynamics in $\mathbb{D}$, other distinct $k-1$ simple fixed points are repelling with real multipliers. All fixed points are located on the unit circle. The unit disk $\mathbb{D}$, the unit circle $\mathbb{S}^1$ and $\hat{\mathbb{C}} \setminus \bar{\mathbb{D}}$ are invariant by f. Since f can not have critical point on $\mathbb{S}^1$, it is a covering map of $\mathbb{S}^1$ of degree $k+1$, and the orbit of every $z \in \hat{\mathbb{C}} \setminus \mathbb{S}^1$ converges to the unique parabolic fixed point on $\mathbb{S}^1$. Thus the Julia set is the unit circle $\mathbb{S}^1$. Alternatively, observe that the Fatou set of the Blaschke product is invariant under the suitable involution. If the Julia set is not connected then the Fatou set of this Blaschke product has a unique component which includes a part of the unit circle $\mathbb{S}^1$ and the complement of $\mathbb{S}^1$. But then on the Fatou points in $\mathbb{S}^1$ the iterates must eventually converge to the parabolic fixed point on $\mathbb{S}^1$ then necessarily they fall on the repelling petal and gets pushed back from the parabolic fixed point, which leads to a contradiction to the convergence to the parabolic fixed points.

By Remark 1, in $\mathcal{A}^\circ$ it suffices to consider a part of homotopies fixing the point at ∞ locally at ∞. The Riemann map $\psi : \mathcal{A}^\circ \to \mathbb{D}$ transports homotopies to the unit disk $\mathbb{D}$. The linearizing coordinates of $k-1$ repelling fixed points of f define $k-1$ invariant accesses among k invariant accesses, and the other invariant access comes from a Fatou coordinate of the parabolic fixed point on $\mathbb{S}^1$. The invariant access associated to the parabolic fixed point defines the *dynamical* access.

Assume that the boundary of $\mathcal{A}^\circ$ is locally connected, which is true if the Newton map is geometrically finite. Carathéodory theorem assures that ψ^{-1} the inverse map to $\psi : \mathcal{A}^\circ \to \mathbb{D}$ extends to the closed unit disk $\overline{\mathbb{D}}$ as a continuous map. Denote the extension again by ψ^{-1}. By continuity, we obtain $\psi^{-1} \circ f = N_{pe^q} \circ \psi^{-1}$ on $\overline{\mathbb{D}}$. Counted with multiplicities, $k+2$ fixed points of f correspond to $k+2$ fixed points of N_{pe^q} on $\partial\mathcal{A}^\circ$. A fixed point of N_{pe^q} that is on the boundary of an immediate basin is the only parabolic fixed point at ∞, so the domain $\mathcal{A}^\circ$ has invariant accesses to ∞ through k distinct directions ($k-1$ directions for the $k-1$ simple and one direction for the triple fixed point of f).

If $\mathcal{A}^\circ$ is not locally connected, so that the inverse to the Riemann map does not extend continuously to the closed unit disk, the statement still holds true. Consider a Koenigs coordinate of a repelling fixed point ξ_j that conjugates f locally near the point ξ_j to the linear map $z \mapsto f'(\xi_j)z$, we take a segment of a straight-line through the origin, which is invariant. We take an invariant curve in the petal associated to the parabolic fixed point of f. Let γ be the preimage of this curve that lands at ξ_j in the dynamical plane of f. Then we have $\gamma \subset f(\gamma)$. Now we pull the curve γ by the Riemann map ψ to $\mathcal{A}^\circ$. The accumulation set of $\psi^{-1}(\gamma)$ in $\partial\mathcal{A}^\circ$ is connected (see Section 17 in [M]) and since γ is invariant we conclude that the accumulation set is pointwise fixed by N_{pe^q}. But ∞ is the only fixed point on the Julia set. This gives us k invariant accesses to ∞ in $\mathcal{A}^\circ$. We need to show that they are all distinct and the only ones.

It is clear that simple curves within $\mathbb{D}$ converging to a given fixed point of f are homotopic so that every fixed point of f defines a unique access in $\mathcal{A}^\circ$. Different fixed points of f lead to non-homotopic curves in $\mathcal{A}^\circ$ and thus to different accesses. Indeed, let $l_i, l_j \subset \mathbb{D}$ be the radial lines converging to $\xi_i \neq \xi_j$ respectively, parametrized by the radius. Assume by contrary that $\psi^{-1}(l_i)$ and $\psi^{-1}(l_j)$ are homotopic curves in $\mathcal{A}^\circ$ by a homotopy fixing end points; $\psi^{-1}(l_i(1)) = \psi^{-1}(l_j(1)) = \infty$, then one of the components bounded by a simple closed curve $\psi^{-1}(l_i) \cup \psi^{-1}(l_j)$ must be contained in $\mathcal{A}^\circ$. Call this component V; then $\psi(V)$ must be one of the sectors bounded by l_i and l_j; call it S. Both V and S are Jordan domains, so ψ^{-1} extends as a homeomorphism from $\bar{S}$ onto $\bar{V}$ by Carathéodory theorem; but then the extension sends the set $\mathbb{S}^1 \cap S$ nowhere.

Conversely, we show that every invariant access to ∞ in $\mathcal{A}^\circ$ comes from a fixed point of the corresponding f. Let $\Gamma : [0,1] \to \mathcal{A}^\circ \cup \infty$ be a curve representing an access. Then $\psi(\Gamma)$ lands at a point $\upsilon \in \mathbb{S}^1$ by Corollary 17.10 in [M], define it as the associated point of Γ. Then for every $n \geq 1$, $N_{pe^q}^{\circ n}(\Gamma)$ represents an access and thus has its associated point $\upsilon_n \in \mathbb{S}^1$. Since the Newton map N_{pe^q} has a parabolic fixed point at ∞, it is locally a homeomorphism there and every fixed point of f gives rise to an access, all υ_n must be contained in the same connected component of $\mathbb{S}^1$ with the fixed points removed; this component is an interval, say I, on which $\{\upsilon_n\}$ must be a monotone sequence converging under f to a fixed point υ of f in $\bar{I}$, i.e. to one of the endpoints. If υ is a one of the repelling fixed points of f then it is impossible. For the remaining possibility, assume υ is a parabolic fixed point of f then the sequence $\{\upsilon_n\} \subset \mathbb{S}^1$ converges to the parabolic fixed point, which is also not possible since $\mathbb{S}^1$ is the Julia set of f and every orbit that converges to a parabolic fixed point must follow the attracting direction which is perpendicular to $\mathbb{S}^1$.

Finally, observe that the unique invariant access corresponding to the parabolic fixed point of f gives rise to the dynamical invariant access and it always exists and unique. $\square$

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