VARIATIONS ON THE THEME OF ZARISKI'S CANCELLATION PROBLEM

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Abstract. This is an expanded version of the talk by the author at the conference *Polynomial Rings and Affine Algebraic Geometry*, February 12–16, 2018, Tokyo Metropolitan University, Tokyo, Japan. Considering a local version of the Zariski Cancellation Problem naturally leads to exploration of some classes of varieties of special kind and their equivariant versions. We discuss several topics inspired by this exploration, including the problem of classifying a class of affine algebraic groups that are naturally singled out in studying the conjugacy problem for algebraic subgroups of the Cremona groups.

1. Introduction. This is an expanded version of the talk by the author at the conference *Polynomial Rings and Affine Algebraic Geometry*, February 12–16, 2018, Tokyo Metropolitan University, Tokyo, Japan.

Our starting point is a local version of the Zariski Cancellation Problem (LZCP). Its consideration naturally leads to distinguishing a class of varieties of a special kind, called here flattenable, and a more general class of locally flattenable varieties. We discuss the relevant examples, including flattenability of affine algebraic groups and the related varieties, in particular, we prove that all smooth spherical varieties are locally flattenable. This is completed by answering (LZCP). We then consider the equivariant versions of flattenability and obtain a series of results on equivariant flattenability of affine algebraic groups. In particular, we prove that a reductive algebraic group is equivariantly flattenable if and only if it is linearly equivariantly flattenable, and we prove that equivariant flattenability of a Levi subgroup of a connected affine algebraic group G implies that of G. The latter yields that every connected solvable affine algebraic group is equivariantly flatenable. As an application, we briefly survey a special role of equivariantly flattenable subgroups in the rational linearization problem. Then we dwell on the classification problem of equivariantly flattenable affine algebraic groups G. We prove that every such G is special in the sense of Serre, which implies that if G is reductive equivariantly flattenable, then its derived group if a product of the groups of types SL and Sp. We complete this discussion with the unexpected recent examples of reductive equivariantly flattenable

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groups, whose derived groups do contain factors of type Sp. In the last section, the local version of equivariantly flattenable varieties and the relevant version of the Gromov problem are briefly considered.

Notation and conventions. We fix an algebraically closed field k of characteristic zero. In what follows, as in [Bo 1991], [PV 1994], variety means algebraic variety over k in the sense of Serre (so algebraic group means algebraic group over k). Unless otherwise specified, all topological terms refer to the Zariski topology. We use freely the standard notation and conventions of loc. cit., where the proofs of the unreferenced claims and/or the relevant references can be found. Action of an algebraic group on an algebraic variety means algebraic (morphic) action; homomorphism of an algebraic group means algebraic homomorphism.

We also use the following notation:

- A^* is the group of units of an associative k-algebra A with identity.
- $\mathrm{Mat}_{n\times m}$ is the k-vector space of all $n\times m$-matrices with entries in k,; for $n=m$, it is naturally endowed with the k-algebra structure.

2. The Zariski Cancellation Problem. So is called the following question:

$$\begin{array}{l}\textit{Are there affine varieties } X \textit{ and } Y \textit{ such that } Y \textit{ and}\\ X\times Y \textit{ are isomorphic respectively to } \mathbf{A}^s \textit{ and } \mathbf{A}^{s+d},\\ \textit{but } X \textit{ is not isomorphic to } \mathbf{A}^d\,?\end{array} \tag{ZCP}$$

At this writing (January 2019), for $d > 2$, it is still open. A historical survey about the Zariski Cancellation Problem is given in [Gu 2015].

Our starting point is a local version of this problem. Making precise its formulation (see (LZCP) in Subsection 5) leads to distinguishing the following class of varieties:

Definition 1. An irreducible variety X is called

(a) *flattenable* if X isomorphic to an open subset of an affine space;
(b) *locally flattenable* if for every point $x\in X$ there is a flattenable open subset of X containing x.

3. Terminology. Under other names, locally flattenable varieties appeared in the literature long ago. The earliest reference known to the author is [Ch 1958, p. 2-09] where Chevalley calls them *special* varieties. In [Ma 1974] Chevalley terminology is used for the definition of R-equivalence. In [Ak 1993] these varieties appear as *algebraic spaces*, in [BHSV 2008] as *plain varieties*, and in [BB 2014] and [Pe 2017] as *uniformly rational varieties.* The term *flattenable variety* is coined in [Po 2013_1], where special properties of linearly equivariantly flattenable algebraic subgroups (see below Definition 18) of the Cremona groups have been revealed (this topic is briefly surveyed in Subsection 9 below).

By Definition 1, every locally flattenable variety is rational. Whether the converse is true is open at this writing (January 2019):

Is every irreducible smooth rational variety locally flattenable? (Gr)

This problem was raised by M. Gromov in [Gr 1989, 3.5.E‴] (for projective varieties).

4. Examples of locally flattenable varieties.

1*. Homogeneous spaces.*

Theorem 2. *Let X be an irreducible variety. If the natural action of* $\operatorname{Aut}(X)$ *on X is transitive, then the following properties are equivalent:*

(a) *X is a rational variety;*
(b) *X is a locally flattenable variety.*

Proof. If (a) holds, then X contains an open flattenable subset U. Since U and gU for any $g \in \operatorname{Aut}(X)$ are isomorphic, (b) follows from the equality $X = \bigcup_{g\in G} gU$ (the latter holds because of the transitivity condition). Definition 1 implies (b)⇒(a). □

Corollary 3. *Let G be a connected affine algebraic group and let H be a closed subgroup of G. Then the following properties are equivalent:*

(a) *G/H is a rational variety;*
(b) *G/H is a locally flattenable variety.*

Remark 4. Maintain the notation of Corollary 3. There are nonrational (and even not stably rational) varieties G/H, where G is

$$\mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_r} \times \mathrm{Sp}_{2m_1} \times \cdots \times \mathrm{Sp}_{2m_s} \tag{1}$$

and H is finite; see [Po 2013_2, Thm. 2]. It is an old problem, still open at this writing (January 2019), whether there are nonrational homogeneous spaces G/H with connected H. For connected H of various special types, rationality of G/H is known; see [CZ 2017] and Remark 34 below. In particular, G is rational as a variety [Ch 1954] (cf. [Po 2013_2, Lem. 2]).

2*. Vector bundles and homogeneous fiber spaces.*

The claim of Theorem 5 below is mentioned in [BB 2014, Ex. 2.1]:

Theorem 5. *Let $X \to Z$ be an* (*algebraic*) *vector bundle over an irreducible variety Z. If Z is locally flattenable, then X is locally flattenable as well.*

Proof. Since the fibers of $X \to Z$ are isomorphic to an affine space, and, by [Se 1958, Thm. 2], algebraic vector bundles are locally trivial in the Zariski topology, the claim follows from Definition 1. □

Let G be a connected algebraic group, H its closed subgroup, and F a quasiprojective variety endowed with a regular action of H. Then we have (see [PV 1994, 4.8]) the algebraic homogeneous fiber space $G \times^H F$ over G/H with fiber F; the natural projection $\pi_{G,H,F}\colon G \times^H F \to G/H$ is locally trivial in the étale topology. If F is a vector space V over k and the action of H on V is linear, then $\pi_{G,H,V}$ is an algebraic vector bundle over G/H with fiber V. Combining Corollary 3 and Theorem 5 yields

Corollary 6. *Maintain the above notation. If G/H is rational, then $G\times^H V$ is locally flattenable.*

Theorem 7. *Let G be a connected reductive algebraic group and let X be a smooth affine variety endowed with an action of G. Assume that*

(a) $k[X]^G = k$;
(b) *the* (*unique*, see, e.g., [PV 1994, Cor. of Thm. 4.7]) *closed G-orbit $\mathcal{O}$ in X is rational.*

Then X is locally flattenable.

Proof. By [Lu 1973, p. 98, Cor. 2] (see also [PV 1994, Thm. 6.7]), (a) and smoothness of X imply that X is G-equivariantly isomorphic to $G \times^H V$, where H is the G-stabilizer of a point of $\mathcal{O}$, and V is a finite-dimensional H-module. The claim then follows from Corollary 6. $\square$

3. *Spherical varieties.*

Let G be a connected reductive algebraic group and let B be a Borel subgroup of G. Recall that a variety X endowed with an action of G is called *spherical variety of* G if there is a dense open B-orbit in X.

Theorem 8. *Every smooth spherical variety is locally flattebable.*

Proof. Let X be a smooth spherical variety of a connected reductive group G.

First, X is rational because every B-orbit is rational (the latter is isomorphic to the complement of a union of several coordinate hyperplanes in some affine space [Gr 1958, Cop. p.5-02]).

Secondly, every G-orbit in X is spherical (see, e.g., [Ti 2011, Prop. 15.14]), hence rational. Therefore, by Theorem 7, if X is affine, then X is locally flattenable.

Thirdly, arbitrary X is covered by open subsets, each of which is isomorphic to a variety of the form $P \times^L Z$, where P and L are respectively a parabolic subgroups of G and a Levi subgroup of P, and Z is an affine spherical variety of L; see, e.g., [Ti 2011, Thm. 15.17]. Since X is smooth, Z is smooth as well. Therefore, as explained above, Z is locally flattenable. The variety $P \times^L Z$ is isomorphic to the product of Z and the underlying variety of the unipotent radical of P. Since this underlying variety is isomorphic to an affine space [Gr 1958, Cor. p. 5-02], we infer that $P \times^L Z$ is locally flattenable. Therefore, X is locally flattenable, too. $\square$

Since every toric variety is spherical, Theorem 8 implies

Corollary 9 ([BB 2014, Expl. 2.2])**.** *Every smooth toric variety is locally flattenable.*

4. *Blow-ups with nonsingular centers.*

Theorem 10 ([Gr1989, p.885, Prop.], [BHSV2008, Thm.4.4], [BB2014, Prop. 2.6])**.** *The blow-up of a locally flattenable variety along a smooth subvariety is locally flattenable.*

5. *Curves and surfaces.*

For varieties of dimension $\leqslant 2$, the answer to (Gr) is affirmative:

Theorem 11 ([BHSV 2008, Prop. 3.2], [BB 2014, Prop. 2.6])**.** *Every irreducible rational smooth algebraic curve or surface X is locally flattenable.*

Proof. If X is a curve, it admits an open embedding in $\mathbf{P}^1$. If X is a surface, it admits an open embedding in a projective smooth surface, which, being rational, is obtained by repeated point blow-ups of a minimal model, i.e., either $\mathbf{P}^2$ or a Hirzebruch surface F_n, $n \neq 1$. Since $\mathbf{P}^1$, $\mathbf{P}^2$, and F_n are toric varieties, the claim follows from Corollary 9 and Theorem 10. □

5. Local version of (ZCP). Given Definition 1, the local version of the Zariski Cancellation Problem mentioned in Subsection 2 is formulated as follows:

Are there affine varieties X and Y such that Y and $X \times Y$ are flattenable, but X is not flattenable? (LZCP)

In Subsection 7 we show that the answer to (LZCP) is affirmative.

6. Flattenable varieties vs. locally flattenable varieties. Flattenable varieties have special properties:

Lemma 12. *Let X be an affine flattenable variety and let $\varphi\colon X \hookrightarrow \mathbf{A}^n$ be an open embedding. If $k[X]^* = k^*$, then $\varphi(X) = \mathbf{A}^n$.*

Proof. Assume that the closed set $\mathbf{A}^n \setminus \varphi(X)$ is nonempty. Then, since X is affine, the dimension of every irreducible component of this set is $n-1$. Therefore, $\mathrm{Pic}(\mathbf{A}^n) = 0$ implies that $\mathbf{A}^n \setminus \varphi(X)$ is the set of zeros of a certain function $f \in k[\mathbf{A}^n]$. Then $f \circ \varphi$ is a nonconstant element of $k[X]^*$,— a contradiction. Hence $\varphi(X) = \mathbf{A}^n$. □

Lemma 13. *For a connected affine algebraic group G, the following properties are equivalent:*

(a) *as a variety, G is isomorphic to an affine space;*
(b) *as a group, G is unipotent.*

Proof. Assume that (a) holds. If G is not unipotent, there exists a nontrivial torus T among the closed subgroups of G. The action of T on G by left multiplication then gives a fixed point free action of T on an affine space, which is impossible by [Bi 1966, Thm. 1]. This contradiction proves (a)$\Rightarrow$(b).

Conversely, (a) follows from (b) by [Gr 1958, Cor. p. 5-02]. □

Theorem 14. *Let G be a connected affine algebraic group, and let $\mathcal{R}G$ be its radical.*

(a) *If G is solvable, then G is flattenable.*
(b) *If G is flattenable and nonsolvable, then $\mathcal{R}G$ is not unipotent.*

Proof. Let G be solvable. Then G, as a variety, is isomorphic to the complement of a union of several coordinate hyperplanes in some affine space [Gr 1958, Cor. p. 5-02]; whence (a).

Assume that G is flattenable and nonsolvable. The latter implies that G is not unipotent, hence, by Lemma 13, as a variety, G is not isomorphic to an affine space. Lemma 12 then implies that there is a nonconstant invertible function $f \in k[G]$. By [Ro 1961, Thm. 3], the map $G \to \mathrm{GL}_1$, $g \mapsto f(g)/f(e)$, is then a nontrivial character. According to [Po 2011, Lem. 1.1], the existence of such a character is equivalent to the property that $\mathcal{R}G$ is not unipotent; whence (b). ☐

Corollary 15. *Let G be a nontrivial connected reductive algebraic group. If G is flattenable, then the dimension of its center is positive. In particular, every semisimple G is not flattenable.*

7. Answering (LZCP).

Theorem 16. *There are affine varieties X and Y such that*

(a) *X is not flattenable;*
(b) *Y and $X \times Y$ are flattenable.*

Proof. As a variety, any SL_n for $n > 1$ is not flattenable by Corollary 15. On the other hand, being open in the affine space $\mathrm{Mat}_{m\times m}$, any GL_m is flatteneble. The morphism

$$\mathrm{SL}_n \times \mathrm{GL}_1 \to \mathrm{GL}_n, \quad (s, a) \mapsto s\,\mathrm{diag}(a, 1, \ldots, 1), \tag{2}$$

is an isomorphism of varieties: its inverse is

$$\mathrm{GL}_n \to \mathrm{SL}_n \times \mathrm{GL}_1, \quad g \mapsto \big(g\,\mathrm{diag}(1/\det(g), 1, \ldots, 1), \det(g)\big).$$

Hence we can take $X = \mathrm{SL}_n$ for $n > 1$, and $Y = \mathrm{GL}_1$. ☐

In Remark 38 below one can find other examples.

8. Equivariantly flattenable varieties.

Definition 17. A variety X endowed with an action of an algebraic group G is called *equivariantly* (respectively, *linearly equivariantly*) *flattenable* if there are

· an action (respectively, a linear action) of G on some $\mathbf{A}^n$;
· a G-equivariant open embedding $X \hookrightarrow \mathbf{A}^n$.

Definition 18. An algebraic group G is called *equivariantly* (respectively, *linearly equivariantly*) *flattenable* if G, as a variety endowed with the G-action by left multiplication, is *equivariantly* (respectively, *linearly equivariantly*) *flattenable.*

Examples 19.

1. Every GL_n is linearly equivariantly flattenable since GL_n is an invariant open set of $\mathrm{Mat}_{n\times n}$ endowed with the GL_n-action by left multiplication.

2. Every (connected) unipotent affine algebraic group G is, as a variety, isomorphic to an affine space; hence G is equivariantly flattenable. In fact, a more general statement,Theorem 22 below, holds. It is easily seen that G is linearly equivariantly flattenable only if it is trivial. On the other hand, the example of a Borel subgroup of SL_2 naturally acting on k^2 shows that there are nontrivial solvable linearly equivariantly flattenable groups.

3. If the $G_1, \ldots, G_m$ are equivariantly (respectively, linearly equivariantly) flattenable groups, then, clearly, $G_1 \times \cdots \times G_m$ is equivariantly (respectively, linearly equivariantly) flattenable as well. In particular, the group

$$\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_s} \tag{3}$$

is linearly equivariantly flattenable for any $n_1, \ldots, n_s$. Taking $n_1 = \ldots = n_s = 1$ yields that every affine algebraic torus is linearly equivariantly flattenable.

4. Generalizing Example 19.1, let A be a finite-dimensional associative k-algebra with identity. The group A^* is a connected affine algebraic group. It is open in A and invariant with respect to the action of A^* on A by left multiplication, cf. [Bo 1991, I,1.6(9)]. Hence A^* is a linearly equivariantly flattenable group. For $A = \mathrm{Mat}_{n\times n}$, we obtain $A^* = \mathrm{GL}_n$. More generally, if A is semisimple, then A^* is a group of type (3), and all groups of type (3) are obtained in this way.

5. Every $G = \mathrm{SL}_n \times \mathrm{GL}_1$ is equivariantly flattenable. Indeed, consider the G-module structure on $V = \mathrm{Mat}_{n\times n}$ defined by the formula

$$G \times V \to V, \quad ((s, a), x) \mapsto s\, x \operatorname{diag}(a, 1, \ldots, 1).$$

For $x = \operatorname{diag}(1, \ldots, 1)$, the orbit map $G \to V$, $g \mapsto g \cdot x$ is then the G-equivariant open embedding (2).

Theorem 20. *The following properties of a connected reductive algebraic group G are equivalent:*

(a) *G is equivariantly flattenable;*
(b) *G is linearly equivariantly flattenable.*

Proof. Let G be equivariantly flattenable. By Definitions 17, 18, we may (and shall) identify G with an open orbit in some $\mathbf{A}^n$ endowed with a regular action of G. Openness of this orbit implies $k[\mathbf{A}^n]^G = k$. Hence, by [Lu 1973, p. 98, Cor. 2] (see also [PV 1994, Thm. 6.7]), there are a closed reductive subgroup L of G and a finite-dimensional algebraic L-module V such that $\mathbf{A}^n$ and $G \times^L V$ are G-equivariantly isomorphic. We claim that this implies

$$L = G. \tag{4}$$

If (4) is proved, then $\mathbf{A}^n$ and V are G-equivariantly isomorphic, which proves (a)$\Rightarrow$(b).

So it remains to prove (4). In view of connectedness of G, to this end it suffices to prove $\dim(L) = \dim(G)$.

Since $\mathrm{char}(k) = 0$, by the Lefschetz principle we may (and shall) assume that $k = \mathbf{C}$; in the remainder of the proof topological terms are related to the Hausdorff $\mathbf{C}$-topology. Since $\mathbf{A}^n$ is simply connected, G/L is simply connected as well, hence L is connected.

We now note that the dimension any connected complex reductive algebraic group R is equal to the maximum m_R of i such that $H_i(R) \neq 0$ (singular homology with complex coefficients). Indeed, if K is a maximal compact subgroup of R, then the Iwasawa decomposition of R shows that R, as a manifold, is a product of K and a Euclidean space. Hence R and K have the same homology. Since K is a compact oriented manifold, this shows that m_R is equal to the dimension of the Lie group K. As R is the complexification of K, the statement follows.

So to prove (4) is the same as to prove $m_G = m_L$. In fact, since L is a subgroup of G, the above equality $m_R = \dim(R)$ yields $m_G \geqslant m_L$, so to prove $m_G = m_L$ we only need to prove the inequality

$$m_G \leqslant m_L. \tag{5}$$

As is known (see, e.g., [Hu 1959, Chap. IX, Thm. 11.1]), the spectral sequence of the natural fiber bundle $G \to G/L$ yields the following inequality for the Betti numbers

$$\dim_{\mathbf{C}}(H_{m_G}(G)) \leqslant \sum_{i+j=m_G} \dim_{\mathbf{C}}(H_i(G/L)) \dim_{\mathbf{C}}(H_j(L)). \tag{6}$$

On the other hand, since $G \times^L V$ is a vector bundle over G/L and $G \times^L V$ is isomorphic to $\mathbf{A}^n$, we have

$$\dim_{\mathbf{C}}(H_i(G/L)) = \dim_{\mathbf{C}}(H_i(\mathbf{A}^n)) = \begin{cases} 1 & \text{for } i = 0, \\ 0 & \text{for } i > 0. \end{cases} \tag{7}$$

From (6), (7) we infer that

$$0 < \dim_{\mathbf{C}}(H_{m_G}(G)) \leqslant \dim_{\mathbf{C}}(H_{m_G}(L)). \tag{8}$$

The definition of m_L and (8) then yield (5). This completes the proof. $\square$

Remark 21. Using the same argument, but (in the spirit of [Bo 1985]) étale cohomology in place of singular homology, one can avoid applying the Lefschetz principle and adapt the above proof to the case of base field of arbitrary characteristic.

Recall that a *Levi subgroup* of a connected affine algebraic group G is its (necessarily reductive) subgroup L such that G is the semi-direct product of L and the unipotent radical $\mathcal{R}_u G$ of G; since $\mathrm{char}(k) = 0$, such L exist and are conjugate in G; see, e.g., [Bo 1991, 11.22].

Theorem 22. *Let G be a connected affine algebraic group and let L be a Levi subgroup of G. If L is equivariantly flattenable, then so is G.*

Proof. Let L be equivariantly flattenable. Then by Theorem 20, we may (and shall) assume that L, as a variety with the L-action by left multiplication, is an open orbit $\mathcal{O}$ of an algebraic L-module V. This implies that $\dim(L) = \dim(V)$, and therefore,

$$\dim(G \times^L V) = \dim(G) - \dim(L) + \dim(V) = \dim(G). \tag{9}$$

We identify V with the fiber of $G \times^L V \to G/L$ over the point of G/L corresponding to L. Since the L-stabilizer of any point $v \in \mathcal{O}$ is trivial, the G-stabilizer of v is trivial as well. From this and (9) we infer that $G \to G \times^L V$, $g \mapsto g \cdot v$, is a G-equivariant (with respect to the action of G on itself by left multiplication) open embedding. Now we note that G/L is isomorphic to the underlying variety of $\mathcal{R}_u G$, therefore, by Lemma 13, to an affine space. Since, by Quillen–Suslin [Qu 1976], [Su 1976], algebraic vector bundles over affine spaces are trivial, we conclude that the variety $G \times^L V$ is isomorphic to an affine space. This completes the proof. □

Corollary 23. *Every connected solvable affine algebraic group is equivariantly flattenable.*

Proof. Levi subgroups of connected solvable affine algebraic groups are tori. As the latter are equivariantly flattenable, the claim follows from Theorem 22. □

9. Flattenable vs. equivariantly flattenable varieties. The following shows that there are affine flattenable varieties endowed with actions of reductive algebraic groups, which are not linearly equivariantly flattenable.

Example 24. As is known (see, e.g., references in survey [Kr 1996]), there are affine spaces endowed with nonlinearizable actions of reductive algebraic groups. By Lemma 12, they are not linearly equivariantly flattenable.

Example 25. Let S be a connected semisimple algebraic group, whose center $\mathcal{C}S$ is nontrivial, and let G be a finite subgroup of S, whose centralizer in S is $\mathcal{C}S$. Note that such pairs S, G do exist. For instance, let F be a nontrivial finite group, satisfying the following conditions:

(a) there are no nontrivial characters $F \to \mathrm{GL}_1$;
(b) there is a faithful irreducible representation $\varphi\colon F \to \mathrm{GL}_n$.

Say, (a) and (b) hold for any nontrivial simple F. By (a), we have $\varphi(F) \subset \mathrm{SL}_n$; therefore, we may (and shall) identify F with $\varphi(F)$ and consider F as a subgroup of SL_n. By (b) and Schur's lemma, the centralizer of F in SL_n is the cyclic group $\{\varepsilon I_n \mid \varepsilon \in k^*, \varepsilon^n = 1\}$ of order n, which is the center of SL_n. We have $n \geqslant 2$ because of (a), so this center is nontrivial. Hence, SL_n, F is an example of the pair of interest.

Returning back to S and G, let B, B^- be a pair of opposite Borel subgroups of S. Then the "big cell" $\Theta := B^-B$ is an open subset of S isomorphic to the complement of a union of several coordinate hyperplanes in $L := \mathbf{A}^{\dim(S)}$; in particular, C is an affine flattenable variety. We have

$$\mathcal{C}S \subset \Theta \tag{10}$$

because $\mathcal{C}S \subset B$ (see, e.g., [Bo 1985, 13.17, Cor. 2(d)]).

Now we consider the conjugating action of G on S. Its fixed point set S^G is $\mathcal{C}S$. The set $X := \bigcap_{g\in G} g\Theta g^{-1}$ is a G-stable open subset of S. In view of (10), it contains $\mathcal{C}S$, therefore,

$$X^G = \mathcal{C}S. \tag{11}$$

Since Θ is an affine flattenable variety, X is such a variety, too.

We claim that X is not linearly equivariantly flattenable. Assume the contrary. Then there is a linear action of G on L such that there is a G-equivariant open embedding $X \hookrightarrow L$; we identify X with the image of this embedding. Hence $X^G = X \cap L^G$. Since L^G is a linear subspace of L, and X is open in L, this implies that X^G is irreducible. The latter contradicts (11), because S is semisimple, and hence $\mathcal{C}S$ is finite (and nontrivial).

Remark 26. For X in Example 25, as a G-variety, the following alternative holds. Either X is not equivariantly flatenable. Or X is equivariantly flattenable, but the action of G on an affine space, extending that on X, is nonlinearizable.

Question 27. Are there flattenable reductive algebraic groups, which are not equivariantly flattenable?

10. Equivariantly flattenable subgroups of the Cremona groups. As an application, below is briefly surveyed a special role of equivariantly flattenable subgroups in the conjugacy problem for algebraic subgroups of the Cremona groups Cr_n (i.e., in the rational linearization problem). We refer to [Po 2013_1], [PV 1994] and references therein regarding the basic definitions and properties of rational algebraic group actions, in particular, the definition of embeddings

$$\mathrm{Cr}_1 \subset \mathrm{Cr}_2 \subset \cdots \subset \mathrm{Cr}_n \subset \mathrm{Cr}_{n+1} \subset \cdots . \tag{12}$$

Theorem 28 ([Po 2013_1, Thm. 2.1])**.** *Let G be a connected algebraic subgroup of the Cremona group Cr_n. Assume that*

(i) *G is linearly equivariantly flattenable;*
(ii) *the natural rational action of G on $\mathbf{A}^n$ is locally free.*

If the field extension $k(\mathbf{A}^n)^G/k$ is purely transcendental, then G is conjugate in Cr_n to a subgroup of GL_n (i.e., the natural rational action of G on $\mathbf{A}^n$ is rationally linearizable).

For tori, the sufficient condition of Theorem 28 for rational linearization is also necessary:

Theorem 29 ([Po 2013_1, Thm. 2.4])**.** *Let T be an affine algebraic torus in the Cremona group Cr_n. The following properties are equivalent:*

(a) *T is conjugate in V to a subgroup of GL_n;*
(b) *the field extension $k(\mathbf{A}^n)^T/k$ is purely transcendental.*

The sufficient condition of Theorem 28 for rational linearization always holds true in the stable range:

Theorem 30 ([Po 2013_1, Lem. 2.2 & Thm. 2.2])**.** *Let G be a connected algebraic subgroup of the Cremona group Cr_n such that assumptions* (i), (ii) *of Theorem* 28 *hold. Then there is an integer $s \geqslant 0$ such that, for the rational action of the group G on $\mathbf{A}^{n+s}$ determined by embedding* (12)*, the field extension $k(\mathbf{A}^{n+s})^G/k$ is purely transcendental.*

Remarks 31. 1. By [Po 2013_1, Thm. 2.6], if G in Theorem 30 is a torus, then one can take $s = \dim(G)$.

2. In general, the integer s in Theorem 30 is strictly positive. For example, by [Po 2013_1, Cor. 2.5], the Cremona group Cr_n for $n \geqslant 5$ contains an $(n-3)$-dimensional affine algebraic torus, which is not conjugate in Cr_n to a subgroup of GL_n.

11. Equivariantly flattenable groups and special groups in the sense of Serre. Recall from [Se 1958, 4.1] that an algebraic group G is called *special* if every principal G-bundle (which, by definition [Se 1958, 2.2], is locally trivial in étale topology) is locally trivial in the Zariski topology. By [Se 1958, Sect. 4.1, Thm. 1] special group is automatically connected and affine. Special groups are classified:

Theorem 32. *The following properties of a connected affine algebraic group G are equivalent:*

(a) *G is special;*
(b) *maximal connected semisimple subgroups of G are isomorphic to a group of type* (1)*.*

Proof. The implications (a)$\Rightarrow$(b) and (b)$\Rightarrow$(a) are proved respectively in [Gr 1958] and [Se 1958]. $\square$

In [Po 2013_1, Lem. 2.2] is sketched a reduction of the following claim to [Po 1994, Thm. 1.4.3]. Below is given the complete self-contained argument.

Theorem 33. *Every linearly equivariantly flattenable affine algebraic group G is special.*

Proof. By Definitions 17, 18, there is a finite-dimensional algebraic G-module V with a G-orbit $\mathcal{O}$ such that

(i) $\mathcal{O}$ is open in V;
(ii) the points of $\mathcal{O}$ have trivial G-stabilizers.

By (ii), V is faithful; hence we may (and shall) identify G with a closed subgroup of $\mathrm{GL}(V)$.

Next, consider the ith direct summand of

$$L := V \oplus \cdots \oplus V \quad (n := \dim(V) \text{ copies}),$$

as a linear subspace V_i of L, denote by π_i the natural projection $L \to V_i$, and identify V with V_1. The $\mathrm{GL}(V)$-module L contains a $\mathrm{GL}(V)$-orbit $\widetilde{\mathcal{O}}$ such that

(iii) $\widetilde{\mathcal{O}}$ is open in L;
(iv) the points of $\widetilde{\mathcal{O}}$ have trivial $\mathrm{GL}(V)$-stabilizers.

From (i), (iii) we infer that $\mathcal{O} \cap \pi_1(\widetilde{\mathcal{O}}) \neq \varnothing$ and $\pi_1^{-1}(\mathcal{O}) \cap \widetilde{\mathcal{O}}$ is an nonempty open subset of $\widetilde{\mathcal{O}}$. Take a point

$$a \in \mathcal{O} \cap \pi(\widetilde{\mathcal{O}}) \tag{13}$$

and consider in L the affine subspace

$$A := \{a + v_2 + \cdots + v_n \in L \mid v_i \in V_i \text{ for all } i\}. \tag{14}$$

From (iii), (13), (14) we deduce that $A \cap \widetilde{\mathcal{O}}$ is a nonempty open subset of A, and from (ii) that the G-orbit of every point of $\pi_1^{-1}(\mathcal{O}) \cap \widetilde{\mathcal{O}}$ intersects $A \cap \widetilde{\mathcal{O}}$ at a single point. This means that the natural action of G on $\widetilde{\mathcal{O}}$ admits a rational section. In view of (iv), this, in turn, means that the natural map $\mathrm{GL}(V) \to \mathrm{GL}(V)/G$ admits a rational section. Since the group $\mathrm{GL}(V)$ is special, this implies, according to [Se 1958, Sect. 4.3, Thm. 2], that the group G is special as well. □

Remark 34. Since A is rational and $A \cap \widetilde{\mathcal{O}}$ is open in A, the above proof of Theorem 33 shows that $\mathrm{GL}(V)/G$ is a rational variety.

12. Classifying equivariantly flattenable groups. Theorem 22 naturally leads to the following

Problem 35. Obtain a classification of equivariantly flattenable reductive algebraic groups.

Theorem 36. *For every nontrivial equivariantly flattenable reductive algebraic group G, the following properties hold:*

(a) *the derived group L of G is a semisimple group of type* (1)*;*
(b) *the radical of G is a central torus C of positive dimension;*
(c) $G = L \cdot C$ *and* $C \cap L$ *is finite.*

Proof. Since L is a maximal connected semisimple subgroup of G, combining Theorems 33 and 32 yields (a). Combining Corollary 15 and [Bo 1991, 14.2, Prop.] implies (b) and (c). □

Problem 35 looks manageable. Initially, being influenced by Example 19.4, the author was even overoptimistic and put forward the conjecture that all equivariantly flattenable reductive algebraic groups are that of type (3) (see [Po 2013_1, p. 221]); this overoptimism was shared by some of the participants of the 2013 Oberwolfach meeting on algebraic groups who even sketched a plan of possible proof. So it came as a surprise, when in [BGM 2017] the examples of equivariantly flattened reductive groups of a type distinct from (3) have been revealed, making Problem 35 even more intriguing. Below we describe them.

Theorem 37 ([BGM 2017, 2.1])**.** *For every positive integer n, the group*

$$G := \mathrm{Sp}_{2n} \times \mathrm{GL}_{2n-1} \times \mathrm{GL}_{2n-2} \times \cdots \times \mathrm{GL}_1 \tag{15}$$

is equivariantly flattenable.

Sketch of proof. Consider the vector space

$$V := \mathrm{Mat}_{2n\times 1} \oplus \mathrm{Mat}_{2n\times(2n-1)} \oplus \mathrm{Mat}_{(2n-1)\times(2n-2)} \oplus \cdots \oplus \mathrm{Mat}_{2\times 1} \tag{16}$$

and the linear action of G on V defined for the elements

$$g := (A,\, B_{2n-1},\, B_{2n-2}, \ldots, B_2,\, B_1) \in G,\ A \in \mathrm{Sp}_n, B_d \in \mathrm{GL}_d,$$

$$v := (X,\, Y_{2n-1},\, Y_{2n-2}, \ldots, Y_1) \in V,\ X \in \mathrm{Mat}_{2n\times 1}, Y_d \in \mathrm{Mat}_{(d+1)\times d}$$

by the formula

$$g \cdot v := (AX,\, AY_{2n-1}B_{2n-1}^{\top},\, B_{2n-1}Y_{2n-2}B_{2n-2}^{\top}, \ldots, B_2Y_1B_1^{\top}).$$

From (15) and (16) we deduce

$$\dim(G) \overset{(15)}{=} 2n^2 + n + \sum_{i=1}^{2n-1} i^2 = 2n + \sum_{i=1}^{2n-1} i(i+1) \overset{(16)}{=} \dim(V). \tag{17}$$

Next, one shows the existence of a point $v_0 \in V$ whose G-stabilizer is trivial. In view of (17), the orbit map $G \to V$, $g \mapsto g\cdot v_0$ is then a G-equivariant open embedding. □

Remark 38. Since the group Sp_{2n} is not flattenable by Corollary 15, but $\mathrm{GL}_{2n-1} \times \mathrm{GL}_{2n-2} \times \cdots \times \mathrm{GL}_1$ is flattenable, Theorem 37 provides other (than that in the proof of Theorem 16) examples, which yield the affirmative answer to (LZCP).

13. Locally equivariantly flattenable varieties. Similarly to flattenability, equivariant flattenability admits an evident local version:

Definition 39 ([Pe 2017, Def. 4(iii)], up to change of terminology)**.** A variety X endowed with an action of an algebraic group G is called *equivariantly* (respectively, *linearly equivariantly*) *locally flattenable* if for every point $x \in X$ there is an equivariantly (respectively, linearly equivariantly) flattenable G-stable open subset of X containing x.

Definition 39 leads to the following equivariant version of Gromov's question (Gr):

Is every irreducible smooth rational G-variety equivariantly locally flattenable? (EqGr)

The examples in [Pe 2017, Sect. 4] show that the answer to (EqGr) is negative.

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