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Unsteady Gerstner waves

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ABSTRACT

We present an analytical description of the class of unsteady vortex surface waves generated by non-uniformly distributed, time-harmonic pressure. The fluid motion is described by an exact solution of the equations of hydrodynamics generalizing the Gerstner solution. The trajectories of the fluid particles are circumferences. The particles on a free surface rotate around circumferences of the same radii, with the centers of the circumferences lying on different horizons. A family of waves has been found in which a variable pressure acts on a limited section of the free surface. The law of external pressure distribution includes an arbitrary function. An example of the evolution of a non-uniform wave packet is considered. The wave and pressure profiles, as well as vorticity distribution are studied. It is shown that, in the case of a uniform traveling wave of external pressure, the Gerstner solution is valid but with a different form of the dispersion relation. A possibility of observing the studied waves in laboratory and in the real ocean is discussed.

1. Introduction

Gerstner waves are steady progressive waves on the surface of a liquid of infinite depth. They are described by the exact solution of the equations of a perfect incompressible fluid [1,2]. The Gerstner solution was rediscovered by Froude [3], Rankine [4] and Reech [5]. This solution is remarkable as it is the only exact solution for gravity waves on deep water. In this sense, for gravitational waves on water studied in the framework of complete hydrodynamic equations the Gerstner waves are analogous to solitons in nonintegrable systems (Kivshar and Malomed [6], Grimshaw et al. [7], Zahibo et al. [8], Stepanyants [9]).

Key contributions to physical understanding and mathematical justification of the Gerstner wave solution were made by Dubreil-Jacotin [10], Lamb [2], Mollo-Christensen [11,12], Constantin and Strauss [13], Constantin [14] and Henry [15], among others. Hydrodynamic stability of the Gerstner wave was investigated by Leblanc [16]. The Gerstner surface wave solution was extended to edge waves in fluids with a free surface and a plane sloping rigid boundary (Yih [17]; Constantin [18]). Gerstner surface and edge waves remain exact solutions in stratified incompressible fluids [10,17] (see also Stuhlmeier [19,20]), including fluids with density and flow velocity discontinuities [12]. Mollo-Christensen developed an analytical model of gravitational and geostrophic billows in the atmosphere by deriving an exact finite-amplitude solution for a wave on an interface of two fluids, with one fluid moving as in the Gerstner wave and the other fluid in a uniform motion at the speed of the wave [11]. Johnson used the Gerstner edge wave solution to construct asymptotic solutions valid for variable but small bottom slopes [21]. Within the f- plane approximation, extensions of Gerstner surface and edge waves to rotating fluids have been proposed by Pollard [22], Matioc [23,24] and Weber [25]. Ionescu–Kruse applied the short-wavelength perturbation method to derive instability criteria for the three-dimensional nonlinear Pollard geophysical waves [26]. Mollo–Christensen obtained the exact solution describing nonlinear edge waves in a rotating fluid in the presence of a mean flow [27]. Constantin and Monismith studied the propagation of Gerstner waves in the presence of mean currents and rotation [28]. Gerstner-type solutions for equatorially trapped surface and internal waves in the ocean were obtained by Constantin [29–31] in the β-plane approximation. The exact solution for equatorially trapped surface waves was further extended by Henry by allowing for a uniform current in the direction of wave propagation [32]. Godin presented the solutions providing an extension of the Gerstner wave in an incompressible fluid with a free boundary to waves in compressible three-dimensionally inhomogeneous moving fluids [33].

At the same time, physical feasibility of Gerstner waves is still disputable. These waves possess vorticity, hence they cannot be generated from rest by conservative forces in an ideal fluid. Lamb pointed that such a wave motion may arise against the background of a shear flow having the same vorticity [2]. This idea was implemented in the laboratory experiments by Monismith et al. [34]. Liquid particles in a Gerstner wave travel around a circumference without drift flow (average over the period). The wave motion

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possessing this property was generated in a flume by Monismith's team. This means that Gerstner waves or their finite-depth relatives must have been observed.

Weber, in turn, hypothesized that a weakly nonlinear Gerstner solution may be realized taking into account the effects of liquid viscosity and surface films [35]. In a linear approximation with respect to the small parameter of wave steepness, liquid particles in a viscous fluid move around circumferences, the radii of which decrease exponentially with time. The wave vorticity is concentrated in a narrow near-surface boundary layer. In the quadratic approximation, the viscosity gives rise to mean drift, which Weber compared with the available experimental data and did not exclude that viscosity-modified Gerstner waves were observed in some wave tank experiments.

The mentioned above studies were performed assuming constant pressure on a free fluid surface. This approximation is justified in the absence of wind. The physical manifestations of wind are variable pressure on the fluid surface and a vortex character of the wave motion. Waseda and Tulin showed experimentally that the wind does not suppress the Benjamin–Feir instability [36]. The investigation of the self-consistent wind-wave interaction is a rather complicated problem. A possible way to simplify it is to choose a definite law of pressure variation on the free surface. In the works (Leblanc [37], Kharif et al. [38], Onorato and Proment [39], Chabchoub et al. [40], Brunetti et al. [41], Eeltink et al. [42]) where the dynamics of weakly nonlinear, narrow bandwidth trains of surface waves was studied, a variable external pressure was specified according to the Miles linear theory of wind wave excitation [43]. Yan and Ma explored the formation of strongly nonlinear waves in the framework of a complete system of equations of hydrodynamics in the presence of a uniform air flow and proposed a phenomenological model for air flow pressure distribution on a free surface [44].

In this paper we will investigate the dynamics of vortex surface waves under the action of time-harmonic pressure non-uniformly distributed on a free surface. The consideration is carried out in Lagrangian variables. The fluid motion is described by a class of exact solutions [45], generalizing the Gerstner solution. Like in the classical Gerstner wave, the trajectories of the fluid particles are circumferences. The particles on a free surface have the same radius of rotation but the centers of the circumferences lie on different horizons, thereby the wave has a variable profile. Such waves are called unsteady Gerstner waves. We will analyze a family of waves, in which variable pressure acts on a bounded section of the free surface. The law of external pressure distribution includes an arbitrary function and corresponds to a wide variety of boundary conditions. We will consider a solution describing the dynamics of a non-uniform packet of unsteady Gerstner waves. Wave and pressure profiles, as well as vorticity distribution will be addressed. A possibility of observing the studied waves in laboratory and in the real ocean will be discussed.

The rest of this paper is organized as follows. In Section 2 we introduce a class of Ptolemaic flows into consideration and show that Gerstner waves are their particular case. An exact solution for Gerstner-type waves with variable pressure on a free surface was obtained in Section 3. In the next section, the dynamics of the Gerstner wave packet is studied, when the variable pressure acts on a bounded interval of the free surface. In Section 5 we show that, in the case of a uniform traveling plane wave of external pressure, the Gerstner solution is valid but with a different form of the dispersion relation. Our findings are summarized in Section 6.

2. Ptolemaic flows and Gerstner wave

Consider gravity waves on the surface of a homogeneous liquid having density ρ. Neglecting viscosity, the equations of 2D hydrodynamics in Lagrangian variables are written in the form [2,46]:

$$\frac{D(X,Y)}{D(a,b)} = \frac{D(X_0,Y_0)}{D(a,b)}\,, \tag{1}$$

$$X_{tt}X_a + Y_{tt}Y_a = -\frac{1}{\rho}p_a - gY_a, \tag{2}$$

$$X_{tt}X_b + Y_{tt}Y_b = -\frac{1}{\rho}p_b - gY_b, \tag{3}$$

where X, Y are the Cartesian coordinates of the liquid particle trajectory, a, b are its Lagrangian coordinates, p is pressure, g is acceleration of gravity, t is time, the subscript "0" denotes the value of the variable at the initial moment of time, and the subscripts in Eqs. (2), (3) stand for differentiation with respect to the corresponding variable. It is assumed that the wave motion occurs in the $b \le 0$ region.

By cross-differentiating the equations of motion (2), (3) we eliminate pressure and obtain the condition of vorticity conservation along the trajectory [2,46]:

$$(X_{ta}X_b + Y_{ta}Y_b - X_{tb}X_a - Y_{tb}Y_a)_t = 0. \tag{4}$$

By introducing complex coordinates of the particle trajectory $W = X + iY$, $\bar{W} = X - iY$ and complex Lagrangian coordinates $\chi = a + ib$, $\bar{\chi} = a - ib$ (the overline in $\bar{\chi}$ denotes the complex conjugate of χ) we can write Eqs. (1), (4) in the form [45]:

$$\begin{aligned} &\tfrac{D(W,\bar{W})}{D(\chi,\bar{\chi})} = \tfrac{D(W_0,\bar{W}_0)}{D(\chi,\bar{\chi})} = D_0(\chi,\bar{\chi})\,, \\ &\tfrac{D(W_t,\bar{W})}{D(\chi,\bar{\chi})} = \tfrac{D(W_{t0},\bar{W}_0)}{D(\chi,\bar{\chi})} = \tfrac{i}{2}D_0\Omega(\chi,\bar{\chi})\,. \end{aligned} \tag{5}$$

Here, Ω is vorticity and the function D_0 determines the dependence of the initial position of liquid particles, W_0, on Lagrangian coordinates. The sign of the function does not change in the flow region due to one-to-one mapping between particle coordinates and their Lagrangian labels. We assume for definiteness $D_0 \ge 0$.

Eq. (5) have an exact solution [45]:

$$W = G(\chi)e^{i\delta t} + F(\bar{\chi})e^{i\mu t}, \tag{6}$$

where F, G are analytical functions and δ, μ are constant frequencies. The particle trajectories for this solution are either epicycloids ($\delta\mu > 0$) or hypocycloids ($\delta\mu < 0$). In the Ptolemaic picture of the world the planets move along such trajectories; that is why the flows (6) were named Ptolemaic. If one of the frequencies is equal to zero, the liquid particles move along the circumference.

The Gerstner wave is a particular case of Ptolemaic flows. The expression for this wave is written in complex variables

$$W = \chi + iA\exp i(k\overline{\chi} - \omega t), \quad \operatorname{Im}\chi = b \le 0, \tag{7}$$

where A is wave amplitude, k is wave number, and ω is frequency. By writing (7) in real form, we obtain the solution proposed by Gerstner [1]:

$$X = a - Ae^{kb}\sin(ka - \omega t); \quad Y = b + Ae^{kb}\cos(ka - \omega t). \tag{8}$$

The liquid particles in a wave rotate around circumferences having radius $A\exp kb$. The free surface corresponds to $b = 0$. It is a trochoid moving at the speed $c = \omega/k$ without changing its shape in the positive direction of the X-axis. The pressure on the free surface of the Gerstner wave is constant. The dispersion equation for the Gerstner wave $\omega^2 = gk$ is identical to that for a linear potential wave.

The Jacobian D_0 for the Gerstner waves is equal to $1 - k^2A^2\exp 2kb$. Since $D_0 \ge 0$ inside the flow region, we have $kA \le 1$. For $kA = 1$, the Gerstner wave crests become cusped, which corresponds to the limiting wave. For the values of steepness $kA > 1$, the wave profile features loops (see also [14,15]).

The Gerstner waves are vortex ones. Making use of Eq. (5), one can readily find an expression for vorticity

$$\Omega = \frac{2ck^3A^2 \exp 2kb}{1 - k^2A^2 \exp 2kb}. \tag{9}$$

In the case $kA << 1$, the vorticity is close to zero and expressions (7), (8) coincide with the solution for a linear potential wave.

The pressure in Gerstner waves depends on the vertical Lagrangian coordinate b only. The isolines $b = const$ are the constant pressure surfaces, and any of them can be chosen as a free one. Obviously, this will change the equilibrium level of the liquid. When the free surface corresponds to $b = 0$, the equilibrium level of the stationary Gerstner wave lies below the surface $Y = 0$ [47].

3. Variable external pressure

Consider Ptolemaic flows of the form

$$W = G(\chi) + iA \exp i(k\overline{\chi} - \omega t), \quad \operatorname{Im}\chi = b \le 0. \tag{10}$$

This expression is a particular case of the relation (6). The function F in (6) is identical to that of the Gerstner wave, and the function G may vary. If G is a linear function, then the relation (10) coincides with Gerstner's solution.

The wave (10) has a number of common properties with the Gerstner wave:

- liquid particles move around circumferences with radius $A\exp kb$;
- liquid particles rotate around circumferences of the same radius on each Lagrangian horizon $\operatorname{Im}\chi = b = const \le 0$;
- there is no averaged drift of liquid particles;
- the waves are vortex ones.

At the same time, these waves have certain differences. For the flow (10), the vorticity is found from system (5) and is written as

$$\Omega = \frac{2\omega (kA)^2 \exp(2kb)}{|G'|^2 - (kA)^2 \exp(2kb)}. \tag{11}$$

For the Gerstner wave, the vorticity depends on b coordinate only, whereas the vorticity (11) is the function of both Lagrangian variables. The function G' determines the vorticity distribution.

The vertical coordinates of the centers of circumferences $Y_c = \operatorname{Im} G(a + ib)$ around which liquid particles in the wave (10) are rotating depends on the form of the G function and is determined by the values of both Lagrangian coordinates. For the particles of the free surface, the vertical Lagrangian coordinate b is equal to zero and the circumcenters are on the $Y_c = \operatorname{Im} G(a)$ line that is no longer horizontal, unlike the case of Gerstner waves. Thus, the particles of the free surface in the wave (10) are rotating around circumferences of the same radii, the centers of which are located at different levels. As a result, the wave has a variable profile. So it is natural to call the wave (10) an unsteady Gerstner wave.

The expression for pressure will be found by substituting formula (10) into (2), (3):

$$\frac{p - p_0}{\rho g} = \frac{\omega^2}{2g} A^2 e^{2kb} + \operatorname{Im}\left[A e^{i\omega t}\left(\frac{\omega^2}{g} \int G' e^{-ik\chi} d\chi - ie^{-ik\chi} \right) - G \right] \tag{12}$$

In the Gerstner wave $G(\chi) = \chi$, $\omega^2 = gk$, hence, the expression in the parentheses is zeroed, the pressure depends only on the vertical Lagrangian coordinate and is constant on the free surface, where $b = 0$. The situation is different for the unsteady Gerstner wave, where pressure on the free surface is variable; it includes both a steady component and an unsteady component varying with time following the harmonic law. We attribute this external pressure to the impact of wind.

The function G has no singularities in the flow region. Besides, for the Jacobian D_0 to have a positive sign (see Section 2), the condition

$$|G'| \ge kA \tag{13}$$

should be met. The equality sign means that the vorticity goes to infinity at free surface (see (11)). There is no derivative of velocity at such points, and the wave profile will contain cusps.

The constraint (13) provides a wide choice of the G function. To conclude, expressions (10), (12) give an exact solution for a large family of exact solutions for waves on water with variable external pressure on a free surface.

4. An example of exact solution

Let a variable external pressure act on a bounded section of a free surface. This requirement corresponds to the following asymptotic behavior of the function G:

$$G(\chi) \to \chi \quad if \quad \operatorname{Re}\chi \to \pm\infty.$$

On both infinities, the wave tends asymptotically to the Gerstner wave, where the pressure is constant. This allows us to assume that the dispersion relation $\omega^2 = gk$ is valid for the waves (10), (12), similarly to the Gerstner wave.

The function G is taken in the form

$$G(\chi) = \chi + \frac{\beta}{\chi - i\alpha}; \quad \alpha, \beta = const > 0, \ \operatorname{Im}\chi \le 0. \tag{14}$$

The parameters α, β have dimensions L and L^2, respectively. The function G has a pole at the point $\chi = i\alpha$ outside the flow region. The derivative G' vanishes to zero at the points $\chi_\pm = \pm\sqrt{\beta} + i\alpha$ that are also outside the flow region. From this follows that G is a bounded, single-valued function.

Consider the condition under which the Jacobian D_0 has a constant sign. The absolute value of the complex function G' reaches its maximum on a free boundary, with the square being equal to

$$|G'(a, b = 0)|^2 = 1 - \frac{2\beta}{a^2 + \alpha^2} + \frac{\beta^2 + 4\beta\alpha^2}{(a^2 + \alpha^2)^2}.$$

The right-hand side of the above expression has extremum at the points satisfying the condition $a_*^2 = \beta + 3\alpha^2$. It is less than unity and can be written as $4\alpha^2/(\beta + 4\alpha^2)$. As at the ends of the interval of variation of the a coordinate, the equalities

$$|G'(a = 0, b = 0)|^2 = \left(1 + \frac{\beta}{\alpha^2}\right)^2 > 1; \quad |G'(a = \infty, b = 0)|^2 = 1$$

hold true, the points with coordinates $\pm a_*$, $b = 0$ are the minimum points of the function G' and the condition (13) may be written in the form

$$\beta \le \frac{4(1 - k^2A^2)\alpha^2}{k^2A^2}$$

The equality sign corresponds to the limiting wave steepness

$$kA = \frac{2\alpha}{\sqrt{\beta + 4\alpha^2}}. \tag{15}$$

Unlike a stationary Gerstner wave this variable is always less than unity. From the relation (15) also follows that, on the intervals where the wave train is uniform, the wave profile will not have cusps.

The dynamics of a free surface over one period is shown in Fig. 1. Its profile $Y(X)$ is specified parametrically as

$$X = a + \frac{\beta a}{a^2 + \alpha^2} - A\sin(ka - \omega t); \quad Y = \frac{\beta\alpha}{a^2 + \alpha^2} + A\cos(ka - \omega t).$$

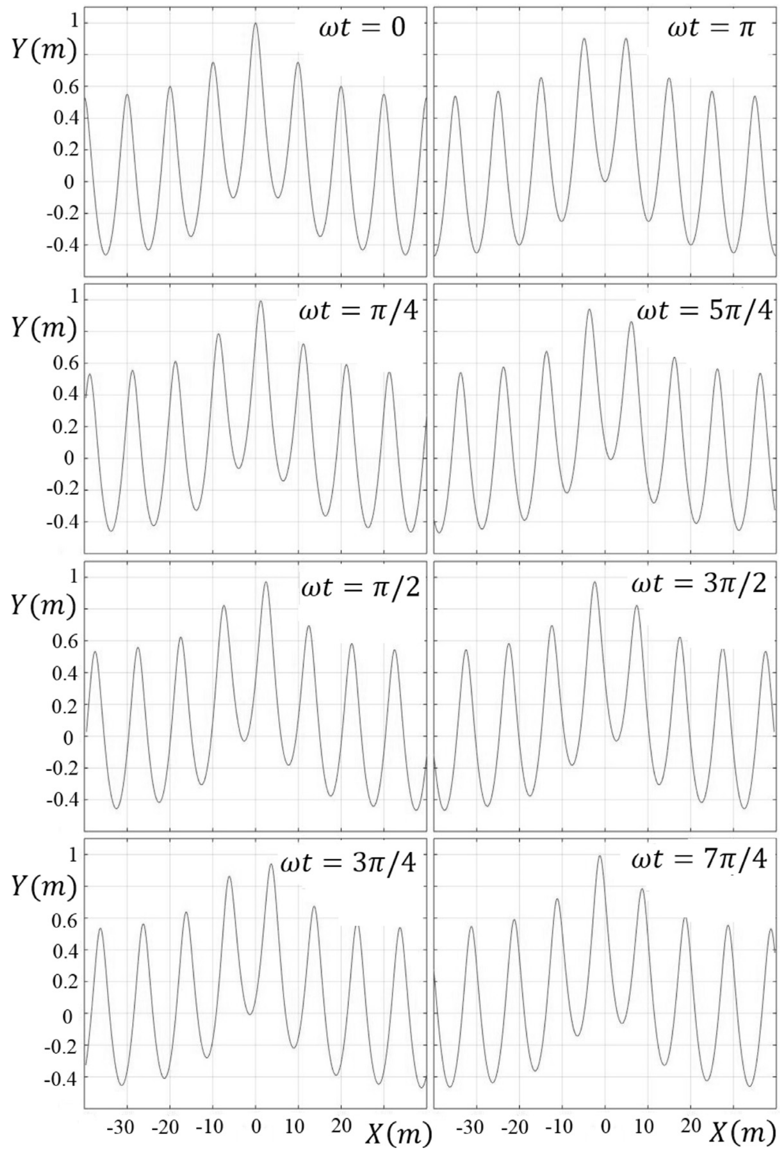

Fig. 1. Wave profile at different moments of time. The origin of the vertical coordinate Y is at the average level of a uniform wave (measured in meters). The $X = 0$ line corresponds to the symmetry axis of the wave at the initial moment of time.

We chose the following values of the parameters: $A = 0.5\ m$; $\lambda = 10\ m$; $\alpha = 10\ m$; $\beta = 5\ m^2$. The curves in Fig. 1 correspond to the change of the phase oscillations $\omega\ t$ by $\pi/4$. The wave profile has a form of a non-uniform wave packet, which is symmetric at the initial moment of time, with the vertical coordinate of the central maximum (at $X = 0$) being 1 m; it is twice the height of a uniform wave packet A. The width of the excited section of the packet, where the amplitudes of the maxima exceed this value, are of order 8λ. The changes occurring within the packet are well seen in the figure. The central maximum shifts to the right and slightly decreases with time. The amplitude of the maximum located behind the central one grows, and that in front decreases. In half a period the wave packet recovers its symmetric shape, but now there is a trough at $X = 0$. The neighboring maxima have the same height of about 0.9 m. Further, the trough starts to shift to the right and the maximum lying behind it begins to grow. At $\omega\ t = 2\pi$, the wave takes on its original shape.

In the region of a non-uniform wave packet, where the height of the maxima exceeds 0.5 m, the average fluid level lies above the $Y = 0$ plane. This is explained by the choice of the function G. The pressure here is different from the constant (atmospheric) pressure. It is convenient to express its relative value through a nondimensional value $\Delta\ p_*$, defined by

$$\Delta\ p_* = \frac{p - p_0}{\rho\, gA} = \frac{1}{2}kA - \frac{\beta\alpha}{A\left(a^2 + \alpha^2\right)} + k\beta \int_{-\infty}^{a} \left[\left(a^2 - \alpha^2\right) \sin\left(ka - \omega\, t\right) + 2a\alpha \cos\left(ka - \omega\, t\right)\right] \times \left(a^2 + \alpha^2\right)^{-2} da\,, \tag{16}$$

where $p_0 = p_a - \frac{1}{2}\rho gkA^2$, and p_a is atmospheric pressure. The dynamics of surface pressure variation in the region of a non-uniform wave packet is demonstrated in Fig. 2.

Its distribution has a form of a “pit” whose depth is varying with time. At the initial moment of time, $\Delta\ p_* = -1$ (in our example, $\rho gA = 0.05\ p_a$). In half a period the magnitude of the relative pressure grows up to -0.7. The values of the maxima $\Delta\ p_*$ at the ends of the non-uniform region increase simultaneously. The overall may be fitted for winds of different intensity. On the plus

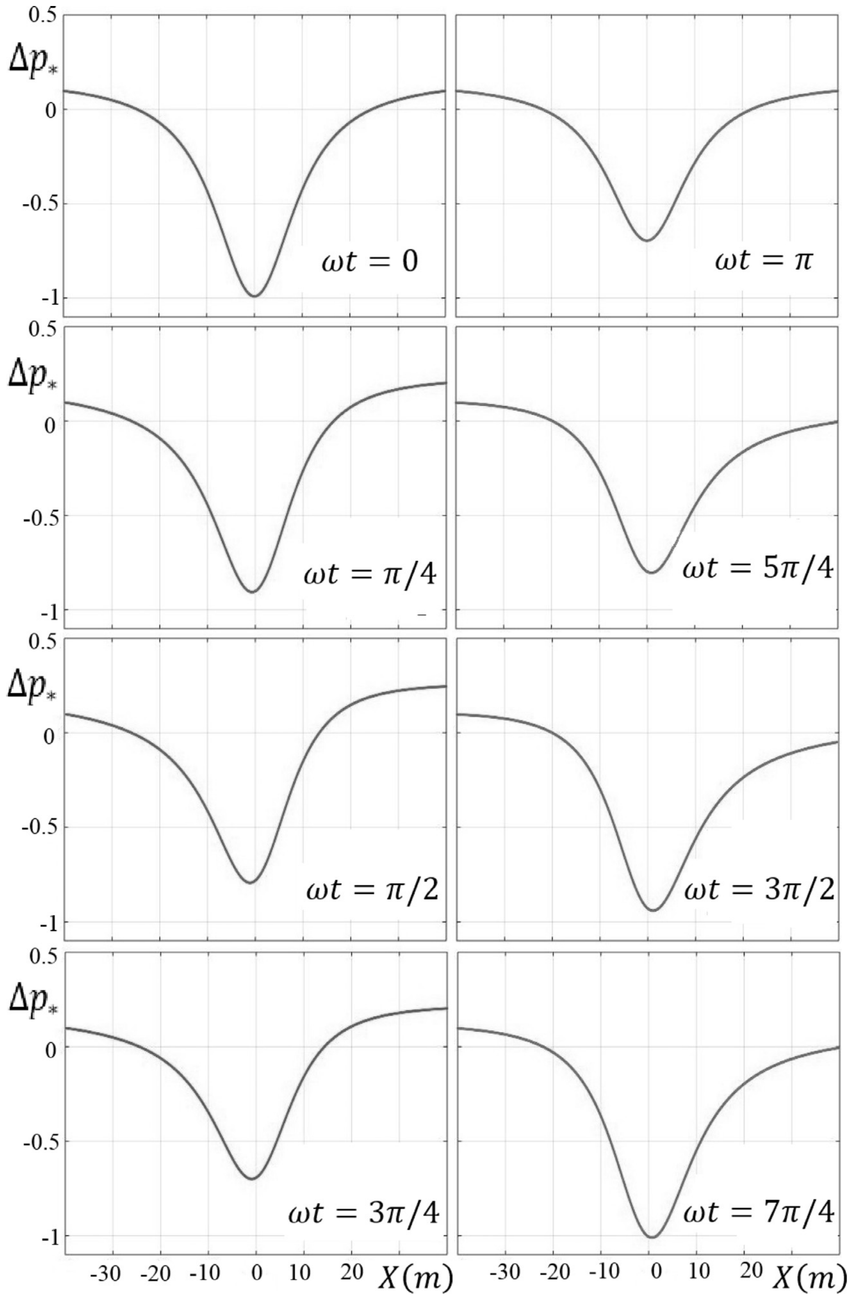

Fig. 2. Relative pressure in the region of a non-uniform wave packet at different moments of time.

and minus infinity the function Δp_* has a horizontal asymptote $\frac{1}{2}kA$.

The wave vorticity is written in the form

$$\Omega = 2\omega(kA)^2 \times \left[1 - \beta\frac{2a^2 - 2(b-\alpha)^2 + \beta}{\left(a^2 + b^2 + \alpha^2\right)^2} - (kA)^2 \exp(2kb)\right]^{-1} \exp(2kb).$$

Unlike a steady Gerstner wave, it depends on both Lagrangian coordinates. The properties of the function $|G'|^2$ (the first two terms in brackets) have been studied above for $b = 0$, which enables us to conclude that the vorticity has a minimum at $a = 0$, and a maximum at $\pm a_*$. In the limiting case (15), when the expression in the brackets vanishes to zero, there appear cusps on the free surface at the points corresponding to the Lagrangian coordinates $(\pm a_*, 0)$. They are of no particular interest in terms of physics, as they exist at all times.

5. Steady Gerstner wave generated by the running harmonic wave pressure

In the previous sections we assumed that the wind forces a bounded area of the free surface. Consequently, outside this interval the waves are regarded to be uniform and their frequency and wave number meet the dispersion relation for Gerstner waves. However, there may arise a situation when a variable external pressure acts along all the free surface. In this case, the wave packet parameters will be determined entirely by the form of pressure distribution.

We will restrict our consideration to the simplest case assuming that, under the action of wind, external pressure p_e in the form of a harmonic traveling wave

$$p_e = p_1 + p_2 \cos\,(ka - \omega t), \tag{17}$$

is maintained on the free surface (p_1, p_2 are constant). Such a boundary condition is satisfied by the classical Gerstner solution. Assuming $G(\chi) = \chi; \quad b = 0$ in Eq. (12) we obtain

$$\frac{p - p_0}{\rho} = \frac{\omega^2}{2}A^2 + \left(\omega^2 k^{-1} - g\right)A\,\cos\,(ka - \omega\,t). \tag{18}$$

Constant pressure on a free surface is a traditional boundary condition for steady Gerstner waves (7). From this follows that zeroing of the multiplier of the cosine in (18) yields a dispersion equation[2]. If, however, the pressure wave (17) is propagating along the free surface, it should be assumed that

$$p_1 = p_0 + \frac{1}{2}\rho\omega^2 A^2, \quad p_2 = \rho\left(\omega^2 k^{-1} - g\right)A. \tag{19}$$

If these conditions are met, we can say that the exact solution (7) corresponds to steady trochoidal waves on the liquid surface maintained by the external pressure (17). The wave amplitude A is found by the known ω and k from the second relation of the system, and p_0 is found from the first equality. The elevation of the free surface is defined by $Y = A\cos\,(ka - \mu\,t)$; hence, for positive values of p_2, the pressure changes in phase with the profile, and for negative values of p_2 in antiphase. The case $p_2 = 0$ corresponds to the Gerstner wave with constant pressure on the profile.

Despite their random nature, the narrow-band wind waves correspond relatively well to the most naive notion of waves as monochromatic formations. In this sense, the excitation of Gerstner waves by a traveling pressure wave seems quite reasonable. As distinct from the classical Gerstner solution, the dispersion equation is written as

$$\omega^2 = \left(g + \frac{p_2}{\rho A}\right)k. \tag{20}$$

Consequently, the phase velocity of Gerstner wind waves changes too. It will be more than $\sqrt{g/k}$ for positive p_2, and less for negative p_2. The additional term in the dispersion Eq. (20) is inversely proportional to amplitude and for small A formula (20) will, evidently, be invalid. Therefore, the assumption that p_2 does not depend on wave amplitude in the boundary condition (17) is incorrect, generally speaking. However, analysis of the $p_2(A)$ dependence is beyond the scope of this study.

Difficulties in interpreting solutions may be avoided by specifying the function G rather than pressure on the profile. For example, for periodic waves it may be represented in a general form as

$$G(\chi) = \chi + \sum_n \gamma_n e^{-ikn\chi},\ n \geq 1,$$

where γ_n are constant values, the sign in the exponent is chosen so that, for $b \to -\infty$, all the terms of the exponent tend to zero. The expression ω^2/g enters the pressure distribution (12) as a parameter having dimension of a half wave, and there is no need to consider it to be equal to the wave number k.

6. Conclusions

The class of exact solutions of two-dimensional hydrodynamics describing vortex gravity surface waves in the presence of an external time-harmonic pressure varying along the coordinate has been found and analyzed. The particles of the free surface in the wave rotate around circumferences of the same radii, like in an ordinary Gerstner wave. Therefore, it is suggested to call this class of waves unsteady Gerstner waves. The pressure distribution on the free surface includes a quite arbitrary analytical function of the complex Lagrangian $G(\chi)$ free for choice. A specific example of variable pressure acting on a limited section of the free surface has been studied. The function $G(\chi)$ has been taken in the form of a simple rational function allowing analytical investigation of the main properties of the flow. Pressure distribution on the free surface depends on the choice of the $G(\chi)$ function. In that way, the obtained solution may meet a wide class of boundary conditions for pressure. This indicates a high probability of such waves in the real conditions.

The easiest way to observe unsteady Gerstner waves is to produce in laboratory conditions a wind flow above traveling surface waves. A direct proof of their existence is absence of a drift flow, like in the experiments by Monismith et al. [34]. It can be assumed that the wind will play the role of a wavemaker generating a shear flow that locally cancels the mass transport associated with the Stokes drift. Using a higher-order nonlinear Schrödinger equation, Curtis et al. [48] showed theoretically that background currents have a significant impact on the mean transport properties of waves. In particular, certain combinations of background shear and carrier wave frequency lead to the disappearance of mean surface mass transport. In the defocusing case, these authors confirmed numerically the theoretical predictions made for the balance between vorticity and carrier wavenumber which is expected to quench the mean-surface drift, thereby providing a possible explanatory mechanism for the results in Monismith et al. [34].

We have studied the dynamics of an unsteady wave packet against the background of a uniform steady Gerstner wave with steepness $kA = 0.314$. These are rather steep waves. However, unsteady Gerstner waves are expected to arise in the real ocean at a small steepness of a uniform background ($kA < < 1$). In this case, background Gerstner waves are identical to linear potential waves, so there is no need to explain their excitation. Whereas in the nonuniform region, the wind generates unsteady Gerstner waves. Such a scenario for the formation of unsteady Gerstner waves seems to be the most probable in the ocean.

In 2006 Smith reported striking field observations of mean currents under wave groups measured in the open ocean [49]: as wave groups pass, Eulerian counterflows occur that cancel the Stokes drift variations at this surface. The mechanism by which these counterflows are generated is not well understood. But if only the wavetrains propagate in the presence of the wind, then we can assume that Smith observed the unsteady Gerstner waves.

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