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On construction of axiom A 3-diffeomorphism with one-dimensional surface attractor-repeller dynamics[1]

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Abstracts. We suggest a method of a construction of axiom A 3-dieomorphisms whose non-wandering set consists of exactly one-dimensional surface attractor and one-dimensional surface repeller. Unlike from examples constructed in [1] and [3], our diffeomorphisms are not structurally stable, however suggested method gives rather simple construction of new types of 3-manifolds, admitting "hyperbolic sink-hyperbolic source" dynamics.

Keywords: A-diffeomorphism, surface basic set

1. Introduction

Let M be a closed n-manifold and $f : M \to M$ be an axiom A diffeomorphism. By Smale's spectral theorem the non-wandering set of f consists of finite number f-invariant closed subsets, named *basic sets*. For a basic set Λ a pair (a, b), where $a = \dim W^u_\Lambda, b = \dim W^s_\Lambda$ is called *type of the basic set.*

A basic set A of the diffeomorphism f is called *attractor* if it has a *trapping region*, that is a compact neighborhood $U_A \subset M$ such that $f(U_A) \subset int\, U_A$ and $\bigcap\limits_{j\in\mathbb{N}} f^j(U_A) = A$. A basic set R is called a *repeller* if it is an attractor for f^{-1}.

A hyperbolic attractor of diffeomorphism f is called *surface* if there exists a compact surface Σ that $A \subset \Sigma$, $f(\Sigma) \subset \Sigma$. A surface repeller is defined as surface attractor for f^{-1}.

There is a natural question: does some manifold admit an A-diffeomorphism with exactly two basic sets of the same dimension and attractor-repeller dynamics? Such a diffeomorphism is automatically Ω-stable. The simplest example of such a system is a Morse-Smale diffeomorphism on n-sphere ($n \geq 1$) whose non-wandering set consists of exactly one sink and one source. These examples are structurally stable and exhaust all possible diffeomorphisms with zero-dimensional attractor and repeller.

For an attractor and repeller with the dimension $2k$ ($k \in \mathbb{N}$) also not so difficult to realize a diffeomorphism on n-manifold ($n > 2k$) as a direct product of Anosov diffeomorphism on $2k$-torus with the type (k, k) by sink-source diffeomorphism on $(n-2k)$-sphere. Moreover, for 3-manifolds by V. Grines, Yu. Levchenko, V. Medvedev,

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O. Pochinka [4] proved that two-dimensional attractor-repeller 3-diffeomorphisms there are only on mapping torus and obtained complete topological classification of such rough systems.

Particular case is a surfaces diffeomorphism with one-dimensional attractor and repeller. Such dynamics is achieved, for example, by taking of the connected sum of two DA-models on 2-tori. However R. Robinson, R. Williams [5] proved that among such diffeomorphisms there are no structural stable one.

B. Jiang, Y. Ni and S. Wang [2] proved that a 3-manifold M admits an axiom A diffeomorphism f whose non-wandering set consists of solenoid attractors and repellers if and only if M is a lens space $L(p, q)$ with $p \neq 0$. They also shown that such f are not structural stable. C. Wang and Y. Zhang [6] got infinitely many genus two 3-manifolds, each admits a diffeomorphism whose non-wandering set consists of two Williams solenoids, one attractor and one repeller. These manifolds contain half of Prism manifolds, Poincare's homology 3-sphere and many other Seifert manifolds, all integer Dehn surgeries on the figure eight knot, also many connected sums.

On the other hand, due to Ch. Bonatti, N. Guelman [1], Shi Yi [3], there are examples of rough 3-diffeomorphisms with one-dimensional attracor and repeller. But all examples have very complicated descriptions.

In this paper we suggest a method of a construction of an axiom A 3-diffeomorphisms whose non-wandering set consists of exactly one-dimensional surface attractor and one-dimensional surface repeller. All known examples were constructed in [1] and [3]. Constructed in this paper diffeomorphisms are not structurally stable, however suggested method gives rather simple construction of new 3-manifolds, admitting "hyperbolic sink-hyperbolic source" dynamics, different from the manifolds constructed in [1] and [3].

2. Construction

The diffeomorphism will be constructed step by step in this section as following:

- take an Anosov diffeomorphism of a 2-torus $\mathbb{T}^2$;
- make a Smale "surgery operation" to obtain the system with one fixed source and one-dimensional attractor on the torus;
- multiply $\mathbb{T}^2$ by $\mathbb{R}$ with contraction to 0, hence one-dimensional surface attractor on $\mathbb{T}^2 \times \mathbb{R}$ will be obtained;
- construct a fundamental domain of the attractor;
- take an analogical sample with one-dimensional surface repeller;
- "glue" the fundamental domains of the diffeomorphisms in the basins of attractor and repeller in accordance with dynamics.

2.1. Anosov diffeomorphism of a 2-torus

Let $C \in GL(2,\mathbb{Z})$ be a hyperbolic matrix with the eigenvalues λ_1, λ_2 so that $\lambda = |\lambda_1| > 1$ and $|\lambda_2| = 1/\lambda$. As the matrix C has the determinant equals 1 then it generates the hyperbolic automorphism $\widehat{C} : \mathbb{T}^2 \to \mathbb{T}^2$ with the fixed point O. This automorphism is Anosov diffeomorphism, so there are two transversal foliations (stable and unstable) which are dense on the 2-torus, and a set of periodical points is also dense.

2.2. Smale "surgery operation"

Let $U(O) \subset \mathbb{T}^2$ be a some neighbourhood of the fixed point O of the diffeomorphism $\widehat{C}$ and x, y be local coordinates such that the diffeomorphism $\widehat{C}$ in these coordinates has a form

$$\widehat{C}(x,y) = (x/\lambda, \lambda y)\,.$$

Then $Ox \subset W^s_O$ and $Oy \subset W^u_O$, also $\{y = const\}$ and $\{x = const\}$ are stable and unstable foliations. A diffeomorphism $\widehat{B} : \mathbb{T}^2 \to \mathbb{T}^2$ with properties described below will be constructed in this section:

- $\widehat{B}$ will be identity on $\mathbb{T}^2 \setminus U(O)$;
- $\widehat{B}$ will keep unstable manifolds of $\widehat{C}$ everywhere;
- $\widehat{B}$ will add additional expansion along stable manifolds of $\widehat{C}$ inside some neighbourhood of O;
- the composition $\widehat{\Psi} = \widehat{B} \circ \widehat{C}$ is DA-diffeomorphism with a fixed source O and one-dimensional attractor A.

Let $\nu : [0,1] \to [0,1]$ be a diffeomorphism defined by the graph on Fig. 1. Then $\nu_t(x) = (1-t)x + t\nu(x)$, $t \in [0,1]$ is an isotopy between the identity map $\nu_0(x)$ on $[0,1]$ and $\nu_1(x) = \nu(x)$. Let $\sigma(x,a,b) : \mathbb{R} \to \mathbb{R}$ be a sigmoid function of the form

$$\sigma(x,a,b) = \begin{cases} \frac{1}{1+\exp\left(\frac{(a+b)/2-x}{(x-a)^2(x-b)^2}\right)}, & a < x < b, \\ 0, & otherwise. \end{cases}$$

It monotonically sends $[a,b]$ to $[0,1]$.

For every $t \in [0,1]$ let us define a diffeomorphism $B_t : [0,1]^2 \to [0,1]^2$ which possess the symmetries with respect to both axis Ox, Oy and in the first quadrant given by the formula

$$B_t(x,y) = \begin{cases} (\nu_t(x), y)\,, & 0 \leqslant y < \frac{1}{2\lambda^3}, \\ \big(\sigma(y, \frac{1}{2\lambda^3}, \frac{1}{2})x + (1-\sigma(y, \frac{1}{2\lambda^3}, \frac{1}{2}))\nu_t(x), y\big)\,, & \frac{1}{2\lambda^3} \leqslant y < \frac{1}{2}, \\ (x,y), & \frac{1}{2} \leqslant y \leqslant 1. \end{cases}$$

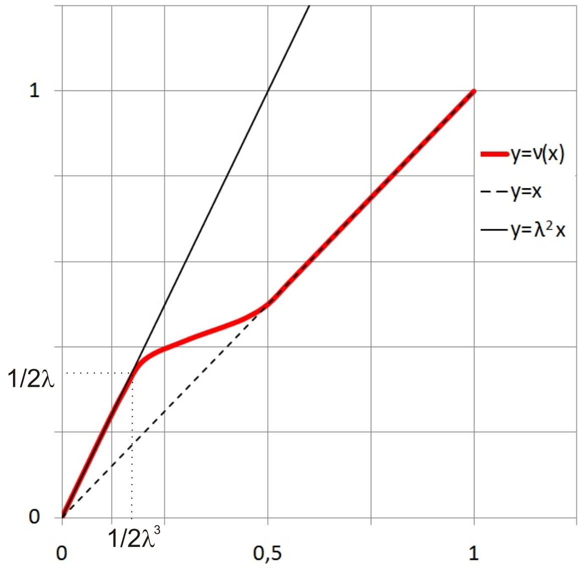

Рис. 1. Graph of function $\nu(x)$

By the construction B_t is an isotopy between the identity map $B_0(x, y)$ on $[0, 1]^2$ and $B_1(x, y) = B(x, y)$. As B preserves y-coordinate, we will use the following designation: $B(x, y) = (\gamma(x, y), y)$. Let $D = \{(x, y) \in U(O) : x^2 + y^2 \leqslant \frac{1}{(2\lambda^3)^2}\}$. By the construction B is identity out of $[0, 1/2]^2$ and for points $(x, y) \in D$ we have $\gamma(x, y) = \lambda^2 x$.

Let $\widehat{B}_t$ be a B_t inside $[0, 1]^2$ and is identity out of it. By arguments like to [7] it is possible to prove that $\widehat{\Psi} = \widehat{B} \circ \widehat{C}$ is a DA-diffeomorphism, whose non-wandering set consists of a one-dimensional attractor and a source.

2.3. One-dimensional surface attractor of $\mathbb{T}^2 \times \mathbb{R}$

Consider a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ by the formula $\varphi(z) = \frac{z}{\lambda}$. Define a diffeomorphism of $\mathbb{T}^2 \times \mathbb{R}$ in coordinates $w \in \mathbb{T}^2, z \in \mathbb{R}$ by the formula

$$\Phi(w, z) = (\widehat{\Psi}(w), \varphi(z)).$$

The diffeomorphism Φ is an A-diffeomorphism whose non-wandering set contains one saddle point $\{O\} \times \{0\}$ and an one-dimensional attractor $\mathcal{A}$ which is placed on a 2-torus $\mathbb{T}^2 \times \{0\}$.

2.4. Fundamental domain of the attractor

To find a fundamental domain of the basin of the attractor $\mathcal{A}$, first of all, notice that $\mathbb{T}^2 \times (-\frac{1}{2\lambda^2}, \frac{1}{2\lambda^2})$ is a trapping neighbourhood of an attractor $\mathbb{T}^2 \times \{0\}$. It is not a trapping neighbourhood for $\mathcal{A}$ because there are a saddle point O and a segment of its stable separatrix $\{O\} \times (-\frac{1}{2\lambda^2}, \frac{1}{2\lambda^2})$ inside. A small neighbourhood of the separatrix will be chosen to remove them.

So $U_1^{\mathcal{A}} = (\mathbb{T}^2 \setminus \widehat{\Psi}^{-1}(D)) \times \left(-\frac{1}{2\lambda^2}, \frac{1}{2\lambda^2}\right)$ is a desired trapping neighbourhood for the attractor $\mathcal{A}$. Indeed, $\Phi(U_1^{\mathcal{A}}) = (\mathbb{T}^2 \setminus D) \times \left(-\frac{1}{2\lambda^3}, \frac{1}{2\lambda^3}\right) = U_2^{\mathcal{A}} \subset int\, U_1^{\mathcal{A}}$ and $\bigcap\limits_{n=1}^{\infty} \Phi^n(U_1^{\mathcal{A}}) = \mathcal{A}$.

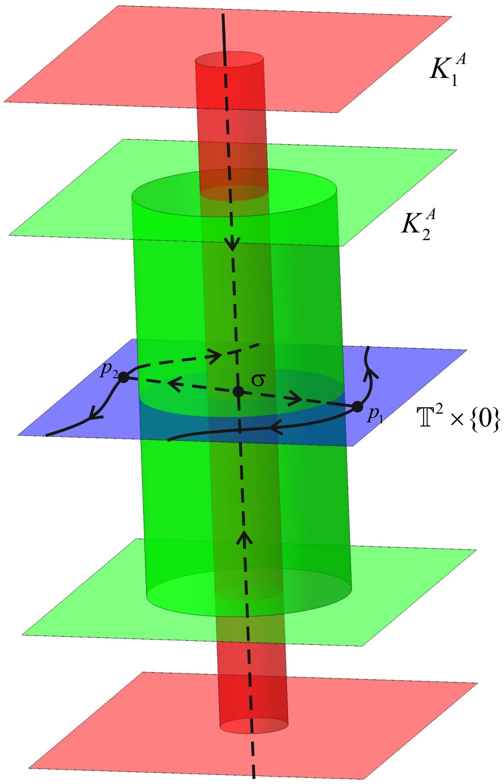

Рис. 2. Fundamental domain of an one-dimensional surface attractor

After that a fundamental domain of a basin of the attractor $\mathcal{A}$ is $K^{\mathcal{A}} = cl\,(U_1^{\mathcal{A}} \setminus U_2^{\mathcal{A}})$ (see Fig. 2). Notice, that $\partial U_1^{\mathcal{A}}$ is a pretzel and $K^{\mathcal{A}}$ is homeomorphic to the direct product of a pretzel by a segment. We will use the following designation: $K_1^{\mathcal{A}} = \partial U_1^{\mathcal{A}}$, $K_2^{\mathcal{A}} = \partial U_2^{\mathcal{A}}$. So $\Phi(K_1^{\mathcal{A}}) = K_2^{\mathcal{A}}$.

2.5. One-dimensional surface repeller of $\mathbb{T}^2 \times \mathbb{R}$

Define a diffeomorphism $\theta : [0,1]^2 \to [0,1]^2$ by the formula $\theta(x,y) = (-y,x)$. For every $t \in [0,1]$ let us define a diffeomorphism $Q_t : [0,1]^2 \to [0,1]^2$ by the formula $Q_t = \theta^{-1} B_t^{-1} \theta$. By the construction Q_t is an isotopy between the identity map $Q_0(x,y)$ on $[0,1]^2$ and $Q_1(x,y) = Q(x,y)$. Let $\widehat{Q}_t$ be a Q_t inside $[0,1]^2$ and is identity out of it. So $\widehat{Q}_t$ keeps stable foliation of $\widehat{C}$ and add additional contraction along unstable manifolds of $\widehat{C}$ inside some neighbourhood of O. Thus $\widehat{\Psi}_- = \widehat{Q} \circ \widehat{C}$ is a DA-diffeomorphism, whose non-wandering set consists of a one-dimensional repeller and a sink.

Consider a copy of $\mathbb{T}^2 \times \mathbb{R}$ with the following diffeomorphisms of it

$$\Phi_-(w, z) = (\widehat{\Psi}_-(w), \varphi^{-1}(z)).$$

The diffeomorphism Φ_- is an A-diffeomorphism whose non-wandering set contains one saddle point $\{O\} \times \{0\}$ and an one-dimensional repeller $\mathcal{R}$ which is placed on a 2-torus $\mathbb{T}^2 \times \{0\}$. Trapping neighbourhoods $U_1^{\mathcal{R}}$, $U_2^{\mathcal{R}}$ of $\mathcal{R}$ and a fundamental domain $K^{\mathcal{R}}$ of a basin of the repeller $\mathcal{R}$ are the same as for diffeomorphism Φ so $\Phi_-(U_2^{\mathcal{R}}) = U_1^{\mathcal{R}}$, $K_1^{\mathcal{R}} = \partial U_1^{\mathcal{R}}$, $K_2^{\mathcal{R}} = \partial U_2^{\mathcal{R}}$, and $\Phi_-(K_2^{\mathcal{R}}) = K_1^{\mathcal{R}}$.

2.6. Gluing of the fundamental domains

The goal of this section is to construct a diffeomorphism $H : K^{\mathcal{R}} \to K^{\mathcal{A}}$ with the properties:

- $H(K_2^{\mathcal{R}}) = H(K_1^{\mathcal{A}})$ and $H(K_1^{\mathcal{R}}) = H(K_2^{\mathcal{A}})$;
- $H = \Phi$ on $K_1^{\mathcal{R}}$ and $H = \Phi_-$ on $K_2^{\mathcal{R}}$.

For every $t \in [0, 1]$ let $\widehat{\xi}_t = \widehat{B}_t \circ \widehat{Q}_t^{-1} : \mathbb{T}^2 \to \mathbb{T}^2$. Notice that $\widehat{\xi}_1 = \widehat{\Psi} \circ \widehat{\Psi}_-^{-1}$. Then $\widehat{\xi}_t \circ \widehat{\Psi}_-$ is an isotopy between $\widehat{\Psi}_-$ and $\widehat{\Psi}$. By the construction $\widehat{\xi}_t \circ \widehat{\Psi}_-$ has a form $\widehat{\xi}_t \circ \widehat{\Psi}_-(x, y) = (k(t)x, k(t)y)$ for $(x, y) \in \widehat{\Psi}(D)$, where

$$k(t) = \left(\lambda - \frac{1}{\lambda}\right) t + \frac{1}{\lambda}.$$

Finally let $r, q : [0, 1] \to \left[\frac{1}{2\lambda^3}, \frac{1}{2\lambda^2}\right]$ be functions given by the formulas

$$r(t) = \frac{\lambda - 1}{2\lambda^3} t + \frac{1}{2\lambda^3}, \;\; q(t) = k(t) \cdot r(t).$$

Consider the fundamental domain $K^{\mathcal{R}}$ of the basin of the repeller $\mathcal{R}$ (all reasoning for the attractor $\mathcal{A}$ are analogical). Let $D_{r(t)} = \{(x, y) \in U(O) : x^2 + y^2 \leqslant r^2(t)\}$. Then $K^{\mathcal{R}}$ can be represented foliated by leaves

$$\{G_{r(t)} = G \times r(t), \, t \in [0, 1]\}$$

such that $G_{r(0)} = K_2^{\mathcal{R}}$, $G_{r(1)} = K_1^{\mathcal{R}}$ and $G_{r(t)}$ coincides with the tori $\mathbb{T}^2 \times \{\pm r(t)\}$ out of $D_{r(t)} \times \mathbb{R}$ and coincides with the cylinder $\partial D_{r(t)} \times [-r(t), r(t)]$ otherwise (see Fig. 3).

Define a map $H_t : G_{r(t)} \to G_{q(t)}$, $t \in [0, 1]$ as follows

$$H_t(w, z) = \left(\widehat{\xi}_t(\widehat{\Psi}_-(w)), \frac{q(t)}{r(t)} z\right).$$

Thus the diffeomorphism $H : K^{\mathcal{R}} \to K^{\mathcal{A}}$, composed by $H_t, t \in [0, 1]$, glues the dynamics of Φ_- and Φ along the fundamental domains. After a smoothing the corners

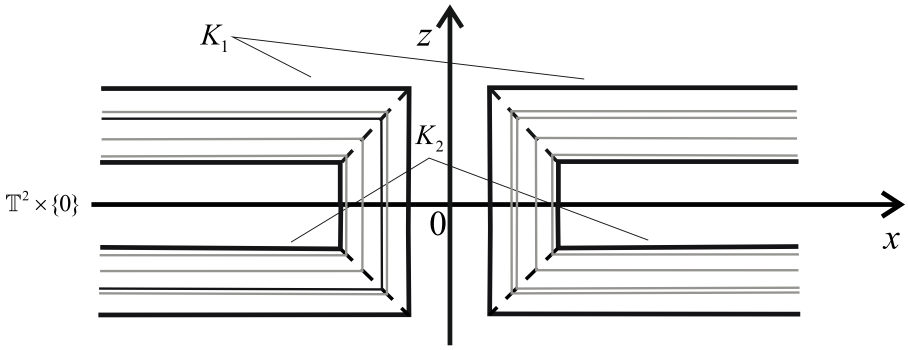

Рис. 3. Foliation of the fundamental domain

we get a new 3-manifold M with the desired A-diffeomorphism whose non-wandering set consists of one-dimensional surface repeller and one-dimensional surface attractor.

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