ISSN 0965-5425, Computational Mathematics and Mathematical Physics, 2018, Vol. 58, No. 12, pp. 1937–1947. © Pleiades Publishing, Ltd., 2018.
Original Russian Text © S.V. Kurochkin, 2018, published in Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki, 2018, Vol. 58, No. 12, pp. 00000–00000.

Existence Conditions of Negative Eigenvalues in the Regular Sturm–Liouville Boundary Value Problem and Explicit Expressions for Their Number

S. V. Kurochkin

Dorodnicyn Computing Center, Federal Research Center "Computer Science and Control," Russian Academy of Sciences, Moscow, 119333 Russia

e-mail: kuroch@ccas.ru

Received May 5, 2018

Abstract—For the regular Sturm–Liouville boundary value problem with general nonseparated self-adjoint boundary conditions, conditions for the existence of zero and negative eigenvalues and expressions for their number are obtained. The conditions are expresses in a closed form, and the coefficient functions of the original equation appear in these conditions indirectly through a single numerical characteristic.

Keywords: Sturm–Liouville equation, boundary value problem, eigenvalue.

DOI: 10.1134/S096554251812014X

1. INTRODUCTION

A string or a rod with free boundary conditions provide an example of vibrating systems that have a zero eigenvalue. The corresponding eigenform is a uniform motion of the system as a rigid body. It was mentioned in [1, Chapter I, §2.4] that, in the case when the directions of forces acting at the endpoints according to the boundary conditions are the same as the direction of motion of the corresponding endpoint, the system can have negative eigenvalues. In various physical statements, the problem of estimating the number of negative eigenvalues is considered in many studies, especially for the Schrödinger equation (see [2] and the papers citing it). The purpose of the current paper is to derive explicit conditions in terms of the potential energy (elastic characteristics) of the system and boundary conditions that enable one to detect the existence and find out the number of negative and zero eigenvalues in the problem indicated in the title of the paper. The closest paper to this one is [3], in which for the Sturm–Liouville problem with separated boundary conditions but with more general coefficients of the equation, relations between the number of negative eigenvalues and other indexes of the Sturm–Liouville operator were obtained.

The paper is organized as follows. In section 2, the specific case of separated boundary conditions is considered. This case is analyzed in a special section for a number of reasons. Firstly, the logic of the analysis is clearly seen already in this case. This logic can be described by the following sequence of steps:

(1) if the spectral parameter λ is zero, all equations—the original one and the equations of the transfer method that transfer the boundary conditions—are solved explicitly;

(2) the limiting position (as $\lambda \to -\infty$) of the subspace corresponding to the boundary condition transferred to the other end of the interval can be found;

(3) the transfer result monotonically depends on λ: in the case of separated boundary conditions, the dependence is interpreted in the direct sense of the word as the angle in the Prüfer transformation; in the more general case, this dependence is interpreted in a more special sense;

(4) the problem solution is reduced to counting signatures.

Secondly, the case of separated boundary conditions does not use sophisticated techniques—in essence, only the Prüfer transformation is used. Thirdly, even for this classical case, it seems that some of the results obtained in Section 2 are new, and they have a meaningful physical interpretation.

Section 3 contains the main results of the paper. First, a summary of necessary results concerning Grassmann varieties formed by linear Lagrangian subspaces and representation of self-adjoint boundary problems for (systems) of ordinary differential equations in standard Hamiltonian form is given. Next, the propositions of Section 2 are obtained for the general case. From the very beginning and in the course of the presentation, a physical interpretation of results is given. In the last section, conclusions are drawn, and additional issues are discussed.

2. THE CASE OF SEPARATED BOUNDARY CONDITIONS

2.1. Statement of the Problem

We consider the equation

$$(p(x)y'(x))' + \lambda\rho(x)y(x) = 0. \tag{1}$$

Here $x \in [0,l]$, $p \in C^1[0,l]$, $\rho \in C[0,l]$, and

$$\min_{x\in[0,l]} \frac{1}{p(x)} > 0, \qquad \min_{x\in[0,l]} \rho(x) > 0.$$

The boundary conditions have the form

$$p(0)y'(0)\sin\alpha - y(0)\cos\alpha = 0, \tag{2a}$$

$$p(l)y'(l)\sin\beta - y(l)\cos\beta = 0. \tag{2b}$$

The anglesα and β in (2) may be chosen up to a multiple of π. To investigate the vibratory properties of the solutions to Eq. (1), it is convenient to use the ranges $\alpha \in [0,\pi)$ and $\beta \in (0,\pi]$; to separate the positive and nonpositive eigenvalues, it is more convenient to use the ranges $\alpha \in [-\pi/2,\pi/2]$ and $\beta \in [-\pi/2,\pi/2]$. Below, we use both ranges, depending on the situation.

Equation (1) describes, in particular, the following problems (see [4]):

(1) small transverse vibrations of a string (in this case, $p = \text{const}$);

(2) small longitudinal vibrations of a rod;

(3) small torsional vibrations of a rod.

The physical interpretation can be given for each of these problems; below, we will use problem (2). In this case, the physical meaning of the quantities appearing in (1), (2), and some related quantities is as follows:

$y(x)$ is the displacement of the point x of the rod,

$p(x)$ is the modulus of elasticity multiplied by the area of the rod cross section,

$\rho(x)$ is the linear density of the rod,

$p(x)y'(x)$ is the strain (according to Hooke's law);

$$U_{int} = \frac{1}{2}\int_0^l p(\xi)(y'(\xi))^2 d\xi$$

is the potential energy of stressed state of the rod.

To simplify the notation and interpretation of the boundary conditions, we will first consider the case of "general position" in which $\alpha, \beta \neq 0$, and $\pi/2(\mathrm{mod}\pi)$ in (2). The other cases (fixed and free ends) can be easily analyzed individually, and they will be taken into account in the final classification.

Condition (2a) corresponds to the elastic attachment of the rod left end, in which case its displacement $y(0)$ causes the constraint to react with the force $p(0)y'(0) = \cot\alpha y(0)$. For $0 < \alpha < \pi/2$, the force acts in the direction opposite to the displacement (below, such a constraint will be called stable); in the case $-\pi/2 < \alpha < 0$ the force acts in the same direction as the displacement (unstable constraint). At the right end of the rod, the sign must be reversed —$0 < \beta < \pi/2$ corresponds to the unstable constraint and $-\pi/2 < \beta < 0$ corresponds to the stable constraint.

The potential energy of constraints (the work done by the constraints when the rod ends are displaced to the positions $y(0)$ and $y(l)$) is

$$U_{0,l} = \frac{1}{2}(\cot\alpha(y(0))^2 - \cot\beta(y(l))^2).$$

Introduce the notation

$$H = \int_0^l \frac{1}{p(\xi)} d\xi. \tag{3}$$

The physical meaning of H is the displacement of the free rod end under the action of the unit force applied to it when the other end is fixed; this is actually the quantity reciprocal of the rod rigidity (compliance in the terminology of [5]).

It is known that problem (1), (2) can have none, one, or two negative or zero eigenvalues (e.g., see [6], Section 4.3).

2.2. Conditions of existence and Expressions for the Number of Zero and Negative Eigenvalues

Below, we will use the Prüfer transformation and the corresponding equation of differential transfer

$$\varphi(x) = \operatorname{Arctan} \frac{y(x)}{p(x)y'(x)},$$

$$\varphi'(x) = \frac{1}{p(x)} \cos^2 \varphi + \lambda \rho(x) \sin^2 \varphi, \quad x \in [0, l]. \tag{4}$$

Let $\varphi(l, \lambda)$ be the result of the transfer of Condition (2a) to the point $x = l$ according to Eq. (4). It is known (e.g., see [7], Chapter 11, Theorem 4.1) that, if $0 \le \alpha < \pi$, then

(1) $\varphi(l, \lambda)$ monotonically increases in λ;

(2) $\varphi(l, \lambda) \to +0$ as $\lambda \to -\infty$.

It is clear that solution (4) always increases when crossing the point $k\pi$, $k = 0, \pm 1, \pm 2, \ldots$.

For $\lambda = 0$, the point $\varphi = \pi/2$ remains fixed relative to Eq. (4). In this case, the mapping $\alpha \to \varphi(l, 0)$ takes the interval $-\pi/2 < \alpha < \pi/2$ into itself, and it is strictly monotone on this interval.

Lemma 1. *Let* $\lambda = 0$ *and let* $\tilde{y}(x)$ *be the solution to Eq. (1) satisfying the conditions* $p(0)\tilde{y}'(0) = \cos\alpha$ *and* $y(0) = \sin\alpha$. *Then,* $p(x)\tilde{y}'(x) \equiv \cos\alpha$,

$$\tilde{y}(l) = \sin\alpha + H\cos\alpha,$$

$$\tan\varphi(l, 0) = \frac{\tilde{y}(l)}{p(l)\tilde{y}'(l)} = \tan\alpha + H.$$

Given this and the properties of the function $\varphi(l, \lambda)$ of λ, conclusions about the existence and number of zero and negative eigenvalues in problem (1), (2) can be immediately drawn.

Proposition 1. *Depending on mutual arrangement of the angles* α *and* β, *the following propositions hold.*

1. $0 \le \alpha < \pi/2$, $-\pi/2 < \beta \le 0$, *i.e., both constraints are stable or an end is fixed. In addition, one (but not both) inequality may be nonstrict (one free end). Then, problem* (1), (2) *has no zero and negative eigenvalues.*

2. $-\pi/2 \le \alpha < 0$, $-\pi/2 \le \beta \le 0$, *i.e., the constraint at the left end is unstable or this end is free, and the constraint at the right end is stable or this end free or it is fixed. Then, if* $\tan\alpha + H > \tan\beta$, *then there is one negative eigenvalue; if* $\tan\alpha + H = \tan\beta$, *then there is a single zero eigenvalue; and if* $\tan\alpha + H < \tan\beta$, *then there are no zero and negative eigenvalues. In this case, we assume that* $\tan(-\pi/2) = -\infty$ *for the free end, and* $-\infty + H = -\infty$.

3. $0 \le \alpha \le \pi/2$, $0 < \beta \le \pi/2$, *i.e., the constraint at the left end is stable or this end is free or it is fixed, and the constraint at the right end is unstable or this end free (case* 3 *is symmetric to case* 2*). If* $\tan\alpha + H > \tan\beta$, *then there is one negative eigenvalue; if* $\tan\alpha + H = \tan\beta$, *then there is a single zero eigenvalue; and if,* $\tan\alpha + H < \tan\beta$, *then there are no zero and negative eigenvalues. In this case, we assume that* $\tan(\pi/2) = +\infty$ *for the free end, and* $+\infty + H = +\infty$.

4. $-\pi/2 < \alpha < 0$, $0 < \beta < \pi/2$, i.e., *both constraints are unstable. Then, if* $\tan\alpha + H > \tan\beta$, *then there are two negative eigenvalues; if* $\tan\alpha + H = \tan\beta$, *then there is one zero and one negative eigenvalue; and if* $\tan\alpha + H < \tan\beta$, *then there is a single negative eigenvalue.*

Remark. In one way or another, these conditions must be known; however, in the literature (e.g., see [8, Section 5.8]), only a part of all possible cases is considered.

The analysis of conditions in these cases shows that, for instance, the existence of at least one negative eigenvalue is not reduced to a single inequality criterion.

Case 2 (or 3) deserves a special remark. If (for α and β in the interval indicated above) $\alpha < \beta$, then the absolute value of the negative rigidity of the unstable constraint at the left end is less than the positive rigidity at the right end; i.e., the destabilizing influence is weaker that the stabilizing influence. However, if the rod is soft (H is large) in this case, then a negative eigenvalue does nevertheless exist.

2.3. Energy Interpretation

Suppose that the rod is put in the position $y(x)$ by the action of external forces (distributed along the rod length or applied at its ends) (clearly, $y(x)$ does not necessarily satisfy the boundary conditions (2)). Then, the total potential energy of the rod and the constraints is

$$U(y) = U_{int} + U_{0,l} = \frac{1}{2}\int_0^l p(\xi)(y'(\xi))^2 d\xi + \frac{1}{2}(\cot\alpha(y(0))^2 - \cot\beta(y(l))^2). \tag{5}$$

Proposition 2. *The number of negative (respectively, zero) eigenvalues of problem* (1), (2) *equals the number of negative (respectively, zero) characteristic numbers of the quadratic form* (5).

This proposition is a consequence of the classical result of the theory of operators in Hilbert space (see [9, Theorem 13' and [10, Section 82, Theorem 2]).

Another possible physical interpretation distinguishes only interesting eigenvalues.

Proposition 3. *The number of negative (respectively, zero) eigenvalues of problem* (1), (2) *equals the number of such eigenvalues in the vibrating system with a finite number of degrees of freedom designed as follows: the rod has the same elastic properties but is weightless, the boundary conditions are the same, and two arbitrary positive masses are attached at the ends of the rod.*

Proof. The matrix of potential energy of the system is positive definite; therefore, the number of eigenvalues is determined by the potential energy. Consider the case of general position ($\alpha, \beta \neq 0$, and $\pi/2 (\mathrm{mod}\, \pi)$ in (2)); the Dirichlet and Neumann boundary conditions at one or both ends are easily investigated individually. The matrix of potential energy is

$$\begin{pmatrix} 1/H + \cot\alpha & -1/H \\ -1/H & 1/H - \cot\beta \end{pmatrix}.$$

Its characteristic equation has the form

$$\lambda^2 - (2/H + \cot\alpha - \cot\beta)\lambda + (\cot\alpha - \cot\beta)/H - \cot\alpha\cot\beta. \tag{6}$$

The free term in (6) is

$$(\tan\beta - \tan\alpha - H)/(H\tan\alpha\tan\beta).$$

Then, we can examine various cases as in Proposition 1. Only one subcase of Case 4 in which $\tan\beta - \tan\alpha - H < 0$ needs special consideration. In this vase, $\lambda_1\lambda_2 > 0$ and

$$\lambda_1 + \lambda_2 = \frac{2(-\tan\alpha)\tan\beta - H(-\tan\alpha + \tan\beta)}{H(-\tan\alpha)\tan\beta} < 0$$

due to the arithmetic mean–geometric mean inequality.

3. GENERAL BOUNDARY CONDITIONS

3.1. Statement of the Problem and Properties of Lagrangian Grassmannians

Without loss of generality, we assume that Eq. (1) is defined on the interval $x \in [-l, l]$. In the general case, the nonseparated boundary conditions are written in the form

$$\Phi\begin{pmatrix} -p(-l)y'(-l) \\ p(l)y'(l) \\ y(-l) \\ y(l) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{7}$$

where Φ is a 2×4 full rank matrix.

Using Moszyński's technique [11], the problem is reduced to the boundary value problem with separated boundary conditions in double dimension:

$$y_1(x) = y(-x), \quad y_2(x) = y(x),$$
$$p_1(x) = p(-x), \quad p_2(x) = p(x),$$
$$\rho_1(x) = \rho(-x), \quad \rho_2(x) = \rho(x), \quad x \in [0, l];$$

this results in the following system with separated boundary conditions in standard Hamiltonian form:

$$\begin{pmatrix} p_1(x)y_1'(x) \\ p_2(x)y_2'(x) \\ y_1(x) \\ y_2(x) \end{pmatrix}' = J \begin{pmatrix} 1/p_1(x) & 0 & 0 & 0 \\ 0 & 1/p_2(x) & 0 & 0 \\ 0 & 0 & \lambda\rho_1(x) & 0 \\ 0 & 0 & 0 & \lambda\rho_2(x) \end{pmatrix} \begin{pmatrix} p_1(x)y_1'(x) \\ p_2(x)y_2'(x) \\ y_1(x) \\ y_2(x) \end{pmatrix},$$
$$J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad x \in [0, l], \tag{8}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} p_1 y_1'(0) \\ p_2 y_2'(0) \\ y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{9}$$

$$\Phi \begin{pmatrix} p_1 y_1'(l) \\ p_2 y_2'(l) \\ y_1(l) \\ y_2(l) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10}$$

In order for the boundary value problem to be self-adjoint (in this paper we consider exactly such problems), it is necessary and sufficient that the differential expression is self-adjoint (this is true for the Sturm–Liouville operator) and the boundary conditions are also self-adjoint. The latter property requires special attention. In [12, Section I.1.6; 13, §9; 14; 15], the self-adjoint property of coupled boundary conditions is defined by four formally different methods; in [12], the definition involves not only the boundary conditions themselves but also the differential expression. The meaning in all the cases is the same: the operator determined by the differential expression and the boundary conditions must be symmetric on its domain; in other words, the out-of-integral terms in the Lagrange formula must vanish for the functions satisfying the boundary conditions. We will use the definition of this concept in the form that is close to [13] (up to the notation and the rearrangement of the components of vectors).

Proposition 4. *The boundary conditions* (7) *are self-adjoint if and only if the plane* Π *that they determine in the space of vectors* $\left(-p(-l)y'(-l), p(l)y'(l), y(-l), y(l)\right)^{\mathrm{T}}$ *has the following property:* $\forall u, v \in \Pi$, $(u, Jv) = 0$ *in the sense of the standard inner product in* $\mathbb{R}^4$.

The proof is based on direct verification by integrating by parts.

Examples of self-adjoint boundary conditions are the separated boundary conditions considered above and periodicity conditions. Examples of non-self-adjoint boundary conditions are the conditions $y(l) = y(-l)$ and $y'(l) = -y'(-l)$ for the equation $y'' + \lambda y = 0$. Every complex λ is an eigenvalue of this boundary value problem (see [16], Chapter 5, problem 5.2).

The planes possessing the property formulated in Proposition 4 are called Lagrangian planes. Below, we will need some other concepts and facts related to these objects, and they are briefly described here (see [17, 18]). The set of (nonoriented) Lagrangian n-dimensional planes in $\mathbb{R}^{2n}$ forms a connected compact and for $n > 1$ nonorientable manifold of dimension $n(n+1)/2$, which is called the Lagrangian Grassmannian $\Lambda(n)$. The standard set of maps in $\Lambda(n)$ can be obtained in the following way. Let $(i_1,\dots,i_k)$, $(j_1,\dots,j_{n-k})$ be an arbitrary partition of the set of indexes $\{1,.,n\}$ into two subsets. The map $U_{(i_1,\dots,i_k)}$ is formed by the Lagrangian planes that can be represented by the linear hull of the columns of the following $2n\times n$ matrix: its rows $(i_1,\dots,i_k)$ are formed by an identity matrix, and the other rows are formed by an arbitrary symmetric (up to the corresponding permutation of rows) matrix. Below, we will use the following two maps in $\Lambda(2)$:

$$U_D = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & b \\ b & c \end{pmatrix}\right\} \tag{11}$$

and

$$U_N = \left\{\begin{pmatrix} a & b \\ b & c \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}, \tag{12}$$

where a, b, and c are arbitrary real numbers. The Lagrangian planes Π_1 and Π_2 in $\Lambda(n)$ are said to be transversal if $\Pi_1+\Pi_2=\mathbb{R}^{2n}$ or, which is equivalent, if $\Pi_1\cap\Pi_2=\{0\}$. The trail of the plane Π is the set of planes in $\Lambda(n)$ that have a nontrivial intersection with Π (are not tansversal to Π). For brevity, we will identify the self-adjoint boundary condition and the element $\Lambda(n)$ determined by it. For instance, map (11) consists of the planes that are transversal to the Neumann conditions, and it is the complement of the trail of the Neumann conditions in $\Lambda(n)$; map (12) contains the planes transversal to the Dirichlet conditions.

If a Lagrangian plane Π is determined at the point $x=0$ (e.g., using boundary conditions), then the set of values of the solution to any other Hamiltonian system (in particular, system (8)) for every other x with the Cauchy data in Π at $x=0$ also forms a Lagrangian plane; therefore, as x moves on $[0,l]$, we have the transfer of the boundary conditions on $\Lambda(n)$.

3.2. Monotonicity Property of the Result o Transferring the Boundary Condition

Denote by $\Pi(\lambda)$ the result of transferring the boundary condition (9) via system (8) to the point $x=l$. Below, we will use the symbol $\sim$ in the following sense: for the matrices A and B of the vertical size n, $A\sim B$ if the linear hulls of the columns of A and B are identical; for the object $\Pi\in\Lambda(n)$, the notation $\Pi\sim A$ means that the corresponding subspaces are identical.

Proposition 5:

$$\Pi(0)\sim\begin{pmatrix} -1 & 0 \\ 1 & 0 \\ -H_1 & 1 \\ H_2 & 1 \end{pmatrix}\sim\begin{pmatrix} -1 & 0 \\ 1 & 0 \\ -H/2 & 1 \\ H/2 & 1 \end{pmatrix},$$

where

$$H_1=\int_{-l}^{0}\frac{1}{p(\xi)}d\xi,\quad H_2=\int_{0}^{l}\frac{1}{p(\xi)}d\xi,$$

$H=H_1+H_2$, *compare with* (3).

Proof. The first relation follows from Lemma 1 applied two times—to the left and to the right halves of the rod. The second equivalence is obvious.

Proposition 6:

$$\lim_{\lambda\to-\infty}\Pi(\lambda)\sim\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix},$$

i.e., to the Dirichlet conditions.

Remark. This proposition does not automatically follow from the analogous fact about a single Sturm–Liouville equation (see Section 2.1) because the corresponding solutions to system (8) stick together as $\lambda\to-\infty$.

Proof. Let, for $i=1,2$, $y_{i,N}(x,\lambda)$ be the solution to the equation

$$(p_i(x)y_i'(x))'+\lambda\rho_i(x)y_i(x)=0,$$

on the interval $x\in[0,l]$ satisfying the conditions $y_{i,N}(0,\lambda)=1$ and $y_{i,N}'(0,\lambda)=0$, and let $y_{i,D}(x,\lambda)$ be the solution to the same equation satisfying the conditions $y_{i,D}(0,\lambda)=0$ and $p(0)y_{i,D}'(0,\lambda)=1$. Then, representing the transfer of the boundary condition in terms of the fundamental system of solutions to (8), we obtain

$$\Pi(\lambda)\sim\begin{pmatrix}-p_1(l)y_{1,D}'(-l,\lambda) & p_1(l)y_{1,N}'(-l,\lambda)\\ p_2(l)y_{2,D}'(l,\lambda) & p_2(l)y_{2,N}'(l,\lambda)\\ -y_{1,D}(-l,\lambda) & y_{1,N}(-l,\lambda)\\ y_{2,D}(l,\lambda) & y_{2,N}(l,\lambda)\end{pmatrix}. \tag{13}$$

To shorten the notation, we denote the elements of matrix (13) by

$$\Pi(\lambda)\sim\begin{pmatrix}-A(\lambda)&C(\lambda)\\B(\lambda)&D(\lambda)\\-a(\lambda)&c(\lambda)\\b(\lambda)&d(\lambda)\end{pmatrix}. \tag{14}$$

The properties of solutions to the Sturm–Liouville equation and the dependence of the Prüfer angle on λ (see Section 2.1) imply that

1. All the quantities $A(\lambda),\dots,d(\lambda)$ are positive for $\lambda<0$ and tend to $+\infty$ as $\lambda\to-\infty$;
2. $a(\lambda)/A(\lambda),\dots,d(\lambda)/D(\lambda)\to0$ as $\lambda\to-\infty$;
3. $a(\lambda)/A(\lambda)<c(\lambda)/C(\lambda)$, $b(\lambda)/B(\lambda)<d(\lambda)/D(\lambda)$ for $\lambda<0$;
4. $b(\lambda)D(\lambda)-d(\lambda)B(\lambda)\equiv a(\lambda)C(\lambda)-c(\lambda)A(\lambda)\equiv-1$ due to the properties of Wronski's determinant.

Multiply matrix (14) by the nonsingular matrix

$$\frac{1}{(A(\lambda)D(\lambda)+B(\lambda)C(\lambda))}\begin{pmatrix}-D(\lambda)&C(\lambda)\\B(\lambda)&A(\lambda)\end{pmatrix}$$

on the right. This operation does not change the linear hull of the columns. As a result, we obtain (for brevity, we omit the dependence on λ)

$$\Pi(\lambda)\sim\begin{pmatrix}1&0\\0&1\\(aD+cB)/(AD+BC)&(-aC+cA)/(AD+BC)\\(-bD+dB)/(AD+BC)&(bC+dA)/(AD+BC)\end{pmatrix},$$

and all elements in the lower half of this matrix are $o(1)$ as $\lambda\to-\infty$.

Proposition 7. 1. $\Pi(\lambda)$ *monotonically depends on* λ *in the following sense*:

(1) *for* $\lambda \le 0$, *the plane* $\Pi(\lambda)$ *belongs to the map* U_N *(see* (12)), *and the upper half of matrix* (12) *is a strictly decreasing matrix function of* λ *(conventionally, for symmetric matrices* A *and* B, $A > B$ *if* $x^{\mathrm{T}}(A - B)x > 0$ *for every vector* $x \neq 0$*)*;

(2) *in this case, this* 2×2 *matrix tends to* $+\infty$ *as* $\lambda \to -\infty$ *in the sense that it becomes greater than any predefined symmetric matrix.*

Proof. The monotonicity in λ is known (e.g., it was proved using somewhat different terms and notation in [19], Theorem 10.2.3 and in [20], Theorem 4).

Proof of item (2). Again, Multiply matrix (14) by the nonsingular matrix

$$\frac{1}{(ad + bc)}\begin{pmatrix} -d & c \\ b & a \end{pmatrix}$$

on the right (again, we omit the dependence on λ) to obtain

$$\Pi(\lambda) \sim \begin{pmatrix} (dA + bC)/(ad + bc) & -1/(ad + bc) \\ -1/(ad + bc) & (aD + cB)/(ad + bc) \\ 1 & 0 \\ 0 & 1 \end{pmatrix}; \tag{15}$$

the diagonal elements in the upper half of the matrix tend to $+\infty$ as $\lambda \to -\infty$, and the off-diagonal elements tend to zero.

3.3. The Case when the Boundary Conditions Are Transversal of the Dirichlet Conditions

Two cases are possible for the boundary conditions (7): the case when the constraints have finite rigidity (it is considered below) and when the boundary conditions contain a component with an infinite rigidity (is considered in the next subsection).

In the first case, the boundary conditions have the form (12). Their physical interpretation is as follows. In response to the unit displacement of the left end (the third component of the phase vector), a force of intensity a in the direction of the displacement acts on the left end, and simultaneously a force of intensity b acts on the right end. Correspondingly, the unit displacement of the right end causes a force of intensity c to be applied to the right end, and a force of intensity b acting on the left end b. Such a mechanical system can be easily designed using three springs (maybe with negative rigidity because a, b, and c are arbitrary numbers). A particular case is the Neumann conditions (all three rigidities are zero).

Thus, let the boundary conditions (10) determine in the space of vectors $(p_1y_1'(l), p_2y_2'(l), y_1(l), y_2(l))^{\mathrm{T}}$ the subspace

$$\begin{pmatrix} S \\ E \end{pmatrix}, \tag{16}$$

where S is an arbitrary symmetric 2×2 matrix and E is the identity 2×2 matrix. Let in the case $\lambda \le 0$ (see Proposition 7)

$$\Pi(\lambda) \sim \begin{pmatrix} Q(\lambda) \\ E \end{pmatrix},$$

where $Q(\lambda)$ is a symmetric 2×2 matrix depending on the spectral parameter. The following two results are obvious.

Lemma 2 (see Proposition 5):

$$Q(0) = \begin{pmatrix} 1/H & -1/H \\ -1/H & 1/H \end{pmatrix}.$$

Lemma 3. *The number* λ *is an eigenvalue of problem* (8)–(10) *if and only if the matrix* $Q(\lambda) - S$ *has a zero eigenvalue. The multiplicity of* λ *is the deficiency of* $Q(\lambda) - S$.

Theorem 1. *The number of negative (respectively, zero) eigenvalues of problem* (8)–(10) *taking into account their multiplicity equals (also with account of multiplicity) to the number of negative (respectively, zero) eigenvalues of the matrix* $Q(0) - S$.

Remark. Given the data of the original problem (8)–(10), the number of negative and zero eigenvalues is calculated constructively without solving differential equations.

Proof. Due to Proposition 7, the matrix $Q(\lambda) - S$ monotonically increases as λ decreases from 0 to $-\infty$, and all its eigenvalues are positive in the limit. Due to the Courant principle, its eigenvalues also change monotonically. Therefore, the number of times they pass through zero (or are equal to zero for $\lambda = 0$ if we deal with zero eigenvalues) equals the corresponding characteristic of the matrix $Q(0) - S$.

The boundary conditions (16) correspond to separated boundary conditions if and only if S is a diagonal matrix. In this particular case, Theorem 1 must give the same result as Proposition 1. Indeed, the characteristic polynomial of $Q(\lambda) - S$ is (in the notation of Section 2)

$$\lambda^2 - (2/H + \cot\alpha - \cot\beta)\lambda + (\cot\alpha - \cot\beta)/H - \cot\alpha\cot\beta,$$

and then the standard analysis using Viète's formulas gives the desired result.

In the case under consideration, combinations of parameters may exist under which the problem has a double zero eigenvalue; this happens when the rod ends are connected by a weightless spring of negative rigidity $-H$ and there are no other constraints. As a result, the rod with such a constraint acquires an infinitely high compliance, which gives one more, in addition to the motion as a rigid body, dimension of the eigenspace. The corresponding forms of free motion are $y(x) \equiv 1$ and $y(x) = \int_0^x d\xi/p(\xi)$.

3.4. The Case when the Boundary Conditions Belong to the Trail of the Dirichlet Conditions

The case when the boundary conditions not only belong to the trail but also coincide with the Dirichlet conditions was considered in Section 2. It remains to examine the situation when the intersection of subspace (10) with the subspace

$$\begin{pmatrix} E \\ 0 \end{pmatrix},$$

is one-dimensional. It is clear that the Lagrangian subspaces of such a form can be determined by column matrices of the form

$$\begin{pmatrix} -\beta & \alpha\cos\gamma \\ \alpha & \beta\cos\gamma \\ 0 & \alpha\sin\gamma \\ 0 & \beta\sin\gamma \end{pmatrix}, \tag{17}$$

where $\alpha^2 + \beta^2 = 1$ and $\sin\gamma \neq 0$.

The question is reduced to finding the conditions under which a linear combination of the columns of matrix (17) belongs to the space $\Pi(\lambda)$, which for this purpose can be conveniently represented in form (15). It is clear that the intersection of subspaces (17) and (15) can be one-dimensional at maximum; i.e., the eigenvalues can be only simple. Furthermore, the coefficient multiplying the second column in the representation of the intersection vector by a linear combination of the columns of (17) must be distinct from zero; therefore, it can be set equal to $1/\sin\gamma$. Then, the coefficients multiplying the columns of (15) are α and β, respectively. Consider two first components of the corresponding linear combination of the columns of (15) (we again use notation (14) and omit the dependence on λ):

$$\begin{pmatrix} u_1(\lambda) \\ u_2(\lambda) \end{pmatrix} = \begin{pmatrix} \alpha(dA + bC)/(ad + bc) - \beta/(ad + bc) \\ -\alpha/(ad + bc) + \beta(cB + aD)/(ad + bc) \end{pmatrix}.$$

As λ decreases from 0 to $-\infty$, the vector $(u_1(\lambda), u_2(\lambda))^{\mathrm{T}}$ varies as follows. Starting from $((\alpha - \beta)/H, -(\alpha - \beta)/H)^{\mathrm{T}}$ at $\lambda = 0$, it then moves in such a way that the inner product of the vectors $(u_1(\lambda), u_2(\lambda))^{\mathrm{T}}$ and $(\alpha, \beta)^{\mathrm{T}}$ strictly increases and tends to $+\infty$. Indeed, this inner product equals the value of the quadratic form determined by the upper half of matrix (15) on the vector $(\alpha,\beta)^{\mathrm{T}}$, and its monotonicity in λ is a consequence of the monotonicity of the matrix itself (see Proposition 7). Therefore, the existence of a negative or zero eigenvalue (in total not more than one with account to multiplicity) depends on which side of the straight line

$$\left\{(\alpha\cot\gamma,\beta\cot\gamma)^{\mathrm{T}} + t(-\beta,\alpha)^{\mathrm{T}};\, t\in\mathbb{R}\right\}$$

is the point $\left((\alpha-\beta)/H, -(\alpha-\beta)/H\right)^{\mathrm{T}}$ (when the direction of the normal $(\alpha,\beta)^{\mathrm{T}}$ is positive).

Thus, we have proved the following result.

Theorem 2. *For the boundary condition belonging to the trail of the Dirichlet conditions, problem* (8)–(10) *can have one zero or negative value or no such eigenvalues at all. The zero eigenvalue exists if* $(\alpha-\beta)^2/H = \cot\gamma$*, and a negative eigenvalue exists if* $(\alpha-\beta)^2/H < \cot\gamma$.

This case includes the periodicity condition when $\alpha = \beta = 1/\sqrt{2}$ and $\gamma = \pi/2$ in (17). Then, Theorem 2 gives the known result: the problem has a simple zero eigenvalue and has no negative eigenvalues.

Consider an example of physical interpretation of the problem with the boundary conditions (17), where we put $\alpha = -\beta = 1/\sqrt{2}$ and $\cot\gamma \neq \infty$ (a generalization of antiperiodic conditions). The structure of the mechanical system is as follows: there is a kinematic constraint between the ends of the rod $y(-l) = -y(l)$, and its ends are connected by a weightless spring of rigidity $k = -\cot\gamma$. Then, Theorem 2 shows that the system has a zero eigenvalue at the negative rigidity $k = -2/H$ and a negative eigenvalue if the rigidity satisfies $k < -2/H$.

Generalization of Propositions 2 and 3 for the case of boundary conditions of general form can be proved similarly. The difference is reduced to the fact that the representation of the constraint energy (the second term in (5)) does not have a diagonal form but is

$$-\frac{1}{2}(y_1(l), y_2(l))S(y_1(l), y_2(l))^{\mathrm{T}},$$

and the analog of the potential energy matrix in Proposition 3 is the matrix $Q(0) - S$ considered above.

4. CONCLUSIONS

Spectral properties of the linear vibrating system are determined by three sets of its parameters: (1) the quadratic form of potential energy, (2) the quadratic form of kinetic energy, and (3) by the linear forms of boundary conditions. The conditions for the existence of negative and zero eigenvalues of the system, which are responsible for its instability, are independent of the distribution of masses and are determined only by the first and third sets of parameters. In this paper, for the regular Sturm–Liouville problem and boundary conditions of general form, such conditions were obtained in a closed form without solving differential equations. It was found that, in the case under examination, the potential energy affects the result only through one numerical characteristic of the system. It seems that both issues—obtaining explicit instability conditions and finding aggregated characteristics of the potential energy responsible for the instability—are of interest for other vibrating systems, including those with a finite number of degrees of freedom.

ACKNOWLEDGMENTS

I am grateful to N.B. Konyukhova for discussions of this paper and to the reviewer of its first version for valuable remarks that enabled me to improve the presentation.

REFERENCES

1. L. Collatz, *Eigenwertaufgaben mit Technischen Anwendungen* (Leipzig, 1963).
2. M. Sh. Birman and M. Z. Solomyak, “Estimates for the number of negative eigenvalues of the Schrödinger operator and its generalizations,” Advances in Soviet Math. **7**, 1–55 (1991).
3. P. Binding and M. Möller, “Negativity indices for definite and indefinite Sturm–Liouville problems,” Math. Nachr. **283**(2), 180–192 (2010).
4. A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics* (Mosk. Gos. Univ., Moscow, 1999) [in Russian].

5. *Vibrations in Engineering: Handbook,* Vol. 1 *Vibrations in Linear Systems,* Ed. by V. V. Bolotin (Mashinostroenie, Moscow, 1999) [in Russian].
6. W. Strauss, *Partial Differential Equations: An Introduction* (Wiley, New York, 2008).
7. P. Hartman, Ordinary Differential Equations (Wiley, New York, 1964).
8. R. Haberman, *Applied Partial Differential Equations* (Pearson, New Jersey, 2012).
9. I. M. Glazman, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators* (Fizmatlit, Moscow, 1963) [in Russian].
10. N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space* (Nauka, Fizmatlit, Moscow, 1966; Dover, New York, 1993).
11. K. Moszyński, "A method of solving the boundary value problem for a system of linear ordinary differential equations," Algorytmy **11**(2), 25–43 (1964).
12. M. A. Naimark, *Linear Differential Operators* (Nauka, Mosow, 1969) [in Russian].
13. L. Greenberg, "A Prüfer method for calculating eigenvalues of self-adjoint systems of ordinary differential equations," Techn. Rep. of University of Maryland, No. TR91-24.1991 (1991).
14. Q. Kong, H. Wu, and A. Zettl, "Dependence of the *n*th Sturm–Liouville eigenvalue on the problem," J. Differ. Equations **156**, 328–354 (1999).
15. G. Wang, Z. Wang, and H. Wu, "Computing the indices of Sturm–Liouville eigenvalues for coupled boundary conditions (the EIGENIND-SLP codes)," J. Comput. Appl. Math. **220**, 490–507 (2008).
16. M. Eastham, *Theory of Ordinary Differential Equations* (Van Nostrand, 1970).
17. V. I. Arnold, "On the characteristic class appearing in quantization conditions," Funk. Anal. Prilozh. **1**(1), 1–14 (1967).
18. V. I. Arnold, "Sturm–Liouville problems and symplectic geometry," Funk. Anal. Prilozh. **19**(4), 1–10 (1985).
19. F. Atkinson, *Discrete and Continuous Boundary Problems* (Academic, New York, 1964).
20. A. A. Abramov, "On finding eigenvalues and eigenfunctions of a self-adjoint differential problem," Zh. Vychisl Mat. Mat. Phys. **31**, 819–831 (1991).

Translated by A. Klimontovich

SPELL: OK