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# Investigation into the Regular and Chaotic States of Twitter

Victor Dmitriev, Andrey Dmitriev†

National Research University Higher School of Economics, 33 Kirpichnaya Street, Moscow, Russia

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**Abstract**

The present paper is devoted to the investigation into the nonlinear dynamics of Twitter. A new model of Twitter as a thermodynamic non-equilibrium system is suggested. Dynamic variables of such system are represented by the variations of tweet/retweet number and instantaneous diversity between the densities of population on different levels around the equilibrium values. Regular and chaotic states of networks are described. It is pointed out, that the system is in a condition of an asymptotically stable equilibrium when the intensity values of an external information are small (the number of tweets eventually tends to its equilibrium value). If the intensity values of external information exceed the critical value, then the chaotic oscillations of tweets are to be observed. We have made the calculations of the correlation dimension and embedding dimension for the dynamics of the 10 most popular @ (TOP 100 by data of Twitter Counter). The results show, that all observed time series have clearly defined chaotic dynamical nature.



## 1 Introduction

From the second half of the XX century the general trend of science development is the penetration of the ideas and methods of physics in natural and traditional humanities disciplines. Methods of physical modeling are often used in sciences such as demography, sociology and linguistics.

In the middle 1990s there was an interdisciplinary research field, known as Econophysics, applying theories and methods originally developed by physicists in order to solve problems in economics, usually those including uncertainty or stochastic processes and nonlinear dynamics. The term "econophysics" was coined by H. Eugene Stanley, to describe the large number of papers written by physicists in the problems of (stock and other) markets (for econophysics reviews see refs. [1–3]).

Physicists' interest in the social sciences is not new; Daniel Bernoulli, as an example, was the originator of utility-based preferences. Sociophysics is the study of social phenomena from a physics perspective, often using the human atom, social atom, or human particle perspective (for sociophysics reviews see refs. [3,4]). The main objective of this new field of natural science consists in the research of objectively measured and formalized laws that define various social processes.

The present paper is devoted to the investigation into the nonlinear dynamics of Twitter as a thermodynamic

†Corresponding author.
Email address: a.dmitriev@hse.ru

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non-equilibrium system. Our interest in this problem mainly stems from the hypothesis that the Twitter time series results from an inherently low-dimensional chaotic process.

The main objective of the present paper is an investigation into the regular and chaotic states of Twitter.

To achieve this goal the following research tasks were defined:

1. Building of the non-equilibrium macroscopic Twitter model in the form of the system of ordinary differential equations of the first order.

2. Investigation into the forming of the regular and chaotic orders of Twitter system operation depending on its control parameters values; definition of essential conditions for the change from one order to another.

3. Determination of the correlation dimension for a supposed chaotic process directly from experimental Twitter time series.

There are a number of works in the field of physical modeling of social networks. The main physical models of the social networks are as follows.

1. Ising model [5–7].
2. Bose-Einstein Condensate model [8, 9].
3. Quantum walk model [10].
4. Ground state and community detection [11].

Other relevant work in this area is that of refs. [12–15].

This paper is organized as follows. In section 2 we present the definition of Twitter as a complex dynamic system of the thermodynamic type and relevant background. In section 3 we present the nonlinear dynamic model of Twitter, including the definition of regular and chaotic state of the social network. In section 4 we give the numerical results of correlation dimension and embedding dimension from Twitter time series. In section 5 we conclude this paper.

## 2 Twitter as a thermodynamic system

In brief, Twitter is an online social networking service that enables users to send and read short messages called “tweets”. Tweets are publicly visible by default, but senders can restrict message delivery to just their followers. Users may subscribe to other users’ tweets - this is known as “following” and subscribers are known as “followers” or “tweeps”, a portmanteau of Twitter and peeps. Individual tweets can be forwarded by other users to their own feed, a process known as a “retweet”.

We suppose that the social network can be considered as such dynamic system of a thermodynamic type [16] that can produce aggregated factors (flows) out of joint activity of individual interests. These flows start to appear on a macro-scale and work according to laws of determined ties and relations. It can be assumed that such network is homeomorphic to the dynamic systems of a hydrodynamic type (from the viewpoint of the generalized macroscopic flows). Therefore, if there are so-called interacting colliding flows in such systems, then as a rule there can appear the phenomenon of generalized turbulence that generates the crisis mode in development of such dynamic systems.

At present time some attempts to include the social dynamics within the scope of theoretic approaches which work well enough within the natural sciences are being implemented. Such attempts were multiple. However, owing to specifics of the research subject, different scientists have different understanding of the question on how to implement the applied mathematics to model the sociological reality. It turned out that the direct application of existing mathematical constructs to the social dynamics was ineffective. Moreover, there is no definition of social dynamics, such as one of, for example, electrodynamics.

The practice of building of mathematical constructions which could efficiently model the dynamics in various physical systems and processes shows that the appropriate mathematical model becomes adequate (to one extent or another) to an original. This happens when all its characteristic properties are being derived out of the features and structure of that kind of movement, which forms the system dynamics. Correctness of such methodological principle when building new theories was brilliantly shown by J. Maxwell by way of creating the classic thermodynamics.

Large and effective experience of the preceding physics-mathematical modeling makes the following methodological specification: there are no ready formalisms in mathematics, which could be immediately applied to description of a new "dynamics". In fact, it turns out that for every new type or class of dynamical structures it is necessary to make a new construction of mathematical formalisms, which could be relevant its matter and peculiarities of its "dynamics".

Further we will concentrate on a special class of complex dynamic systems, which is called thermodynamic systems (TS) [16].

Complex thermodynamic systems are the systems made of the large number of approximately single-type elements ("atoms" or "users" for Twitter). Interaction between such elements occurs under the definite ontogenetic law. Kinematics and dynamics of such systems depend on a "life story" (or individual "life lines") of every element within such system.

There are 3 structural levels of functioning for this system type. The first one is a level of micro-local dynamics $S_0$, where the local interaction of every element ("user") $a_i \in \Lambda$ with any other system element is being under consideration. This level is an ontogenetic level of a system, which generates all other dynamic effects there.

In second, aggregative ("reductive") information structure $S_1$ follows the $S_0$-level. It is reasonable to call it the system mesodynamics [17]. It is shown by the averaged kinematics and dynamics of any its element. An intermediate dynamics of such system type is as a rule represented by the kinetic equations of the Boltzmann type in the molecular-kinetic theory.

Finally, if there are the functions of the system state that are defined in a phase space of a complex dynamic system, then it is possible to get the macroscopically observed expressions of explored systems by means of averaging them over the individual and meso-stories. Such empirically defined artifacts are usually marked by the term "observed".

Let the macroscopic observable that is numerated by index $\alpha \in \Lambda$ and related to the time point $t \in T$, be $x_\alpha(t)$. Let's unite the set of all simultaneously observed dynamic variables $\{x_\alpha(t)\}$ into a single complex:

$$\mathbf{x}(t) = \langle x_1(t), x_2(t), ..., x_\alpha(t) \rangle . \tag{1}$$

And let's consider it as a factor characterizing the global macroscopic state of a system under consideration.

In particular, dynamic variables of Twitter are represented by the variations of tweet/retweet number and instantaneous diversity between the densities of population on different levels around the equilibrium values.

Let's assume that every individual element $a_i \in \Lambda$, belonging to the base set of system elements $\Lambda$, has its own space of "internal states" $A_i$. Let the eigenstate of $a_i-$element related to the time point $t_k$, be marked as $\omega_i^k$. When all $\omega_i^k$ are gathered in a single complex

$$\boldsymbol{\omega}^k = \langle \omega_1^k, \omega_2^k, ..., \omega_N^k \rangle, \tag{2}$$

we shall get the phase microscopic states of the whole system, related to the time moment $t_k$.

In particular, Twitter includes users, which can have just two states: a ground state and an excited state. Those users, who didn't get sufficient amount of information from the mass media and other sources to be able to send tweets, stay in the ground state. Those users, who got sufficient amount of external information to be able to send tweets, are in the excited state.

Then, let's unite all the possible observable macroscopic system states $\mathbf{X}_\beta$ into the single aggregate

$$\boldsymbol{\Phi} = \left\{ \mathbf{X}_\beta \right\}, \beta \in B \tag{3}$$

and call it a macroscopic phase space of a system. By analogy, if all possible micro-local system states $\boldsymbol{\omega}^\alpha$ are united into

$$\boldsymbol{\Omega} = \{ \boldsymbol{\omega}^\alpha \}, \alpha \in A, \tag{4}$$

then we'll get a micro-local reference space of a system.

It should be taken into account that the space $\mathbf{\Omega}$ represents an abstract hypothetical mathematical construct. Its states $\omega^{\alpha}$ are not observable. However there are some reasonable assumptions that can be brought into the mathematical structure of the space $\mathbf{\Omega}$ and factors of interaction between the system elements. These assumptions allow the abstract theoretical defining of the stochastic dynamics on the micro-level of the selected system.

If such the hypothetical micro-local system dynamics is successfully assumed, than applying some folding techniques to the micro-local dynamic information it is possible to theoretically reconstruct the macroscopic global system dynamics in the language of phase space $\mathbf{\Phi}$ as well as of related dynamic operator generating the macroscopic system dynamics. Match making process between the hypothetical micro-local dynamics of the $\mathbf{\Omega}$ -space and the observable macro-global dynamics unfolding in the $\mathbf{\Phi}$-space is the main task of the stochastic dynamics of TS.

It is determined that any TS being left on her own always tends (both in micro-local and macro-global representations) to its equilibrium state. If a deviation (by observed parameters) from the equilibrium state doesn't equal 0, than the system will definitely tend to the nearest local equilibrium. Such a transition period is called relaxation [18].

There has been discovered a peculiar phenomenon of self-organization and spontaneous appearance of ordered structures of one or another type in large physicochemical TS during the investigation into the transition processes. There are necessary conditions of appearance of the new valuable dynamic information in a complex dynamic TS. They are the strong non-equilibrium of the related system states and the non-linear character of interaction between the system elements.

Discovery of this phenomenon that works universally within the large amount of complex dynamic TS has become a cause of appearance of the interdisciplinary trend, which is called "synergetics". Thus synergism proves itself in sociology because of the fact that the social systems (like, for example, microblogging social networks) are complex dynamic TS. And the following content of the present paper is aimed to show, how the thermodynamic conceptualism works in the dynamics of the complex social systems.

Let's mark out the essential terms and propositions that we shall use later on.

Let's consider that any system is organized by elements $\{a_i\}$ that are parts of a reference underlying set $\Lambda \equiv \{a_i\}$. We shall examine large, but finite systems. These are the systems where the power of set $\Lambda$ (or the value $|\Lambda| = N$) goes to infinity, but never equals it. This proposition is fundamental. Within the TS theory the $N \to \infty$ abstraction is called thermodynamic limit. If $N$-factor is not large enough, then some of stochastic dynamics effects have a risk not to appear in the system. At the same time, the limit dividing large and small systems is very relative, but we shall not consider this problem at the present paper.

When considering any complex dynamic system, the system factors are being in the foreground. Among them are: ontological nature of the system elements ("users"); nature and type of interaction (on the "user" level) between the system elements; nature of "movements" arising in the system (user interaction).

Let's consider as uniting symbol $S$ the totality of algebra-geometric structural properties of interaction $V$ in the system and "internal" state spaces $A_i$. If the basic defining factors $\langle \Lambda_\Sigma, S_\Sigma, V_\Sigma \rangle$ are specified for some particular complex dynamic system $\Sigma$, then from the abstract-theoretical point of view such system can be considered as a definite complex

$$\Sigma \equiv \langle \Lambda_\Sigma, S_\Sigma, V_\Sigma \rangle . \tag{5}$$

The modern theory of complex systems has allowed to discover the type of "movement" appearing in a system dynamics. Dynamic system class related to the stochastic dynamics paradigm is huge. Leaving out of consideration such universalism, it is possible to notice that it is important to understand the structural possibilities of applying the methods of stochastic dynamics for solving the definite range of problems in social dynamics.

One of the essential peculiarities of stochastic dynamics consists in the fact that it is able to represent a dynamic mode of operation with the definite degree of uncertainty. The quantitative measure of different uncertainty types is a "physical" value of entropy. From the other hand, there is also a measure of certainty, or order, in the system. That is information. Thus, the stochastic dynamics can be considered as the informational dynamics. Synergetic conjugacy of these two poles of binary opposition lies as a basis of another binary opposition – atomism and holism. There is a fully developed theory of stochastic dynamics nowadays, and we shall not go deep into its details.

## 3 Nonlinear dynamic model of Twitter

Twitter includes users, which can have just two states: a ground state and an excited state. Those users, who didn't get sufficient amount of information from the mass media and other sources to be able to send tweets, stay in the ground state. Those users, who got sufficient amount of external information to be able to send tweets, are in the excited state. By sending tweets the network users transfer from the excited state to the ground state. Let's consider the amount of users being in the ground state at the time point $t$ as $\omega_0(t)$ and the amount of users in the excited state as $\omega_1(t)$.

Let's also introduce a parameter characterizing an instantaneous diversity between the densities of population on different levels (i.e. inversion):

$$x_3(t) = \omega_1(t) - \omega_0(t). \tag{6}$$

This variable will be further considered as one of the dynamic variables of the system.

As the second dynamic variable we shall consider the variation of the tweet amount around the equilibrium position $x_1(t) = T(t) - T_0$, where $T_0$ is a number of tweets in the network that stays in a state of the stable equilibrium.

Let's represent the variation speed of the tweet number as following:

$$\dot{x}_1(t) = -\alpha x_1(t) + \beta x_2(t). \tag{7}$$

Equation (7) has the following terms: the first term of the right side corresponds to the decrease in the number of tweets in the system due to the system relaxation; the second term correlates to the increase in the number of tweets in the network due to the increase in variation of the number of retweets $x_2(t) = R(t) - R_0$, where $R_0$ is the number of tweets in a network being in the state of the stable equilibrium; $\alpha = 1/\tau_1$ is a relaxation parameter; $\tau_1$ is a relaxation time.

According to the Le Chatelier-Braun principle [19], if a system deviates from the state of the stable equilibrium, then the forces arise and try to return the system back to the equilibrium state. If $|T(t) - T_0| \ll T_0$, then as a first approximation it can be considered that these forces are proportionate to deviation.

As the third dynamic variable we shall consider the variation of the number of retweets, which can be represented as follows:

Let's represent the variation speed of the tweet number as following:

$$\dot{x}_2(t) = -\gamma x_2(t) + c x_1(t) x_3(t). \tag{8}$$

The first term of the right side (8) corresponds to the decrease in the number of tweets due to the system relaxation with the relaxation parameter $\gamma = 1/\tau_2$. The nature of the second term in the right side (8) can be explained in the following way. The number of retweets (created by the network users in the presence of the tweet flow) is proportionate to the number of tweets and depends on the level at which the user is (i.e. ground or excited level). An average contribution to the speed of the retweet change is proportionate to the product of the number of tweets in a network and the difference (6).

The third equation describes the change of difference in a density of population on different layers (9) and looks like the following:

$$\dot{x}_3(t) = \varepsilon (I_0 - N(t)) - k x_1(t) x_2(t), \tag{9}$$

where $\varepsilon$ is a population relaxation parameter; $I_0$ characterizes the intensity of the external flow of information incoming to the system. The term $x_1(t)x_2(t)$ corresponds to the power that is spent by the tweet flow for the retweet creation.

Let's introduce some new variables and parameters:

$$x = \sqrt{kc}x_1/\gamma, y = \beta\sqrt{kc}x_2/(\alpha\gamma), z = \beta c x_3/(\alpha\gamma), b = \varepsilon/\gamma, \sigma = \alpha/\gamma, \rho = \beta c I_0/(\alpha\gamma). \tag{10}$$

Equation (7)-(8) can be reduced to:

$$\dot{x} = \sigma(y-x), \dot{y} = -y + xz, \dot{z} = b(\rho - z) - xy. \tag{11}$$

The system (11) describes a dynamics of a Twitter as a point-dissipative system. When it is considered that by the way of changing the variable $w = \rho - z$ the system (11) can be reduced to the well-known Lorenz system [20], one can assume that the parameter $\rho$ is a control parameter of the system (11).

Let's examine the system behavior depending on the control parameter $\rho$ or on the intensity of the external information flow $I_0\,(\rho \sim I_0)$.

The Lorenz system is a well examined dynamic system. The main properties of such system are presented in works [21, 22]. Let's consider only those properties that will be required to analyze the change of the Twitter state due to the external informational influence $I_0$.

Let's use the dynamic system to discuss a question of existence of the asymptotically stable Twitter state.

In the context of the earlier derived model, there is a condition of existence of the equilibrium (which is not obligatory stable) Twitter state. It is the equality of the tweet amount to its equilibrium value. For the tweet amount variation, this condition is equivalent to the following equality: $T(t) = T_0$.

System (11) has 3 stationary points:

$$O(0,0,0), E(\pm\sqrt{b(\rho-1)}, \pm\sqrt{b(\rho-1)}, 0). \tag{12}$$

If $\rho < 1$ (i.e. low external intensity of information $I_0$), then the null point $O$ is asymptotically stable. At the same time another $E$-points do not exist. The value of parameter $\rho = 1$ is a bifurcation value of the supercritical forked bifurcation of a system. If $\rho > 1$, then the stationary point $O$ is unstable. Therefore, at a small quantity of external information the number of tweets asymptotically goes to its stable equilibrium value $T_0$. It's one of the regular state of Twitter.

If $\rho = \rho_c$, where $\rho_c = \sigma(\sigma + b + 3)/(\sigma - b - 1)$ is a critical value of external information, there arose the limit cycles around the non-zero equilibrium points. These are the fluctuations of the number of tweets; their characteristics do not depend on initial values of a system.

When $\rho > \rho_c$, the limit cycle fails turning into the chaotic deviations. A chaotic attractor appears.

For example, Fig. 1 shows the observed dynamics of tweets, derived with the help of the Twitter analysis service TWITONOMY (https://www.twitonomy.com/).

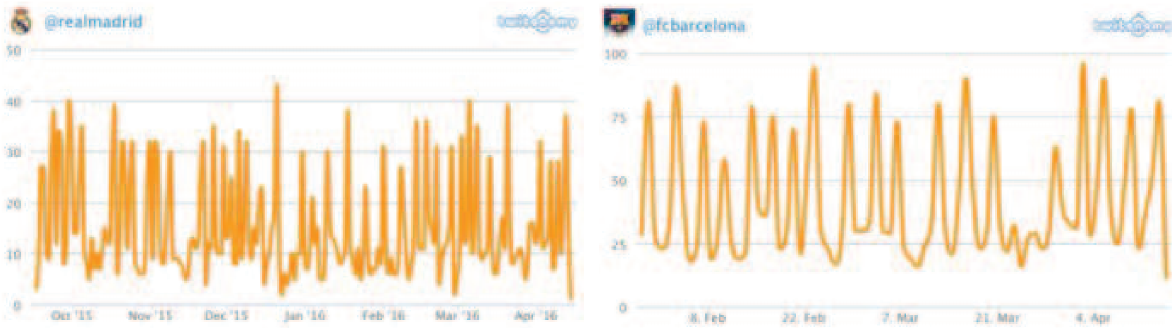
It is obvious, that the dynamics of @realmadrid and @fcbarcelona are chaotic. Existence of the chaotic dynamics is connected with the huge amount of external information entering the microblogging network. In fact, the presented time intervals correspond to the increased interest of football fans to the football events - Spanish La Liga and UEFA Champions League.

Lorenz attractor, defined by the system (11), has also another, mathematically equivalent representation. By changing of variables $\langle x,z\rangle \to \langle x,u\rangle$, where

$$u(x,z) = 2\sigma z - x^2, z(x,u) = \left(u + x^2\right)/2\sigma \tag{13}$$

and considering that $y = \dot{x}/\sigma + x, \dot{y} = \ddot{x}/\sigma + \dot{x}$, the Lorenz equation set (11) can be reduced to the following:

$$\ddot{x} = -(\sigma+1)\dot{x} - x^3/2 + \left(\sigma(\rho-1) - u/2\right)x, \tag{14}$$




(a) (b)

**Fig. 1** Dynamics of tweets: (a) @realmadrid and (b) @fcbarcelona.

$$\dot{u} = -bu + (2\sigma - b)\,x^2. \tag{15}$$

Stationary states of the system (14-15) in coordinates $\langle x, u\rangle$ can be represented by the three points in the coordinate space $xOu$:

$$O\,(0,0)\,, E\,(\pm\sqrt{b(\rho-1)}, (2\sigma-b)(\rho-1)) \tag{16}$$

Equations set (9-10) allows to notice in this dynamics a range of well-known standard factors discovered in a theory of nonlinear non-harmonic oscillations.

Let's introduce the following symbols:

$$\gamma = \sigma + 1, k = \sigma\,(\rho - 1)\,, \omega = k - u/2 \tag{17}$$

and characteristic potential function of the dynamic system (14-15)

$$V\,(u,x) = x^4/8 - \omega x^2/2. \tag{18}$$

Let's define the generalized "force" $F\,(u,x)$, acting in direction of the coordinate $Ox$ according to the rule

$$F\,(u,x) = -x^3/2 + \omega x. \tag{19}$$

Considering (14), the equation (9) can be represented in the following form:

$$x = f\,(\gamma, \omega, \dot{x}, x)\,, \tag{20}$$

where

$$f = -\gamma x + F\,(u,x)\,. \tag{21}$$

Considering (21), (14) turns into the Duffing equation [23, 24]. This equation is well-known and used in the theory of non-linear oscillations.

Duffing equation

$$\ddot{x} = -\gamma\dot{x} - x^3/2 + ((k - u\,(x))/2)\,x \tag{22}$$

represents the dynamics (damped by the term $-\gamma\dot{x}$) along the coordinate $Ox$ in a double-humped potential well, that is described by the function (18). This function has an essential non-standard feature: $\omega = (k - u\,(x))/2$ is not constant. Because of the dependance of this function from $x$, the complex astable ties are generated between $u\,(x)$ and $x\,(u)$. These ties are defined by the equation:

$$\dot{u} = -bu + (2\sigma - b)\,x^2. \tag{23}$$

The system (11) that is equivalent to the equation set (14-15) generates in the $\langle u, x\rangle$-plane some complex irregular movements, which cause the dynamic chaos at the definite correlation of parameters $\langle \sigma, b, \rho\rangle$.

**Table 1** Calculated correlation dimension and embedding dimension for financial time series.

| Twitter users | Correlation dimension | Embedding dimension |
|---|---|---|
| @katyperry | 3.80 | 4 |
| @justionbieber | 4.17 | 5 |
| @taylorswift13 | 3.49 | 4 |
| @BarackObama | 4.46 | 5 |
| @rihanna | 3.99 | 6 |
| @YouTube | 3.74 | 4 |
| @ladygaga | 3.58 | 4 |
| @TheEllenShow | 3.53 | 5 |
| @picazomario | 3.95 | 4 |
| @jtimberlake | 4.44 | 5 |

## 4 Correlation dimension of Twitter time series

The determination of the correlation dimension [25] for a supposed chaotic process directly from experimental time series is an often used means of gaining information about the nature of the underlying dynamics (see, for example, contributions in ref. [26]; for reviews on dimension measurements see ref. [27]). In particular, such analyses have been used to support the hypothesis that the time series results from an inherently low-dimensional chaotic process [26].

The geometry of chaotic attractors can be complex and difficult to describe. It is therefore useful to have quantitative characterizations of such geometrical objects. One of these characterizations is the correlation dimension $D_2$. The correlation dimension have several advantages comparing to other dimensional measures: $D_2$ is easy to compute from Twitter experimental data (from Twitter analysis service TWITONOMY); using $D_2$ one can distinguish chaotic dynamical system ($D_2$ is finite) from stochastic system ($D_2 \to \infty$); $D_2$ is the lower bound estimate of attractor's dimension $d$ ($d \geq D_2$).

The correlation dimension of the attractor of dynamical system can be estimated using the Grassberger-Procaccia algorithm (GP algorithm) [25]. The implementation of GP Algorithm in this work is written in Mathematica Language for Wolfram Mathematica® 10.2.

Table 1 provides summary of calculated correlation dimension and embedding dimension ($k$) for Twitter time series. We have made the calculations of the correlation dimension and embedding dimension for the dynamics of the 10 most popular @ (TOP 100 by data of Twitter Counter http://twittercounter.com/pages/100).

The results show, that all observed time series have clearly defined chaotic dynamical nature. It is also noticeable, that Twitter time series have correlation dimension close to each other and equal to 4.

It is not strictly proven, that any Twitter time series has a chaotic dynamical nature. However, we can consider the dynamic system (11) as an approximate model of the Twitter. So, we also suggest, that the estimation of correlation dimension shall be a preliminary step of Twitter time series analysis. Moreover, the implementation of GP algorithm, including all optimizations, is computationally effective.

## 5 Conclusions

The main contributions of this paper are as follows.

1. We present the nonlinear dynamic model of Twitter as a thermodynamic non-equilibrium system, including the definition of regular and chaotic state of the social network. It is pointed out, that the system is in a condition of an asymptotically stable equilibrium when the intensity values of an external information are small (the number of tweets eventually tends to its equilibrium value).

2. We report new numerical results of the correlation dimension and embedding dimension from Twitter time series. The results show, that all observed time series have clearly defined chaotic dynamical nature. It is also noticeable, that Twitter time series have correlation dimension close to each other and equal to 4.

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