On Embedding of Multidimensional Morse-Smale Diffeomorphisms into Topological Flows

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Abstract

J. Palis found necessary conditions for a Morse-Smale diffeomorphism on a closed n-dimensional manifold M^n to embed into a topological flow and proved that these conditions are also sufficient for $n = 2$. For the case $n = 3$ a possibility of wild embedding of closures of separatrices of saddles is an additional obstacle for Morse-Smale cascades to embed into topological flows. In this paper we show that there are no such obstructions for Morse-Smale diffeomorphisms without heteroclinic intersection given on the sphere S^n, $n \geq 4$, and Palis's conditions again are sufficient for such diffeomorphisms.

1 Introduction and statements of results

Let M^n be a smooth connected closed n-manifold. Recall that *a C^m-flow* ($m \geq 0$) on the manifold M^n is a continuously depending on $t \in \mathbb{R}$ family of C^m-diffeomorphisms $X^t : M^n \to M^n$ that satisfies the following conditions:

1) $X^0(x) = x$ for any point $x \in M^n$;

2) $X^t(X^s(x)) = X^{t+s}(x)$ for any $s, t \in \mathbb{R}$, $x \in M^n$.

A C^0-flow is also called a *topological flow.* One says that a homeomorphism (diffeomorphism) $f : M^n \to M^n$ *embeds* into a C^m-flow on M^n if f is the time one map of this flow.

Obviously, if a homeomorphism embeds in a flow then it is isotopic to identity. For a homeomorphism of the line and a connected subset of the line this condition also is necessary (see [6],[8]). If an orientation preserving homeomorphism f of the circle satisfies either one of the three conditions: 1) f has a fixed point, 2) f has a dense orbit, or 3) f is periodic then it embeds in a flow (see [7]). Sufficient conditions of embedding in topological flow for a homeomorphisms of a compact two-dimensional disk and of the plane one can find in review [35]. An analytical, $\varepsilon-$closed to the identity diffeomorphism $f : M^n \to M^n$ can be approximated with accuracy $e^{-\frac{c}{\varepsilon}}$ by a diffeomorphism which embeds in an analytical flow, see [34].

Due to [27] the set of C^r-diffeomorphisms ($r \geq 1$) which embed in C^1-flows is a subset of the first category in $Diff^r(M^n)$. As Morse-Smale diffeomorphisms are structurally stable (see [26], [28]) then for any manifold M^n there exists an open set (in $Diff^1(M^n)$) of Morse-Smale diffeomorphisms embeddable in topological flows. This set contains neighborhoods of time one maps of Morse-Smale flows without periodic trajectories (according to [30] such flows exist on an arbitrary smooth manifold).

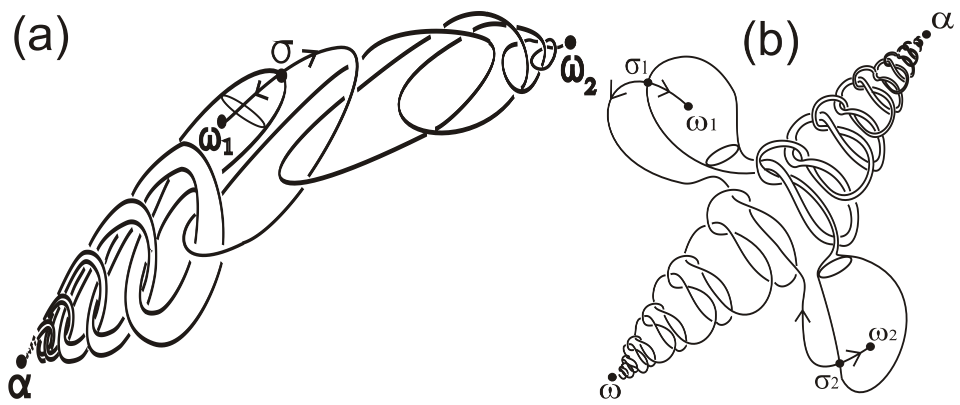

Figure 1: Phase portraits of Morse-Smale diffeomorphisms on S^3 which do not embed in topological flows

Recall that a diffeomorphism $f : M^n \to M^n$ is called *a Morse-Smale diffeomorphism* if it satisfies the following conditions:

- the non-wandering set Ω_f is finite and consists of hyperbolic periodic points;
- for any two points $p, q \in \Omega_f$ the intersection of the stable manifold W^s_p of the point p and the unstable manifold W^u_q of the point q is transversal[1].

In [26] J. Palis established the following necessary conditions of the embedding of a Morse-Smale diffeomorphism $f : M^n \to M^n$ into a topological flow (we call them *Palis conditions*):

(1) the non-wandering set Ω_f coincides with the set of fixed points of f;

(2) the restriction of the diffeomorphism f to each invariant manifold of a fixed point $p \in \Omega_f$ preserves the orientation of the manifold;

(3) if for two distinct saddle points $p, q \in \Omega_f$ the intersection $W^s_p \cap W^u_q$ is not empty then it contains no compact connected components.

According to [26] these conditions are not only necessary but also sufficient for the case $n = 2$. For the case $n = 3$ a possibility of wild embedding of closures of separatrices of saddles is another obstruction for Morse-Smale cascades to embed in topological flows (phase portraits of such diffeomorphisms are shown on the Figure 1). In [12] examples of such cascades are described and a criteria for embedding of Morse-Smale 3-diffeomorphisms in topological flows is provided. In the present paper we establish that the Palis conditions are sufficient for Morse-Smale diffeomorphisms on S^n, $n \geq 4$, such that for any distinct saddle points $p, q \in \Omega_f$ the intersection $W^s_p \cap W^u_q$ is empty.

Theorem 1. *Suppose that a Morse-Smale diffeomorphism $f : S^n \to S^n$, $n \geq 4$ satisfies the following conditions:*

i) the non-wandering set Ω_f of the diffeomorphism f coincides with the set of its fixed points;

[1]Definitions of stable and unstable manifolds and of transversality are given in the section 4; see also the book [15] for references.

ii) the restriction of f to each invariant manifold of a fixed point $p \in \Omega_f$ preserves the orientation of the manifold;

iii) the invariant manifolds of distinct saddle points of f do not intersect.

Then f embeds into a topological flow.

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2 Comments to Theorem 1

Due to [26] the conditions $i)$ and $ii)$ are necessary for embedding a Morse-Smale diffeomorphism into a flow. Our condition that the ambient manifold is the sphere S^n and the absence of heteroclinic intersections (condition iii)) are not necessary but violation of each of them allows to construct examples of Morse-Smale diffeomorphisms which do not embed in topological flows. Below we describe such examples.

In [23] V. Medvedev and E. Zhuzhoma constructed a Morse-Smale diffeomorphism $f_0 : M^4 \to M^4$ satisfying conditions $i) - iii)$ on a projective-like manifold M^4 (different from S^4) whose non-wandering set consists of exactly three fixed points: a source, a sink and a saddle. Invariant manifolds of the saddle are two-dimensional and the closure of each of them is a wild sphere (see [23], Theorem 4, item 2). Assume that f_0 embeds in a topological flow X_0^t. Then X_0^t is a topological flow whose the non-wandering set consists of three equilibrium points with locally hyperbolic behavior. According to [36, Theorem 3] the closures of the invariant manifolds of the saddles are locally flat spheres. That is a contradiction because the closures of the invariant manifolds of the saddle singularities of X_0^t and f_0 coincide. Thus, f_0 does not embed into a flow.

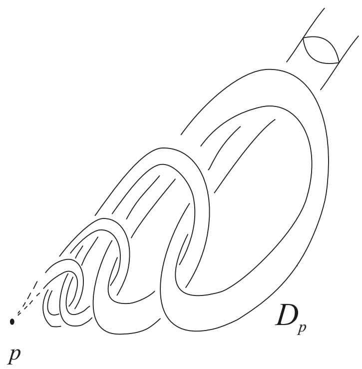

Figure 2: The disk $D_p \subset W_p^s$

In [24] T.Medvedev and O. Pochinka constructed an example of Morse-Smale diffeomorphism $f_1 : S^4 \to S^4$ satisfying to the conditions $i) - ii)$ of the Theorem 1. The non-wandering set of the diffeomorphism f_1 consists of two sources, two sinks and two saddles

p, q such that $dim\ W^s_p = dim\ W^u_q = 3$. The intersection $W^s_p \cap W^u_q$ is not empty and its closure in W^s_p is a wildly embedded open disk D_p (see Fig. 2). If $S^2 \subset W^s_p$ is a 2-sphere which bounds an open ball containing the point p then the intersection $S^2 \cap D_p$ contains at least three connected components. Assume that f_1 embeds into a topological flow X^t_1. Then due to [12] the restriction X^t_p of X^t_1 to $W^s_p \setminus p$ is topologically conjugated by means of a homeomorphism $h : W^s_p \setminus p \to \mathbb{S}^2 \times \mathbb{R}$ to a shift flow $\chi^t(s, r) = (s, r + t)$, $(s, r) \in \mathbb{S}^2 \times \mathbb{R}$. Let $\Sigma = h^{-1}(\mathbb{S}^2 \times \{0\})$. Then every trajectory of the flow X^t_p intersects the sphere Σ at a unique point. Since the disk D_p is invariant with respect to the flow X^t_p the intersection $D_p \cap \Sigma$ consists of a unique connected component and that is a contradiction. Thus, f_1 does not embed into a flow.

3 The scheme of the proof of Theorem 1

The proof of Theorem 1 is based on the technique developed for classification of Morse-Smale diffeomorphisms on orientable manifolds in a series of papers [2], [3], [4], [9], [17], [18], [11],[13]. The idea of the proof consists of the following.

In section 4 we introduce a notion of Morse-Smale homeomorphism on a topological n-manifold and define the subclass $G(S^n)$ of such homeomorphisms satisfying to conditions similar to $i) - iii)$ of Theorem 1.

Let $f \in G(S^n)$. In [13, Theorem 1.3] it is shown that the dimension of the invariant manifolds of the fixed points of f can be only one of $0, 1, n-1$ or n. Denote by Ω^i_f the set of all fixed points of f whose unstable manifolds have dimension $i \in \{0, 1, n-1, n\}$, and by m_f the number of all saddle points of f.

Represent the sphere S^n as the union of pairwise disjoint sets

$$A_f = (\bigcup_{\sigma\in\Omega^1_f} W^u_\sigma) \cup \Omega^0_f,\ R_f = (\bigcup_{\sigma\in\Omega^{n-1}_f} W^s_\sigma) \cup \Omega^n_f,\ V_f = S^n \setminus (A_f \cup R_f).$$

Similar to [16] one can prove that the sets A_f, R_f, V_f are connected, the set A_f is an attractor, R_f is a repeller[2] and V_f consists of wandering orbits of f moving from R_f to A_f.

Denote by $\widehat{V}_f = V_f/f$ the orbit space of the action of f on V_f and by $p_f : V_f \to \widehat{V}_f$ the natural projection. Let

$$\hat{L}^s_f = \bigcup_{\sigma\in\Omega^1_f} p_f(W^s_\sigma \setminus \sigma),\ \ \hat{L}^u_f = \bigcup_{\sigma\in\Omega^{n-1}_f} p_f(W^u_\sigma \setminus \sigma).$$

Definition 3.1. *The collection $S_f = (\widehat{V}_f, \hat{L}^s_f, \hat{L}^u_f)$ is called the scheme of the homeomorphism $f \in G(S^n)$.*

Definition 3.2. *Schemes S_f and $S_{f'}$ of homeomorphisms $f, f' \in G(S^n)$ are called equivalent if there exists a homeomorphism $\hat{\varphi} : \widehat{V}_f \to \widehat{V}_{f'}$ such that $\hat{\varphi}(\hat{L}^s_f) = \hat{L}^s_{f'}$ and $\hat{\varphi}(\hat{L}^u_f) = \hat{L}^u_{f'}$.*

The next statement follows from paper [13, Theorem 1.2] (in fact, Theorem 1.2 was proven for Morse-Smale diffeomorphisms but the smoothness plays no role in the proof).

[2]A set A is called an attractor of a homeomorphism $f : M^n \to M^n$ if there exists a closed neighborhood $U \subset M^n$ of the set A such that $f(U) \subset int\ U$ and $A = \bigcap\limits_{n\geq 0} f(U)$. A set R is called a repeller of a homeomorphism f if it is an attractor for the homeomorphism f^{-1}.

Statement 3.1. *Homeomorphisms $f, f' \in G(S^n)$ are topologically equivalent if and only if their schemes S_f, $S_{f'}$ are equivalent.*

The possibility of embedding of $f \in G(S^n)$ into a topological flow follows from triviality of the scheme in the following sense.

Let a^t be the flow on the set $\mathbb{S}^{n-1} \times \mathbb{R}$ defined by $a^t(x, s) = (x, s+t)$, $x \in S^{n-1}, s \in \mathbb{R}$ and let a be the time-one map of a^t. Let $\mathbb{Q}^n = \mathbb{S}^{n-1} \times \mathbb{S}^1$. Then the orbit space of the action a on $\mathbb{S}^{n-1} \times \mathbb{R}$ is $\mathbb{Q}^n$. Denote by $p_{_{\mathbb{Q}^n}} : \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{Q}^n$ the natural projection. Let $m \in \mathbb{N}$ and $c_1, ..., c_m \subset \mathbb{S}^{n-1}$ be a collection of smooth pairwise disjoint $(n-2)$-spheres. Let $Q_i^{n-1} = \bigcup\limits_{t \in \mathbb{R}} a^t(c_i)$, $\mathbb{L}_m = \bigcup\limits_{i=1}^{m} Q_i^{n-1}$ and $\widehat{\mathbb{L}}_m = p_{_{\mathbb{Q}^n}}(\mathbb{L}_m)$.

Definition 3.3. *The scheme $S_f = (\widehat{V}_f, \hat{L}_f^s, \hat{L}_f^u)$ of a homeomorphism $f \in G(S^n)$ is called trivial if there exists a homeomorphism $\hat{\psi} : \widehat{V}_f \to \mathbb{Q}^n$ such that $\hat{\psi}(\hat{L}_f^s \cup \hat{L}_f^u) = \widehat{\mathbb{L}}_{m_f}$.*

In the section 5 we prove the following key lemma.

Lemma 3.1. *If $f \in G(S^n)$ then its scheme S_f is trivial.*

In the section 6 we construct a topological flow X_f^t whose time one map belongs to the class $G(S^n)$ and has the scheme equivalent to S_f. According to Statement 3.1 there exists a homeomorphism $h : S^n \to S^n$ such that $f = hX_f^1h^{-1}$. Then the homeomorphism f embeds into the topological flow $Y_f^t = hX_f^th^{-1}$.

4 Morse-Smale homeomorphisms

This section contains some definitions and statements which was introduced and proved in [14].

4.1 Basic definitions

Remind that a linear automorphism $L : \mathbb{R}^n \to \mathbb{R}^n$ is called hyperbolic if its matrix has no eigenvalues with absolute value equal one. In this case a space $\mathbb{R}^n$ have a unique decomposition into the direct sum of L-invariant subsets E^s, E^u such that $||L|_{E^s}|| < 1$ and $||L^{-1}|_{E^u}|| < 1$ in some norm $|| \cdot ||$ (see, for example, Propositions 2.9, 2.10 of Chapter 2 in [25]).

According to Proposition 5.4 of the book [25] any hyperbolic automorphism L is topologically conjugated with a linear map of the following form:

$$a_{\lambda,\mu,\nu}(x_1, x_2, ..., x_\lambda, x_{\lambda+1}, x_{\lambda+2}, ..., x_n) = (2\mu x_1, 2x_2, ..., 2x_\lambda, \frac{1}{2}\nu x_{\lambda+1}, \frac{1}{2}x_{\lambda+2}, ..., \frac{1}{2}x_n), \quad (1)$$

where $\lambda = dim\, E^u \in \{0, 1, ..., n\}$, $\mu = -1$ ($\mu = 1$) if the restriction $L|_{E^u}$ reverses (preserves) an orientation of E^u, and $\nu = -1$ ($\nu = 1$) if the restriction $L|_{E^s}$ reverses (preserves) an orientation of E^s.

Put $\mathbb{E}_\lambda^s = \{(x_1, ..., x_n) \in \mathbb{R}^n |\, x_1 = x_2 = \cdots = x_\lambda = 0\}$, $\mathbb{E}_\lambda^u = \{(x_1, ..., x_n) \in \mathbb{R}^n |\, x_{\lambda+1} = x_{\lambda+2} = \cdots = x_n = 0\}$ and denote by $P_x^s(P_y^u)$ a hyperplane that parallel to the hyperplane $\mathbb{E}_\lambda^s$ ($\mathbb{E}_\lambda^u$) and contain a point $x \in \mathbb{E}_\lambda^u$ ($y \in \mathbb{E}_\lambda^s$). Unions $\mathcal{P}_\lambda^s = \{P_x^s\}_{x \in \mathbb{E}_\lambda^u}, \mathcal{P}_\lambda^u = \{P_y^u\}_{y \in \mathbb{E}_\lambda^s}$ form the $a_{\lambda,\mu,\nu}$-invariant foliation.

Suppose that M^n is an n-dimensional topological manifold, $f : M^n \to M^n$ is a homeomorphism and p is a fixed point of the homeomorphism f. We will call the point p *topologically hyperbolic point of index* λ_p, if there exists its neighborhood $U_p \subset M^n$, numbers $\lambda_p \in \{0, 1, ..., n\}, \mu_p, \nu_p \in \{+1, -1\}$, and a homeomorphism $h_p : U_p \to \mathbb{R}^n$ such that $h_p f|_{U_p} = a_{\lambda_p,\mu_p,\nu_p} h_p|_{U_p}$ when the left and right parts are defined. Call the sets $W^s_{p,loc} = h_p^{-1}(E^s), W^u_{p,loc} = h_p^{-1}(E^u)$ *the local invariant manifolds* of the point p, and the sets $W^s_p = \bigcup\limits_{i\in\mathbb{Z}} f^i(W^s_{p,loc})$, $W^u_p = \bigcup\limits_{i\in\mathbb{Z}} f^i(W^u_{p,loc})$ *the stable and unstable invariant manifolds of the point* p.

It follows form the definition that $W^s_p = \{x \in M^n : \lim\limits_{i\to+\infty} f^i(x) = p\}, W^u_p = \{x \in M^n : \lim\limits_{i\to+\infty} f^{-i}(x) = p\}$ and $W^u_p \cap W^u_q = \emptyset$ ($W^s_p \cap W^s_q = \emptyset$) for any distinct hyperbolic points p, q. Moreover, there exists an injective continuous immersion $J : \mathbb{R}^{\lambda_p} \to M^n$ such that $W^u_p = J(\mathbb{R}^{\lambda_p})$[3].

A hyperbolic fixed point is called *the source* (*the sinks*) if its indice equals n (0), a hyperbolic fixed point p of index $0 < \lambda_p < n$ is called *the saddle point*.

A periodic point p of period m_p of a homeomorphism f is called *a topologically hyperbolic sink* (*source, saddle*) *periodic point* if it is the topologically hyperbolic (source, saddle) fixed point for the homeomorphism f^{m_p}. The stable and unstable manifolds of the periodic point p considered as the fixed point of the homeomorphism f^{m_p} are called the stable and unstable manifolds of the point p. Every connected component of the set $W^s_p \setminus p$ ($W^u_p \setminus p$) is called *the stable* (*the unstable*) *separatrix* and is denoted by l^s_p (l^u_p).

The linearizing homeomorphism $h_p : U_p \to \mathbb{R}^n$ induces a pair of transversal foliations $\mathcal{F}^s_p = h_p^{-1}(\mathcal{P}^s_{\lambda_p})$, $\mathcal{F}^u_p = h_p^{-1}(\mathcal{P}^u_{\lambda_p})$ on the set U_p. Every leaf of the foliation $\mathcal{F}^s_p$ ($\mathcal{F}^u_p$) is an open disk of dimension λ_p ($n - \lambda_p$). For any point $x \in U_p$ denote by $F^s_{p,x}, F^u_{p,x}$ the leaf of the foliation $\mathcal{F}^s_p, \mathcal{F}^u_p$, correspondingly, containing the point x.

The invariant manifolds W^s_p and W^u_q of saddle periodic points p, q of a homeomorphism f intersect *consistently transversally* if one of the following conditions holds:

1. $W^s_p \cap W^u_q = \emptyset$;
2. $W^s_p \cap W^u_q \neq \emptyset$ and $F^s_{q,x} \subset W^s_p$; $F^u_{p,y} \subset W^u_q$ for any points $x \in W^s_p \cap U_q$, $y \in W^u_q \cap U_p$.

Definition 4.1. *A homeomorphism $f : M^n \to M^n$ is called the Morse-Smale homeomorphism if it satisfies the next conditions:*

1. *its non-wandering set Ω_f finite and any point $p \in \Omega_f$ is topologically hyperbolic;*
2. *invariant manifolds of any two saddle points $p, q \in \Omega_f$ intersect consistently transversally.*

4.2 Properties of Morse-Smale homeomorphisms

Statement 4.1. *Let $f : M^n \to M^n$ be a Morse-Smale homeomorphism. Then:*

1. *$W^u_p \cap W^s_p = p$ for any saddle point $p \in \Omega_f$;*
2. *for any saddle points $p, q, r \in \Omega_f$ the conditions $(W^s_p \setminus p) \cap (W^u_q \setminus q) \neq \emptyset$, $(W^s_q \setminus q) \cap (W^u_r \setminus r) \neq \emptyset$ imply $(W^s_p \setminus p) \cap (W^u_r \setminus r) \neq \emptyset$;*

[3]A map $J : \mathbb{R}^m \to M^n$ is called immersion if for any point $x \in \mathbb{R}^m$ there exists a neighborhood $U_x \in \mathbb{R}^m$ such that the restriction $J|_{U_x}$ of the map J on the set U_x is a homeomorphism.

3. *there are no sequence of distinct saddle points* $p_1, p_2, ..., p_k \in \Omega_f$, $k > 1$, *such that* $(W^s_{p_i} \setminus p_i) \cap (W^u_{p_{i+1}} \setminus p_{i+1}) \neq \emptyset$ *for* $i \in \{1, ..., k-1\}$ *and* $(W^s_{p_k} \setminus p_k) \cap (W^u_{p_1} \setminus p_1) \neq \emptyset$.

Statement 4.2. *Let* $f : M^n \to M^n$ *be a Morse-Smale homeomorphism. Then:*

1) $M^n = \bigcup\limits_{p \in \Omega_f} W^u_p$;

2) *for any point* $p \in \Omega_f$ *the manifold* W^u_p *is a topological submanifold of the manifold* M^n;

3) *for any point* $p \in \Omega_f$ *and any connected component* l^u_p *of the set* $W^u_p \setminus p$ *the following equality holds:* $cl\, l^u_p \setminus (l^u_p \cup p) = \bigcup\limits_{q \in \Omega_f : W^s_q \cap l^u_p \neq \emptyset} W^u_q$ [4].

Corollary 4.1. *If* $f : M^n \to M^n$ *is a Morse-Smale homeomorphism and* $p \in \Omega_f$ *is a saddle point such that* $l^u_p \cap W^s_q = \emptyset$ *for any saddle point* $q \neq p$, *then there exists a unique sink* $\omega \in \Omega_f$ *such that* $cl\, l^u_p = l^u_p \cup p \cup \omega$ *and* $cl\, l^u_p$ *is either a compact arc in case* $\lambda_p = 1$ *or a sphere of dimension* λ_p *in case* $\lambda_p > 1$.

For an arbitrary point $q \in \Omega_f$ and $\delta \in \{u, s\}$ put $V^\delta_q = W^\delta_q \setminus q$ and denote by $\widehat{V}^\delta_q = V^s_q / f$ the orbit space of the action of the homeomorphism f on the set V^δ_q. The following statement is proved in the book [9] (Proposition 2.1.5).

Statement 4.3. *The space* $\widehat{V}^u_q$ *is homeomorphic to* $\mathbb{S}^{\lambda_q - 1} \times \mathbb{S}^1$ *and the space* $\widehat{V}^s_q$ *is homeomorphic to* $\mathbb{S}^{n - \lambda_q - 1} \times \mathbb{S}^1$.

Remark that $\mathbb{S}^0 \times \mathbb{S}^1$ means a union of two disjoint closed curves.

Proposition 4.1. *Suppose* $f : M^n \to M^n$ *is a Morse-Smale homeomorphism,* $n \geq 4$, *and* $\sigma \in \Omega_f$ *is a saddle point of index* $(n-1)$ *such that* $l^u_\sigma \cap W^s_q = \emptyset$ *for any saddle point* $q \neq p$. *Then the sphere* $cl\, l^u_\sigma$ *is bicollared.*

Proof: Let $\omega \in \Omega^0_f$ be a sink point such that $l^u_\sigma \subset W^s_\omega$. Due to Corollary 4.1 and the item 2 of Statement 4.2 the set $cl\, l^u_\sigma = l^u_\sigma \cup \omega$ is an $(n-1)$-sphere which is locally flat embedded in M^n at all its points apart possibly one point ω. According to [5], [20] an $(n-1)$-sphere in a manifold M^n of dimension $n \geq 4$ is either locally flat or have more than countable set of points of wildness. Therefore the sphere $cl\, l^u_\sigma$ is locally flat at point ω. According to [1] a locally flat sphere is bicollared. $\diamond$

By $G(S^n)$ we denoted a class of Morse-Smale homeomorphism on the sphere S^n such that any $f \in G(S^n)$ satisfy the following conditions:

i) Ω_f consists of fixed points;

ii) $W^s_p \cap W^u_q = \emptyset$ for any distinct saddle points $p, q \in \Omega_f$;

iii) the restriction of a homeomorphism f on every invariant manifolds of an arbitrary fixed point $p \in \Omega_f$ preserves its orientation.

Proposition 4.2. *If* $f \in G(S^n)$, *then any saddle fixed point has index* 1 *and* $(n-1)$.

[4]Here $cl\, l^u_p$ means the closure of the set l^u_p.

Proof: Suppose that, on the contrary, there exists a point $\sigma \in \Omega_f$ of index $j \in (1, n-1)$. According to Corollary 4.1 the closures $cl\,W^u_\sigma, cl\,W^s_\sigma$ of the stable and unstable manifolds of the point σ are spheres of dimensions j and $n-j$ correspondingly. Due to item 1 of Statements 4.1, the spheres $S^j = cl\,W^u_\sigma, S^{n-j} = cl\,W^s_\sigma$ intersect at a single point σ. Therefore their intersection index equals either 1 or -1 (depending on the choice of orientations of the spheres S^j, S^{n-j} and S^n). Since homology groups $H_j(S^n), H_{n-j}(S^n)$ are trivial it follows that there is a sphere $\tilde{S}^j$ homological to the sphere S^j and having the empty intersection with the sphere S^{n-j}. Then the intersection number of the spheres S^j, S^{n-j} must be equal to zero as the intersection number is the homology invariant (see, for example, [32], § 69). This contradiction proves the statement. $\diamond$

4.3 Canonical manifolds connected with saddle fixed points of a homeomorphism $f \in G(S^n)$

It follows from Statement 4.2 that for each saddle point of a homeomorphism $f \in G(S^n)$ there exists a neighborhood where f is topologically conjugated either with the map $a_1 : \mathbb{R}^n \to \mathbb{R}^n$ defined by $a_1(x_1, x_2, \ldots, x_n) = (2x_1, \frac{1}{2}x_2, \ldots, \frac{1}{2}x_n)$ or with the map a_1^{-1}. In this section we describe canonical manifolds defined by the action of the map a_1 and prove Proposition 4.3 allowing to define similar canonical manifolds for the homeomorphism $f \in G(S^n)$.

Put $\mathbb{U}_\tau = \{(x_1, ..., x_n) \in \mathbb{R}^n |\ x_1^2(x_2^2 + ... + x_n^2) \leq \tau^2\}$, $\tau \in (0, 1]$, $\mathbb{U} = \mathbb{U}_1$; $\mathbb{U}_0 = \{(x_1, ..., x_n) \in \mathbb{R}^n |\ x_1 = 0\}$, $\mathbb{N}^s = \mathbb{U} \setminus Ox_1$, $\mathbb{N}^u = \mathbb{U} \setminus \mathbb{U}_0$, $\widehat{\mathbb{N}}^s = \mathbb{N}^s/a_1$, $\widehat{\mathbb{N}}^u = \mathbb{N}^u/a_1$. Denote by $p_s : \mathbb{N}^s \to \widehat{\mathbb{N}}^s$, $p_u : \mathbb{N}^u \to \widehat{\mathbb{N}}^u$ the natural projections and put $\widehat{V}^s = p_s(\mathbb{U}_0)$.

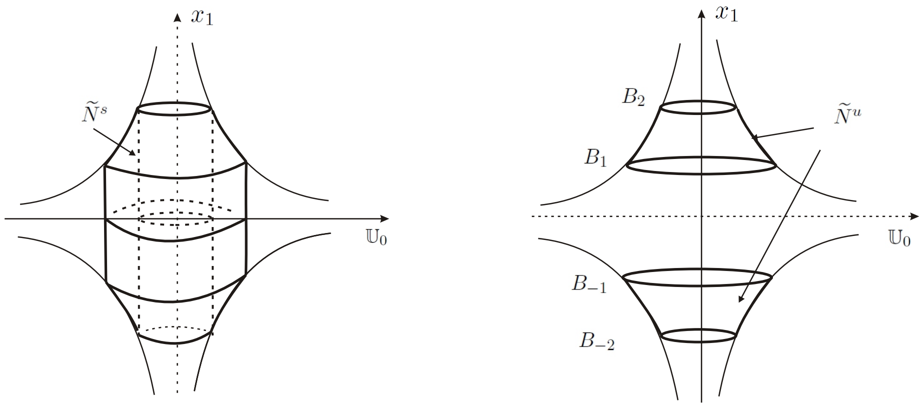

Figure 3: Fundamental domains $\widetilde{N}^s, \widetilde{N}^u$ of the action of the homeomorphism a_1 on the sets $\mathbb{N}^s, \mathbb{N}^u$

The following statement is proved in [11] (Propositions 2.2, 2.3).

Statement 4.4. *The space $\widehat{\mathbb{N}}^s$ is homeomorphic to the direct product $\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [-1, 1]$, the space $\widehat{\mathbb{N}}^u$ consists of two connected components each of which is homeomorphic to the direct product $\mathbb{B}^{n-1} \times \mathbb{S}^1$.*

Recall that *an annulus* of dimension n is a manifold homeomorphic to $\mathbb{S}^{n-1}\times[0,1]$.

On the Figure 3 we present the neighborhoods $\mathbb{N}^s,\mathbb{N}^u$ and the fundamental domains $\widetilde{N}^s=\{(x_1,\dots,x_n)\in\mathbb{N}^s|\frac{1}{4}\le x_2^2+\cdots+x_n^2\le 1\}$, $\widetilde{N}^u=\{(x_1,\dots,x_n)\in\mathbb{N}^u||x_1|\in[1,2]\}$ of the action of the diffeomorphism a_1[5]. Put $\mathcal{C}=\{\{(x_1,\dots,x_n)\in\mathbb{R}^n|\frac{1}{4}\le x_2^2+\cdots+x_n^2\le 1\}$. The set $\mathbb{N}^s$ is the union of the hyperplanes $\mathcal{L}_t=\{(x_1,...,x_n)\in N^s|x_1^2(x_2^2+\cdots+x_n^2)=t^2\},t\in[-1,1]$. Then the fundamental domain $\widetilde{N}^s$ is the union of the pairs of annuli $\mathcal{K}_t=\mathcal{L}_t\cap\mathcal{C},t\in[-1,1]$ and the space $\widehat{\mathbb{N}}^s$ can be obtained from $\widetilde{N}^s$ by gluing the connected components of the boundary of each annulus by means of the diffeomorphism a_1. The set $\widetilde{N}^u$ consist of two connected components each of which is homeomorphic to the direct product $\mathbb{B}^{n-1}\times[0,1]$. The space $\widehat{\mathbb{N}}^u$ is obtained from $\widetilde{N}^u$ by gluing the disk $B_1=\{(x_1,\dots,x_n)\in\mathbb{N}^u|x_1=1\}$ to the disk $B_2=\{(x_1,\dots,x_n)\in\mathbb{N}^u|x_1=2\}$ and the disk $B_{-1}=\{(x_1,\dots,x_n)\in\mathbb{N}^u|x_1=-1\}$ to the disk $B_{-2}=\{(x_1,\dots,x_n)\in\mathbb{N}^u|x_1=-2\}$ by means of the diffeomorphism a_1.

Proposition 4.3. *Suppose $f\in G(S^n)$; then there exists a set of pair-vise disjoint neighborhoods $\{N_\sigma\}_{\sigma\in\Omega_f^1\cup\Omega_f^{n-1}}$ such that for any neighborhood N_σ there exists a homeomorphism $\chi_\sigma:N_\sigma\to\mathbb{U}$ such that $\chi_\sigma f|_{N_\sigma}=a_1\chi_\sigma|_{N_\sigma}$ whenever $\lambda_\sigma=1$ and $\chi_\sigma f|_{N_\sigma}=a_1^{-1}\chi_\sigma|_{N_\sigma}$ whenever $\lambda_\sigma=n-1$.*

Proof: Put $V^\delta_{\Omega^i_f}=\bigcup\limits_{q\in\Omega^i_f}V^\delta_q$, $\widehat{V}^\delta_{\Omega^i_f}=\bigcup\limits_{q\in\Omega^i_f}\widehat{V}^\delta_q$, $i\in\{0,1,n-1,n\}$, $\delta\in\{s,u\}$ and denote by $p^\delta_{\Omega^i_f}:V^\delta_{\Omega^i_f}\to\widehat{V}^\delta_{\Omega^i_f}$ the natural projection such that $p^\delta_{\Omega^i_f}|_{V^\delta_q}=p^\delta_q|_{V^\delta_q}$ for any point $q\in\Omega_f$.

Put $\Sigma_f=\Omega^1_f\cup\Omega^{n-1}_f$, $\widehat{L}^u_{\Sigma_f}=p^s_{\Omega^0_f}(V^u_{\Omega^1_f}\cup V^u_{\Omega^{n-1}_f})$.

The set $\widehat{L}^u_{\Sigma_f}$ consists of finite number of compact topological submanifolds. Then there is a set of pair-vise disjoint compact neighborhoods $\{\widehat{K}^u_\sigma,\sigma\in\Sigma_f\}$ of these manifolds in $\widehat{V}^s_{\Omega^0}$. For every point $\sigma\in\Sigma_f$ put $K^u_\sigma=(p^s_{\Omega^0_f})^{-1}(\widehat{K}^u_\sigma)$ and $\widetilde{N}_\sigma=K^u_\sigma\cup W^s_\sigma$.

Let $U_\sigma\subset\widetilde{N}_\sigma$ be a neighborhood of the point σ such that a homeomorphism $g_\sigma:U_\sigma\to\mathbb{R}^n$ satisfying the condition $g_\sigma f|_{U_\sigma}=a_{\lambda_\sigma}g_\sigma|_{U_\sigma}$ is defined.

Put $u_\tau=\{(x_1,...,x_n)\in\mathbb{U}_\tau|\ x_2^2+...+x_n^2\le 1,|x_1|\le 2\tau\}$, $D^u_\tau=\{(x_1,...,x_n)\in\mathbb{U}_\tau|\ \tau<|x_1|\le 2\tau\}$, $D^s_\tau=\{(x_1,...,x_n)\in\mathbb{U}_\tau|\ \frac{1}{4}\le x_2^2+...+x_n^2\le 1\}$, $\tilde{u}_\tau=g_\sigma^{-1}(u_\tau)$, $\widetilde{D}^\delta_\tau=g_\sigma^{-1}(D^\delta_\tau)$, $\delta\in\{s,u\}$, and $N_\tau=\bigcup\limits_{i\in\mathbb{Z}}f^i(\tilde{u}_\tau)$.

Let us show that there is a number $\tau_1>0$ such that for any $i\in\mathbb{N}$ the intersection $f^i(\widetilde{D}^u_{\tau_1})\cap\tilde{u}_{\tau_1}$ is empty. Suppose $\sigma\in\Omega^{n-1}_f$ (the argument for the case $\sigma\in\Omega^1_f$ is similar). By the Statement 4.2, the set $\bigcup\limits_{i\in\mathbb{N}}f^i(\widetilde{D}^u_\tau)$ lies in the stable manifold of a unique sink point ω. Since the homeomorphism f is locally conjugated with the linear compression a_0 in a neighborhood of the point ω, we have that there exists a ball $B^n\subset W^s_\omega\setminus U_\sigma$ such that $\omega\subset B^n$ and $f(B^n)\subset int\ B^n$. Since $\widetilde{D}^u_\tau$ is compact, there is $i^*>0$ such that $f^i(\widetilde{D}^u_\tau)\cap U_\sigma\subset B^n$ for all $i>i^*$. Hence the set of numbers i_j such that $f^{i_j}(\widetilde{D}^u_\tau)\cap\tilde{u}_\tau\neq\emptyset$ is finite. Then one can choose $\tau_1\in(0,\tau)$ such that $\tilde{u}_{\tau_1}\cap f^i(\widetilde{D}^u_\tau))=\emptyset$ and therefore $\tilde{u}_{\tau_1}\cap f^i(\widetilde{D}^u_{\tau_1}))=\emptyset$ for any $i\in\mathbb{N}$. Similarly one can show that there exists a number $\tau_2\in(0,\tau_1]$ such that for any $i\in\mathbb{N}$ the intersection of $f^{-i}(\widetilde{D}^s_{\tau_2})\cap\tilde{u}_{\tau_2}$ is empty.

[5]A fundamental domain of the action of a group G on a set X is a closed set $D_G\subset X$ containing a subset $\tilde{D}_G$ with the following properties: 1) $cl\,\tilde{D}_G=D_G$; 2) $g(\tilde{D}_G)\cap\tilde{D}_G=\emptyset$ for any $g\in G$ distinct from the neutral element; 3) $\bigcup\limits_{g\in G}g(\tilde{D}_G)=X$.

Suppose $\lambda_\sigma = 1$, put $N_\sigma = \bigcup\limits_{i\in\mathbb{Z}} f^i(\tilde{u}_{\tau_2})$, and define a homeomorphism ${\chi^*}_\sigma : N_\sigma \to U_{\tau_2}$ by the following: $\chi^*_\sigma(x) = g_\sigma(x)$ whenever $x \in \tilde{u}_{\tau_2}$, and $\chi^*_\sigma(x) = a^{-k}_{\lambda_\sigma}(g_\sigma(f^k(x)))$ whenever $x \in N_\sigma \setminus (\tilde{u}_{\tau_2})$, where $k \in \mathbb{Z}$ is such that $f^k(x) \in \tilde{u}_{\tau_2}$. The homeomorphism ${\chi^*}_\sigma$ conjugates the homeomorphism $f|_{N_\sigma}$ with the linear diffeomorphism $a_1|_{\mathbb{U}_{\tau_2}}$. Since the homeomorphism $a_1|_{\mathbb{U}_{\tau_2}}$ is topologically conjugated with $a_1|_{\mathbb{U}}$ by means of the diffeomorphism $g(x_1, ..., x_n) = \left(\frac{x_1}{\sqrt{\tau_2}}, ..., \frac{x_n}{\sqrt{\tau_2}}\right)$, we see that the superposition $\chi_\sigma = g\chi^*_\sigma : N_\sigma \to \mathbb{U}$ topologically conjugates $f|_{N_\sigma}$ with $a_1|_{\mathbb{U}}$. A homeomorphism χ_σ for the case $\lambda_\sigma = n-1$ can be constructed in the same way.

$\diamond$

Put $N^u_\sigma = N_\sigma \setminus W^s_\sigma$, $N_{\tau,\sigma} = \chi^{-1}_\sigma(\mathbb{U}_\tau)$, $N^s_\sigma = N_\sigma \setminus W^u_\sigma$, $\widehat{N}^s_\sigma = N^s_\sigma/f$, $\widehat{N}^u_\sigma = N^u/f$.

5 Triviality of the scheme of the homeomorphism $f \in G(S^n)$

This section is devoted to the proof of Lemma 3.1. In subsections 5.1-5.3 we establish some axillary results.

5.1 Introduction results on the embedding of closed curves and their tubular neighborhoods in a manifold M^n

Further we denote by M^n a topological manifold possibly with non-empty boundary.

Recall that a manifold $N^k \subset M^n$ of dimension k without boundary is *locally flat in a point* $x \in N^k$ if there exists a neighborhood $U(x) \subset M^n$ of the point x and a homeomorphism $\varphi : U(x) \to \mathbb{R}^n$ such that $\varphi(N^k \cap U(x)) = \mathbb{R}^k$, where $\mathbb{R}^k = \{(x_1, ..., x_n) \in \mathbb{R}^n |\ x_{k+1} = x_{k+2} = ... = x_n = 0\}$.

A manifold N^k is *locally flat* in M^n or *the submanifold* of the manifold M^n if it is locally flat at each its point.

If the condition of local flatness fails in a point $x \in N^k$ then the manifold N^k is called *wild* and the point x is called *the point of wildness.*

A topological space X is called *m-connected* (for $m > 0$) if it is non-empty, path-connected and its first m homotopy groups $\pi_i(X)$, $i \in \{1, \ldots, m\}$ are trivial. The requirements of being non-empty and path-connected can be interpreted as (-1)-connected and 0-connected correspondingly.

A topological space P generated by points of a simplicial complex K with the topology induced from $\mathbb{R}^n$ is called *the polyhedron.* The complex K is called *the partition* or *the triangulation* of the polyhedron P.

A map $h : P \to Q$ of polyhedra is called *piece-vise linear* if there exists partitions K, L of polyhedra P, Q correspondingly such that h move each simplex of the complex K into a simplex of the complex L (see for example [29]).

A polyhedron P is called *the piece-vise linear manifold* of dimension n with boundary if it is a topological manifold with boundary and for any point $x \in int\, P$ ($y \in \partial P$) there is a neighborhood U_x (U_y) and a piece-vise linear homeomorphism $h_x : U_x \to \mathbb{R}^n$ ($h_y : U_y \to \mathbb{R}^n_+ = \{(x_1, ..., x_n) \subset \mathbb{R}^n |\ x_1 \geq 0\}$).

The following important statement follows from Theorem 4 of [19].

Statement 5.1. *Suppose that N^k, M^n are compact piece-vise linear manifolds of dimension k, n correspondingly, N^k is the manifold without boundary, M^n possibly has a non-empty boundary, $\tilde{e}, e : N^k \to int\ M^n$ are homotopic piece-vise linear embeddings, and the following conditions hold:*

1. $n - k \geq 3$;
2. N^k *is* $(2k - n + 1)$*-connected;*
3. M^n *is* $(2k - n + 2)$*-connected.*

Then there exists a family of piece-vise linear homeomorphisms $h_t : M^n \to M^n$, $t \in [0, 1]$, such that $h_0 = id$, $h_1\tilde{e} = e$, $h_t|_{\partial M^n} = id$ for any $t \in [0, 1]$.

We will say that a topological submanifold $N^k \subset M^n$ of the manifold M^n is *an essential* if a homomorphism $e_{\gamma_*} : \pi_1(N^k) \to \pi_1(M^n)$ induced by an embedding $e_{_{N^k}} : N^k \to M^n$ is the isomorphism. We will call an essential manifold β homeomorphic to the circle $\mathbb{S}^1$ *the essential knot.*

Let $\beta \in M^n$ be an essential knot and $h : \mathbb{B}^{n-1} \times \mathbb{S}^1 \to M^n$ be a topological embedding such that $h(\{O\} \times \mathbb{S}^1) = \beta$. Call the image $N_\beta = h(\mathbb{B}^{n-1} \times \mathbb{S}^1)$ *the tubular neighborhood* of the knot β.

Proposition 5.1. *Suppose that $\mathbb{P}^{n-1}$ is either $\mathbb{S}^{n-1}$ or $\mathbb{B}^{n-1}$, $\beta_1, ..., \beta_k \subset int\,\mathbb{P}^{n-1} \times \mathbb{S}^1$ are essential knots and $x_1, ..., x_k \subset int\ \mathbb{P}^{n-1}$ are arbitrary points. Then there is a homeomorphism $h : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$ such that $h(\bigcup\limits_{i=1}^{k} \beta_i) = \bigcup\limits_{i=1}^{k} \{x_i\} \times \mathbb{S}^1$ and $h|_{\partial\,\mathbb{P}^{n-1} \times \mathbb{S}^1} = id$.*

Proof: Put $b_i = \{x_i\} \times \mathbb{S}^1$, $i \in \{1, ..., k\}$. Choose pair-vise disjoint neighborhoods $U_1, \ldots, U_k$ of knots $\beta_1, \ldots, \beta_k$ in $int\,\mathbb{P}^{n-1} \times \mathbb{S}^1$. It follows from Theorem 1.1 of the paper [10] that there exists a homeomorphism $g : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$ that is identity outside the set $\bigcup\limits_{i=1}^{k} U_i$ and such that for any $i \in \{1, ..., k\}$ the set $g(\beta_i)$ is a subpolyhedron.

By assumption, piece-vise linear embeddings $\tilde{e} : \mathbb{S}^1 \times \mathbb{Z}_k \to \mathbb{P}^{n-1} \times \mathbb{S}^1$, $e : \mathbb{S}^1 \times \mathbb{Z}_k \to \mathbb{P}^{n-1} \times \mathbb{S}^1$ such that $\tilde{e}(\mathbb{S}^1 \times \mathbb{Z}_k) = \bigcup\limits_{i=1}^{k} g(\beta_i)$, $e(\mathbb{S}^1 \times \mathbb{Z}_k) = \bigcup\limits_{i=1}^{k} b_i$ are homotopic. By Statement 5.1, there exists a family of piece-vise linear homeomorphisms $h_t : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$, $t \in [0, 1]$, such that $h_0 = id$, $h_1\tilde{e} = e$, $h_t|_{\partial\,\mathbb{P}^{n-1} \times \mathbb{S}^1} = id$ for any $t \in [0, 1]$. Then h_1 is the desired homeomorphism. $\diamond$

The following Statement 5.2 is proved in the paper [11] (see Lemma 2.1).

Statement 5.2. *Let $h : \mathbb{B}^{n-1} \times \mathbb{S}^1 \to int\ \mathbb{B}^{n-1} \times \mathbb{S}^1$ be a topological embedding such that $h(\{O\} \times \mathbb{S}^1) = \{O\} \times \mathbb{S}^1$. Then a manifold $\mathbb{B}^{n-1} \times \mathbb{S}^1 \setminus int\ h(\mathbb{B}^{n-1} \times \mathbb{S}^1)$ is homeomorphic to the direct product $\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0, 1]$.*

Proposition 5.2. *Suppose that Y is a topological manifold with boundary, X is a closed component of its boundary, Y_1 is a manifold homeomorphic to $X \times [0, 1]$, and $Y \cap Y_1 = X$. Then a manifold $Y \cup Y_1$ is homeomorphic to Y. Moreover, if the manifold Y is homeomorphic to the direct product $X \times [0, 1]$ then there exists a homeomorphism $h : X \times [0, 1] \to Y \cup Y_1$ such that $h(X \times \{\frac{1}{2}\}) = X$.*

Proof: By [1] (Theorem 2), there exists a topological embedding $h_0 : X \times [0,1] \to Y$ such that $h_0(X \times \{1\}) = X$. Put $Y_0 = h_0(X \times [0,1])$. Let $h_1 : X \times [0,1] \to Y_1$ be a homeomorphism such that $h_1(X \times \{0\}) = X = h_0(X \times \{1\})$.

Define homeomorphisms $g : X\times[0,1] \to X\times[0,1]$, $\tilde{h}_1 : X\times[0,1] \to Y_1$, $h : X\times[0,1] \to Y_0 \cup Y_1$ by $g(x,t) = (h_1^{-1}(h_0(x,1)),t)$, $\tilde{h}_1 = h_1 g$,

$$h(x,t) = \begin{cases} h_0(x,2t), \ t \in [0,\frac{1}{2}]; \\ \tilde{h}_1(x,2t-1), \ t \in (\frac{1}{2};1], \end{cases}$$

and define a homeomoprhism $H : Y \cup Y_1 \to Y$ by

$$H(x) = \begin{cases} h_0(h^{-1}(x)), \ x \in Y_0 \cup Y_1; \\ x, \ x \in Y \setminus Y_0. \end{cases}$$

To prove the second item of the statement it is enough to put $Y = Y_0$. Then the homeomorphism $h : X \times [0,1] \to Y \cup Y_1$ defined above is the desired one. ⋄

Proposition 5.3. *Suppose that $\mathbb{P}^{n-1}$ is either the ball $\mathbb{B}^{n-1}$ or the sphere $\mathbb{S}^{n-1}$, $\beta_1,...,\beta_k \subset int\,\mathbb{P}^{n-1} \times \mathbb{S}^1$ are essential knots, $N_{\beta_1},...,N_{\beta_k} \subset \mathbb{P}^{n-1} \times \mathbb{S}^1$ are their pair-vise disjoint neighborhoods, $D_1^{n-1},...,D_k^{n-1} \subset \mathbb{P}^{n-1}$ are pair-vise disjoint disks, and $x_1,...,x_k$ are inner points of the disks $D_1^{n-1},...,D_k^{n-1}$ correspondingly. Then there exist a homeomorphism $h : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$ such that $h(\beta_i) = \{x_i\} \times \mathbb{S}^1, h(N_{\beta_i}) = D_i^{n-1} \times \mathbb{S}^1, i \in \{1,...,k\}$ and $h|_{\partial\,\mathbb{P}^{n-1}\times\mathbb{S}^1} = id$.*

Proof: By Proposition 5.1, there exists a homeomorphism $h_0 : \mathbb{P}^{n-1}\times\mathbb{S}^1 \to \mathbb{P}^{n-1}\times\mathbb{S}^1$ such that $h_0(\beta_i) = \{x_i\} \times \mathbb{S}^1$, $h_0|_{\partial\,\mathbb{P}^{n-1}\times\mathbb{S}^1} = id$. Put $\tilde{N}_i = h_0(N_{\beta_i})$. By [1], there exist topological embeddings $e_i : \mathbb{S}^{n-2}\times\mathbb{S}^1\times[0,1] \to int\,\mathbb{P}^{n-1}\times\mathbb{S}^1$ such that $e_i(\mathbb{S}^{n-2}\times\mathbb{S}^1\times\{1\}) = \partial\tilde{N}_{\beta_i}$, $e_i(\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0,1]) \cap e_j(\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0,1]) = \emptyset$ for $i \neq j, i,j \in \{1,...,k\}$. Put $U_i = e_i(\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0,1]) \cup \tilde{N}_i$.

Suppose that $D_{0,1}^{n-1},...,D_{0,k}^{n-1},D_{1,1}^{n-1},...,D_{1,k}^{n-1} \subset \mathbb{P}^{n-1}$ are disks such that $x_i \subset int\,D_{j,i}^{n-1}$, $D_{j,i}^{n-1} \subset int\,D_i^{n-1}$, $j \in \{0,1\}$, $D_{0,i}^{n-1} \subset int\,D_{1,i}^{n-1}$, and $D_{1,i}^{n-1} \times \mathbb{S}^1 \subset int\,\tilde{N}_i$.

By Proposition 5.2, every set $\tilde{N}_i \setminus (int\,D_{1,i}^{n-1} \times \mathbb{S}^1)$, $(D_{1,i}^{n-1} \setminus int\,D_{0,1}^{n-1}) \times \mathbb{S}^1$ is homeomorphic to the direct product $\mathbb{S}^{n-2}\times\mathbb{S}^1\times[0,1]$. By Proposition 5.2, there exists a homeomorphism $g_i : \mathbb{S}^{n-2}\times\mathbb{S}^1\times[0,1] \to U_i \setminus int\,D_{0,i}^{n-1}\times\mathbb{S}^1$ such that $g_i(\mathbb{S}^{n-2}\times\mathbb{S}^1\times\{t_1\}) = \partial\,\tilde{N}_i$, $g_i(\mathbb{S}^{n-2} \times \mathbb{S}^1 \times \{t_2\}) = \partial\,D_{1,i}^{n-1} \times \mathbb{S}^1$ for some $t_1, t_2 \subset (0,1)$. Let $\xi : [0,1] \to [0,1]$ be a homeomorphism that is identity on the ends of the interval $[0,1]$ and such that $\xi(t_1) = t_2$. Define a homeomorphism $\tilde{g}_i : \mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0,1] \to \mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0,1]$ by $\tilde{g}_i(x,t) = (x,\xi(t))$.

Define a homeomorphism $h_i : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$ by

$$h_i(x) = \begin{cases} g_i(\tilde{g}_i(g_i^{-1}(x))), \ x \in U_i \setminus int\,D_{0,i}^{n-1} \times \mathbb{S}^1; \\ x, \ x \in (\mathbb{P}^{n-1} \times \mathbb{S}^1 \setminus U_i). \end{cases}$$

The superposition $\eta = h_k \cdots h_1 h_0$ maps every knot β_i into the knot $\{x_i\} \times \mathbb{S}^1$, the neighborhood N_{β_i} into the set $D_{1,i}^{n-1} \times \mathbb{S}^1$, and keeps the set $\partial\,\mathbb{P}^{n-1} \times \mathbb{S}^1$ fixed. Construct a homeomorphism $\Theta : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1$ that be identity on the set $\partial\,\mathbb{P}^{n-1} \times \mathbb{S}^1$ and on the knots $\{x_1\}\times\mathbb{S}^1,...,\{x_k\}\times\mathbb{S}^1$ and move the set $D_{1,i}^{n-1}\times\mathbb{S}^1$ into the set $D_i^{n-1}\times\mathbb{S}^1$ for every $i \in \{1, ..., k\}$. It follows from the Annulus Theorem[6] that the set $D_i^{n-1} \setminus int\, D_{1,i}^{n-1}$ is homeomorphic to the annulus $\mathbb{S}^{n-2} \times [0,1]$. Then apply the construction similar to one described above to define a homeomorphism $\theta : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ such that $\theta(x_i) = x_i$, $\theta(D_i^{n-1}) = D_{1,i}^{n-1}$, $\theta|_{\partial \mathbb{P}^{n-1}} = id$. Put $\Theta(x,t) = (\theta^{-1}(x), t)$, $x \in \mathbb{P}^{n-1}, t \in \mathbb{S}^1$. Then $h = \Theta\eta$ is the desired homeomorphism. ◇

Corollary 5.1. *If $N \subset \mathbb{S}^{n-1} \times \mathbb{S}^1$ is a tubular neighborhood of an essential knot than the manifold $(\mathbb{S}^{n-1} \times \mathbb{S}^1) \setminus int\ N$ is homeomorphic to the direct product $\mathbb{B}^{n-1} \times \mathbb{S}^1$.*

5.2 A surgery of the manifold $\mathbb{S}^{n-1} \times \mathbb{S}^1$ along an essential submanifold homeomorphic to $\mathbb{S}^{n-2} \times \mathbb{S}^1$

Recall that we put $\mathbb{Q}^n = \mathbb{S}^{n-1} \times \mathbb{S}^1$. Suppose that $N \subset \mathbb{Q}^n$ is an essential submanifold homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$, $T = \partial N$, and $e_T : \mathbb{S}^{n-2} \times \mathbb{S}^1 \times [-1; 1] \to \mathbb{Q}^n$ is a topological embedding such that $e_T(\mathbb{S}^{n-2} \times \mathbb{S}^1 \times \{0\}) = T$. Put $K = e_T(\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [-1; 1])$ and denote by N_+, N_- connected components of the set $\mathbb{Q}^n \setminus int\, K$. It follows from Propositions 5.3, 5.2 that the manifolds N_+, N_- are homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$. Let N'_+, N'_- manifolds homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$. Denote by $\psi_\delta : \partial N_\delta \to \partial N'_\delta$ an arbitrary homeomorphism reversing the natural orientation, by Q_δ a manifold obtained by gluing the manifolds N_δ and N'_δ by means of homeomorphism ψ_δ, and by $\pi_\delta : (N_\delta \cup N'_\delta) \to Q_\delta$ the natural projection, $\delta \in \{+, -\}$.

We will say that the manifolds Q_+, Q_- are obtained from $\mathbb{Q}^n$ *by the surgery along the submanifold T.*

Note that $\mathbb{S}^{n-2} \times \mathbb{S}^1$ is the boundary of $\mathbb{B}^{n-1} \times \mathbb{S}^1$. By [22] (Theorem 2), the following statement holds.

Statement 5.3. *Let $\psi : \mathbb{S}^{n-2} \times \mathbb{S}^1 \to \mathbb{S}^{n-2} \times \mathbb{S}^1$ be an arbitrary homeomorphism. Then there exists a homeomorphism $\Psi : \mathbb{B}^{n-1} \times \mathbb{S}^1 \to \mathbb{B}^{n-1} \times \mathbb{S}^1$ such that $\Psi|_{\mathbb{S}^{n-2} \times \mathbb{S}^1} = \psi|_{\mathbb{S}^{n-2} \times \mathbb{S}^1}$.*

Proposition 5.4. *The manifolds Q_+, Q_- are homeomorphic to $\mathbb{Q}^n$.*

Proof: Let $D^{n-1} \subset \mathbb{S}^{n-1}$ be an arbitrary disk, $\mathbb{N}_\delta = D^{n-1} \times \mathbb{S}^1$ and $h_\delta : \pi_\delta(N_\delta) \to \mathbb{N}_\delta$ be an arbitrary homeomorphism. Put $\tilde{\psi}_\delta = h_\delta \pi_\delta \psi_\delta \pi_\delta^{-1} h_\delta^{-1}|_{\partial \mathbb{N}_\delta}$. Due to Proposition 5.3 a homeomorphism $\tilde{\psi}_\delta$ can extend up to a homeomorphism $h'_\delta : \pi_\delta(N'_\delta) \to \mathbb{Q}^n \setminus int\, \mathbb{N}_\delta$. Then a map $H_\delta : Q_\delta \to \mathbb{Q}^n$ defined by $H_\delta(x) = h_\delta(x)$ whenever $x \in \pi_\delta(N_\delta)$ and $H_\delta(x) = h'_\delta(x)$ whenever $x \in \pi_\delta(N'_\delta)$ is the desired homeomorphism. ◇

5.3 A surgery of manifolds homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$ along essential knots

Let $Q_1^n, \ldots, Q_{k+1}^n$ be manifolds homeomorphic to $\mathbb{Q}^n$. Denote by $\beta_1, ..., \beta_{2k} \subset \bigcup\limits_{i=1}^{k+1} Q_i^n$ essential knots such that for any $j \in \{1, ..., k\}$ knots β_{2j-1}, β_{2j} belongs to distinct manifolds

[6] The Annulus Theorem states that the closure of an open domain on the sphere S^{n+1} bounded by two disjoint locally flat spheres S_1^n, S_2^n is homeomorphic to the annulus $\mathbb{S}^n \times [0,1]$. In dimension 2 it was proved by Rado in 1924, in dimension 3 — by Moise in 1952, in dimension 4 — by Quinn in 1982, and in dimension 5 and greater — by Kirby in 1969.

from the union $\bigcup\limits_{i=1}^{k+1} Q_i^n$ and every manifold Q_i^n contains at least one knot from the set $\beta_1, ..., \beta_{2k}$. Let $N_{\beta_1}, ..., N_{\beta_{2k}}$ be tubular neighborhoods of the knots $\beta_1, ..., \beta_{2k}$ correspondingly.

Let $K_1, ..., K_k$ be manifolds homeomorphic to the direct product $\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [-1; 1]$. For every $j \in \{1, \dots, k\}$ denote by $T_j \subset K_j$ a manifold homeomorphic to $\mathbb{S}^{n-2} \times \mathbb{S}^1$ that cuts K_j into two connected components whose closures are homeomorphic to $\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0; 1]$, and by $\psi_j : \partial N_{2j-1} \cup \partial N_{2j} \to \partial K_j$ an arbitrary reversing the natural orientation homeomorphism.

Glue manifolds $\widetilde{Q} = \bigcup\limits_{i=1}^{k+1} Q_i^n \setminus \bigcup\limits_{\nu=1}^{2k} int\ N_\nu$ and $K = \bigcup\limits_{j=1}^{k} K_j$ by means of the homeomorphisms $\psi_1, ..., \psi_k$, denote by Q the obtained manifold and by $\pi : \widetilde{Q} \cup K \to Q$ the natural projection. We will say that the manifold Q is obtained from $Q_1^n, ..., Q_{k+1}^n$ by *the surgery along knots* $\beta_1, ..., \beta_{2k}$ and call every pair β_{2j-1}, β_{2j} *the binding pair*, $j \in \{1, 2, ..., k\}$.

Proposition 5.5. *The manifold Q is homeomorphic to $\mathbb{Q}^n$ and every manifold $\pi(T_j)$ cuts Q into two connected components whose closures are homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$.*

Proof: Prove the proposition by induction on k. Consider the case $k = 1$. Due to Propositions 5.3, 5.2 manifolds $\widetilde{N}_1 = Q_1^n \setminus int\, N_1$, $\widetilde{N}_2 = Q_2^n \setminus int\, N_2$, $\widetilde{N}_1 \bigcup_{\psi_1|_{\partial N_1}} K_1$ are homeomorphic to the direct product $\mathbb{B}^{n-1} \times \mathbb{S}^1$. By definition, the manifold T_1 cuts the manifold K_1 into two connected components whose closures are homeomorphic to $\mathbb{Q}^{n-1} \times [0, 1]$. It follows from Proposition 5.2 that T_1 cuts $\widetilde{N}_1 \bigcup_{\psi_1|_{\partial N_1}} K_1$ into two connected components such that the closure of one of which, denote it by N, is homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$ and the closure of another is homeomorphic to $\mathbb{Q}^{n-1} \times [0, 1]$. Suppose that $D_0^{n-1} \subset \mathbb{S}^{n-1}$ is an arbitrary disk, $N_0 = D_0^{n-1} \times \mathbb{S}^1$ and $h_0 : \pi(\widetilde{N}_1 \bigcup K_1) \to N_0$ is an arbitrary homeomorphism. Put $\tilde{\psi}_1 = h_0 \pi \psi_1^{-1} \pi^{-1} h_0^{-1}|_{\partial N_0}$. In virtue of Proposition 5.3 a homeomorphism $\tilde{\psi}$ can be extended up to a homeomorphism $h_1 : \pi(\widetilde{N}_2) \to \mathbb{Q}^n \setminus int\, N_0$. Then the map $h : Q \to \mathbb{Q}^n$ defined by $h(x) = h_0(x)$ for $x \in \pi(\widetilde{N}_1 \bigcup K_1)$ and $h(x) = h_1(x)$ for $x \in \pi(\widetilde{N}_2)$ is the desired homeomorphism. The manifold $\pi(T_1)$ cuts Q into two connected components such that the closure of one of them is $\pi(N)$ which is homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$. By Corollary 5.1, the closure of another connected component is also homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$.

Suppose that the statement is true for all $\lambda = k$ and show that it is true also for $\lambda = k + 1$. Since $2k \geq k + 1$ we have that there exists at least one manifold among the manifolds $Q_1^n, ..., Q_{\lambda+1}^n$, say $Q_{\lambda+1}^n$, containing exactly one knot from the set $\beta_1, ..., \beta_{2k}$ (if every of that manifolds would contain no less than two knots, then the total number of all knots be no less than $2k + 2$). Let $\beta_{2\lambda} \subset Q_{\lambda+1}^n$, $\beta_{2\lambda-1} \subset Q_i^n$, $i \in \{1, \dots, \lambda\}$, be a binding pair. By the induction hypothesis and Corollary 5.1, the manifold Q_λ obtained by the surgery of manifolds $Q_1^n, ..., Q_\lambda^n$ along knots $\beta_1, ..., \beta_{2\lambda-2}$ is homeomorphic to $\mathbb{Q}^n$; the projection of every manifold (T_j) cuts Q_λ into two connected components such that the closure of each of which is homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$; and the projection of the knot $\beta_{2\lambda-1}$ is the essential knot. Now apply the surgery to manifolds Q_λ, $Q_{\lambda+1}^n$ along knots $\pi(\beta_{2\lambda-1})$, $\beta_{2\lambda}$ and use the first step arguments to obtain the desired statement. $\diamond$

5.4 Proof of Lemma 3.1

Step 1. *Proof of the fact that the manifold $\widehat{V}_f$ is homeomorphic to $\mathbb{Q}^n$ and every connected* *component $\mathcal{Q}^{n-1}$ of the set $\hat{L}^u_f \cup \hat{L}^s_f$ cuts $\widehat{V}_f$ into two connected components whose closures are homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$.*

Put $k_i = |\Omega^i_f|$, $i \in \{0, 1, n-1, n\}$. Due to Statement 4.2 and the fact that the closure of every separatrix of dimension $(n-1)$ cuts the ambient sphere S^n into two connected components one gets $k_0 = k_1 + 1$, $k_n = k_{n-1} + 1$.

Denote by $\beta_1, ..., \beta_{2k_1}$ the essential knots in the set $\widehat{V} = \bigcup\limits_{\omega \in \Omega^0_f} \widehat{V}^s_\omega$ which are projections (by means of $p_{\widehat{V}}$) of all one-dimension unstable separatrices of the diffeomorphism f. Without loss of generality assume that knots β_{2j-1}, β_{2j} are the projection of the separatrices of the same saddle point $\sigma_j \in \Omega^1_f$, $j \in \{1, ..., k_1\}$.

It follows from Statement 4.2 that every manifold $\widehat{V}^s_\omega$ contains at least one knot from the set $\beta_1, ..., \beta_{2k_1}$. Since stable and unstable manifolds of different saddle points do not intersect we have that for any $j \in \{1, ..., k_1\}$ knots β_{2j-1}, β_{2j} belong to distinct connected components of $\widehat{V}$. Indeed, if one suppose that $\beta_{2j-1}, \beta_{2j} \subset \widehat{V}^s_\omega$ for some j, ω, then the set $cl\, W^u_{\sigma_j} = W^u_{\sigma_j} \cup \omega$ is homeomorphic to the circle. Since $cl\, W^s_{\sigma_j}$ divides the sphere S^n into two parts and intersect the circle $cl\, W^u_{\sigma_j}$ at the point σ_j we have that there exists at least one point in $cl\, W^s_{\sigma_j} \cap cl\, W^u_{\sigma_j}$ different from σ_j. This fact contradicts to the item 1 of Statement 4.1.

Let N_{σ_j}, $\chi_{\sigma_j} : N_{\sigma_j} \to \mathbb{U}$ be the neighborhood of the point σ_j and the homeomorphism defined in Proposition 4.3. Further we use denotations of the sections 4.2, 4.3. Denote by N_{2j-1}, N_{2j} the connected components of the set $\widehat{N}^u_{\sigma_j}$ containing knots β_{2j-1}, β_{2j} correspondingly. Let $\psi : \partial\widehat{\mathbb{N}}^u \to \partial\widehat{\mathbb{N}}^s$ be a homeomorphism such that $\psi p_u|_{\partial\mathbb{U}} = p_s|_{\partial\mathbb{U}}$. Put $K_j = \widehat{N}^s_{\sigma_j}$, $T_j = \widehat{V}^s_{\sigma_j}$ and define homeomorphisms $\varphi_{u,j} : N_{2j-1} \cup N_{2j} \to \widehat{\mathbb{N}}^u$, $\varphi_{s,j} : K_j \to \widehat{\mathbb{N}}^s$, $\psi_j : \partial N_{2j-1} \cup \partial N_{2j} \to \partial K_j$ by

$$\varphi_{u,j} = p_u \chi_{\sigma_j} p^{-1}_{\widehat{V}_f}|_{N_{2j-1} \cup N_{2j}},$$

$$\varphi_{s,j} = p_s \chi_{\sigma_j} p^{-1}_{\widehat{V}_f}|_{K_j},$$

$$\psi_j = \varphi^{-1}_{s,j} \psi \varphi_{u,j}|_{\partial N_{2j-1} \cup \partial N_{2j}},$$

and denote by

$$\Psi : \bigcup_{j=1}^{k_1} (\partial N_{2j-1} \cup \partial N_{2j}) \to \bigcup_{j=1}^{k_1} K_j$$

the homeomorphism such that

$$\Psi|_{\partial N_{2j-1} \cup \partial N_{2j}} = \psi_j|_{\partial N_{2j-1} \cup \partial N_{2j}}.$$

Since

$$V_f = \left(\bigcup_{\omega \in \Omega^0_f} V^s_\omega \setminus \left(\bigcup_{\sigma \in \Omega^1_f} V^u_\sigma \right) \right) \bigcup \left(\bigcup_{\sigma \in \Omega^1_f} V^s_\sigma \right) = \left(V_f \setminus \left(\bigcup_{\sigma \in \Omega^1_f} N^u_\sigma \right) \right) \bigcup \left(\bigcup_{\sigma \in \Omega^1_f} N^s_\sigma \right)$$

it follows that

$$\widehat{V}_f = \left(\widehat{V}_f \setminus \left(\bigcup_{\sigma\in\Omega^1_f} \widehat{N}^u_\sigma\right)\right) \cup_\Psi \left(\bigcup_{\sigma\in\Omega^1_f} \widehat{N}^s_\sigma\right) = \left(\widehat{V}_f \setminus \left(\bigcup_{j=1}^{2k_1} N_j\right)\right) \cup_\Psi \left(\bigcup_{j=1}^{k_1} K_j\right).$$

So, the manifold $\widehat{V}_f$ is obtained from $\bigcup\limits_{\omega\in\Omega^0_f} \widehat{V}^s_\omega$ by the surgery along knots $\beta_1, ..., \beta_{2k_1}$. Due to Proposition 5.5, the manifold $\widehat{V}_f$ is homeomorphic to $\mathbb{Q}^n$ and every connected component of the set $\hat{L}^s_f$ cuts the set $\widehat{V}_f$ into two connected components such that the closure of each of which is homeomorphic to $\mathbb{B}^{n-1}\times\mathbb{S}^1$.

From the other hand

$$V_f = \left(\bigcup_{\alpha\in\Omega^n_f} V^u_\alpha \setminus \left(\bigcup_{\sigma\in\Omega^{n-1}_f} V^s_\sigma\right)\right) \bigcup \left(\bigcup_{\sigma\in\Omega^{n-1}_f} V^u_\sigma\right) = \left(V_f \setminus \left(\bigcup_{\sigma\in\Omega^{n-1}_f} N^s_\sigma\right)\right) \bigcup \left(\bigcup_{\sigma\in\Omega^{n-1}_f} N^u_\sigma\right).$$

Similar to previous arguments one can conclude that the set $\widehat{V}_f$ is obtained from $\bigcup\limits_{\alpha\in\Omega^n_f} \widehat{V}^u_\alpha$ by the surgery along the projections of all one-dimensional stable separatrices of the saddle points of the diffeomorphism f. In virtue of Proposition 5.5 every connected component of the set $\hat{L}^u_f$ cuts the set $\widehat{V}_f$ into two connected components such that the closure of each of which is homeomorphic to $\mathbb{B}^{n-1}\times\mathbb{S}^1$.

Step 2. *Proof of the fact that there is a set $\widehat{\mathbb{L}}_{m_f} \subset \mathbb{Q}^n$ and a homeomorphism $\hat{\varphi}$: $\widehat{V}_f \to \mathbb{Q}^n$ such that $\hat{\varphi}(\hat{L}^s_f \cup \hat{L}^u_f) = \widehat{\mathbb{L}}_{m_f}$.*

Denote by $\mathcal{Q}^{n-1}_1, ..., \mathcal{Q}^{n-1}_{k_1+k_{n-1}}$ all elements of the set $\hat{L}^s_f \cup \hat{L}^u_f$ and suppose that $\mathcal{Q}^{n-1}_1$ is an element such that all elements of the set $\hat{L}^s_f \cup \hat{L}^u_f \setminus \mathcal{Q}^{n-1}_1$ are contained exactly in one of the connected component of the manifold $\widehat{V}_f \setminus \mathcal{Q}^{n-1}_1$. Denote by N_1 the closure of this connected component. By Step 1, N_1 is homeomorphic to $\mathbb{B}^{n-1}\times\mathbb{S}^1$. By Proposition 5.3, there exists a disk $D^{n-1}_1 \subset \mathbb{S}^{n-1}$ and a homeomorphism $\psi_0 : \hat{V}_f \to \mathbb{Q}^n$ such that $\psi_0(N_1) = D^{n-1}_1 \times \mathbb{S}^1$. If $k_1 + k_{n-1} = 1$ then the proof is complete and $\hat{\varphi} = \psi_0$, $\widehat{\mathbb{L}}_{m_f} = \partial D^{n-1}_1 \times \mathbb{S}^1$.

Let $k_1 + k_{n-1} > 1$. Denote the images of $\mathcal{Q}^{n-1}_1, ..., \mathcal{Q}^{n-1}_{k_1+k_{n-1}}$ under the homeomorphism ψ_0 by the same symbols as their originals. For $i \in \{2, \dots, k_1 + k_{n-1}\}$ denote by N_i the connected component of the set $\mathbb{Q}^n \setminus \mathcal{Q}^{n-1}_i$ contained in the set $D^{n-1}_1 \times \mathbb{S}^1$. Without loss of generality suppose that the numeration of the sets $\mathcal{Q}^{n-1}_1, ..., \mathcal{Q}^{n-1}_{k_1+k_{n-1}}$ is chosen in such a way that there exist a number $l_1 \in [2, k_1 + k_{n-1}]$ and pair-vise disjont sets $N_2, \dots, N_{l_1}$ such that $\bigcup\limits_{i=2}^{l_1} N_i = \bigcup\limits_{i=2}^{k_1+k_{n-1}} N_i$. Choose in the interior of the disk D^{n-1}_1 arbitrary pair-vise disjoint disks $D^{n-1}_2, \dots, D^{n-2}_{l_1}$. Due to Proposition 5.3 there exists a homeomorphism $\psi_1 : \mathbb{Q}^n \to \mathbb{Q}^n$ such that $\psi_1|_{\mathbb{Q}^n \setminus int\, D^{n-1}_1 \times \mathbb{S}^1} = id$, $\psi_1(N_i) = D^{n-1}_i \times \mathbb{S}^1$, $i \in \{2, \dots, l_1\}$. If $l_1 = k_1 + k_{n-1}$ then the proof is complete and $\hat{\varphi} = \psi_1\psi_0$, $\widehat{\mathbb{L}}_{m_f} = \bigcup\limits_{i=1}^{l_1} \partial\, D^{n-1}_i \times \mathbb{S}^1$.

Let $l_1 < k_1 + k_{n-1}$. Denote the images of $\mathcal{Q}^{n-1}_1, \dots, \mathcal{Q}^{n-1}_{k_1+k_{n-1}}$ and $N_1, \dots, N_{k_1+k_{n-1}}$ under the homeomorphism ψ_1 by the same symbols as their originals. Put $\mathcal{N} = \bigcup\limits_{i=l_1+1}^{k_1+k_{n-1}} N_i$.

If for fixed $i \in \{2, ..., l_1\}$ the set N_i has non-empty intersection with the set $\mathcal{N}$, then denote by $l_i, \tilde{k}_i$, $l_i \leq \tilde{k}_i$, the positive numbers such that $N_{i,1}, ..., N_{i,\tilde{k}_i}$ are all elements from $N_i \cap \mathcal{N}$ and $N_{i,1}, ..., N_{i,l_i}$ are pair-vise disjoint elements from $N_i \cap \mathcal{N}$ such that $\bigcup\limits_{j=1}^{l_i} N_{i,j} = \bigcup\limits_{j=2}^{\tilde{k}_i} N_{i,j}$. Choose in the interior of the every disk D_i^{n-1} pair-vise disjoint disks $D_{i,1}^{n-1}, \ldots D_{i,l_i}^{n-1}$. It follows from Proposition 5.3 that there exists a homeomorphism $\psi_i : \mathbb{Q}^n \to \mathbb{Q}^n$ such that $\psi_i|_{\mathbb{Q}^n \setminus int\, N_i} = id$, $\psi_i(N_{i,j}) = D_{i,j}^{n-1} \times \mathbb{S}^1$, $j \in \{1, \ldots, l_i\}$, $i \in \{2, ..., l_1\}$. If $N_i \cap \mathcal{N} = \emptyset$, put $\psi_i = id$.

If $l_i = \tilde{k}_i$ for any $i \in \{2, \ldots, l_1\}$ such that the numbers $l_i, \tilde{k}_i$ are defined, then the proof is complete and $\hat{\varphi} = \psi_{l_1} \psi_{l_1 - 1} \cdots \psi_1$, $\widehat{\mathbb{L}}_{m_f} = \bigcup\limits_{i=1}^{l_1} \bigcup\limits_{j=1}^{l_i} \partial D_{i,j}^{n-1} \times \mathbb{S}^1$. Otherwise, continue the process and after finite number of steps get the desired set $\widehat{\mathbb{L}}_{m_f}$ and the desired homeomorphism $\hat{\varphi}$ as a superposition of all constructed homeomorphisms.

6 Embedding of diffeomorphisms from the class $G(M^n)$ into topological flows

6.1 Free and properly discontinuous action of a group of maps

In this section we collect an axillary facts on properties of the transformation group $\{g^n, n \in \mathbb{Z}\}$ which is an infinite cyclic group acting freely and properly discontinuously on a topological (in general, non-compact) manifold X and generated by a homeomorphism $g : X \to X$[7].

Denote by X/g the orbit space of the action of the group $\{g^n, n \in \mathbb{Z}\}$ and by $p_{_{X/g}} : X \to X/g$ the natural projection. In virtue of [33] (Theorem 3.5.7 and Proposition 3.6.7) the natural projection $p_{_{X/g}} : X \to X/g$ is a covering map and the space X/g is a manifold.

Denote by $\eta_{_{X/g}} : \pi_1(X/g) \to \mathbb{Z}$ a homeomorphism defined in the following way. Let $\hat{c} \subset X/g$ be a loop non-homotopic to zero in X/g and $[\hat{c}] \in \pi_1(X/g)$ be a homotopy class of $\hat{c}$. Choose an arbitrary point $\hat{x} \in \hat{c}$, denote by $p_{_{X/g}}^{-1}(\hat{x})$ the complete inverse image of $\hat{x}$, and fix a point $\tilde{x} \in p_{_{X/g}}^{-1}(\hat{x})$. As $p_{_{X/g}}$ is the covering map then there is a unique path $\tilde{c}(t)$ beginning at the point $\tilde{x}$ ($\tilde{c}(0) = \tilde{x}$) and covering the loop c (such that $p_{_{X/g}}(\tilde{c}(t)) = \hat{c}$). Then there exists the element $n \in \mathbb{Z}$ such that $\tilde{c}(1) = f^n(\tilde{x})$. Put $\eta_{_{X/g}}([\hat{c}]) = n$. It follows from [21] (гл. 18) that the homomorphism $\eta_{_{X/g}}$ is an epimorphism.

The next statement 6.1 can be found in [21] (Theorem 5.5) and [4] (Propositions 1.2.3 и 1.2.4).

Statement 6.1. *Suppose that X, Y are connected topological manifolds and $g : X \to X$, $h : Y \to Y$ are homeomorphisms such that groups $\{g^n,\ n \in \mathbb{Z}\}$, $\{h^n,\ n \in \mathbb{Z}\}$ acts freely*

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[7] A group $\mathcal{G}$ *acts* on the manifold X if there is a map $\zeta : \mathcal{G} \times X \to X$ with the following properties:
1) $\zeta(e, x) = x$ for all $x \in X$, where e is the identity element of the group $\mathcal{G}$;
2) $\zeta(g, \zeta(h, x)) = \zeta(gh, x)$ for all $x \in X$ and $g, h \in \mathcal{G}$.
A group $\mathcal{G}$ acts *freely* on a manifold X if for any different $g, h \in X$ and for any point $x \in X$ an inequality $\zeta(g, x) \neq \zeta(h, x)$ holds.
A group $\mathcal{G}$ acts *properly discontinuously* on the manifold X if for every compact subset $K \subset X$ the set of elements $g \in \mathcal{G}$ such that $\zeta(g, K) \cap K \neq \emptyset$ is finite.

and properly discontinuously on X, Y correspondingly. Then:

1) *if $\varphi : X \to Y$ is a homeomorphism such that $h = \varphi g \varphi^{-1}$ and $\varphi_* : \pi_1(X/g) \to \pi_1(Y/h)$ is the induced homomorphism, then a map $\widehat{\varphi} : X/g \to Y/h$ defined by $\widehat{\varphi} = p_{_{Y/h}} \varphi p^{-1}_{_{X/g}}$ is a homeomorphism and $\eta_{_{X/g}} = \eta_{_{Y/h}} \varphi_*$;*
2) *if $\widehat{\varphi} : X/g \to Y/h$ is a homeomorphism such that $\eta_{_{X/g}} = \eta_{_{Y/h}} \varphi_*$ and $\hat{x} \in X/g$, $\tilde{x} \in p^{-1}_{_{X/g}}(x)$, $y = \widehat{\varphi}(x)$, $\tilde{y} \in p^{-1}{}_{Y/h}(y)$, then there exists a unique homeomorphism $\varphi : X \to Y$ such that $h = \varphi g \varphi^{-1}$ and $\varphi(\tilde{x}) = \tilde{y}$.*

6.2 Proof of Theorem 1

Suppose that a Morse-Smale diffeomorphism $f : S^n \to S^n$ has no heteroclinic intersection and satisfy Palis conditions. To prove the theorem it is enough to construct a topological flow X^t_f such that its time one map X^1_f belongs to the class $G(S^n)$ and the scheme $S_{X^1_f}$ is equivalent to the scheme S_f (see Section 3).

Step 1. It follows from Lemma 3.1 and Proposition 6.1 that there exists a homeomorphism $\psi_{_f} : V_f \to \mathbb{S}^{n-1} \times \mathbb{R}$ such that:

1) $f|_{V_f} = \psi^{-1}_{_f} a \psi_{_f}$, where a is the time one map of the flow $a^t(x, s) = (x, s + t)$, $x \in S^{n-1}, s \in \mathbb{R}$;

2) for $(n-1)$-dimensional separatrix l_σ of an arbitrary saddle point $\sigma \in \Omega_f$ there exists a sphere $S^{n-2}_\sigma \subset \mathbb{S}^{n-1}$ such that $\psi_{_f}(l_\sigma) = \bigcup\limits_{t \in \mathbb{R}} a^t(S^{n-2}_\sigma)$.

Recall that we denote by L^s_f and L^u_f the union of all $(n-1)$-dimensional stable and unstable separatrices of the diffeomorphism f correspondingly. Put $\mathbb{L}^s = \psi_f(L^s_f)$, $\mathbb{L}^u = \psi_f(L^u_f)$. Then $\mathbb{L}^\delta$ is the union of pair-vise disjoint cylinders $\tilde{Q}^\delta_1 \cup \dots \cup \tilde{Q}^\delta_{k^\delta}$, $\delta \in \{s, u\}$. Denote by $N(\mathbb{L}^\delta) = N(\tilde{Q}^\delta_1) \cup \dots \cup N(\tilde{Q}^\delta_{k^\delta})$ the set of their pair-vise disjoint closed tubular neighborhoods such that $N(\tilde{Q}^\delta_i) = K^\delta_i \times \mathbb{R}$, where $K^\delta_i \subset \mathbb{S}^{n-1}$ is an annulus of dimension $(n-1)$, $i = 1, \dots, k^\delta$.

Define a flow a^t_1 on the set $\mathbb{U} = \{(x_1, ..., x_n) \in \mathbb{R}^n | \ x^2_1(x^2_2 + ... + x^2_n) \leq 1\}$ by $a^t_1(x_1, x_2, ..., x_n) = (2^t x_1, 2^{-t} x_2, ..., 2^{-t} x_n)$. It follows from Statements 4.4, 6.1 that there exists a homeomorphism $\chi^s_i : N(\tilde{Q}^s_i) \to \mathbb{N}^s$ such that $a^1_1|_{\mathbb{N}^s} = \chi^s_i a^1 (\chi^s_i)^{-1}|_{\mathbb{N}^s}$. Denote by $\chi^s : N(\mathbb{L}^s) \to \mathbb{U} \times \mathbb{Z}_{k^s}$ a homeomorphism such that $\chi^s|_{N(\tilde{Q}^s_i)} = \chi^s_i$ for any $i \in \{1, ..., k^s\}$. Put $\mathbb{Q}^s = (\mathbb{S}^{n-1} \times \mathbb{R}) \cup_{\chi^s} (\mathbb{U} \times \mathbb{Z}_{k^s})$. A topological space $\mathbb{Q}^s$ is a connected oriented n-manifold without boundary.

Denote by $\pi_s : (\mathbb{S}^{n-1} \times \mathbb{R}) \cup (\mathbb{U} \times \mathbb{Z}_{k^s}) \to Q^s$ a natural projection. Put $\pi_{_{s,1}} = \pi_{_s}|_{\mathbb{S}^{n-1} \times \mathbb{R}}$, $\pi_{_{s,2}} = \pi_{_s}|_{\mathbb{U} \times \mathbb{Z}_{k^s}}$. Define a flow $\tilde{Y}^t_s$ on the manifold Q^s by

$$\tilde{Y}^t_s(x) = \begin{cases} \pi_{_{s,1}}(a^t(\pi^{-1}_{_{s,1}}(x))), \ x \in \pi_{_{s,1}}(\mathbb{S}^{n-1} \times \mathbb{R}); \\ \pi_{_{s,2}}(a^t_1(\pi^{-1}_{_{s,2}}(x))), \ x \in \pi_{_{s,2}}(\mathbb{U} \times \{i\}), \ i \in \mathbb{Z}_{k^s} \end{cases}.$$

By construction the non-wandering set of the flow $\tilde{Y}^t_s$ consists of k^s equilibria such that the flow $\tilde{Y}^t_s$ is locally topologically conjugated with the flow a^t_1 at the neighborhood of each equilibrium.

Step 2. Denote the images of the sets $\mathbb{L}^u$, $N(\mathbb{L}^u)$ by means of the projection $\pi_{_s}$ by the same symbols as their originals. Due to Statements 4.4, 6.1 there exists a homeomorphism $\chi^u_i : N(\tilde{Q}^u_i) \to \mathbb{N}^u$ such that $a^{-1}_1|_{\mathbb{N}^u} = \chi^u_i \tilde{Y}^1_s (\chi^u_i)^{-1}$, $i = 1, \dots, k^u$. Denote by $\chi^u : N(\mathbb{L}^u) \to \mathbb{U} \times \mathbb{Z}_{k^u}$ the homeomorphism such that $\chi^u|_{N(\tilde{Q}^u_i)} = \chi^u_i|_{N(\tilde{Q}^u_i)}$ for any $i = 1, \dots, k^u$. Put $\mathbb{Q}^u = \mathbb{Q}^s \cup_{\chi^u} (\mathbb{U} \times \mathbb{Z}_{k^u})$. A topological space $\mathbb{Q}^u$ is a connected oriented n-manifold without boundary.

Denote by $\pi_u : \mathbb{Q}^s \cup (\mathbb{U} \times \mathbb{Z}_{k^u}) \to \mathbb{Q}^u$ the natural projection. Put $\pi_{u,1} = \pi_u|_{\mathbb{Q}^s}$, $\pi_{u,2} = p_u|_{\mathbb{U}\times\mathbb{Z}_{l^u}}$. Define a flow $\tilde{Y}^t_u$ on the manifold $\mathbb{Q}^u$ by

$$\tilde{Y}^t_u(x) = \begin{cases} \pi_{u,1}(\tilde{Y}^t_s(\pi^{-1}_{u,1}(x))), \ x \in \pi_{u,1}(\mathbb{Q}^s); \\ \pi_{u,2}(a^{-t}_1(\pi^{-1}_{u,2}(x))), \ x \in \pi_{u,2}(\mathbb{U} \times \{i\}), \ i \in \mathbb{Z}_{k^u} \end{cases}.$$

The non-wandering set $\Omega_{\tilde{Y}^t_u}$ of the flow $\tilde{Y}^t_u$ consists of k^s equilibria such that the flow $\tilde{Y}^t_u$ is locally topological conjugated with the flow a^t_1 in each of their neighborhoods and k^u equilibria such that the flow $\tilde{Y}^t_u$ is locally topologically conjugated with the flow a^{-t}_1 in each of their neighborhoods.

Step 3. Put $R^s = Q^u \setminus W^s_{\Omega_{\tilde{Y}^t_u}}$, denote by $\rho^s_1, \dots, \rho^s_{n^s}$ connected components of the set R^s and put $\hat{\rho}^s_i = \rho^s_i/_{\tilde{Y}^1_u}$. A union of the orbit spaces $\bigcup\limits_{i=1}^{n^s} \hat{\rho}^s_i$ is obtained from the manifold $\widehat{V}_a$ by a sequence of the surgeries along essential submanifolds of codimension 1. In virtue of Proposition 5.4 for any $i \in \{1, ..., n^s\}$ the manifold $\hat{\rho}^s_i$ is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$, the manifold ρ^s_i is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{R}$ and the flow $\tilde{Y}^t_u|_{\rho^s_i}$ is topologically conjugated with the flow $a^t|_{\mathbb{R}^n\setminus O}$ by means of a homeomorphism ν^s_i. Denote by $\nu^s : R^s \to (\mathbb{R}^n \setminus \{0\}) \times \mathbb{Z}_{n^s}$ the homeomorphism consisting of the homeomorphisms $\nu^s_1, \dots, \nu^s_{n^s}$. Put $M^s = Q^u \cup_{\nu^s} (\mathbb{R}^n \times \mathbb{Z}_{n^s})$. Then M^s is a connected oriented n-manifold without boundary.

Put $\bar{M}^s = Q^u \cup (\mathbb{R}^n \times \mathbb{Z}_{n^s})$ and denote by $q_s : \bar{M}^s \to M^s$ the natural projection. Put $q_{s,1} = q_s|_{Q^u}$, $q_{s,2} = q_s|_{\mathbb{R}^n\times\mathbb{Z}_{n^s}}$. Define a flow $\tilde{X}^t_s$ on the manifold M^s by

$$\tilde{X}^t_s(x) = \begin{cases} q_{s,1}(\tilde{Y}^t_u(q^{-1}_{s,1}(x))), \ x \in q_{s,1}(Q^u); \\ q_{s,2}(a^t(q^{-1}_{s,2}(x))), \ x \in q_{s,2}(\mathbb{R}^n \times \{i\}), \ i \in \mathbb{Z}_{n^s} \end{cases}.$$

By construction the non-wandering set of the time one map of the flow $\tilde{X}^t_s$ consists of k^s saddle topologically hyperbolic fixed points of index 1, k^u saddle topologically hyperbolic fixed points of index $(n-1)$ and n^s sink topologically hyperbolic fixed points.

Step 4. Put $R^u = M^s \setminus W^u_{\Omega_{\tilde{X}^t_s}}$ and denote by $\rho^u_1, \dots, \rho^u_{n^u}$ connected components of the set R^u. Similar to Step 3 one can prove that every component ρ^u_i is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{R}$ and the flow $\tilde{X}^t_s|_{\rho^u_i}$ is conjugated with the flow $a^{-t}|_{\mathbb{R}^n\setminus\{O\}}$ by a homeomorphism μ^u_i. Denote by $\mu^u : R^u \to (\mathbb{R}^n \setminus \{O\}) \times \mathbb{Z}_{n^u}$ a homeomorphism consisting of the homeomorphisms $\mu^u_1, \dots, \mu^u_{n^u}$. Put $M^u = M^s \cup_{\mu^u} (\mathbb{R}^n \times \mathbb{Z}_{n^u})$. M^u is a connected closed oriented n-manifold.

Put $\bar{M}^u = M^s \cup (\mathbb{R}^n \times \mathbb{Z}_{n^u})$, denote by $q_u : \bar{M}^u \to M^u$ the natural projection, and put $q_{u,1} = q_u|_{M^s}$, $q_{u,2} = q_u|_{\mathbb{R}^n\times\mathbb{Z}_{n^u}}$. Define a flow $\tilde{X}^t_u$ on the manifold M^u by

$$\tilde{X}^t_u(x) = \begin{cases} q_{u,1}(\tilde{X}^t_s(q^{-1}_{u,1}(x))), \ x \in q_{u,1}(M^s); \\ q_{u,2}(a^{-t}_0(q^{-1}_{u,2}(x))), \ x \in q_{u,2}(\mathbb{R}^n \times \{i\}), \ i \in \mathbb{Z}_{n^u} \end{cases}.$$

By construction the non-wandering set of the time one map of the flow $\tilde{X}^t_u$ consists of k^s saddle topologically hyperbolic fixed points of index 1, k^u saddle topologically hyperbolic fixed points of index $(n-1)$, n^s sink and n^u source topologically hyperbolic fixed points.

Step 5. Put $\tilde{f} = \tilde{X}^1_u$. By construction $\tilde{f}$ is a Morse-Smale homeomorphism on the manifold M^u and its restriction $\tilde{f}|_{V_{\tilde{f}}}$ is topologically conjugated with the diffeomorphism $f|_{V_f}$ by a homeomorphism mapping the $(n-1)$-dimensional separatrices of the diffeomorphism $\tilde{f}$ to the $(n-1)$-dimensional separatrices of the diffeomorphism f and preserving their stability. Due to Statement 3.1 homeomorphisms $\tilde{f}$ and f are topologically conjugated. Hence $M^u = S^n$ and $X^t = \tilde{X}^t_u$ is the desired flow.

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