Functional Galois connections and a classification of symmetric conservative clones with a finite carrier

Nikolay L. Polyakov
Higher School of Economics, niknikols0@gmail.com

Abstract

We propose a classification of symmetric conservative clones with a finite carrier. For the study, we use the functional Galois connection $(\mathrm{Inv}_Q, \mathrm{Pol}_Q)$, which is a natural modification of the connection $(\mathrm{Inv}, \mathrm{Pol})$ based on the preservation relation between functions f on a set A (of all finite arities) and sets of functions $h \in A^Q$ for an arbitrary set Q.

1 Introduction

Clones of conservative functions on an arbitrary set A are a naturally generalization of closed classes of Boolean functions preserving $\mathbf{0}$ and $\mathbf{1}$. Some important information about conservative clones can be found in the papers [1] and [2]. In 2005, S. Shelah [3] found an unexpected application of conservative clones to Computational Social Choice. Using the methods of [3], a complete classification of symmetric classes of selection functions with the Arrow property was obtained in [4]. A further development of this approach requires an explicit classification of symmetric conservative clones with a finite carrier, as well as a description of the corresponding fragment of the functional Galois connection $(\mathrm{Inv}_Q, \mathrm{Pol}_Q)$. Functional Galois connection $(\mathrm{Inv}_Q, \mathrm{Pol}_Q)$ is a natural modification of the connection $(\mathrm{Inv}, \mathrm{Pol})$ based on the preservation relation between functions f on a set A (of all finite arities) and sets of functions $h \in A^Q$ for an arbitrary set Q. The $(\mathrm{Inv}_Q, \mathrm{Pol}_Q)$-connection is a convenient tool for studying closed classes of discrete functions since, on the one hand, it is easily transformed into a $(\mathrm{Inv}, \mathrm{Pol})$-connection and, on the other hand, it is closely related to the functional closure.

2 Notation and basic definitions

Let A be an arbitrary non-empty set. For any set Q the symbol A^Q denotes the set of all function $f\colon Q \to A$. Elements of the Cartesian power A^n, $n < \omega$, are identified with sequences $(a_0, a_1, \ldots, a_{n-1})$, $a_i \in A$, $i < n$, i.e. with functions $\mathbf{a}\colon n = \{\, 0, 1, \ldots, n-1 \,\} \to A$; therefore, for any sequence $\mathbf{a} \in A^{<\omega}$ the standard notations $\operatorname{dom} \mathbf{a}$ and $\operatorname{ran} \mathbf{a}$ denote the domain and the range of $\mathbf{a}$, respectively. The set $\{\, \mathbf{a} \in A^n \colon |\operatorname{ran} \mathbf{a}| = k \,\}$ is denoted by A^n_k. In a natural sense, we use the notations $A^n_{<m} = \bigcup_{k<m} A^n_k$, $A^n_{\leqslant m} = \bigcup_{k \leqslant m} A^n_k$, etc. Usually,

when writing sequences we omit special characters, i.e. instead of $(a_0, a_1, \dots, a_{n-1})$ we write $a_0a_1 \dots a_{n-1}$.

The symbol $\mathcal{O}(A)$ denotes the set of all functions over A (of all arities), i.e. $\mathcal{O}(A) = \bigcup_{n<\omega} A^{A^n}$. For any $n < \omega$ and $\mathcal{F} \subseteq \mathcal{O}(A)$, the symbol $\mathcal{F}_{[n]}$ denotes the set of all n-ary functions in $\mathcal{F}$, i.e. $\mathcal{F}_{[n]} = \mathcal{F} \cap A^{A^n}$. For every natural number n, function $f \in \mathcal{O}(A)_{[n]}$ and functions $h_0, h_1, \dots, h_{m-1} \in A^Q$ the symbol $f(h_0, h_1, \dots, h_{m-1})$ denotes the *composition* of these functions defined by

$$f(h_0, h_1, \dots, h_{m-1})(q) = f(h_0(q)h_1(q) \dots h_{m-1}(q))$$

for each $q \in Q$. We hope that it is always clear from the context what the expression $f(a_0a_1 \dots a_{n-1})$ denotes, the value of the function on the sequence $a_0a_1 \dots a_{n-1}$ or the composition f and $a_0a_1 \dots a_{n-1}$. For example, for any $f\colon A \to A$ and $\mathbf{a} = a_0a_1 \dots a_{n-1} \in A^n$, we have $f(\mathbf{a}) = f(a_0)f(a_1) \dots f(a_{n-1})$ (if $n = 1$, we identify an element $a_0 \in A$ and the corresponding one-element sequence).

The function $f \in \mathcal{O}(A)_{[n]}$ which assigns to each sequence $\mathbf{a} \in A^n$, $\mathbf{a} = a_0a_1 \dots a_{n-1}$, the element a_i for some fixed number $i < n$, is called the n-ary i-th *projection* and is denoted by e_i^n. The set of all projection $e \in \mathcal{O}(A)$ is denoted by $\mathcal{E}(A)$. In accordance with the standard notation, we sometimes write x_i instead of e_i^n. For example, for any binary function f, we write $f(x_2, x_1)$ instead of $f(e_2^2, e_1^2)$.

The restriction of a function $h\colon Q \to A$ to a set P is denoted by $h|_P$, i.e. $h|_P = h \cap (P \times A)$ (we do not assume that $P \subseteq Q$). The symbol $h|^P$ denotes the set $\{\, f \in A^{P \cup \operatorname{dom} f} \colon f|_{\operatorname{dom} h} = h \,\}$. For any set H of functions (perhaps with different domains), we denote $H|_P = \{\, h|_P \colon h \in H \,\}$ and $H|^P = \bigcup_{h \in H} h|^P$.

For any $H \subseteq A^Q$, $\mathbf{h} = h_0h_1 \dots h_{n-1} \in H^n$ and $q \in Q$ we denote $H(q) = \{\, h(q) \colon h \in H \,\}$ and $\mathbf{h}(q) = h_0(q)h_1(q) \dots h_{n-1}(q)$.

The set of all subsets of A is denoted by $\mathscr{P}(A)$. The set of all k-element subsets of A is denoted by $[A]^k$, i.e. $[A]^k = \{\, B \subseteq A \colon |B| = k \,\}$. The symbol S_A denotes the set of all permutations of A.

Definition 1. A set $\mathcal{F} \subseteq \mathcal{O}(A)$ is called a *clone* (with the carrier A) if it is closed with respect to composition and contains all projections.

Definition 2. A *natural isomorphism* from clone $\mathcal{F}$ to clone $\mathcal{G}$ with carriers A and B, respectively, is a pair of one-to-one functions $\sigma\colon A \to B$ and $\tau\colon \mathcal{F} \to \mathcal{G}$ for which

$$f \in \mathcal{F}_{[n]} \Rightarrow \tau(f) \in \mathcal{G}_{[n]}$$

for all $f \in \mathcal{F}$ and natural number n, and

$$\sigma(f(\mathbf{a})) = \tau(f)(\sigma(\mathbf{a}))$$

for all $f \in \mathcal{F}$ and $\mathbf{a} \in \operatorname{dom} f$.

Clones $\mathcal{F}$ and $\mathcal{G}$ are *naturally isomorphic* if there is a natural isomorphism from $\mathcal{F}$ to $\mathcal{G}$.

Definition 3. A clone $\mathcal{F} \subseteq \mathcal{O}(A)$ is called *symmetric* if for any function $f \in \mathcal{O}(A)$ and permutation σ of A

$$f \in \mathcal{F} \Rightarrow f_\sigma \in \mathcal{F},$$

where for any $f \in \mathcal{O}(A)$ the function f_σ is defined by

$$f_\sigma(\mathbf{a}) = \sigma^{-1}(f(\sigma(\mathbf{a})))$$

for all $\mathbf{a} \in \operatorname{dom} f$.

3 Galois connections $(\mathrm{Inv}_Q, \mathrm{Pol}_Q)$

The Galois connection $(\mathrm{Inv}, \mathrm{Pol})$ is one of the basic concepts in the theory of discrete functions, see [5], [6]. Galois connection $(\mathrm{Inv}, \mathrm{Pol})$ allows one to characterize clones using their invariant sets. Recall that a function $f \in \mathcal{O}(A)_{[n]}$ *preserves* a predicate $P \subseteq A^m$ if for all $a_0^0 a_1^0 \dots a_{m-1}^0$, $a_0^1 a_1^1 \dots a_{m-1}^1$, …, $a_0^{n-1} a_1^{n-1} \dots a_{m-1}^{n-1}$ from P we have

$$f(a_0^0 a_0^1 \dots a_0^{n-1}) f(a_1^0 a_1^1 \dots a_1^{n-1}) \dots f(a_{m-1}^0 a_{m-1}^1 \dots a_{m-1}^{n-1}) \in P.$$

For any set $\mathcal{F} \subseteq \mathcal{O}(A)$ the set of all predicates P such that any function $f \in \mathcal{F}$ preserves P is denoted by $\mathrm{Inv}\,\mathcal{F}$. In the opposite direction, for any set $\mathbb{P}$ of predicates the set of all functions $f \in \mathcal{O}(A)$ such that f preserves any predicate $P \in \mathbb{P}$ is denoted by $\mathrm{Pol}\,\mathbb{P}$. The pair $(\mathrm{Inv}, \mathrm{Pol})$ is a Galois connection between boolean lattices $\mathscr{P}\left(\bigcup_{m<\omega} \mathscr{P}(A^m)\right)$ and $\mathscr{P}(\mathcal{O}(A))$. Any Galois closed set $\mathcal{F} \in \mathcal{O}(A)$ is a clone. If A is a finite set, the class of Galois closed sets $\mathcal{F} \in \mathscr{P}(\mathcal{O}(A))$ coincides with the class of all clones with the carrier A.

However, it will be more convenient for us to use the concept of functional Galois connections.

Definition 4. Let A and Q be non-empty sets. A function $f \in \mathcal{O}(A)_{[n]}$ *preserves* a set $H \subseteq A^Q$ if for all $h_0, h_1 \dots, h_{n-1} \in H$ the set H contains the function $f(h_0, h_1 \dots, h_{n-1})$.

For any set $\mathcal{F} \subseteq \mathcal{O}(A)$ the set of all sets $H \subseteq A^Q$ such that any function $f \in \mathcal{F}$ preserves H is denoted by $\mathrm{Inv}_Q\,\mathcal{F}$. Any set $H \in \mathrm{Inv}_Q\,\mathcal{F}$ is called a *Q-invariant set* of $\mathcal{F}$.

For any set $\mathbb{H} \subseteq \mathscr{P}(A^Q)$ the set of all functions $f \in \mathcal{O}(A)$ such that f preserves any set $H \in \mathbb{H}$ is denoted by $\mathrm{Pol}_Q\,\mathbb{D}$.

Proposition 1. *For all sets $A \neq \varnothing$, $\mathcal{F} \subseteq \mathcal{O}(A)$, Q, $H, H' \subseteq A^Q$, P and function $f\colon P \to Q$*

1. $H \in \mathrm{Inv}_Q \mathcal{F} \Rightarrow \{\, h(f) \colon h \in H \,\} \in \mathrm{Inv}_P \mathcal{F}$,
2. $H \in \mathrm{Inv}_Q \mathcal{F} \Rightarrow H|_P \in \mathrm{Inv}_{P \cap Q} \mathcal{F}$,
3. $H \in \mathrm{Inv}_Q \mathcal{F} \Rightarrow H|^P \in \mathrm{Inv}_{P \cup Q} \mathcal{F}$,
4. $H, H' \in \mathrm{Inv}_Q \mathcal{F} \Rightarrow H \cap H' \in \mathrm{Inv}_Q \mathcal{F}$,

Proof. By a direct verification. □

There is a simple relationship between these two preservation relations. Consider an m-ary predicate P over A as a set of functions $\mathbf{a}\colon m \to A$. It is easy to verify that for any function $f \in \mathcal{O}(A)$, $P \in \mathrm{Inv}\,\{\, f \,\}$ if and only if $P \in \mathrm{Inv}_m\,\{\, f \,\}$. On the other hand, suppose that Q is a finite set of cardinality m, and let some numbering $Q = \{\, q_0, q_1, \dots, q_{m-1} \,\}$ be fixed. It easy to check that a function $f \in \mathcal{O}(A)$ preserves a set $H \subseteq A^Q$ if and only if

f preserves the predicate $P = \{\, h(q_0)h(q_1)\dots h(q_{m-1}) \colon h \in H \,\}$. These arguments allow us immediately to obtain some statements for $(\mathrm{Inv}_Q, \mathrm{Pol}_Q)$ connection. In particular, the following proposition holds.

Proposition 2. *For any non-empty sets A and Q the pair $(\mathrm{Inv}_Q, \mathrm{Pol}_Q)$ is a Galois connection between the Boolean latices $\mathscr{P}(\mathscr{P}(A^Q))$ and $\mathscr{P}(\mathcal{O}(A))$. Any Galois closed set $\mathcal{F} \in \mathcal{O}(A)$ is a clone.*

Also note that for any clone $\mathcal{F} \subseteq \mathcal{O}(A)$, $\mathcal{F}_{[n]} \in \mathrm{Inv}_{[A^n]} \mathcal{F}$. It immediately follows that each clone $\mathcal{F}$ with a finite carrier is uniquely characterized by the set $\mathrm{Inv}\, \mathcal{F}$. Thus, a family of Galois connections $(\mathrm{Inv}_Q, \mathrm{Pol}_Q)$ unites the concepts of invariant sets and functional closure. In some papers, other Galois connections $(\mathrm{Inv}_Q, \mathrm{Pol}_Q)$ are considered. E.g., the case $Q = [A]^r$ is studied in [3], [4].

4 Decomposition theorems

Now we will show that under certain conditions, the set $\mathrm{Inv}_Q \mathcal{F}$ is arranged quite simply. Let A, Q be arbitrary sets, and let $H \subseteq A^Q$ and $\mathscr{R} \subseteq \mathscr{P}(Q)$. The set

$$H_{(\mathscr{R})} = \{\, (H|_R)|^Q \colon R \in \mathscr{R} \,\}$$

is called a *decomposition* of H over $\mathscr{R}$. It is easy to verify that the following proposition is true.

Proposition 3. *For all $H \subseteq A^Q$ and $\mathscr{R} \subseteq \mathscr{P}(Q)$*

1. *$H \subseteq \bigcap H_{(\mathscr{R})}$,*
2. *$\left(\bigcap H_{(\mathscr{R})}\right)|_R = H|_R$ for any set $R \in \mathscr{R}$.*

Definition 5. A set $H \subseteq A^Q$ is *decomposable* over $\mathscr{R}$ if $H = \bigcap H_{(\mathscr{R})}$.

We show that clones satisfying the conditions Δ^∂, Δ^e_r, and Δ^2 defined below have Q-invariant sets that are decomposable over non-trivial sets $\mathscr{R}$.

Definition 6. Let $\mathcal{F}$ be a clone with a carrier A and n a natural number, $n \geqslant 2$. The clone $\mathcal{F}$ satisfies the condition

- Δ^s_n if there is a natural number $i < n$ such that for all $\mathbf{a} \in A^n_n$ and $a \in \mathrm{ran}\, \mathbf{a}$ there is a function $s \in \mathcal{F}_{[n]}$ for which
$$s(\mathbf{a}) = a \text{ and } s(\mathbf{x}) = x_i \text{ for all } \mathbf{x} = x_0x_1\dots x_{n-1} \in A^n_{<n};$$
- Δ^∂ if for all $\mathbf{a} \in A^3_3$ and $a \in \mathrm{ran}\, \mathbf{a}$ there is a function $\partial \in \mathcal{F}_{[3]}$ for which
$$\partial(\mathbf{a}) = a \text{ and } \partial(xxy) = \partial(xyx) = \partial(yxx) = x \text{ for all } x, y \in A;$$
- Δ^2 if for all $\mathbf{a}, \mathbf{b} \in A^2_2$ such that $\mathrm{ran}\, \mathbf{a} \neq \mathrm{ran}\, \mathbf{b}$, and for all $a \in \mathrm{ran}\, \mathbf{a}$, and $b \in \mathrm{ran}\, \mathbf{b}$ there is a function $w \in \mathcal{F}_{[2]}$ for which
$$w(\mathbf{a}) = a,\ w(\mathbf{b}) = b \text{ and } w(xx) = x \text{ for all } x \in A.$$

Definition 7. Let A, $B \subseteq A$, Q, $H \subseteq A^Q$ be non-empty sets and n a natural number. We denote by

(1) $[Q]^{2,0}_H$ the set of all sets $P = \{ p, q \} \in [Q]^2$ such that there is a permutation $\sigma \in S_A$ for which $h(q) = \sigma(h(p))$ for all $h \in H$,

(2) $[Q]^{2,id}_H$ the set of all sets $P = \{ p, q \} \in [Q]^2$ such that $h(q) = h(p)$ for all $h \in H$,

(3) $[Q]^{2,1}_H$ the set of all sets $P = \{ p, q \} \in [Q]^2$ such that there are $a, b \in A$ for which $h(p) = a \vee h(q) = b$ for all $h \in H$,

(4) $Q^{(n)}_H$ the set $\{ q \in Q \colon |H(q)| < n \}$,

(5) $Q^{[B]}_H$ the set $\{ q \in Q \colon |H(q)| \subseteq B \}$.

We will use the following technical definition.

Definition 8. Let $H \subseteq A^Q$, $p, q \in Q$ and $a \in H(p)$. We say that H *weakly separates p from q at the point a* if H contains functions h_1 and h_2 such that

$$h_1(p) = h_2(p) = a \text{ and } h_1(q) \neq h_2(q)$$

If H weakly separates p *from* q or q *from* p at least at one point, we will simply say that it *weakly separates p and q*.

We say that H *strongly separates p from q at the point a* if for each $b \in H(q)$, H contains a function h such that

$$h(p) = a \text{ and } h(q) = b.$$

Any function $\partial \in \mathcal{O}(A)_{[3]}$ satisfying $\partial(xxy) = \partial(xyx) = \partial(yxx) = x$ is called a *∂-function* (the terms *majority function* and *discriminator* are also often used). Any function $w \in \mathcal{O}(A)$ satisfying $w(xx \dots x) = x$ is called an *idempotent function*.

Theorem 1. *Let A, Q and $H \subseteq A^Q$ be non-empty finite sets. Let $\mathcal{F}$ be a clone with the carrier A, and let at least one of the following two conditions hold:*

(a) $\mathcal{F}$ satisfies Δ^∂,

(b) $\mathcal{F}$ contains a ∂-function, and $|H(q)| \leqslant 2$ for all $q \in Q$.

Then $H \in \mathrm{Inv}_Q\mathcal{F}$ if and only if the following conditions hold:

1. $H|_P \in \mathrm{Inv}_P\mathcal{F}$ for all $P \in [Q]^1 \cup [Q]^{2,0}_H \cup [Q]^{2,1}_H$,

2. H is decomposable over $[Q]^1 \cup [Q]^{2,0}_H \cup [Q]^{2,1}_H$.

Proof. If $|Q| = 1$, the theorem is obvious. Assume $|Q| \geqslant 2$. In the *if* direction the theorem follows immediately from Proposition 1. Let us prove the *only if* direction. Let $H \in \mathrm{Inv}_Q\mathcal{F}$. Item 1 again follows from Proposition 1. To prove item 2, we first prove the following Lemmas. We assume that all the premises of the theorem hold.

Lemma 1. *Let $p, q \in Q$, $a \in H(p)$ and H weakly separate p from q at the point a. Then H strongly separates p from q at the point a.*

Proof. Obviously, the lemma is true if $H(q) \leqslant 2$. Suppose, on the contrary, that $H(q) \geqslant 3$. Let $b \in H(q)$. Suppose that H does not contain a function h such that $h(p) = a$ and $h(q) = b$. Choose functions $h_0, h_1, h_2 \in H$ and distinct elements $d, c \in A$ for which

$$h_0(p) = h_1(p) = a,\ h_0(q) = c,\ h_1(q) = d,\ h_2(q) = b.$$

By the above supposition, $b \notin \{ c, d \}$, so $cdb \in A_3^3$. By the premises of the theorem there is a ∂-function $\partial \in \mathcal{F}$ such that $\partial(cdb) = b$. Consider the function $f = \partial(h_0, h_1, h_2)$. Since $H \in \mathrm{Inv}_Q\mathcal{F}$ we have $f \in H$. However, it is easy to calculate that $f(p) = a$ and $f(q) = b$, a contradiction. □

Lemma 2. *Let $P = \{ p, q \} \in [Q]^2$. Then one of the following three cases holds:*

1. *$H|_P$ is the set of all functions $h \in A^Q$ such that $h(p) \in H(p)$ and $h(q) \in H(q)$,*
2. *$H|_P$ is the set of all functions $h \in A^Q$ such that $h(p) \in H(p)$ and $h(q) \in H(q)$, and $h(q) = \sigma(h(p))$ for some $\sigma \in S_A$,*
3. *$H|_P$ is the set of all functions $h \in A^Q$ such that $h(p) \in H(p)$ and $h(q) \in H(q)$, and $(h(p) = a \vee h(q) = b)$ for some $a, b \in A$.*

Proof. Let H do not weakly separate p and q. Since H does not weakly separates p from q, there is a function $\sigma\colon H(p) \to H(q)$ such that $h(q) = \sigma(h(p))$ for all $h \in H$. Obviously, σ is a surjective function. Suppose that $\sigma(a) = \sigma(b)$ for some distinct $a, b \in H(p)$. Choose a function $h_0, h_1 \in H$ such that $h_0(p) = a$ and $h_1(p) = b$. We have $h_0(q) = h_1(q)$. This means that H weakly separates q from p at the point $h_1(q)$, a contradiction. Therefore, σ is a one-to-one mapping. We can extend σ to some permutation of A. So, we have the case 2.

Let H weakly separate one of the elements p, q from the other at least at two distinct points. Without loss of generality, we assume that H weakly separates p from q at least at two distinct points. By Lemma 1, H strongly separates p from q at least at two distinct points. So, for any $b \in H(q)$ there are functions $h_0, h_1 \in H$ such that $h_0(q) = h_1(q) = b$ and $h_0(p) \neq h_1(p)$. Hence, H weakly (and, by Lemma 1, strongly) separates q from p at any point $b \in H(q)$. We have the case 1.

Now let the two previous assumptions do not satisfied. Without loss of generality, we assume that H weakly separates p from q at the unique point a. If $|H(p)| = 1$, by Lemma 1, we have case 1. Let $|H(p)| \geqslant 2$, a' be an arbitrary element of $H(p) \setminus \{ a \}$, and h an arbitrary function in H for which $h(p) = a'$. By Lemma 1 there is a function $h_0 \in H$ such that $h_0(p) = a$ and $h_0(q) = b$. So, H weakly separates q from p at the point b. By assumption, there is at most one such point $b \in H(q)$. We have $h(q) = b$ for any function h satisfying $h(p) \neq a$, the case 3. □

It follows from Lemma 2 that it suffices to prove that H is decomposable over $[Q]^2$. Define

$$H^* = \bigcap_{P \in [Q]^2} (H|_P)|^Q.$$

By Proposition 3, $H \subseteq H^*$. Therefore, it suffices to prove the reverse inclusion.

We prove that for any set $Q' \subseteq Q$ and function $f \in H^*$ there exists a function $h \in H$ such that $f|_{Q'} = h|_{Q'}$. By induction on the cardinality of Q'. If $|Q'| = 1$, this follows from Proposition 3.

Let $|Q'| \geqslant 2$ and $f \in H^*$. Choose two distinct $p, q \in Q'$. By the induction hypothesis there are two functions $h_p, h_q \in H$ that coincide with f on $Q' \setminus \{ p \}$ and $Q' \setminus \{ q \}$, respectively. In addition, by definition of H^*, there is a function $h_{pq} \in H$ that coincides with f on $\{ p, q \}$, i.e. $h_{pq}(p) = f(p)$ and $h_{pq}(q) = f(q)$. Choose an arbitrary ∂-function $\partial \in \mathcal{F}$, and consider the function $h = \partial(f_p, f_q, f_{p,q})$. We have $h \in H$ because $H \in \mathrm{Inv}_Q \mathcal{F}$. Moreover, the following equalities are true:

$$\begin{aligned} h(p) &= \partial(f_p(p) f_q(p) f_{p,q}(p)) = \partial(f_p(p) f(p) f(p)) = f(p), \\ h(q) &= \partial(f_p(q) f_q(q) f_{p,q}(q)) = \partial(f(q) f_q(q) f(q)) = f(q), \\ h(x) &= \partial(f_p(x) f_q(x) f_{p,q}(x)) = \partial(f(x) f(x) f_{p,q}(x)) = f(x) \end{aligned}$$

for all $x \in Q' \setminus \{ p, q \}$. The induction step is proved.

□

Theorem 2. *Let A, Q and $H \subseteq A^Q$ be non-empty finite sets and n a natural number, $n \geqslant 3$. Let $\mathcal{F}$ be a clone with the carrier A satisfying Δ_n^s.*

Then $H \in \mathrm{Inv}_Q \mathcal{F}$ if and only if

1. *$H|_P \in \mathrm{Inv}_P \mathcal{F}$ for all $P \in [Q]^1 \cup [Q]_H^{2,0} \cup \{ Q_H^{(n)} \}$,*
2. *H is decomposable over $[Q]^1 \cup [Q]_H^{2,0} \cup \{ Q_H^{(n)} \}$.*

Proof. If $|Q| = 1$, the theorem is obvious. Assume $|Q| \geqslant 2$. In the *if* direction the theorem follows immediately from Proposition 1. Let us prove the *only if* direction. Let $H \in \mathrm{Inv}_Q \mathcal{F}$. Item 1 again follows from Proposition 1. To prove item 2, we first prove the following Lemmas. We assume that all the premises of the theorem hold.

Lemma 3. *Let $j < n$, $\mathbf{b} \in A_n^n$ and $b \in \operatorname{ran} \mathbf{b}$. Then there is a function $t \in \mathcal{F}$ for which $t(\mathbf{b}) = b$ and $t(\mathbf{x}) = x_j$ for all $\mathbf{x} = x_0 x_1 \dots x_{n-1} \in A_{<n}^n$.*

Proof. By condition Δ_n^s we have that there is a natural number $i < n$ such that for all $\mathbf{a} \in A_n^n$ and $a \in \operatorname{ran} \mathbf{a}$ there is a function $s \in \mathcal{F}_{[n]}$ for which $s(\mathbf{a}) = a$ and $s(\mathbf{x}) = x_i$ for all $\mathbf{x} = x_0 x_1 \dots x_{n-1} \in A_{<n}^n$. If $j = i$, the lemma is proved. Let $j \neq i$, and τ be the transposition $(i, j) \in S_n$. Choose a function $s \in \mathcal{F}_{[n]}$ for which $s(\mathbf{b}(\tau)) = b$ and $s(\mathbf{x}) = x_i$ for all $\mathbf{x} = A_{<n}^n$. It is easy to see that we can put $t = s(x_{\tau(0)}, x_{\tau(1)}, \dots, x_{\tau(n-1)})$. □

Lemma 4. *Let $p, q \in Q$, $a \in H(p)$, $|H(q)| \geqslant n$ and H weakly separate p from q at the point a. Then H strongly separates p from q at the point a.*

Proof. For an arbitrary element $b \in H(q)$, assume that H does not contain a function h such that $h(p) = a$ and $h(q) = b$. Choose a functions $h_0, h_1, h_2 \in H$, and distinct elements $c, d \in A$ such that

$$h_0(p) = h_1(p) = a,\ h_0(q) = c,\ h_1(q) = d,\ h_2(q) = b.$$

By the above assumption, $b \notin \{c, d\}$. Using the inequality $|H(q)| \geqslant n$, choose $n-3$ functions $h_3, h_4, \dots, h_{n-1}$ from H such that $\mathbf{b} = h_0(q)h_1(q)h_2(q)\dots h_{n-1}(q)$ is a repetition-free sequence. The sequence $\mathbf{a} = h_0(p)h_1(p)h_2(p)\dots h_{n-1}(p)$ belongs to $A^n_{<n}$. By Lemma 3, there exists a function $t \in \mathcal{F}$ for which $t(\mathbf{a}) = h_0(p) = a$ и $t(\mathbf{b}) = h_2(q) = b$. Consider the function $h = t(h_0, h_1, \dots, h_{n-1})$. Since $H \in \mathrm{Inv}_Q\mathcal{F}$ we have $h \in H$. However, it is easy to calculate that $h(p) = t(\mathbf{a}) = a$ и $h(q) = t(\mathbf{b}) = b$; a contradiction. □

Lemma 5. *Let $P = \{p, q\} \in [Q]^2$, $P \nsubseteq Q^{(n)}_H$. Then one of the following two cases holds:*

1. *$H|_P$ is the set of all functions $h \in A^P$ such that $h(p) \in H(p)$ and $h(q) \in H(q)$,*
2. *$H|_P$ is the set of all functions $h \in A^P$ such that $h(p) \in H(p)$ and $h(q) \in H(q)$, and $h(q) = \sigma(h(p))$ for some $\sigma \in S_A$.*

Proof. If H does not weakly separate p and q then we have the case 2, which can be proved in the same way as in Theorem 1.

Otherwise, we show that the case 1 holds. Without loss of generality, we assume that $|H(p)| \leqslant |H(q)|$. Therefore, we have $|H(q)| \geqslant n$. We show that without loss of generality we can assume that H weakly separates p from q. In fact, otherwise there exists a surjective function $\sigma\colon H(p) \to H(q)$. Hence, $|H(p)| = |H(q)|$, and p and q can be interchanged if necessary.

Let H weakly separate p from q at the point $a \in H(p)$. By Lemma 4, it suffices to show that H weakly separates p from q at each point $a' \in H(p) \setminus \{a\}$.

Choose an arbitrary element $a' \in H(p) \setminus \{a\}$. Using Lemma 4, choose some functions $h_0, h_1, \dots, h_{n-1} \in H$ such that $h_0(p) = a'$, $h_1(p) = h_2(p) = \dots = h_{n-1}(p) = a$, and $\mathbf{b} = h_0(q)h_1(q)h_2(q)\dots h_{n-1}(q)$ is a repetition-free sequence. Since the sequence $\mathbf{a} = h_0(p)h_1(p)h_2(p)\dots h_{n-1}(p) = a'a\dots a$ belongs to $A^n_{<n}$, it follows from Lemma 3 that $\mathcal{F}$ contains a function t such that $t(\mathbf{a}) = a'$ and $t(\mathbf{b}) \neq h_0(q)$. Then the values of the functions h_0 and $h = t(h_0, h_1, h_2, \dots, h_{n-1})$ coincide (and are equal to a') on p and are different on q. The function h belongs to H because $H \in \mathrm{Inv}_Q\mathcal{F}$. So, H weakly separates p from q at the point a'. □

It follows from Lemma 5 that it suffices to prove that H is decomposable over $[Q]^2 \cup \{Q^{(n)}_H\}$. Define

$$H^* = (H|_{Q^{(n)}_H})|^Q \cap \bigcap_{P \in [Q]^2} (H|_P)|^Q.$$

By Proposition 3, $H \subseteq H^*$. Therefore, to prove the second part of the theorem it suffices to prove the reverse inclusion.

We prove that for any set $Q' \subseteq Q$ and function $f \in H^*$ there exists a function $h \in H$ such that $f|_{Q'} = h|_{Q'}$. By induction on the cardinality of Q'. If $Q' \subseteq Q^{(n)}_H$ or $|Q'| = 1$, the statement follows from Proposition 3.

Let $|Q'| \geqslant 3$, $Q' \nsubseteq Q^{(n)}_H$, and $f \in H^*$. Choose two distinct $p, q \in Q'$ such that $H(q) \geqslant n$. By the induction hypothesis there are two functions $h_p, h_q \in H$ that coincide with f on $Q' \setminus \{p\}$ and $Q' \setminus \{q\}$, respectively. In particulary,

$$h_q(p) = f(p) \text{ and } h_p(q) = f(q).$$

If $h_q(q) = f(q)$, we put $h = f_q$.

Let $f_q(q) \neq h(q)$. Consider the set $H|_P$ where $P = \{\, p, q \,\}$. If the case 2 of the Theorem holds, we have

$$h_q(q) = \sigma(h_q(p)) = \sigma(f(p)) = f(q),$$

a contradiction.

Then the case 1 of the Theorem holds. Choose $n-2$ functions $h_2, h_3, \ldots, h_{n-1}$ from H for which $h_2(p) = h_3(p) = \ldots = h_{n-1}(p) = f(p)$ and

$$\mathbf{b} = h_q(q)h_p(q)h_2(q)h_3(q)\ldots h_{n-1}(q)$$

is a repetition-free sequence. Using Lemma 3 choose a function $t \in \mathcal{F}$ such that $t(\mathbf{b}) = h_p(q) = f(q)$ and $t(\mathbf{x}) = x_0$ for all $\mathbf{x} = x_0x_1\ldots x_{n-1} \in A^n_{<n}$. Consider the function $h = t(h_q, h_p, h_2, h_3, \ldots, h_{n-1})$. We have $h \in H$ because $H \in \mathrm{Inv}_Q\mathcal{F}$. Note that the sequence

$$\mathbf{a} = h_q(p)h_p(p)h_2(p)h_3(p)\ldots h_{n-1}(p)$$

belongs to $A^n_{<n}$. Moreover, for all $x \in Q' \setminus \{\, p, q \,\}$ we have $h_p(x) = h_q(x) = f(x)$, and, therefore, the sequence

$$h_q(x)h_p(x)h_2(x)h_3(x)\ldots h_{n-1}(x)$$

also belongs to $A^n_{<n}$. Consequently,

$$\begin{aligned}
&h(p) = t(\mathbf{a}) = h_q(p) = f(p),\\
&h(q) = t(\mathbf{b}) = h_p(q) = f(q),\\
&h(x) = t(h_q(x)h_p(x)h_2(x)h_3(x)\ldots h_{n-1}(x)) = h_q(x) = f(x)
\end{aligned}$$

for all $x \in Q' \setminus \{\, p, q \,\}$. The induction step is proved. $\square$

Theorem 3. *Let A, Q and $H \subseteq A^Q$ be non-empty finite sets. Let $\mathcal{F}$ be a clone with the carrier A satisfying Δ^2.*

Then $H \in \mathrm{Inv}_Q\mathcal{F}$ if and only if

1. *$H|_P \in \mathrm{Inv}_P\mathcal{F}$ for all $P \in [Q]^1 \cup \{\, Q_H^{[B]} \colon B \in [A]^2 \,\}$,*
2. *H is decomposable over $[Q]^1 \cup [Q]^{2,id}_H \cup \{\, Q_H^{[B]} \colon B \in [A]^2 \,\}$.*

Proof. As in the previous theorems, in the direction *if* the theorem follows immediately from Proposition 1 (note that for any $P \in [Q]^{2,id}_H$ the set $H|_P$ is preserved by any function $f \in \mathcal{O}(A)$). Let us prove the theorem in the opposite direction. Let $H \in \mathrm{Inv}_Q\mathcal{F}$. Item 1 again follows from Proposition 1.

Define

$$H^* = \bigcap_{B\in[A]^2} (H|_{Q_H^{[B]}})|^Q \cap \bigcap_{P\in[Q]^{2,id}_H} (H|_P)|^Q \cap \bigcap_{q\in Q} (H|_{\{\, q \,\}})|^Q.$$

By Proposition 3, $H \subseteq H^*$. Therefore, to prove the theorem it suffices to prove the reverse inclusion.

We prove that for any set $Q' \subseteq Q$ and function $f \in H^*$ there exists a function $h \in H$ such that $f|_{Q'} = h|_{Q'}$. Induction on the cardinality of Q'. If $|Q'| = 1$ or $Q' \subseteq Q_H^{[B]}$ for some $B \in [A]^2$, the statement follows from Proposition 3.

Let $|Q'| \geqslant 2$, $Q' \nsubseteq Q_H^{[B]}$ for all $B \in [A]^2$, and $f \in H^*$. By the induction hypothesis, for any $q \in Q'$ there is a function $h_q \in H$ which coincides with f on $Q' \setminus \{ q \}$. Fixe some family $\{ h_q \}_{q \in Q'}$ of such functions.

Lemma 6. *At least one of the following conditions holds:*

1. *There is $q \in Q'$ for which $h_q(q) = f(q)$.*
2. *$[Q]_H^{2,id} \cap [Q']^2 = \varnothing$, and for all $q \in Q'$ and $a \in H(q)$ there is a function $f_{q,a} \in H^*$ satisfying*
$$f_{q,a}|_{Q' \setminus \{ q \}} = h_q|_{Q' \setminus \{ q \}} \text{ and } f_{q,a}(q) = a.$$

Proof. Let $q \in Q'$, $a \in H(q)$ and $f^-_{q,a}$ be a function from Q' to A, satisfying

$$f^-_{q,a} = h_q|_{Q' \setminus \{ q \}} \text{ and } f^-_{q,a}(q) = a.$$

Let condition 1 do not hold. If $[Q]_H^{2,id} \cap [Q']^2 \neq \varnothing$, we have $h_{q'}(q') = h_{q'}(p') = f(p') = f(q')$ for some distinct $p', q' \in Q'$, a contradiction.

Hence, it suffices to show that $f^-_{q,a}$ belongs to each of the sets $H|_{Q_H^{[B]} \cap Q'}$ where $B \in [A]^2$.

Suppose that there is $B \in [A]^2$ such that

$$f^-_{q,a} \notin H|_{Q_H^{[B]} \cap Q'}. \tag{$*$}$$

This means that for all $g \in H$, if g and $f^-_{q,a}$ coincide on the set $(Q_H^{[B]} \cap Q') \setminus \{ q \}$, then $g(q) \neq a$. Recall that h_q and $f^-_{q,a}$ coincide on $Q' \setminus \{ q \}$. Therefore, $h_q(q) \neq a$.

On the other hand, f and $f^-_{q,a}$ coincide on $Q' \setminus \{ q \}$. Since $f \in H^*$, we have

$$f|_{Q_H^{[B]}} \in H|_{Q_H^{[B]}}.$$

If $q \notin Q_H^{[B]}$ or $f(q) = f^-_{q,a}(q) = a$ these conditions contradict supposition ($*$). So, $q \in Q_H^{[B]}$, and, consequently, $|H(q)| \leqslant 2$. Moreover, $f(q) \neq a$. Therefore, we have $h_q(q) = f(q)$, a contradiction. $\square$

We continue the proof of the induction step. If $h_q(q) = f(q)$ for some $q \in Q'$ we can put $h = h_q$. In the sequel, we assume that

$$h_q(q) \neq f(q) \tag{$**$}$$

for all $q \in Q'$. Then by Lemma 6 and the induction hypothesis, for any distinct $p, q \in Q'$ and $a \in H(q)$ the set H contains a function $h_{p,q,a}$ that coincides with $f_{q,a}$ on $Q' \setminus \{ p \}$. Thus, for all distinct $p, q \in Q'$ and $a \in H(q)$ we have

$$\begin{aligned}
&h_{p,q,a}|_{Q' \setminus \{ p,q \}} = h_q|_{Q' \setminus \{ p,q \}} = f|_{Q' \setminus \{ p,q \}} \\
&h_p(q) = f(q), h_q(p) = f(p) \\
&h_{p,q,a}(q) = a \\
&h_p, h_q, h_{p,q,a} \in H.
\end{aligned}$$

Case 1: $\operatorname{ran} f(p)h_p(p) \neq \operatorname{ran} h_q(q)f(q)$ for some distinct $p, q \in Q'$.

Choose such p and q. Using Δ^2 choose an idempotent function $w_0 \in \mathcal{F}_{[2]}$ such that

$$w_0(f(p)h_p(p)) = f(p) \text{ and } w_0(h_q(q)f(q)) = f(q).$$

We can put $h = w_0(h_q, h_p)$.

Case 2: $H(p) \neq H(q)$ for some $p, q \in Q'$.

Choose such p and q. Without loss of generality, assume $H(p) \setminus H(q) \neq \varnothing$. Choose $a \in H(p) \setminus H(q)$. Using Δ^2 choose idempotent functions $w_0, w_1 \in \mathcal{F}_{[2]}$ such that

$$\begin{array}{ll} w_0(f(p)a) = f(p), & w_0(h_q(q)f(q)) = f(q) \\ w_1(h_p(p)a) = a, & w_1(f(q)h_{q,p,a}(q)) = f(q). \end{array}$$

We can put $h = w_0(h_q, w_1(h_p, h_{q,p,a}))$.

Case 3: Case 1 and Case 2 do not hold.

We have $H(p) = H(q)$ for all $p, q \in Q'$. Denote $C = H(p)$ for some $p \in Q'$. If $|C| \leqslant 2$ we go back to the induction base. Further, we assume that $|C| \geqslant 3$.

Subcase 3.1: there exist distinct $p, q \in Q'$ for which $f(p) = f(q)$. Choose such p and q and denote $f(p) = a$. Since the assumption $(**)$ is true and the Case 1 does not hold, we have $h_p(p) = h_q(q) = b$ for some $b \in A \setminus \{ a \}$. Choose $c \in C \setminus \{ a, b \}$. Using Δ^2 choose idempotent functions $w_0, w_1 \in \mathcal{F}_{[2]}$ such that

$$w_0(ac) = w_0(ba) = a, \; w_1(bc) = c \text{ and } w_1(af_{q,p,c}(q)) = a.$$

We can put $h = w_0(h_q, w_1(h_p, h_{q,p,c}))$.

Subcase 3.2: $f(p) \neq f(q)$ for all distinct $p, q \in Q'$.

First, let Q' contain only two elements p and q. Denote $f(p) = a$ and $f(q) = b$. Since the assumption $(**)$ is true and the Case 1 does not hold, we have $f_q(q) = a$ and $f_p(p) = b$.

Further, by the assumption $(**)$ and Lemma 6 we have that there is a function $h' \in H$ such that $h'(p) \neq h'(q)$. Denote $h'(p) = c$ and $h'(q) = d$.

We show that we can choose $c \neq b$. Consider the set

$$D = \{ x \in C \colon \text{for all } h \in H \; h(p) = x \to h(q) = x \}.$$

It suffices to prove $|D| \leqslant 1$. Let $x \in D \setminus \{ d \}$. Choose a function $h'' \in H$ such that $h''(p) = x$. Using Δ^2 choose an idempotent functions $w \in \mathcal{F}_{[2]}$ for which $w(cx) = x$ and $w(dx) = d$. Then we have $h''' = w(h', h'') \in H$, $h'''(p) = x$ and $h'''(q) = d$, a contradiction. So, $D \subseteq \{ d \}$.

Now using Δ^2 choose idempotent functions $w_0, w_1 \in \mathcal{F}_{[2]}$ such that

$$w_0(ac) = a, \; w_0(ab) = b, \; w_1(bc) = c, \; w_1(bd) = b.$$

We can put $h = w_0(h_q, w_1(h_p, h'))$.

Now let $|Q'| \geqslant 3$. Choose pairwise different $p, q, r \in Q'$. Denote $f(p) = a$, $f(q) = b$ и $f(r) = c$. By by the assumption $(**)$ and Lemma 6 the function $f_{p,b}$ and $f_{q,c}$ belong to H^*. In addition, each of them satisfies subcase 3.1. Consequently, there are functions $h^*_{p,b}, h^*_{q,c} \in H$ which coincide with the functions $f_{p,b}$ and $f_{q,c}$ on the whole set Q', respectively. Using Δ^2 choose an idempotent function $w_0 \in \mathcal{F}_{[2]}$ for which

$$w_0(ba) = a, \; w_0(bc) = b.$$

We can put $h = w_0(h^*_{p,b}, h^*_{q,c})$. The induction step is proved. $\square$

Remarks Theorem 1 is close to the main result of paper [8]. In addition, Theorems 1 and 2 in a particular case are proved in [3]. In our proofs, we used some ideas of [3].

5 Conservative clones

In this section, we apply decomposition theorems to the classification of symmetric conservative clones with a finite carrier A.

Definition 9. A function $f \in \mathcal{O}(A)$ is called *conservative* if

$$f(\mathbf{a}) \in \operatorname{ran} \mathbf{a}$$

for each $\mathbf{a} \in \operatorname{dom} f$. .

Note that in terms of $(\mathrm{Inv}, \mathrm{Pol})$-connection a conservative function can be characterized as a function preserving any unary predicate.

It easy to check that the set of all conservative function $f \in \mathcal{O}(A)$ is a clone. We denote this clone by the symbol $\mathcal{C}(A)$. Any clone $\mathcal{F} \subseteq \mathcal{C}(A)$ is called *conservative.* The case $|A| = 1$ is of no interest. Therefore, we further believe that $|A| \geqslant 2$.

Now we introduce the concept of *characteristic* of conservative clone.

Definition 10. Let $\mathcal{F}$ be a clone with a carrier A. Then $\mathrm{r}(\mathcal{F})$ is the minimal natural number r for which there exists a r-ary function $f \in \mathcal{F}$ that is not a projection. If $\mathcal{F} = \mathcal{E}(A)$ we put $\mathrm{r}(\mathcal{F}) = \omega$.

Obviously, if a clone $\mathcal{F}$ is conservative, then it does not contain 0-ary functions and any unary function $f \in \mathcal{F}$ is a projection. So, $\mathrm{r}(\mathcal{F}) \geqslant 2$ for each conservative clone $\mathcal{F}$.

Definition 11. *Let $\mathcal{F}$ be a clone with a carrier A. Then, for any natural number n, $\mathrm{R}_n(\mathcal{F})$ is a binary relation on A^n defined by*

$$\mathbf{a}\,\mathrm{R}_n(\mathcal{F})\,\mathbf{b} \leftrightarrow (\exists \sigma \in S_A)(\forall f \in \mathcal{F}_{[n]})\, f(\mathbf{b}) = \sigma f(\mathbf{a}).$$

We define $\mathrm{R}(\mathcal{F}) = \bigcup_{n<\omega} \mathrm{R}_n(\mathcal{F})$.

Therefore, $\mathrm{R}(\mathcal{F})$ is a binary relation on $A^{<\omega}$.

Definition 12. Let $\mathcal{F}$ be a clone with a carrier A. Then, for any natural number n, $\mathrm{D}_n(\mathcal{F})$ is a binary relation on A^n defined by

$$\mathbf{a}\,\mathrm{D}_n(\mathcal{F})\,\mathbf{b} \leftrightarrow (\exists a, b \in a)(\forall f \in \mathcal{F}_{[n]})\, f(\mathbf{a}) = a \vee f(\mathbf{b}) = b.$$

We define $\mathrm{D}(\mathcal{F}) = \bigcup_{n<\omega} \mathrm{D}_n(\mathcal{F})$.

Therefore, $\mathrm{D}(\mathcal{F})$ is a binary relation on $A^{<\omega}$.

Now let $\mathcal{F}$ is a conservative clone with a carrier A. It easy to check that for any set $B \in [A]^2$ the set $\mathcal{F}|_{B^{<\omega}}$ is a clone with the carrier B, and there exists a natural isomorphism (σ_B, τ_B) from $\mathcal{F}|_{B^{<\omega}}$ to some Post's class Π_B of Boolean functions preserving **0** and **1** (A Boolean function f *preserves* **0** *and* **1** if $f(0, 0, \ldots, 0) = 0$ and $f(1, 1, \ldots, 1) = 1$, i.e. if f is conservative). Note that Π_B is defined up to a natural isomorphism of Post's classes. For definiteness, we choose some maximal by inclusion set $\mathbb{P}$ of pairwise non natural isomorphic Post's classes and assume that Π_B belongs to $\mathbb{P}$ for any $B \in [A]^2$. In this case the class Π_B is uniquely determined for each $B \in [A]^2$. Also, in this case the natural isomorphism (σ_B, τ_B) is uniquely determined if Π_B is not closed with respect to duality (Post's class P is closed with respect to duality if for all natural number n and n-ary function $f \in P$ the function $f^* = \overline{f(\overline{x}_0, \overline{x}_1, \dots, \overline{x}_{n-1})}$ belongs to P, i.e. if P is symmetric). If Π_B is closed with respect to duality, there are two distinct natural isomorphisms from $\mathcal{F}|_{B^{<\omega}}$ to Π_B. For definiteness, we assume that (σ_B, τ_B) is any one of these two natural isomorphisms.

Definition 13. Let $\mathcal{F}$ be a conservative clone with a carrier A. The family of natural isomorphisms $\{ (\sigma_B, \tau_B) \}_{B \in [A]^2}$ from $\mathcal{F}|_{B^{<\omega}}$ to Π_B is denoted by $\Pi(\mathcal{F})$.

Definition 14. The quadruple $\chi(\mathcal{F}) = (\mathrm{r}(\mathcal{F}), \mathrm{R}(\mathcal{F}), \mathrm{D}(\mathcal{F}), \Pi(\mathcal{F}))$ is called the *characteristic* of a conservative clone $\mathcal{F}$.

We show that the characteristic $\chi(\mathcal{F})$ uniquely determines a conservative symmetric clone $\mathcal{F}$ with a finite carrier. In addition to decomposition theorems, Post's classification of closed classes of Boolean functions is used in the following proofs. Post's classification can be found in [7] or (in a more modern version) in [6]. We will not refer to these works every time we use them. We need only the classification of closed classes of Boolean functions that are closed with respect to duality and consist of functions preserving $\mathbf{0}$ and $\mathbf{1}$. Recall that there are only six such classes: O_1, D_1, D_2, L_4, A_4 and C_4 (in Post's notation). Of these, the classes O_1, D_1, D_2, and L_4 consist of self-dual functions. They are generated by the functions x, $\overline{x}y \vee \overline{x}z \vee yz$, $xy \vee yz \vee xz$ и $x \oplus y \oplus z$, respectively. Moreover, $L_4 \cup D_2 \subseteq D_1$.

We begin with the following lemma, which concerns not only symmetric clones.

Lemma 7. *Let $|A| \geqslant 2$. Let $\mathcal{F} \subseteq \mathcal{O}(A)$ be a conservative clone, and $\mathrm{r}(\mathcal{F}) = r \geqslant 3$. Then there is Post's class $P \in \{ O_1, D_1, D_2, L_4 \}$ and a surjective mapping $\tau \colon \mathcal{F} \to P$ such that for each set $B \in [A]^2$, one-to-one mapping $\sigma \colon B \to 2$, natural number n, function $f \in \mathcal{F}_{[n]}$ and n-tuple $\mathbf{a} \in B^n$,*

$$f(\mathbf{a}) = \sigma^{-1}(\tau(f)(\sigma(\mathbf{a}))).$$

If $r \geqslant 4$ then $P = O_1$ and any n-ary function $f \in \mathcal{F}$ coincides with some projection on a set $A^n_{<r}$.

Proof. The proof follows easily from the following claim.

Claim 1. *Let $\mathcal{F} \subseteq \mathcal{O}(A)$ be a conservative clone, and $\mathrm{r}(\mathcal{F}) = r \geqslant 3$. Let $\mathbf{a} = a_0 a_1 \dots a_{n-1} \in A^n_{<r}$, $f \in \mathcal{F}_{[n]}$, and $\sigma \colon A \to A$.*

Then $f(\sigma \cdot \mathbf{a}) = \sigma(f(\mathbf{a}))$.

Proof. It is easy to see that $f(\sigma(\mathbf{a})) = \sigma(f(\mathbf{a}))$ for any projection $f \in \mathcal{E}_{[n]}(A)$. Further, let $\mathbf{b} = (b_0, b_1, \dots, b_{t-1})$ be some repetition-free sequence of all elements from $\operatorname{ran} \mathbf{a}$. Denote $\xi = \mathbf{b}^{-1}(\mathbf{a})$ and consider the function $f' = f(x_{\xi(0)}, x_{\xi(1)}, \dots, x_{\xi(n-1)}) \in \mathcal{F}_{[t]}$. For any $\mathbf{c} \in A^t$ we have

$$f'(\mathbf{c}) = f(\mathbf{c}(\xi)).$$

Since $t < r$, we have that f' is a projection. Therefore,

$$\sigma(f(\mathbf{a})) = \sigma(f'(\mathbf{b})) = f'(\sigma(\mathbf{b})) = f(\sigma(\mathbf{b}(\xi))) = f(\sigma(\mathbf{a})).$$

$\square$

Without loss of generality, assume $2 = \{0, 1\} \subseteq A$. So, $\mathcal{F}|_{2^{<\omega}}$ is some of Post's classes of Boolean functions preserving $\mathbf{0}$ and $\mathbf{1}$. Denote $P = \mathcal{F}|_{2^{<\omega}}$. Claim 1 implies that any $g \in P$ is self-dual function (should be considered a function $\sigma\colon A \to A$ such that $\sigma(0) = 1$ and $\sigma(1) = 0$). Consequently, $P \in \{\mathrm{O}_1, \mathrm{D}_1, \mathrm{D}_2, \mathrm{L}_4\}$.

Define $\tau(f) = f|_{2^{<\omega}}$. Then τ is a surjective function from $\mathcal{F}$ to P. Claim 1 implies that for any $B \in [A]^2$, one-to-one mapping $\sigma\colon B \to 2$, natural number n, function $f \in \mathcal{F}_{[n]}$ and n-tuple $\mathbf{a} \in B^n$, we have

$$f(\mathbf{a}) = \sigma^{-1}(\tau(f)(\sigma(\mathbf{a}))).$$

Now let $r \geqslant 4$. Any class $P \in \{D_1, D_2, L_4\}$ contains some ternary function which is not a projection. Hence, $P = O_1$, i.e. P consists only of projections e_i^n ($1 \leqslant n < \omega$, $0 \leqslant i < n$). Let f be an arbitrary function from $\mathcal{F}_{[n]}$ for some natural number n. Let i be the number for which $\tau(f) = e_i^n$. For any $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1}) \in A^n_{<r}$ choose a function $\sigma_{\mathbf{a}}\colon A \to A$ for which $\sigma_{\mathbf{a}}(a_i) = 1$ and $\sigma_{\mathbf{a}}(x) = 0$ for all $x \in A \setminus \{a_i\}$. By Claim 1 we have

$$\sigma_{\mathbf{a}}(f(\mathbf{a})) = f(\sigma_{\mathbf{a}}(\mathbf{a})) = (\tau(f))(\sigma_{\mathbf{a}}(\mathbf{a})) = 1,$$

whence $f(\mathbf{a}) = a_i$. Therefore, f coincides with the i-th projection on $A^n_{<r}$. $\square$

It follows from Lemma 7 that for any conservative clone $\mathcal{F}$ with $\mathrm{r}(\mathcal{F}) \geqslant 3$ each natural isomorphism (σ_B, τ_B), $B \in [A]^2$, maps the clone $\mathcal{F}|_{B^{<\omega}}$ to the same Post's class P. Obviously, this statement is also true for any symmetric conservative clone. We will denote this Post's class P by the symbol $\Pi_0(\mathcal{F})$. It is easy to see that for any symmetric conservative clone $\mathcal{F}$ the class $\Pi_0(\mathcal{F})$ is closed with respect to duality. Thus, the case $\mathrm{r}(\mathcal{F}) = 2$ adds two more possibilities: $\Pi_0(\mathcal{F}) = A_4$ and $\Pi_0(\mathcal{F}) = C_4$.

For any set A of cardinality 3 we denote

$$R_{\uparrow}(A) = \{ (ab, cd) \in A_2^2 \times A_2^2\colon ab = cd \vee (b = c \wedge a \neq d) \vee (a = d \wedge b \neq c) \}.$$

For any set A of cardinality 4 we denote

$$R_{\pm}(A) = \{ (\mathbf{x}, \mathbf{y}) \in A_2^2 \times A_2^2\colon \operatorname{ran}\mathbf{x} = \operatorname{ran}\mathbf{y} \text{ or } \operatorname{ran}\mathbf{x} \cap \operatorname{ran}\mathbf{y} = \varnothing \}.$$

For any set A of cardinality 4 and Post's class P we say that a function $f \in \mathcal{O}(A)_{[n]}$ is Klein's P-function if

1. for any $B \in [A]^2$, $f|_{B^n}$ belongs to the clone $\mathcal{F} \subseteq \mathcal{O}(B)$ which is naturally isomorphic to P,
2. for any $\mathbf{a} \in A_2^n$ and permutation σ from Klein four-group of permutations of A, $\sigma(f(\mathbf{a})) = f(\sigma(\mathbf{a}))$.

Theorem 4. *Let A be a finite set, $|A| \geqslant 2$. Then any symmetric conservative clone $\mathcal{F} \subseteq \mathcal{O}(A)$ is uniquely defined by by its characteristics, i.e.*

$$\chi(\mathcal{F}) = \chi(\mathcal{G}) \Rightarrow \mathcal{F} = \mathcal{G} \tag{$***$}$$

for all symmetric conservative clones $\mathcal{F}, \mathcal{G} \subseteq \mathcal{O}(A)$.

Let Q and $H \subseteq A^Q$ be non-empty finite sets and $\mathcal{F} \subseteq \mathcal{O}(A)$ a symmetric conservative clone. Then $\mathrm{r}(\mathcal{F}) \geqslant 2$, and

1. *If $\mathrm{r}(\mathcal{F}) \geqslant 4$, then $H \in \mathrm{Inv}_Q\mathcal{F}$ if and only if*
   (a) *$H|_P \in \mathrm{Inv}_P\mathcal{F}$ for all $P \in [Q]^{2,0}_H$,*
   (b) *H is decomposable over $[Q]^1 \cup [Q]^{2,0}_H \cup \{ Q^{(r)}_H \}$.*
2. *If $\mathrm{r}(\mathcal{F}) = 3$ and $\Pi_0(\mathcal{F}) \neq L_4$, then $H \in \mathrm{Inv}_Q\mathcal{F}$ if and only if*
   (a) *$H|_P \in \mathrm{Inv}_P\mathcal{F}$ for all $P \in [Q]^{2,0}_H \cup [Q]^{2,1}_H$,*
   (b) *H is decomposable over $[Q]^1 \cup [Q]^{2,0}_H \cup [Q]^{2,1}_H$.*
3. *If $\mathrm{r}(\mathcal{F}) = 3$ and $\Pi_0(\mathcal{F}) = L_4$, then $H \in \mathrm{Inv}_Q\mathcal{F}$ if and only if*
   (a) *$H|_P \in \mathrm{Inv}_P\mathcal{F}$ for all $P \in [Q]^{2,0}_H$,*
   (b) *$H|_{Q^{(3)}_H}$ is preserved by a function $\ell \in \mathcal{O}(A)_{[3]}$ satisfying $\ell(x,y,y) = \ell(y,x,y) = \ell(y,y,x) = x$,*
   (c) *H is decomposable over $[Q]^1 \cup [Q]^{2,0}_H \cup \{ Q^{(3)}_H \}$.*
4. *If $\mathrm{r}(\mathcal{F}) = 2$ and the following cases do not hold: (i) $|A| = 4$ and $\mathrm{R}_2(\mathcal{F}) = R_\pm$, (ii) $|A| = 3$ and $\mathrm{R}_2(\mathcal{F}) = R_\uparrow$, then $H \in \mathrm{Inv}_Q\mathcal{F}$ if and only if*
   (a) *$H|_P \in \mathrm{Inv}_P\mathcal{F}|_{B^{<\omega}}$ for all $P \in Q^{[B]}_H$ and $B \in [A]^2$,*
   (b) *H is decomposable over $[Q]^1 \cup [Q]^{2,id}_H \cup \{ Q^{[B]}_H \colon B \in [A]^2 \}$.*
5. *If $\mathrm{r}(\mathcal{F}) = 2$ and $|A| = 4$, and $\mathrm{R}_2(\mathcal{F}) = R_\pm(A)$, then $H \in \mathrm{Inv}_Q\mathcal{F}$ if and only if*
   (a) *$H|_P \in \mathrm{Inv}_P\mathcal{F}$ for all $P \in [Q]^{2,0}_H$,*
   (b) *$H|_{Q^{(3)}_H}$ is preserved by all Klein's $\Pi_0(\mathcal{F})$-functions,*
   (c) *H is decomposable over $[Q]^1 \cup [Q]^{2,0}_H \cup \{ Q^{(3)}_H \}$.*
6. *If $r = 2$ and $|A| = 3$, and $\mathrm{R}_2(\mathcal{F}) = R_\uparrow(A)$, then $H \in \mathrm{Inv}_Q\mathcal{F}$ if and only if*
   (a) *$H|_P \in \mathrm{Inv}_P\mathcal{F}$ for all $P \in [Q]^{2,0}_H \cup [Q]^{2,1}_H$,*
   (b) *H is decomposable over $[Q]^1 \cup [Q]^{2,0}_H \cup [Q]^{2,1}_H$.*

Proof. The proof is based on decomposition theorems and the following Lemma.

Lemma 8. *Let $|A|$ be a set of cardinality $|A| \geqslant 2$, and $\mathcal{F}$ be a symmetric conservative clone with the carrier A. Let $\mathrm{r}(\mathcal{F}) = r < \omega$. Then*

1. *If* $r \geqslant 4$*, then* $\mathcal{F}$ *satisfies* Δ_r^s*;*

2. *If* $\mathrm{r}(\mathcal{F}) = 3$ *and* $\Pi_0(\mathcal{F}) \neq L_4$*, then* $\mathcal{F}$ *satisfies* Δ^∂*;*

3. *If* $\mathrm{r}(\mathcal{F}) = 3$ *and* $\Pi_0(\mathcal{F}) = L_4$*, then* $\mathcal{F}$ *satisfies* Δ_3^s*;*

4. *If* $r = 2$*, then one of the following cases holds:*

   (a) $\mathcal{F}$ *satisfies* Δ^2*,*

   (b) $|A| = 4$ *and* $\mathrm{R}_2(\mathcal{F}) = R_\pm(A)$*, and* $\mathcal{F}$ *satisfies* Δ_3^s*, and for all* $B \in [A]^3$*,* $\mathcal{F}|_{B^{<\omega}}$ *satisfies* Δ^2*,*

   (c) $|A| = 3$ *and* $\mathrm{R}_2(\mathcal{F}) = R_\uparrow(A)$*, and* $\mathcal{F}$ *satisfies* Δ^∂*.*

Proof. First we prove the following claim. All function $\ell \in \mathcal{O}(A)_{[3]}$ satisfying

$$\ell(x, y, y) = \ell(y, x, y) = \ell(y, y, x) = x$$

is called an ℓ-function.

Claim 2. *Let* $\mathcal{F} \subseteq \mathcal{O}(A)$ *be a conservative symmetric clone and* n *a natural number.*

1. *If* $\mathcal{F}$ *contains an* n*-ary function* $f \notin \mathcal{E}(A)$ *such that* $f|_{A^n_{<n}} \in \mathcal{E}(A)|_{A^n_{<n}}$*, then* $\mathcal{F}$ *satisfies* Δ_n^s*.*

2. *If* $\mathcal{F}$ *contains a* ∂*-function, then* $\mathcal{F}$ *satisfies* Δ^∂*.*

3. *If* $\mathcal{F}$ *contains a* ℓ*-function, then* $\mathcal{F}$ *satisfies* Δ_3^s*.*

Proof. Let $f \in F_{[n]} \setminus \mathcal{E}(A)$ coincide with e_i^n on $A^n_{<n}$. Then $f(\mathbf{a}) = a_j$ for some $\mathbf{a} = a_0 a_1 \dots a_{n-1} \in A^n_n$ and $j \in n \setminus \{ i \}$. For any $k \in n \setminus \{ i, j \}$ denote the transposition $(a_j, a_k) \in S_A$ by σ_k, and the transposition (j, k) by τ_k. For any $k < n$ denote

$$s^k = \begin{cases} f_{\sigma_k}(x_{\tau_k(0)}, x_{\tau_k(1)}, \dots, x_{\tau_k(n-1)}), & \text{if } k \in n \setminus \{ i, j \}, \\ f, & \text{if } k = j, \\ e_i^n, & \text{if } k = i. \end{cases}$$

For all $k < n$ we have

$$s^k(\mathbf{a}) = a_k \text{ and } s^k|_{A^n_{<n}} = e_i^n|_{A^n_{<n}}.$$

Let $\mathbf{b}$ be an arbitrary n-tuple from A^n_n and $\sigma_\mathbf{b}$ an arbitrary permutation of A for which $\sigma_\mathbf{b}(\mathbf{b}) = \mathbf{a}$. Then for all $k < n$ we have

$$s^k_{\sigma_\mathbf{b}}(\mathbf{b}) = b_k \text{ and } s^k_{\sigma_\mathbf{b}}|_{A^n_{<n}} = e_i^n|_{A^n_{<n}}.$$

All the functions $s^k_{\sigma_\mathbf{b}}$ belong to $\mathcal{F}$. Therefore, Δ_n^e holds.

Now let ∂ be a ∂-function from $\mathcal{F}$, and $\mathbf{a} = a_0 a_1 a_2$ an arbitrary triple from A^3_3. Let $\partial(\mathbf{a}) = a_i$ and $j \in 3 \setminus \{ i \}$. Denote the transposition $(a_i, a_j) \in S_A$ by σ_j, and the transposition (i, j) by τ_j. Let $\partial^j = \partial_{\sigma_j}(x_{\tau_j(0)}, x_{\tau_j(1)}, x_{\tau_j(2)})$. Obviously, $\partial^j \in \mathcal{F}$. Moreover, it easy to check that ∂^j is a ∂-function and $\partial^j(\mathbf{a}) = a_j$. Therefore, Δ^∂ holds.

Arguing similarly, we find that if $\mathcal{F}$ contains an ℓ-function, then for any $\mathbf{a} = a_0a_1a_2 \in A_3^3$ and $a \in \operatorname{ran}\mathbf{a}$ there is an ℓ-function $\ell \in \mathcal{F}_{[3]}$ such that $\ell(\mathbf{a}) = a$. We show that for any $i \in \{0, 1, 2\}$ there exists a function $s_i \in \mathcal{F}_{[3]}$ such that

$$s_i(\mathbf{a}) = a_i \text{ and } s(\mathbf{x}) = x_0 \text{ for all } \mathbf{x} = x_0x_1x_2 \in A_{<3}^3.$$

If $i = 0$, we put $s_0 = e_0^3$. If $i \neq 0$, choose ℓ-functions $\ell_0, \ell_i \in \mathcal{F}$ such that $\ell_0(\mathbf{a}) = a_0$ and $\ell_i(\mathbf{a}) = a_i$. It is easy to verify that the function

$$s_i = \ell_0(x_0, \ell_0, \ell_i)$$

satisfies the required conditions. □

Now, for $r \geqslant 3$, Lemma 8 follows immediately from Lemma 1 and Claim 2.

Before considering the Case $r = 2$, we prove several auxiliary assertions.

For any pair $(\mathbf{a}, \mathbf{b}) \in A_2^2 \times A_2^2$ we define the *type* $\mathrm{t}(\mathbf{a}, \mathbf{b}) \in 2 \cup 2^2 \cup \{2\}$ of $(\mathbf{a}, \mathbf{b})$ as follows. Let $\mathbf{a} = a_0a_1$ and $\mathbf{b} = b_0b_1$. Then for all $i, j \in \{0, 1\}$

$$\mathrm{t}(\mathbf{a}, \mathbf{b}) = \begin{cases} 0, & \text{if } \mathbf{a} = \mathbf{b} \\ 1, & \text{if } a_0 = b_1 \text{ and } a_1 = b_0 \\ ij, & \text{if } a_i = b_j \text{ and } a_{1-i} \neq b_{1-j} \\ 2, & \text{if } \mathrm{ran}\mathbf{a} \cap \mathrm{ran}\mathbf{b} = \varnothing \end{cases}$$

Obviously, the type $\mathrm{t}(\mathbf{a}, \mathbf{b})$ is defined for every $(\mathbf{a}, \mathbf{b}) \in A_2^2 \times A_2^2$.

For any $i \in \{0, 1\}$ we denote the binary relation $\rhd_i$ on A_2^2 by

$$\mathbf{a} \rhd_i \mathbf{b} \leftrightarrow \big((\forall f \in \mathcal{F}_{[2]})\, f(\mathbf{a}) = a_i \to f(\mathbf{b}) = b_i\big)$$

for all $\mathbf{a} = a_0a_1, \mathbf{b} = b_0b_1 \in A_2^2$.

For any pair $\mathbf{a} = a_0a_1 \in A_2^2$, the symbol $\overline{\mathbf{a}}$ denote the pair a_1a_0.

Claim 3. *For every $i \in \{0, 1\}$ the binary relation $\rhd_i$ is reflexive and transitive. Besides, for every $i \in \{0, 1\}$, pairs $\mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}' \in A_2^2$ and permutation $\sigma \in S_A$*

1. $\mathbf{a} \rhd_i \mathbf{b} \Rightarrow \sigma(\mathbf{a}) \rhd_i \sigma(\mathbf{b})$,
2. $\mathbf{a} \rhd_i \mathbf{b} \Rightarrow \overline{\mathbf{a}} \rhd_{1-i} \overline{\mathbf{b}}$,
3. $\mathbf{a} \rhd_i \mathbf{b} \Rightarrow \mathbf{b} \rhd_{1-i} \mathbf{a}$,
4. $t(\mathbf{a}, \mathbf{b}) = t(\mathbf{a}', \mathbf{b}') \Rightarrow (\mathbf{a} \rhd_i \mathbf{b} \to \mathbf{a}' \rhd_i \mathbf{b}')$.

Proof. The reflexivity and transitivity of $\rhd_0$ and $\rhd_1$ are obvious.

Let $\mathbf{a} = a_0a_1$, $\mathbf{b} = b_0b_1$ and $\mathbf{a} \rhd_i \mathbf{b}$. Without loss of generality, $i = 0$.

Assume $f(\sigma(\mathbf{a})) = \sigma(a_0)$ for some $f \in \mathcal{F}_{[2]}$ and $\sigma \in S_A$. Then we have $\sigma(f_\sigma(\mathbf{a})) = f(\sigma(\mathbf{a})) = \sigma(a_0)$ and, so, $f_\sigma(\mathbf{a}) = a_0$, whence $f(\sigma(\mathbf{b})) = \sigma(f_\sigma(\mathbf{b})) = \sigma(b_0)$. Item 1 is proved.

Assume $f(\overline{\mathbf{a}}) = f(a_1a_0) = a_0$ for some $f \in \mathcal{F}_{[2]}$. Let $g = f(x_1, x_0)$. Then we have $g(a_0a_1) = f(a_1a_0) = a_0$ and, so, $f(\overline{\mathbf{b}}) = f(b_1b_0) = g(b_0b_1) = b_0$. Item 2 is proved.

Assume $f(\mathbf{b}) = f(b_0b_1) = b_1$ for some $f \in \mathcal{F}_{[2]}$. If $f(\mathbf{a}) = f(a_0a_1) = a_0$, we have $f(\mathbf{b}) = b_0$, a contradiction. So, $f(\mathbf{a}) = a_1$. Item 3 is proved.

Item 4 follows immediately from item 1.

□

Claim 4. *One of the following five cases holds.*

1. $(\forall i < 2)\,(\forall \mathbf{x}, \mathbf{y} \in A_2^2)\, \mathbf{x} \rhd_i \mathbf{y}$,
2. $(\forall i < 2)\,(\forall \mathbf{x}, \mathbf{y} \in A_2^2)\, \mathbf{x} \rhd_i \mathbf{y} \leftrightarrow \mathrm{t}(\mathbf{x}, \mathbf{y}) = 0$,
3. $(\forall i < 2)\,(\forall \mathbf{x}, \mathbf{y} \in A_2^2)\, \mathbf{x} \rhd_i \mathbf{y} \leftrightarrow \mathrm{t}(\mathbf{x}, \mathbf{y}) \in \{\,0, 1\,\}$,
4. $|A| = 4 \wedge (\forall i < 2)\,(\forall \mathbf{x}, \mathbf{y} \in A_2^2)\, \mathbf{x} \rhd_i \mathbf{y} \leftrightarrow \mathrm{t}(\mathbf{x}, \mathbf{y}) \in \{\,0, 1, 2\,\}$,
5. $|A| = 3 \wedge (\forall i < 2)\,(\forall \mathbf{x}, \mathbf{y} \in A_2^2)\, \mathbf{x} \rhd_i \mathbf{y} \leftrightarrow \mathrm{t}(\mathbf{x}, \mathbf{y}) \in \{\,0, 01, 10\,\}$.

Proof. Let i be a fixed number in $\{\,0, 1\,\}$.

Let $\rhd_i$ contain some pair $(\mathbf{a}, \mathbf{b}) \in A_2^2$ of type 00. Let $\mathbf{a} = a_0a_1$ and $\mathbf{b} = a_0b_1$. Then we have

(a) $a_0a_1 \rhd_i a_0b_1$ (supposition),
(b) $a_0b_1 \rhd_{1-i} a_0a_1$ from (a) by item 2 of Claim 3,
(c) $b_1a_0 \rhd_i a_1a_0$ from (b) by item 3 of Claim 3,
(d) $a_0b_1 \rhd_i a_1b_1$ from (c) by item 4 of Claim 3,
(e) $a_0a_1 \rhd_i a_1b_1$ from (a) and (c) by transitivity,
(f) $a_1b_1 \rhd_i b_1a_0$ from (e) by item 4 of Claim 3,
(g) $a_0a_1 \rhd_i b_1a_0$ from (a) and (f) by transitivity,
(h) $a_0b_1 \rhd_i a_1a_0$ from (g) by item 4 of Claim 3,
(i) $a_0a_1 \rhd_i a_1a_0$ from (a) and (h) by transitivity.

The pairs from $(c), (e), (g), (i)$ have the types 11, 10, 01 and 1, respectively. Given the reflexivity of $\rhd_i$ and item 4 of Claim 3, we have

$$\mathbf{x} \rhd_i \mathbf{y} \text{ for all } (\mathbf{x}, \mathbf{y}) \text{ such that } \mathrm{t}(\mathbf{x}, \mathbf{y}) \neq 2.$$

If $|A| = 3$, we have the case 1. If $|A| \geqslant 4$, choose $c \in A \setminus \{\,a_0, a_1, b_1\,\}$, and continue.

(j) $a_0b_1 \rhd_i b_1c$ from (e) by item 4 of Claim 3,
(k) $a_0a_1 \rhd_i b_1c$ from (a) and (j) by transitivity.

The pair (a_0a_1, b_1c) has the type 2. So, we have the case 1.

If $\rhd_i$ contains some pair $(\mathbf{a}, \mathbf{b}) \in A_2^2$ of type 11, arguments are similar. Furthermore, we assume that $\rhd_i$ does not contain pairs $(\mathbf{a}, \mathbf{b})$ of type 00 or 11.

Let $\rhd_i$ contain some pair $(\mathbf{a}, \mathbf{b}) \in A_2^2$ of type 01. Let $\mathbf{a} = a_0a_1$ and $\mathbf{b} = a_1b_1$. We have

(a') $a_0a_1 \rhd_i a_1b_1$ (supposition),
(b') $a_1b_1 \rhd_i b_1a_0$ from (a') by item 4 of Claim 3,
(c') $a_0a_1 \rhd_i b_1a_0$ from (a') и (b') by transitivity.

So, $\mathbf{x} \rhd_i \mathbf{y}$ for all $(\mathbf{x}, \mathbf{y})$ of type 10.

If $\mathbf{x} \rhd_i \mathbf{y}$ for some pair $(\mathbf{x}, \mathbf{y})$ of type 1, we have

(d') $a_1a_0 \rhd_i a_0a_1$ (supposition),
(e') $a_1a_0 \rhd_i a_1b_1$ from (a') and (d') by transitivity.

A contradiction because $\mathrm{t}(a_1a_0, a_1b_1) = 00$.

Therefore, $\rhd_i$ contains all pairs of types $0, 01, 10$, and does not contain any pair of types $1, 00, 11$. If $|A| = 3$ we have a case 5. If $|A| \geqslant 4$, choose $c \in A \setminus \{ a_0, a_1, b_1 \}$, and continue.

(f') $a_1b_1 \rhd_i b_1c$ from (a') by item 4 of Claim 3,

(g') $a_0a_1 \rhd_i b_1c$ from (a') and (f') by transitivity,

(h') $b_1c \rhd_i a_1a_0$ from (g') by item 4 of Claim 3,

(i') $a_0a_1 \rhd_i a_1a_0$ from (a') and (h') by transitivity.

A contradiction because $\mathrm{t}(a_0a_1, a_1a_0) = 1$.

If $\rhd_i$ contains some pair of type 10, arguments are similar. Now we can assume that all pairs $(\mathbf{x}, \mathbf{y}) \in \rhd_i$ have the type 0, 1, or 2.

Let $\rhd_i$ contain some pair $(\mathbf{a}, \mathbf{b}) \in A^2_2$ of type 2. Let $\mathbf{a} = a_0a_1$ and $\mathbf{b} = b_0b_1$. We have

(a'') $a_0a_1 \rhd_i b_0b_1$ (supposition),

(b'') $b_0b_1 \rhd_i a_1a_0$ from (a'') by item 4 of Claim 3,

(c'') $a_0a_1 \rhd_i a_1a_0$ from (a'') and (b'') by transitivity.

Note that $\mathrm{t}(a_0a_1, a_1a_0) = 1$. If $|A| = 4$, we have the case 4. If $|A| \geqslant 5$, choose $c \in A \setminus \{ a_0, a_1, b_0, b_1 \}$, and continue.

(d'') $b_0b_1 \rhd_i ca_0$ from (a'') by item 4 of Claim 3,

(e'') $a_0a_1 \rhd_i ca_0$ from (a'') and (d'') by transitivity, a contradiction because $\mathrm{t}(a_0a_1, ca_0) = 01$.

Now we can assume that all pairs $(\mathbf{x}, \mathbf{y}) \in \rhd_i$ have the type 0 or 1. By item 4 of Claim 3 we have that one of the cases 2, 3 holds. $\square$

Claim 5. $\rhd_0 = \rhd_1 = \mathrm{R}_2(\mathcal{F})$.

Proof. By Claim 4, we have $(\mathbf{x}, \mathbf{y}) \in \rhd_i \Leftrightarrow (\mathbf{y}, \mathbf{x}) \in \rhd_i$. It remains to use item 3 of Claim 3. $\square$

We continue the proof of Lemma 8. Let $r = 2$. Consider the cases of Claim 4, taking into account Claim 5. The case 1 implies $r \geqslant 3$, a contradiction. The cases 2 and 3 imply Δ^2.

Let the case 4 holds. We have $\mathrm{R}_2(\mathcal{F}) = R_{\pm}(A)$ by Claim 5. Choose an arbitrary set $B \in [A]^3$. It's easy to see that the clone $\mathcal{G} = \mathcal{F} \upharpoonright B^{<\omega}$ satisfies Δ^2. Since $\mathcal{G}_{[3]} \in \mathrm{Inv}_{A^3}\mathcal{G}$, Theorem 3 implies that $\mathcal{G}$ contains some ternary function f for which

$$f|_{B^3_{<3}} = e^3_0|_{B^3_{<3}} \text{ and } f(\mathbf{a}) = a_1$$

for some $\mathbf{a} = a_0a_1a_2 \in B^3_3$.

For any $\mathbf{b} \in A^3_2 \setminus B^3_2$ there is $\mathbf{c} \in B^3_2$ and $\sigma \in S_A$ such that $\sigma(\mathbf{b}) = \mathbf{c}$. Without loss of generality we take $\mathbf{b} = b_0b_0b_1$ and $\mathbf{c} = c_0c_0c_1$. Denote $f' = f(x_0, x_0, x_1)$. Since $b_0b_1 \rhd_i c_0c_1$ for any $i \in \{ 0, 1 \}$, we have

$$f(b_0b_0b_1) = b_0 \leftrightarrow f'(b_0b_1) = b_0 \leftrightarrow f'(c_0c_1) = c_0 \leftrightarrow f(c_0c_0c_1),$$

so, f coincides with the projection e^3_0 on the whole set $A^3_{<3}$. It remains to use Claim 2.

Finally, let the case 5 holds. We have $\mathrm{R}_2(\mathrm{F}) = R_{\uparrow}(A)$ by Claim 5. It is easy to see that each binary function $f \in \mathcal{F} \setminus \mathcal{E}(A)$ is uniquely determined by its value on any $\mathbf{a} \in A^2_2$. Let $A = \{ a, b, c \}$. Then $\mathcal{F}$ contains exactly two functions u и v that are not projections defined by

u	a	b	c
a	a	b	a
b	b	b	c
c	a	c	c

v	a	b	c
a	a	a	c
b	a	b	b
c	c	b	c

Consider the function

$$f = v(v(u(x_0, x_1), u(x_0, x_2)), u(x_1, x_2)).$$

It easy to check that f is ∂-function (for this it is convenient to note that both functions u and v are commutative and $u(x, y) = x \leftrightarrow v(x, y) = y$). It is remain to use Claim 2. $\square$

Now the proof of the theorem reduces to an analysis of the cases of Lemma 8. We will collect some useful observations in one claim.

Claim 6. *Let A and Q be non-empty sets.*

1. *For any clone $\mathcal{F} \subseteq \mathcal{O}(A)$ and natural number n, $\mathcal{F}_{[n]} \in \operatorname{Inv}_{A^n}\mathcal{F}$.*
2. *For any sets $H \subseteq A^Q$ and $P \in [Q]^1$, the set $H|_P$ belongs to $\operatorname{Inv}_P\mathcal{F}$ for any conservative clone $\mathcal{F} \subseteq \mathcal{O}(A)$. Moreover, for all conservative clones $\mathcal{F}, \mathcal{G} \subseteq \mathcal{O}(A)$, $[A^n]^1_{\mathcal{F}_{[n]}} = [A^n]^1_{\mathcal{G}_{[n]}} = \{\, \{\boldsymbol{x}\} \colon \boldsymbol{x} \in A^n \,\}$, and $\mathcal{F}_n|_P = \mathcal{G}_n|_P$ for all $P \in [A^n]^1_{\mathcal{F}_{[n]}}$.*
3. *For all clones $\mathcal{F}, \mathcal{G} \subseteq \mathcal{C}(A)$ and a set $H \subseteq A^Q$ if $\mathcal{F}|_{A^{<\omega}_{<r}} = \mathcal{G}|_{A^{<\omega}_{<r}}$ and $(\forall q \in Q)\, |H(q)| < r$, then $H \in \operatorname{Inv}_Q\mathcal{F} \Leftrightarrow H \in \operatorname{Inv}_Q\mathcal{G}$.*
4. *For any conservative clone $\mathcal{F} \subseteq \mathcal{O}(A)$, set $B \subseteq A$ and set $H \subseteq B^Q$, $H \in \operatorname{Inv}_Q\mathcal{F} \Leftrightarrow H \in \operatorname{Inv}_Q\mathcal{F}|_{B^{<\omega}}$.*
5. *For any clone $\mathcal{F} \subseteq \mathcal{O}(A)$ and natural number n,*
$$[A^n]^{2,0}_{\mathcal{F}_{[n]}} = \{\, \{\mathbf{x}, \mathbf{y}\} \in [A^n]^2 : \mathbf{x} \neq \mathbf{y} \text{ and } (\mathbf{x}, \mathbf{y}) \in \mathrm{R}_n(\mathcal{F}) \,\}.$$
Moreover, for all $\mathbf{x} = x_0x_1 \dots x_{n-1}$ and $\mathbf{y} = y_0, y_1 \dots y_{n-1}$ in A^n, if $(\mathbf{x}, \mathbf{y}) \in \mathrm{R}_n(\mathcal{F})$, then
$$f(\mathbf{x}) = x_i \Leftrightarrow f(\mathbf{y}) = y_i$$
for any $f \in \mathcal{F}_{[n]}$ and $i < n$. Therefore, for all conservative clones $\mathcal{F}, \mathcal{G} \subseteq \mathcal{O}(A)$ if $\mathrm{R}_n(\mathcal{F}) = \mathrm{R}_n(\mathcal{G})$, then $[A^n]^{2,0}_{\mathcal{F}_{[n]}} = [A^n]^{2,0}_{\mathcal{G}_{[n]}}$ and for all $P \in [A^n]^{2,0}_{\mathcal{F}_{[n]}}$, $\mathcal{F}_{[n]}|_P = \mathcal{G}_{[n]}|_P$.
6. *For any clone $\mathcal{F} \subseteq \mathcal{O}(A)$ and natural number n,*
$$[A^n]^{2,1}_{\mathcal{F}_{[n]}} = \{\, \{\mathbf{x}, \mathbf{y}\} \in [A^n]^2 : \mathbf{x} \neq \mathbf{y} \text{ and } (\mathbf{x}, \mathbf{y}) \in \mathrm{D}_n(\mathcal{F}) \,\}.$$
Moreover, if $\mathcal{F}$ is a conservative clone, then for all $\mathbf{x}$ and $\mathbf{y}$ in A^n if $(\mathbf{x}, \mathbf{y}) \in \mathrm{D}_n(\mathcal{F}) \setminus \mathrm{R}_n(\mathcal{F})$ and $\max(|\operatorname{ran}\mathbf{x}|, |\operatorname{ran}\mathbf{y}|) \geqslant 2$, then
   (a) *there is the unique pair $(a, b) \in A^2$ for which $f(\mathbf{x}) = a \vee f(\mathbf{y}) = b$, and*

(b) $\mathcal{F}_n|_{\{\mathbf{x},\mathbf{y}\}}$ *contains all functions* $f \in A^{\{\mathbf{x},\mathbf{y}\}}$ *satisfying* $f(\mathbf{x}) \in \operatorname{ran} \mathbf{x}$, $f(\mathbf{y}) \in \operatorname{ran} \mathbf{y}$, *and* $f(\mathbf{x}) = a \vee f(\mathbf{y}) = b$.

Therefore, for all conservative clones $\mathcal{F}, \mathcal{G} \subseteq \mathcal{O}(A)$ *if* $\mathrm{D}_n(\mathcal{F}) = \mathrm{D}_n(\mathcal{G})$, *then* $[A^n]^{2,1}_{\mathcal{F}_{[n]}} = [A^n]^{2,1}_{\mathcal{G}_{[n]}}$ *and* $\mathcal{F}_{[n]}|_P = \mathcal{G}_{[n]}|_P$ *for all* $P \in [A^n]^{2,0}_{\mathcal{F}_{[n]}} \setminus [A^n]^{2,1}_{\mathcal{F}_{[n]}}$.

7. *For any conservative clone* $\mathcal{F} \subseteq \mathcal{O}(A)$ *with* $\mathrm{r}(\mathcal{F}) = 3$ *and* $\Pi_0(\mathcal{F}) = L_4$, $\mathcal{F}|_{A^{\leq\omega}_{<3}} = \mathcal{L}(A)|_{A^{\leq\omega}_{<3}}$ *where* $\mathcal{L}(A)$ *is the clone generated by all conservative functions* $\ell \in \mathcal{O}(A)_{[3]}$ *satisfying* $\ell(x,y,y) = \ell(y,x,y) = \ell(y,y,x) = x$.

8. *For any conservative symmetric clone* $\mathcal{F} \subseteq \mathcal{O}(A)$, *if* $|A| = 4$, $\mathrm{r}(\mathcal{F}) = 2$ *and* $\mathrm{R}_2(\mathcal{F}) = R_\pm$, *then* $\mathcal{F}|_{A^{\leq\omega}_{<3}} = \mathcal{K}(A)|_{A^{\leq\omega}_{<3}}$ *where* $\mathcal{K}(A)$ *is the set (in fact, the clone) of all conservative Klein's* $\Pi_0(\mathcal{F})$*-functions.*

Proof. By a direct verification. □

Case 1 ($\mathrm{r}(\mathcal{F}) \geqslant 4$). By Lemma 7 any function $f \in \mathcal{F}_{[n]}$ coincides with a projection on $A^n_{<r}$. Therefore, by item 3 of Claim 6, any set $H \subseteq A^P$ satisfying $(\forall p \in P)\,|H(p)| < r$ belongs to $\mathrm{Inv}_P \mathcal{F}$. So, item 1 of Theorem 4 follows from item 2 of Claim 6 and Theorem 2. To prove the statement $(***)$ in this case, it suffices to note that $(A^n)^{(r)}_{\mathcal{F}_{[n]}} = A^n_{<r}$ and $\mathcal{F}_{[n]}|_{A^n_{<r}} = \mathcal{E}_{[n]}|_{A^n_{<r}}$ for any conservative clone $\mathcal{F} \subseteq \mathcal{O}(A)$ with $\mathrm{r}(\mathcal{F}) \geqslant 4$, and use items 1, 2, 5 of Claim 6, and Theorem 2.

Case 2 ($\mathrm{r}(\mathcal{F}) = 3$ and $\Pi_0(\mathcal{F}) \neq L_4$). Item 2 of Theorem 4 follows from item 2 of Claim 6 and Theorem 1. The statement $(***)$ follows from items 1, 2, 5, 6 of Claim 6, and Theorem 1.

Case 3 ($\mathrm{r}(\mathcal{F}) = 3$ and $\Pi_0(\mathcal{F}) = L_4$). Item 3 of Theorem 4 follows from items 2, 3, 7 of Claim 6 and Theorem 2. To prove the statement $(***)$ in this case, it suffices to note that $(A^n)^{(3)}_{\mathcal{F}_{[n]}} = A^n_{<3}$ for any conservative clone $\mathcal{F} \subseteq \mathcal{O}(A)$, and use items 1, 2, 5, 7 of Claim 6, and Theorem 2.

Case 4 ($\mathrm{r}(\mathcal{F}) = 2$ and the following cases do not hold: (i) $|A| = 4$ and $\mathrm{R}_2(\mathcal{F}) = R_\pm$, (ii) $|A| = 3$ and $\mathrm{R}_2(\mathcal{F}) = R_\uparrow$). Item 4 of Theorem 4 follows from items 2, 4 of Claim 6 and Theorem 3. To prove the statement $(***)$ in this case, note that for any conservative symmetric clones $\mathcal{F}, \mathcal{G} \subseteq \mathcal{O}(A)$ and set $B \in [A]^2$, $(A^n)^{[B]}_{\mathcal{F}_{[n]}} = (A^n)^{[B]}_{\mathcal{G}_{[n]}} = B^n$ and, if $\Pi_0(\mathcal{F}) = \Pi_0(\mathcal{G})$, $\mathcal{F}_{[n]}|_{B^n} = \mathcal{G}_{[n]}|_{B^n}$. Besides, $[A^n]^{2,id}_{\mathcal{F}_{[n]}} = \varnothing$ for any clone $\mathcal{F} \subseteq \mathcal{O}(A)$. Now it is sufficient to use items 1, 2 of Claim 6, and Theorem 3.

Case 5 ($\mathrm{r}(\mathcal{F}) = 2$, $|A| = 4$ and $\mathrm{R}_2(\mathcal{F}) = R_\pm$). Item 5 of Theorem 4 follows from items 2, 3, 8 of Claim 6 and Theorem 2. To prove the statement $(***)$ in this case, it suffices to note that $(A^n)^{(3)}_{\mathcal{F}_{[n]}} = A^n_{<3}$ for any conservative clone $\mathcal{F} \subseteq \mathcal{O}(A)$, and use items 1, 2, 5, 8 of Claim 6, and Theorem 2.

Case 6 ($\mathrm{r}(\mathcal{F}) = 2$, $|A| = 3$ and $\mathrm{R}_2(\mathcal{F}) = R_\uparrow$). Item 6 of Theorem 4 follows from item 2 of Claim 6 and Theorem 1. The statement $(***)$ follows from items 1, 2, 5, 6, and Theorem 1.

□

Remarks. The classification of symmetric conservative clones $\mathcal{F}$ with a finite carrier can be further detailed, since the relations $\mathrm{R}(\mathcal{F})$ and $\mathrm{D}(\mathcal{F})$ have a very special form in this case, see [10]. The parameter $\mathrm{r}(\mathcal{F})$ is introduced in [3]. Symmetric clones with a finite carrier containing all constants are described in [9] (note that conservative clones, on the contrary, do not contain any constants).

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