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ON ALGORITHMIC DECIDABILITY OF THE SQUARE-FREE WORD PROBLEM RELATIVE TO A SYSTEM OF TWO DEFINING RELATIONS

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Abstract. We prove that the square-free word problem relative to a system of two defining relations is algorithmic decidable.

Let A be a finite alphabet. Elements of A are called *letters*. We denote by the symbol A^* the set of all finite sequences of letters and also the free monoid generated by A. Elements of A^* are called *words*. The *empty word* is denoted by the symbol ε. For any word $w = a_1a_2\dots a_n \in A^*$, the natural number n is called the *length of word* w and is denoted by $|w|$. A word w is said to be *square-free* if

$$w = ussv \to s = \varepsilon \quad \text{for all words } u, s, v \in A^*.$$

We denote the set of all square-free words in an alphabet A by $SF(A)$. A well-known result of Thue (see [4]) states that a free monoid over a three-letter alphabet has infinitely many square-free words (see also [2]). In [3], Carpi and de Luca introduces the notion of a *square-free word relative to a rewriting system* π. We will use the notion *square-free word relative to a system of defining relations* π (these notions are equivalent if π is symmetric).

A *system of defining relations* π on A is arbitrary binary irreflexive relation on the set A^*. We denote by $\overset{\pi}{\leftrightarrow}$ the regular closure of $\pi \cup \pi^{-1}$. In other terms, we have $x \overset{\pi}{\leftrightarrow} y$ $(x, y \in A^*)$ if and only if

$$x = rus \ \&\ y = rvs \ \&\ ((u, v) \in \pi \ \vee\ (v, u) \in \pi)$$

for some $r, s \in A^*$. The reflexive and transitive closure of the relation $\overset{\pi}{\leftrightarrow}$ is denoted by $\overset{\pi}{=}$. The relation $\overset{\pi}{=}$ is a congruence of the monoid A^*. The equivalence class on the congruence $\overset{\pi}{=}$ of the word $w \in A^*$ is denoted by $[w]_\pi$.

Let π be a system of defining relations on A. A word $w \in A^*$ is said to be *square-free relative to* π if and only if $[w]_\pi \subseteq SF(A)$. The set of all square-free words relative to π will be denoted by $SF(A, \pi)$. If the set $SF(A, \pi)$ is recursive, then we say that *the square-free word problem relative to* π *is* (*algorithmically*) *decidable.* Carpi and de Luca proved (see [3]) the following theorem.

Theorem 1. *Let A be an alphabet and π be a finite system of defining relations on A defined by*

$$\pi = \{(u_i, u_j) \mid 1 \le i < j \le n\}, \tag{1}$$

where $u_i \in A^*$ and $u_i = u_j \to i = j$ for all i and j, $1 \le i \le n$, $1 \le j \le n$. Then the square-free word problem relative to π is decidable.

In particular, the square-free word problem relative to the system of one defining relation is decidable.

For any system of defining relations π, the square-free word problem relative to π is decidable if the following condition holds:

$$\forall w \in A^* \ w \in SF(A, \pi) \to \operatorname{card}([w]_\pi) < \aleph_0. \tag{2}$$

In fact, the following proposition was proved in [3].

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Proposition 2. *Any finite system of defining relations π of the form* (1) *satisfies the condition* (2).

We prove the following generalization of Theorem 1.

Theorem 3. *Let A be an alphabet and π be a finite system of defining relations on A defined by*

$$\pi = \{(u_i, u_j) \mid 1 \le i < j \le n\} \cup \{(v_k, v_l) \mid 1 \le k < l \le m\},$$

where $u_i, v_k \in A^*$ and

$$u_i = u_j \to i = j \quad \text{and} \quad v_k = v_l \to k = l \quad \text{for each numbers } i, j, k, l,\ 1 \le i, j \le n,\ 1 \le k, l \le n.$$

Then condition (2) *holds.*

Proof. For each system θ of defining relations on A, the set of all words in θ is called the *set of defining words* and denoted by D_θ:

$$x \in D_\theta \leftrightarrow \exists y \in A^* \ ((x, y) \in \theta \vee (y, x) \in \theta).$$

Let

$$\sigma = \{(u_i, u_j) \mid 1 \le i < j \le n\}, \quad \rho = \{(v_k, v_l) \mid 1 \le k < l \le m\}.$$

Let the symbol $\overline{D}_\sigma$ denote the closure of D_σ by $\overset{\rho}{=}$ and let the symbol $\overline{D}_\rho$ denote the closure of D_ρ by $\overset{\sigma}{=}$, i.e.,

$$\overline{D}_\sigma = \bigcup_{w \in D_\sigma} [w]_\rho, \quad \overline{D}_\rho = \bigcup_{w \in D_\rho} [w]_\sigma.$$

Let

$$\tau = (\overline{D}_\sigma \times \overline{D}_\sigma \cup \overline{D}_\rho \times \overline{D}_\rho) \setminus \{(u, u) \mid u \in A^*\}.$$

It is clear that the relations $\overset{\pi}{=}$ and $\overset{\tau}{=}$ on the set A^* coincide. We write $x \sim y$ instead of $x, y \in D_\tau$.

The following proposition follows from Proposition 2.

Proposition 4. *Condition* (2) *holds if one of the following conditions is valid*:

(1) $\operatorname{card}(\overline{D}_\sigma) = \aleph_0$ *or* $\operatorname{card}(\overline{D}_\rho) = \aleph_0$;

(2) *$\overline{D}_\sigma$ is not closed by $\overset{\sigma}{=}$ or $\overline{D}_\rho$ is not closed by $\overset{\rho}{=}$*;

(3) *$D_\sigma = \emptyset$ or $D_\rho = \emptyset$ or $\overline{D}_\sigma \cap \overline{D}_\rho \ne \emptyset$.*

Proof. Assume that $w \in SF(A, \pi)$.

Assume that the set $\overline{D}_\sigma$ is infinite. Then there is a word $d \in D_\sigma$ such that

$$\operatorname{card}([d]_\rho) = \aleph_0.$$

If

$$[w]_\pi \cap A^* D_\sigma A^* \ne \emptyset,$$

then there is a word $w' \in [w]_\pi \cap A^* d A^*$. Obviously,

$$\operatorname{card}([w']_\rho) = \aleph_0,$$

so, $w' \notin SF(A, \rho)$ and now $w \notin SF(A, \pi)$; a contradiction.

If, conversely,

$$[w]_\pi \cap A^* D_\sigma A^* = \emptyset,$$

then $[w]_\pi = [w]_\rho$ and

$$\operatorname{card}([w]_\pi) < \aleph_0.$$

The symmetric case

$$\operatorname{card}(D_\rho) = \aleph_0$$

is examined similarly.

Now let the set $\overline{D}_\sigma$ is not closed by $\stackrel{\sigma}{=}$. Then there are words $r, s \in A^*$ $(rs \neq \varepsilon)$ and a word $u \in D_\sigma$ for which $rus \in \overline{D}_\sigma$. So,

$$u \stackrel{\pi}{=} rus \stackrel{\pi}{=} rruss \notin SF(A)$$

and, therefore,

$$A^* D_\sigma A^* \cap SF(A, \pi) = \emptyset.$$

Then $[w]_\pi = [w]_\rho$ and hence

$$\operatorname{card}\left([w]_\pi\right) < \aleph_0.$$

The symmetric case is is examined similarly.

Further, assume that condition 3 is valid. It suffices to verify the proposition in the assumption

$$\operatorname{card}\left(\overline{D}_\sigma\right), \operatorname{card}\left(\overline{D}_\rho\right) < \aleph_0.$$

Then it suffices to note that the relation $\stackrel{\pi}{=}$ coincides with the relation $\stackrel{\pi'}{=}$ for some system π' of the form (1). □

In the sequel, we assume that sets $\overline{D}_\sigma$ and $\overline{D}_\rho$ are finite, nonempty, closed by $\stackrel{\sigma}{=}$ and $\stackrel{\rho}{=}$, respectively, and that their intersection is empty. In particular, the set D_τ does not contain the empty word.

Definition 1. A finite sequence of nonempty words $x_1, x_2, \ldots, x_n$, $n \geq 1$, is called a *linear decomposition* of a word $x = x_1 x_2 \ldots x_n$ if and only if there are words p_i, q_i, u_i, and v_i, $1 \leq i \leq n$, such that the following conditions holds:

(1) $p_i x_i q_i \sim q_{i-1} u_i$ for all numbers i, $1 < i \leq n$;
(2) $p_i x_i q_i \sim v_i p_{i+1}$ for all numbers i, $1 \leq i < n$;
(3) $q_i = \varepsilon \vee p_{i+1} = \varepsilon$ for all numbers i, $1 \leq i < n$;
(4) $p_1 = q_n = \varepsilon$;
(5) $n = 1 \rightarrow x_1 \in D_\tau$.

For each number i, $1 \leq i \leq n$, the words p_i and q_i are called respectively the ith left and ith right *supplementary members* of the linear decomposition $x_1, x_2, \ldots, x_n$ of the word x. Words having linear decomposition with at most n members are called *linear decomposable of order* n. The set of all linear decomposable words of order n is denoted by $\operatorname{Lin}(n)$. Let

$$\operatorname{Lin} = \bigcup_{n<\omega} \operatorname{Lin}(n).$$

Elements of the set Lin are called *linear decomposable words*.

We show that the relation $\stackrel{\pi}{=}$ preserves the structure of occurrences of linear decomposable words in a word $w \in SF(A, \pi)$.

Proposition 5. *Let a finite sequence $x_1, x_2, \ldots, x_n$, $n \geq 1$, be a linear decomposition of a word x, and let words p_i and q_i, $1 \leq i \leq n$, be the ith left and ith right supplementary members of the linear decomposition. Then for each number i, $1 \leq i \leq n$,*

(1) *$x_1 x_2 \ldots x_{i-1} \stackrel{\pi}{=} f p_i$ and $x_{i+1} x_{i+2} \ldots x_n \stackrel{\pi}{=} q_i g$ for some words f and g;*
(2) *the sequence of words*

$$x_1,\ x_2,\ \ldots,\ x_i q_i$$

is a linear decomposition of the word $x_1 x_2 \ldots x_i q_i$ and the sequence of words

$$p_i x_i,\ x_{i+1},\ \ldots,\ x_n$$

is a linear decomposition of the word $p_i x_i x_{i+1} \ldots x_n$;

(3) *the sequence*

$$x_1,\ x_2,\ \ldots,\ x_{i-1},\ x',\ x_{i+1},\ \ldots,\ x_n$$

is a linear decomposition of the word $x_1x_2\ldots x_{i-1}x'x_{i+1}\ldots x_n$ *if* $x_i \stackrel{\pi}{=} x'$ *for some word* x';

(4) *let also the finite sequence of words*

$$y_1,\ y_2,\ \ldots,\ y_m, \quad m \geq 1,$$

be a linear decomposition of a word y. *Let* $x_n = x'e$ *and* $y_1 = ey'$ *for some words* e, x', *and* y'. *Let*

$$z = x_1x_2\ldots x_{n-1}x'ey'y_2\ldots y_{m-1}y_m.$$

Then

$$z \in \mathrm{Lin}\left(n + m + \mathrm{sign}(|x'y'|) - 1\right)$$

and the following sequence is a linear decomposition of z:

$$x_1,\ x_2,\ \ldots,\ x_{n-1},\ x',\ y_1,\ \ldots,\ y_{m-1},\ y_m$$

if $x' \neq \varepsilon$;

$$x_1,\ x_2,\ \ldots,\ x_n,\ y',\ y_2,\ \ldots,\ y_{m-1},\ y_m$$

if $y' \neq \varepsilon$;

$$x_1,\ x_2,\ \ldots,\ x_{n-1},\ e,\ y_2,\ \ldots,\ y_{m-1},\ y_m$$

if $x' = y' = \varepsilon$;

(5) *for each natural* n, *the set* $\mathrm{Lin}(n)$ *is finite.*

The proof is immediate.

Proposition 6. *Let a finite sequence of words*

$$x_1,\ x_2,\ \ldots,\ x_n, \quad n \geq 1,$$

be a linear decomposition of a word $x \in SF(A,\pi)$ *and the words* p_i *and* q_i, $1 \leq i \leq n$, *be the ith left and ith right supplementary members of the linear decomposition. Then*

$$\neg(p_ix_iq_i \sim p_{i+1}x_{i+1}q_{i+1})$$

for each number i, $1 \leq i < n$.

Proof. Assume the contrary. By Proposition 5(1), for some words f and g we have

$$x \stackrel{\pi}{=} fp_ix_ix_{i+1}q_{i+1}g.$$

If $p_{i+1} \neq \varepsilon$, then $q_i = \varepsilon$ and

$$x \stackrel{\pi}{=} fp_{i+1}\underbrace{x_{i+1}q_{i+1}x_{i+1}q_{i+1}}g \notin SF(A,\pi).$$

Otherwise,

$$x \stackrel{\pi}{=} f\underbrace{p_ix_ip_ix_i}q_{i+1}g \notin SF(A,\pi),$$

a contradiction. □

Proposition 7. *Let a finite sequence of words*

$$x_1,\ x_2,\ \ldots,\ x_n, \quad n \geq 1,$$

be a linear decomposition of a word x *and words* p_i *and* q_i, $1 \leq i \leq n$ *be the ith left and ith right supplementary member of the linear decomposition.*

(1) *Let also for some number i, $1 \le i \le n$, and some words x', x'', u, s, r, h, and y, the following conditions hold*:

$$x_i = x'x'' \;\&\; x'' \neq \varepsilon,$$
$$xr = x_1x_2\dots x_{i-1}x'us = ys \in SF(A,\pi),$$
$$u = x''h \in D_\tau.$$

Then there is a linear decomposition $y_1, y_2, \dots, y_m$ of the word y for which

$$\big(y_m = u\big) \;\&\; \big(m \le i+1\big) \;\&\; \big(u \sim p_ix_iq_i \to m \le i\big);$$

(2) *Let, otherwise, for some number i, $1 \le i \le n$, and some words x', x'', u, s, r, h, and y, the following conditions hold*:

$$x_i = x'x'' \;\&\; x' \neq \varepsilon,$$
$$rx = sux''x_{i+1}\dots x_{n-1}x_n = sy \in SF(A,\pi),$$
$$u = hx' \in D_\tau.$$

Then there is a linear decomposition $y_1, y_2, \dots, y_m$ of the word y for which

$$\big(y_1 = u\big) \;\&\; \big(m \le n-i+2\big) \;\&\; \big(u \sim p_ix_iq_i \to m \le n-i+1\big).$$

Proof. Let all conditions of item (1) hold. If $u \sim p_ix_iq_i$, then by Proposition 5(1) for some word f we have

$$ys = x_1x_2\dots x_{i-1}x'us \overset{\pi}{=} fp_ix'us \overset{\pi}{=} f\,\underbrace{p_ix'p_ix'}\,x''q_is,$$

and hence the word p_ix' is empty. Then by Proposition 5(2) we conclude have that the sequence

$$x_1,\ x_2,\ \dots,\ x_{i-1},\ x_iq_i$$

is a linear decomposition of the word $x_1x_2\dots x_{i-1}x_iq_i$. Next, from $u \sim p_ix_iq_i = x_iq_i$ by Proposition 5(3) we obtain that the sequence

$$x_1,\ x_2,\ \dots,\ x_{i-1},\ u$$

is a linear decomposition of the word $y = x_1x_2\dots x_{i-1}u$.

Now let $\neg(u \sim p_ix_iq_i)$. If the word q_i is empty, then by Proposition 5(2) the sequence $x_1, x_2, \dots, x_i$ is a linear decomposition of the word $x_1x_2\dots x_i$. Next, since $x''h \in \mathrm{Lin}(1)$, the sequence

$$x_1,\ x_2,\ \dots,\ x_{i-1},\ x',\ u$$

or the sequence

$$x_1,\ x_2,\ \dots,\ x_{i-1},\ u$$

(if $x' = \varepsilon$) is a linear decomposition of the the word $y = x_1x_2\dots x_{i-1}x'u$ by Proposition 5(4).

If $q_i \neq \varepsilon$, then $i < n \;\&\; p_{i+1} = \varepsilon$ by Definition 1. Let us show that this leads to a contradiction. Indeed, by Proposition 6, in this case we have $u \sim x_{i+1}q_{i+1}$, which implies, by Proposition 5(1), the following chain of relations:

$$ys \overset{\pi}{=} x_1x_2\dots x_ix_{i+1}q_{i+1}gr \overset{\pi}{=} x_1x_2\dots x_iugr \overset{\pi}{=} x_1x_2\dots x'\,\underbrace{x''x''}\,hgr \notin SF(A,\pi)$$

for some word g.

Item 2 of the proposition can be proved similarly. □

Proposition 8. *Let $x = yuz \in SF(A,\pi) \cap \mathrm{Lin}(n)$ and $u \sim v$ for some words x, y, z, and u. Then $yvz \in Lin(n)$.*

Proof. Let us consider a linear decomposition

$$x_1,\ x_2,\ \ldots,\ x_n$$

of the word x. Then there are numbers i and j, $1 \le i \le j \le n$, and words x_i', x_i'', x_j', and x_j'' such that

$$x_i = x_i'x_i'' \ \&\ x_i'' \ne \varepsilon \ \&\ y = x_1 \ldots x_{i-1}x_i' \ \&\ x_j = x_j'x_j'' \ \&\ x_j' \ne \varepsilon \ \&\ z = x_j''x_{j+1} \ldots x_n.$$

If $i = j$, then the proposition immediately follows from Proposition 5(3). Otherwise, by Proposition 7, there exists a linear decomposition

$$y_1,\ y_2,\ \ldots,\ y_{m_1-1},\ y_{m_1}$$

of the word yu for which $y_{m_1} = u$ and $m_1 \le i + 1$, and also there exists a linear decomposition

$$z_1,\ z_2,\ z_3,\ \ldots,\ z_{m_2}$$

of the word uz for which $z_1 = u$ and $m_2 \le n - j + 2$. Then by Proposition 5(3), the sequence

$$y_1,\ y_2,\ \ldots,\ y_{m_1-1},\ v$$

is a linear decomposition of the word $y_1y_2 \ldots y_{m_1-1}v$ and the sequence

$$v,\ z_2,\ z_3,\ \ldots,\ z_{m_2}$$

is a linear decomposition of the word $vz_2z_3 \ldots z_{m_2}$. Then

$$yvz = y_1y_2 \ldots y_{m_1-1}vz_2z_3 \ldots z_{m_2} \in \mathrm{Lin}(m_1 + m_2 - 1)$$

by Proposition 5(4). Obviously, if $j \ge i + 2$, this yields $yvz \in \mathrm{Lin}(n)$. Let $j = i + 1$. Then

$$u \sim p_ix_iq_i \ \vee\ u \sim p_jx_jq_j$$

by Proposition 6 and hence Proposition 7 implies one of the inequalitis $m_1 \le i$ or $m_2 \le n - i$. □

Proposition 9. *Let $xe \in \mathrm{Lin}$, $ey \in \mathrm{Lin}$, and $xey \in SF(A, \pi)$ for some words x, y, and e. Then $xey \in \mathrm{Lin}$.*

Proof. If e is the empty word, the proposition is a particular case of Proposition 5(4). If $e \ne \varepsilon$, let us consider a linear decomposition

$$y_1,\ y_2,\ \ldots,\ y_m$$

of ey. Then for some number i, $1 \le i \le m$, for some words y', y'', and e', we have

$$e = y_1y_2 \ldots y_{i-1}y' \ \&\ y_i = y'y'' \ \&\ y = y''y_{i+1}y_{i+2} \ldots y_m \ \&\ y' \ne \varepsilon.$$

By Proposition 5(1), there exist words f and g such that

$$y_1y_2 \ldots y_{i-1} \overset{\pi}{=} fp_i \quad \text{and} \quad y_{i+1}y_{i+2} \ldots y_m \overset{\pi}{=} q_ig,$$

where p_i and q_i are respectively the ith left and ith right supplementary members of the linear decomposition. Note that the words $p_iy_iy_{i+1} \ldots y_m$ and $xfp_iy_iq_ig$ (and all their subwords) are square-free relative to π.

Let us show that $xfp_iy_iq_i \in \mathrm{Lin}$, and some linear decomposition of the word $xfp_iy_iq_i$ has the last member $p_iy_iq_i$.

The word $xfp_iy' \overset{\pi}{=} xe$ is in Lin by Proposition 8. Consider a linear decomposition $x_1, x_2, \ldots, x_n$ of the word xfp_iy'. Since $p_iy' \ne \varepsilon$, there is a number j, $1 \le j \le n$, such that

$$x_j = x'x'' \ \&\ x'' \ne \varepsilon,$$

$$xfp_iy'y''q_i = x_1x_2 \ldots x_{j-1}x'p_ix_iq_i.$$

Also, $p_ix_iq_i \in D_\tau$ and, as was noted,

$$xfp_iy_iq_i \in SF(A, \pi).$$

Then the fact follows immediately from Proposition 7.

Let us consider the word $p_i y_i y_{i+1} \dots y_m$. It is linearly decomposable by Proposition 5(2) and, as was noted, square-free relative to π. So, by Proposition 8, the set Lin contains the word

$$p_i y_i q_i g \overset{\pi}{=} p_i y_i y_{i+1} \dots y_m,$$

and the word $p_i y_i q_i$ is the first member of some linear decomposition of the word $p_i y_i q_i g$ (this fact follows, for example, from Proposition 7).

Then $x f p_i y_i q_i g \in \mathrm{Lin}$ by Proposition 5(4). Obviously,

$$x f p_i y_i q_i g \overset{\pi}{=} x e y.$$

Then $xey \in \mathrm{Lin}$ by Proposition 8. □

In what follows, it is convenient to use the *occurrence* concept (see [1]).

An occurrence of a word $e \in A^*$ in a word $x = peq \in A^*$ is the word $p * e * q$ of the alphabet $A \cup \{*\}$, where the symbol $*$ does not belong to A. Let $x = peq = rds$ for some words q, p, e, q, r, d, and s in the alphabet A. Let us say that the occurrence $p * e * q$ *is contained* in the occurrence $r * d * s$ if $|r| \le |p|$ and $|s| \le |q|$. Occurrences $p * e * q$ and $r * d * s$ of words e and d, respectively, in the word $x = peq = rds$ are said *to intersect* if there exists an occurrence $v * f * w$ of a nonempty word f in the word $x = vfs$, which is contained in the occurrence $p * e * q$ and the occurrence $r * d * s$.

Definition 2. An occurrence φ of a word $e \in \mathrm{Lin}$ in a word $w \in A^*$ is said to be *maximal* if it is not contained in other (different) occurrence of a linearly decomposable words in the words x. For each natural number $n \ge 1$, the set of all maximal occurrences φ of different words $e \in \mathrm{Lin}(n)$ in the word w will be denoted by $\mathrm{MaxLin}(n, w)$.

Proposition 9 implies the following.

Proposition 10. *The different maximal occurrences of a linearly decomposable words in a word $w \in SF(A, \pi)$ do not intersect.*

Let $w \in SF(A, \pi)$. It follows from Proposition 10 that w can be uniquely represented in the form

$$w = r_1 x_1 r_2 x_2 \dots r_n x_n r_{n+1},$$

where for each i, $1 \le i \le n$, the word r_i does not contain subwords from D_π, and each occurrences $r_1 x_1 r_2 x_2 \dots r_i * x_i * r_{i+1} \dots r_n x_n r_{n+1}$ is in $\mathrm{MaxLin}(n_i, w)$ for some natural numbers $n_i \ge 1$ $(1 \le i \le n)$. Considering naturals numbers n_i, $1 \le i \le n$, as fixed, let us define the set T_w by

$$T_w = \Big\{ r_1 y_1 r_2 y_2 \dots r_n y_n r_{n+1} \;\Big|\; y_i \in \mathrm{Lin}(n_i) \;\&\; r_1 y_1 r_2 y_2 \dots r_i * y_i * r_{i+1} \dots r_n y_n r_{n+1} \in \mathrm{MaxLin}\big(n_i, r_1 y_1 r_2 y_2 \dots r_n y_n r_{n+1}\big),\; 1 \le i \le n \Big\}.$$

By Proposition 5(5), this set is finite. Thus, to prove the theorem, it suffices to prove the following implication:

$$\forall w \in A^* \; w \in SF(A, \pi) \to [w]_\pi \subseteq T_w.$$

Let the word

$$rus = r_1 y_1 r_2 y_2 \dots r_n y_n r_{n+1}$$

is in $T_w \cap SF(A, \pi)$ and for each number i, $1 \le i \le n$, the occurrence

$$r_1 y_1 r_2 y_2 \dots r_i * y_i * r_{i+1} \dots r_n y_n r_{n+1}$$

is in $\mathrm{MaxLin}(n_i, rus)$. Let $u \sim v$. Let us prove that rvs is in T_w.

By Proposition 10, the occurrence $r * u * s$ is contained in some occurrence $r_1 y_1 r_2 y_2 \dots r_i * y_i * r_{i+1} \dots r_n y_n r_{n+1}$, i.e., there are words r' and s' such that

$$r = r_1 y_1 r_2 y_2 \dots r_i r'$$

and

$$y_i = r' u s'.$$

Then $r' v s' \in \mathrm{Lin}(n_i)$ by Proposition 8. Assume that the occurrence

$$r_1 y_1 r_2 y_2 \dots r_i * r' v s' * r_{i+1} \dots r_n y_n r_{n+1}$$

are not maximal. This means that for some words a, b, c, and d, we have

$$r_1 y_1 r_2 y_2 \dots r_i = ab, \quad r_{i+1} y_{i+1} \dots r_n y_n r_{n+1} = cd,$$

$br' v s' c \in \mathrm{Lin}$, and $bc \neq \varepsilon$. Then $br' u s' c \in \mathrm{Lin}$ by Proposition 8 and hence

$$r_1 y_1 r_2 y_2 \dots r_i * r' u s' * r_{i+1} \dots r_n y_n r_{n+1} \notin \mathrm{MaxLin}(n_i, rus),$$

a contradiction. □

Corollary 11. *The square-free word problem relative to a system of two defining relations is decidable.*

Concluding remarks and questions. In the general case, the square-free word problem relative to a system of defining relations π ($\mathrm{card}(\pi) < \aleph_0$) does not decidable (A. Carpi). What is the minimal cardinality of the system of defining relations π? What is the minimal cardinality of a system of defining relations π, such that condition (2) does not hold?

It is interesting to compare the square-free word problem with the *word problem* relative to a system of defining relations. The word problem relative to a system of defining relationsπ is decidable if and only if the relation $\overset{\pi}{=}$ is recursive. It is unknown whether there exist a system of defining relations π, $\mathrm{card}(\pi) < 3$, with an undecidable word problem.

REFERENCES

1. S. I. Adian, *The Burnside Problem and Identities in Groups* [in Russian], Nauka, Moscow (1975).
2. D. R. Bean, A. Ehrenfeucht, and G. F. McNulty, "Avoidable patterns in strings of symbols," *Pac. J. Math.*, **85**, No. 2 (1979).
3. A. Carpi and A. de Luca, "Non-repetitive words relative to a rewriting system," *Theor. Comput. Sci.*, **72**, No. 1, 39–53 (1990).
4. A. Thue, "Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen," *Norske Vid. Selsk. Skr. I, Mat. Nat. Kl. Christiana*, **1**, 1–67 (1912).

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