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MATHEMATICS

On a Generalization of Arrow's Impossibility Theorem

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A complete classification of symmetric classes of choice functions on r-element subsets of an arbitrary finite set possessing the Arrow property is obtained. This result strengthens Shelah's theorem on the Arrow property and is a generalization of Arrow's impossibility theorem.

Arrow's impossibility theorem (see [1]) is a rigorous mathematical result or, more precisely, a collection of similar results concerning social choice theory (see [2]). The available versions of Arrow's impossibility theorem differ in nonessential details. The following formulation was given in [3].

Theorem 1 (Arrow's impossibility theorem)**.** *Given a finite set A with $|A| \geq 3$, every aggregation rule for complete preorders on A that satisfies the independence of irrelevant alternatives and unanimity is a dictatorship.*

Here, we use the following system of concepts. Any complete preorder $\pi \subseteq A \times A$ (i.e., any transitive binary relation π on A) that satisfies the condition $\pi \cup \pi^{-1} = A \times A$ is called a preference relation on A. Let $n = \{0, 1, \dots, n-1\}$, $n \geq 1$, be a fixed set, whose elements are called individuals. An arbitrary n-tuple of preference relations is called a profile (of the individuals). Given a profile $\Pi = (\pi_0, \pi_1, \dots, \pi_{n-1})$, each constituent relation π_i, $i < n$, is called an individual preference relation (of individual i), while the relation $\pi_i \backslash \pi_i^{-1}$ is called a strict individual preference relation (of individual i). An aggregation rule (for complete preorders on A; other aggregation rules are not considered in [3]) is a function from the set of all profiles to the set of all preference relations on A. The value π of an aggregation rule on a profile Π is called a social preference relation (for Π), and the relation $\pi \backslash \pi^{-1}$ is called a strict social preference relation (for Π).

An aggregation rule f is said to satisfy

(i) the independence of irrelevant alternatives if the social relative ranking of any two alternatives depends only on their relative ranking by every individual, in other words, if for any profiles $\Pi = (\pi_0, \pi_1, \dots, \pi_{n-1})$ and $\Pi' = (\pi'_0, \pi'_1, \dots, \pi'_{n-1})$ and a pair $(a, b) \in A \times A$,

$$((\forall i < n)\ (a, b) \in \pi_i \leftrightarrow (a, b) \in \pi'_i)$$
$$\to ((a, b) \in f(\Pi) \leftrightarrow (a, b) \in f(\Pi'));$$

(ii) unanimity if each pair of alternatives belonging to each strict individual preference relation also belongs to the strict social preference relation, i.e., if for any profile $\Pi = (\pi_0, \pi_1, \dots, \pi_{n-1})$ and a pair $(a, b) \in A \times A$, it holds that

$$((\forall i < n)(a, b) \in \pi_i \backslash \pi_i^{-1}) \to (a, b) \in f(\Pi) \backslash f(\Pi)^{-1}.$$

An aggregation rule f is said to be dictatorial (dictatorship) if each pair of alternatives belongs to the strict social preference relation as soon as it belongs to the strict individual preference relation of some fixed individual (dictator), i.e., if there exists an index $i < n$ such that, for any profile $\Pi = (\pi_0, \pi_1, \dots, \pi_{n-1})$ and a pair $(a, b) \in A \times A$, it holds that

$$(a, b) \in \pi_i \backslash \pi_i^{-1} \to (a, b) \in f(\Pi) \backslash f(\Pi)^{-1}.$$

Aggregation rules for complete preorders are frequently replaced by aggregation rules for strict linear orders defined as functions f: $(\mathrm{Ord}(A))^n \to \mathrm{Ord}(A)$, where $\mathrm{Ord}(A)$ is the set of all strict linear orders on A (see [1, 4]). In this case, the independence on irrelevant alternatives, unanimity, and a dictatorial rule are defined in a similar manner (with natural simplifications using the equality $\pi \cap \pi^{-1} = \emptyset$ for any strict linear order π).

It is easy to show (see the strict neutrality lemma in [3]) that each aggregation rule satisfying the assumptions of Theorem 1 preserves strict linear orders in the sense that a binary relation $f(\pi_0, \pi_1, \dots, \pi_{n-1})$ is a strict linear order for any strict linear orders $\pi_0, \pi_1, \dots, \pi_{n-1}$. Therefore, Theorem 1 can easily be derived from the following proposition, which is known as a version of Arrow's impossibility theorem (see [4]).

Proposition 1. *Given a finite set A with $|A| \geq 3$, every aggregation rule for strict linear orders on A that satisfies*

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the independence of irrelevant alternatives and unanimity is a dictatorship.

This proposition can be formulated as an equivalent assertion in terms of choice functions (see [2]), which makes it possible to consider a natural generalization of the used concepts and obtain a result similar to Proposition 1 in a much more general situation (see [5]).

We need the following system of definitions.

Suppose that we are given a set A (of alternatives) and a positive integer r. The set of all r-element subsets of A is denoted by $[A]^r$, and the set of all choice functions defined on $[A]^r$ is denoted by the symbol $\mathfrak{C}_r(A)$; i.e.,

$$[A]^r = \{B \subseteq A: |B| = r\} \text{ and}$$

$$\mathfrak{C}_r(A) = \left\{\mathfrak{c} \in {}^{[A]^r}A: (\forall p \in [A]^r)\mathfrak{c}(p) \in p\right\}.$$

A function $\mathfrak{c} \in \mathfrak{C}_r(A)$ is said to be rational if it associates every set $p \in [A]^r$ with the maximal element of p for some linear ordering of the set A. The set of all rational functions $\mathfrak{c} \in \mathfrak{C}_r(A)$ is denote by $\mathfrak{R}_r(A)$.

Let $n \geq 1$ be a positive integer. Without fear of falling into confusion, any function f: $(\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$ is referred to as (n-place) aggregation rule. The set of all n-place aggregation rules is designated as $\mathbb{O}(A, r)_{[n]}$. Let $\mathbb{O}(A, r) = \bigcup\limits_{1 \leq n < \omega} \mathbb{O}(A, r)_{[n]}$. An aggregation rule $f \in \mathbb{O}(A, r)_{[n]}$ is said to preserve a set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ if

$$f(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1}) \in \mathfrak{D} \text{ for all}$$

$$\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1} \in \mathfrak{D}.$$

The set of all aggregation rules $f \in \mathbb{O}(A, r)$ that preserve a set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ is denoted by the symbol $\operatorname{pol}\mathfrak{D}$.

An aggregation rule $f \in \mathbb{O}(A, r)_{[n]}$ is called normal if, for every set $p \in [A]^r$, there is a function f_p: $p^n \to p$ such that

(i) $f(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1})(p) = f_p(\mathfrak{c}_0(p), \mathfrak{c}_1(p), \ldots, \mathfrak{c}_{n-1}(p))$ for all $\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1} \in \mathfrak{C}_r(A)$;

(ii) $\bigvee\limits_{i<n} f_p(a_0, a_1, \ldots, a_{n-1}) = a_i$ for all $a_0, a_1, \ldots, a_{n-1} \in p$.

The set of all normal aggregation rules $f \in \mathbb{O}(A, r)$ is designated as $\mathcal{N}(A, r)$.

An aggregation rule $f \in \mathbb{O}(A, r)_{[n]}$ is said to be dictatorial if it is a projection, i.e., if

$$\bigvee_{i<n}(\forall \mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1} \in \mathfrak{C}_r(A))f(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1}) = \mathfrak{c}_i.$$

The set of all dictatorial aggregation rules $f \in \mathbb{O}(A, r)$ is denoted by $\mathcal{M}(A, r)$.

A set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ is said to have the Arrow property if

$$\operatorname{pol}\mathfrak{D} \cap \mathcal{N}(A, r) = \mathcal{M}(A, r).$$

It is easy to show that Proposition 1 is equivalent to the following result.

Proposition 2. *The set $\mathfrak{R}_2(A)$ has the Arrow property for any finite set A with $|A| \geq 3$.*

In [5] this proposition was extended to any symmetric set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ under certain additional conditions.

A set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ is said to be symmetric if for any function $\mathfrak{c} \in \mathfrak{D}$ and a permutation σ of the set A, the function $\mathfrak{c}_\sigma$ defined as

$$\mathfrak{c}_\sigma(p) = \sigma^{-1}\mathfrak{c}(\sigma p) \text{ for all } p \in [A]^r$$

belongs to $\mathfrak{D}$.

The following result was proved in [5].

Theorem 2 (Shelah's theorem on Arrow's property). *There exist numbers r_1^* and r_2^* (which can be set to $r_1^* = r_2^* = 7$) such that, for every finite set A and a positive integer r satisfying the inequalities $r_1^* \leq r \leq |A| - r_2^*$, any symmetric nonempty proper subset $\mathfrak{D}$ of the set $\mathfrak{C}_r(A)$ has the Arrow property.*

We refine this result and give a complete description of symmetric sets $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ having the Arrow property. Specifically, for $|A| \geq 5$, Theorem 2 holds if $r_1^* = 3$ and $r_2^* = 0$. To describe the cases $r = 2$ and $r = 3 \wedge |A| = 4$, we define the sets $\mathfrak{C}_3^K(A)$, $\mathfrak{C}_2^0(A)$, and $\mathfrak{C}_2^1(A)$.

Let $|A| = 4$. Denote by K the Klein four-group of permutations of A. For every pair of sets $p, q \in [A]^3$, there is a unique permutation $\sigma_{p,q} \in K$ such that

$$q = \sigma_{p,q}p.$$

Let $\mathfrak{C}_3^K(A)$ denote the set of all functions $\in \mathfrak{C}_3(A)$ for which

$$\mathfrak{c}(q) = \sigma_{p,q}\mathfrak{c}(p) \text{ for all } p, q \in [A]^3.$$

It is easy to see that the set $\mathfrak{C}_3^K(A)$ is symmetric, nonempty, and does not coincide with the entire set $\mathfrak{C}_3(A)$.

Now let A be an arbitrary finite set. For every function $\mathfrak{c} \in \mathfrak{C}_2(A)$, every element $a \in A$, and every index $i \in \{0, 1\}$, define

$$Z_a^{\mathfrak{c}} = \{b \in A \backslash \{a\}: \mathfrak{c}(\{a, b\}) = a\},$$

$$W_i^{\mathfrak{c}} = \{a \in A: |Z_a^{\mathfrak{c}}| = i \pmod 2\},$$

$$\mathfrak{C}_2^i(A) = \{\mathfrak{c} \in \mathfrak{C}_2(A): W_{(1-i)}^{\mathfrak{c}} = \phi\}.$$

Remark. Either of the sets $\mathfrak{C}_2^0(A)$ and $\mathfrak{C}_2^1(A)$ can also be defined as follows. Every function $\mathfrak{c} \in \mathfrak{C}_2(A)$ can be put in a one-to-one correspondence with a complete directed graph (tournament) $\Gamma_{\mathfrak{c}} = (A, E)$, where $E = \{(a, b) \in A \times A: a \neq b \wedge \mathfrak{c}(\{a, b\}) = b\}$. The function $\mathfrak{c} \in \mathfrak{C}_2(A)$ belongs to the set $\mathfrak{C}_2^0(A)$ (to the set $\mathfrak{C}_2^1(A)$) if and only if each vertex of $\Gamma_{\mathfrak{c}}$ has an even (respectively, odd) indegree.

The result below is easy to prove.

Proposition 3. *Let A be a given finite set. Then the following assertions hold:*

(i) *The sets* $\mathfrak{C}_2^0(A)$, $\mathfrak{C}_2^1(A)$, *and* $\mathfrak{C}_2^0(A) \cup \mathfrak{C}_2^1(A)$ *are symmetric.*

(ii) $\mathfrak{C}_2^0(A) \neq \phi$ *if and only if* $|A|$ *is equal to* 0 *or* 1 (mod 4).

(iii) $\mathfrak{C}_2^1(A) \neq \phi$ *if and only if* $|A|$ *is equal to* 0 *or* 3 (mod 4).

(iv) $\mathfrak{C}_2^0(A) \cup \mathfrak{C}_2^1(A) \neq \mathfrak{C}_2(A)$.

Now the main result can be stated.

Theorem 3. *For any finite set A and a positive integer r, any nonempty symmetric proper subset* $\mathfrak{D}$ *of the set* $\mathfrak{C}_r(A)$ *does not have the Arrow property if and only if*

(i) $r = 2$, $|A|$ *is equal to* 0 *or* 1 (mod 4), *and* $\mathfrak{D} = \mathfrak{C}_2^0(A)$;

(ii) $r = 2$, $|A|$ *is equal to* 0 *or* 3 (mod 4), *and* $\mathfrak{D} = \mathfrak{C}_2^1(A)$;

(iii) $r = 2$, $|A| = 0$ (mod 4), *and* $\mathfrak{D} = \mathfrak{C}_2^0(A) \cup \mathfrak{C}_2^1(A)$;

(iv) $r = 3$, $|A| = 4$, *and* $\mathfrak{D} = \mathfrak{C}_3^K(A)$.

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